# Calibrating affine stochastic volatility models with jumps
# An asymptotic approach

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Based on joint works with M. Forde, J. Gatheral, M. Keller-Ressel, R. Lee and A. Mijatović,

3rd SMAI European Summer School
in Financial Mathematics, Paris, August 2010

## A short (non) fictitious story

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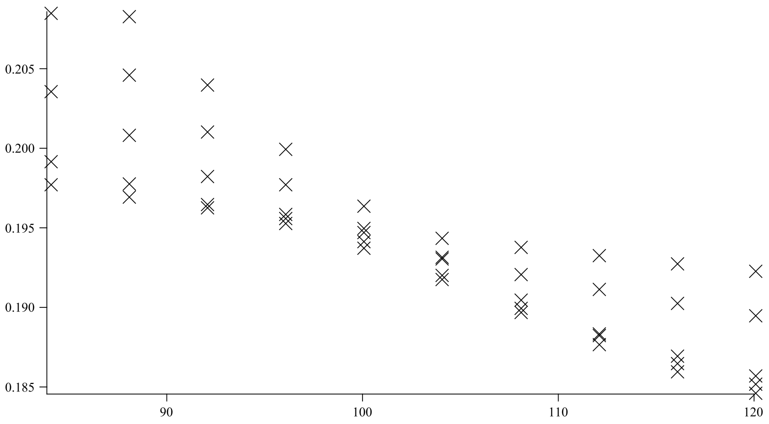


Figure: Market implied volatilities for different strikes and maturities.

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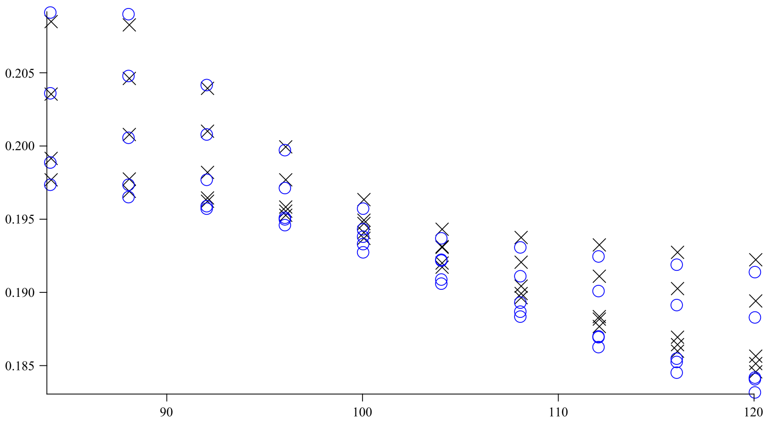


Figure: Sum of squared errors: 4.53061E-05

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Boss: 'Classic mistake!! You should take $a_2$ instead.'

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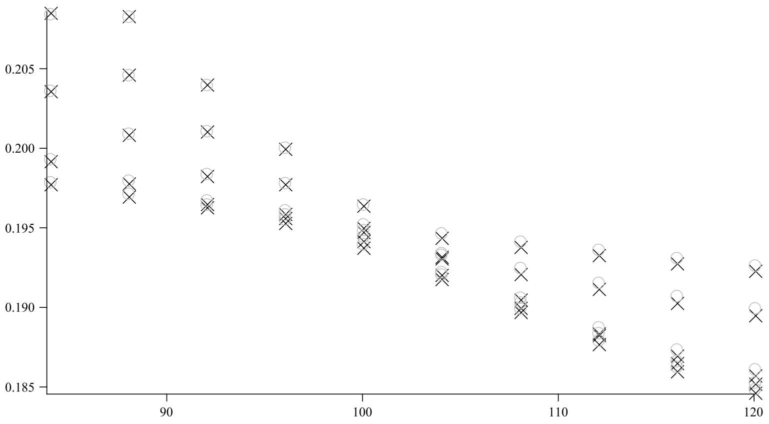


Figure: Sum of squared errors: 2.4856E-06

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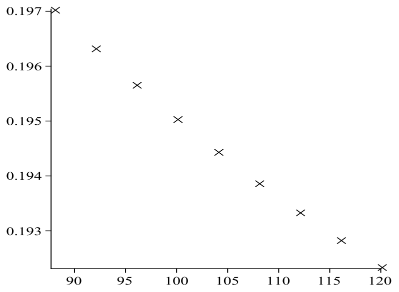
**”Start every day off with a smile and get it over with.” (W.C. Fields)**

# So let us start off with a smile (one maturity slice)



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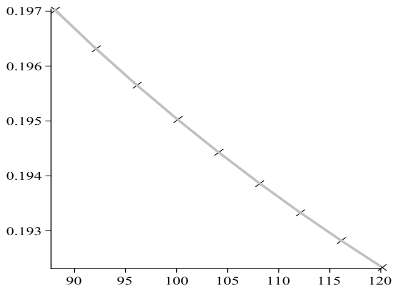


Solid blue: $x \mapsto g(x) := C_{\mathrm{BS}}^{-1}\left(\mathcal{F}^{-1}\Re\left\{f(x,z)\,\phi_a(z)\right\}\right)$

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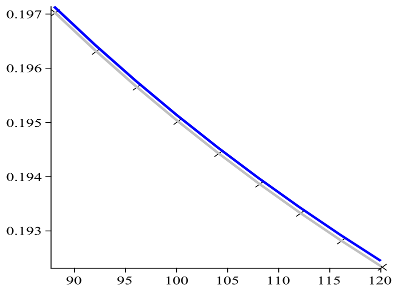


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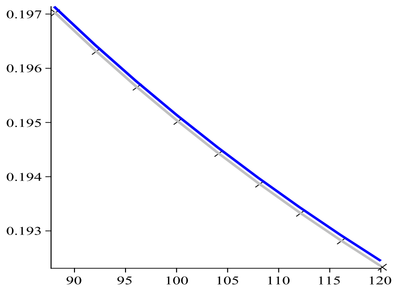


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**Easier to calibrate $\hat{g}$ than $g$.**

# Motivation and goals

- Obtain closed-form formulae for the implied volatility under ASVM in the short and in the large-maturity limits.
- Propose an accurate starting point for calibration purposes.
- Discuss conditions on jumps for a model to be usable in practice.

**Definition:** The implied volatility is the unique parameter $\sigma \geq 0$ such that

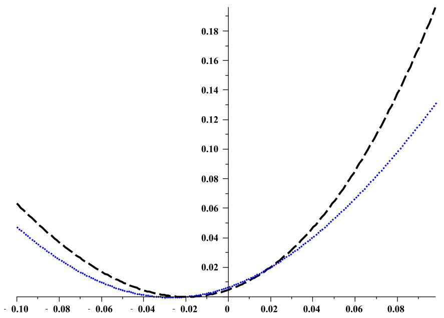
$$C_{\mathrm{BS}}\left(S_0, K, T, \sigma\right) = C_{\mathrm{obs}}\left(S_0, K, T\right).$$

# Large deviations theory

## Lemma

The family of random variables $(Z_t)_{t\geq 1}$ satisfies the large deviations principle (LDP) with the good rate function $\Lambda^*$ if for every Borel measurable set $B$ in $\mathbb{R}$

$$-\inf_{x\in B^o}\Lambda^*(x) \leq \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(Z_t\in B\right) \leq \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(Z_t\in B\right) \leq -\inf_{x\in\overline{B}}\Lambda^*(x),$$



## The Gärtner-Ellis theorem

**Assumption A.1:** For all $u \in \mathbb{R}$, define the limiting cumulant generating function

$$\Lambda(u) := \lim_{t \to \infty} t^{-1} \log \mathbb{E}\left(\mathrm{e}^{utX_t}\right) = \lim_{t \to \infty} t^{-1} \Lambda_t\left(ut\right)$$

as an extended real number. Denote $\mathcal{D}_\Lambda := \{u \in \mathbb{R} : \Lambda(u) < \infty\}$. Assume further that

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(i) the origin belongs to $\mathcal{D}_\Lambda^0$;

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**Theorem (Gärtner-Ellis)** (*special case of the general th. Dembo & Zeitouni*)

Under Assumption A.1, the family of random variables $(X_t)_{t \geq 0}$ satisfies the LDP with rate function $\Lambda^*$, defined as the Fenchel-Legendre transform of $\Lambda$,

$$\Lambda^*(x) := \sup_{u \in \mathbb{R}} \{ux - \Lambda(u)\}, \quad \text{for all } x \in \mathbb{R}.$$

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- Translate the tail behaviour of $X$ into an asymptotic behaviour of Call prices.
- Translate these Call price asymptotics into implied volatility asymptotics.

# Option price and Share measure

Define the **Share** measure $\widetilde{\mathbb{P}}$ by $\widetilde{\mathbb{P}}(A) := \mathbb{E}\left(\left(X_t - X_0\right)\mathbb{1}_A\right)$.
A European call option price reads

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{X_t}-\mathrm{e}^{x}\right)_+ &= \mathbb{E}\left(\left(\mathrm{e}^{X_t}-\mathrm{e}^{x}\right)\mathbb{1}_{X_t\geq x}\right)\\ &= \mathbb{E}\left(\mathrm{e}^{X_t}\mathbb{1}_{X_t\geq x}\right)-\mathrm{e}^{x}\mathbb{P}\left(X_t\geq x\right)\\ &= \widetilde{\mathbb{P}}\left(\mathbf{X_t}\geq\mathbf{x}\right)-\mathrm{e}^{x}\mathbb{P}\left(\mathbf{X_t}\geq\mathbf{x}\right).\end{aligned}$$

Denote $\widetilde{\Lambda}$ and $\widetilde{\Lambda}^*$ the corresponding limiting cgf and Fenchel-Legendre transform under $\widetilde{\mathbb{P}}$. They satisfy the following relations:

$$\widetilde{\boldsymbol{\Lambda}}(\mathbf{u}) = \boldsymbol{\Lambda}(\mathbf{u}+\mathbf{1}), \quad \text{if } (1+u)\in\mathcal{D}_\Lambda, \quad \text{and} \quad \widetilde{\boldsymbol{\Lambda}}^*(\mathbf{x}) = \boldsymbol{\Lambda}^*(\mathbf{x})-\mathbf{x}, \quad \text{for all } x\in\mathbb{R}.$$

## Theorem

Let $x$ be a fixed real number.

(i) If $(X_t/t)_{t\geq 1}$ satisfies a full LDP under the measure $\mathbb{P}$ with the good rate function $\Lambda^*$, the asymptotic behaviour of a put option with strike $\exp(xt)$ reads

$$\lim_{t\to\infty} t^{-1} \log \mathbb{E}\left[\left(\mathrm{e}^{xt} - \mathrm{e}^{X_t}\right)_+\right] = \begin{cases} x - \Lambda^*(x) & \text{if } x \leq \Lambda'(0), \\ x & \text{if } x > \Lambda'(0). \end{cases}$$

(ii) If $(X_t/t)_{t\geq 1}$ satisfies a full LDP under the measure $\widetilde{\mathbb{P}}$ with the good rate function $\widetilde{\Lambda}^*$, the asymptotic behaviour of a call option struck at $\mathrm{e}^{xt}$ is given by the formula

$$\lim_{t\to\infty} t^{-1} \log \mathbb{E}\left[\left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)_+\right] = \begin{cases} x - \Lambda^*(x) & \text{if } x \geq \Lambda'(1), \\ 0 & \text{if } x < \Lambda'(1), \end{cases}$$

(iii) If $(X_t/t)_{t\geq 1}$ satisfies a full LDP under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ with good rate functions $\Lambda^*$ and $\widetilde{\Lambda}^*$, the covered call option with payoff $\mathrm{e}^{X_t} - \left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)_+$ satisfies

$$\lim_{t\to\infty} t^{-1} \log\left(1 - \mathbb{E}\left[\left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)_+\right]\right) = x - \Lambda^*(x) \quad \text{if } x \in \left[\Lambda'(0), \Lambda'(1)\right].$$

# Idea of the proof

The following inequalities hold for all $t \geq 1$ and $\varepsilon > 0$:

$$\mathrm{e}^{xt}\left(1-\mathrm{e}^{-\varepsilon}\right)\mathbb{1}_{\{X_t/t<x-\varepsilon\}} \leq \left(\mathrm{e}^{xt}-\mathrm{e}^{X_t}\right)_+ \leq \mathrm{e}^{xt}\,\mathbb{1}_{\{X_t/t<x\}}.$$

Taking expectations, logarithms, dividing by $t$ and applying the LDP for $(X_t/t)_{t\geq 1}$

$$x - \inf_{y<x-\varepsilon}\Lambda^*(y) \leq \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\left(\mathrm{e}^{xt}-\mathrm{e}^{X_t}\right)_+\right] \leq \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\left(\mathrm{e}^{xt}-\mathrm{e}^{X_t}\right)_+\right] \leq x - \inf_{y\leq x}\Lambda^*(y).$$

## Black-Scholes intermezzo

Consider the Black-Scholes model: $\mathrm{d}X_t = -\Sigma^2/2\,\mathrm{d}t + \Sigma\mathrm{d}W_t$, with $\Sigma > 0$. Then

$$\begin{aligned} \Lambda_{\mathrm{BS}}(u) &= u\,(u-1)\,\Sigma^2/2, && \text{for all } u \in \mathbb{R}, \\ \Lambda^*_{\mathrm{BS}}(x,\Sigma) &:= \left(x + \Sigma^2/2\right)^2 / \left(2\Sigma^2\right), && \text{for all } x \in \mathbb{R},\ \Sigma \in \mathbb{R}^*_+, \end{aligned}$$

**Lemma**

Under the Black-Scholes model, we have the following option price asymptotics.

$$\lim_{t\to\infty} t^{-1} \log \mathbb{E}\left(\mathrm{e}^{xt} - \mathrm{e}^{X_t}\right)_+ = \begin{cases} x - \Lambda^*_{\mathrm{BS}}(x) & \text{if } x \leq -\Sigma^2/2, \\ x & \text{if } x > -\Sigma^2/2, \end{cases}$$

$$\lim_{t\to\infty} t^{-1} \log \mathbb{E}\left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)_+ = \begin{cases} x - \Lambda^*_{\mathrm{BS}}(x) & \text{if } x \geq \Sigma^2/2, \\ 0 & \text{if } x < \Sigma^2/2, \end{cases}$$

$$\lim_{t\to\infty} t^{-1} \log \left(1 - \mathbb{E}\left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)_+\right) = x - \Lambda^*_{\mathrm{BS}}(x) \quad \text{if } x \in \left[-\Sigma^2/2, \Sigma^2/2\right].$$

# Implied volatility asymptotics

Define the function $\hat{\sigma}_{\infty}^2 : \mathbb{R} \to \mathbb{R}_+$ by

$$\hat{\sigma}_{\infty}^2(x) := 2\left(2\Lambda^*(x) - x + 2\mathcal{I}(x)\sqrt{\Lambda^*(x)\left(\Lambda^*(x) - x\right)}\right),$$

where

$$\mathcal{I}(x) = \left(1\!\!1_{x\in(\Lambda'(0),\Lambda'(1))} - 1\!\!1_{x\in\mathbb{R}\setminus(\Lambda'(0),\Lambda'(1))}\right).$$

Note that $\Lambda^*\left(\Lambda'(0)\right) = 0$ and $\Lambda^*\left(\Lambda'(1)\right) = \Lambda'(1)$ (equivalently $\widetilde{\Lambda}^*\left(\Lambda'(1)\right) = 0$).

**Theorem**

If the random variable $(X_t/t)_{t\geq 1}$ satisfies a full large deviations principle under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, then the function $\hat{\sigma}_{\infty}$ is continuous on the whole real line and is the uniform limit of $\hat{\sigma}_t$ as $t$ tends to infinity.

## Affine stochastic volatility models

Let $(S_t)_{t\geq 0}$ represent a share price process and a martingale. Define $X_t := \log S_t$ and assume that $(X_t, V_t)_{t\geq 0}$ is a stochastically continuous, time-homogeneous Markov process satisfying

$$\Phi_t(u,w) := \log \mathbb{E}\left( \mathrm{e}^{uX_t + wV_t} \middle| X_0, V_0 \right) = \phi(t,u,w) + \psi(t,u,w)\, V_0 + uX_0,$$

for all $t, u, w \in \mathbb{R}_+ \times \mathbb{C}^2$ such that the expectation exists.
Define $F(u,w) := \partial_t \phi(t,u,w)|_{t=0^+}$, and $R(u,w) := \partial_t \psi(t,u,w)|_{t=0^+}$. Then

$$F(u,w) = \left\langle \frac{a}{2}\begin{pmatrix} u \\ w \end{pmatrix} + b, \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle + \int_{D\setminus\{0\}} \left( \mathrm{e}^{xu+yw} - 1 - \left\langle \omega_F(x,y), \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle \right) \mathrm{m}(\mathrm{d}x, \mathrm{d}y),$$

$$R(u,w) = \left\langle \frac{\alpha}{2}\begin{pmatrix} u \\ w \end{pmatrix} + \beta, \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle + \int_{D\setminus\{0\}} \left( \mathrm{e}^{xu+yw} - 1 - \left\langle \omega_R(x,y), \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle \right) \mu(\mathrm{d}x, \mathrm{d}y),$$

where $D := \mathbb{R} \times \mathbb{R}_+$, and $\omega_F$ and $\omega_R$ are truncation functions.
See Duffie, Filipović, Schachermayer (2003) and Keller-Ressel (2009).

## Why this class of models?

- They feature most market characteristics: jumps, stochastic volatility, ...
- Their analytic properties are known (Duffie, Filipović & Schachermayer).
- They are tractable and pricing can be performed using Carr-Madan or Lewis inverse Fourier transform method.
- Most models used in practice fall into this category: Heston, Bates, exponential Lévy models (VG, CGMY), pure jump process (Merton, Kou), Barndorff-Nielsen & Shephard, ...

# Continuous case

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\left(a + V_t\right)\mathrm{d}t + \rho\sqrt{V_t}\,\mathrm{d}W_t + \sqrt{a + \left(1-\rho^2\right)V_t}\,\mathrm{d}Z_t, & X_0 = x \in \mathbb{R},\\
\mathrm{d}V_t &= \left(b + \beta V_t\right)\mathrm{d}t + \sqrt{\alpha V_t}\,\mathrm{d}W_t, & V_0 = v \in (0,\infty),
\end{aligned}
$$

with $a \geq 0$, $b \geq 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, and $\rho \in [-1,1]$.
In the Heston model: $a = 0$, $b = \kappa\theta > 0$, $\beta = -\kappa < 0$, $\alpha = \sigma^2$.

**Theorem**

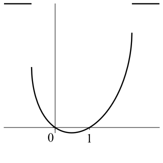
$$
\Lambda(u) = -\frac{b}{\alpha}\left(\chi(u) + \gamma(u)\right) + \frac{a}{2}u(u-1) \quad \text{for all } u \in \mathcal{D}_\Lambda,
$$

where $\chi(u) := \beta + u\rho\sqrt{\alpha}$ and $\gamma(u) := \left(\chi(u)^2 + \alpha u(1-u)\right)^{1/2}$ and

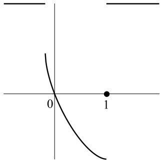
(i) If $\chi(0) \leq 0$,
  (a) if $\chi(1) \leq 0$ then $\mathcal{D}_\Lambda = [u_-, u_+]$;
  (b) if $\chi(1) > 0$ then $\mathcal{D}_\Lambda = [u_-, 1]$.

(ii) If $\chi(0) > 0$,
  (a) if $\chi(1) \leq 0$ then $\mathcal{D}_\Lambda = [0, u_+]$;
  (b) if $\chi(1) > 0$ then $\mathcal{D}_\Lambda = [0, 1]$.

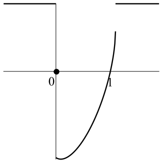
$u_-$ and $u_+$ are explicit and $u_- \leq 0$ and $u_+ \geq 1$.



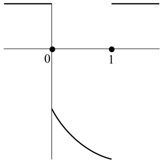
(a) Case (i)(a)



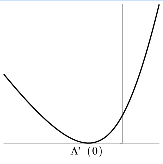
(b) Case (i)(b)



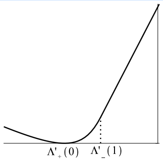
(c) Case (ii)(a)



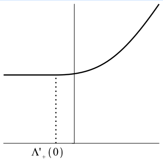
(d) Case (ii)(b)



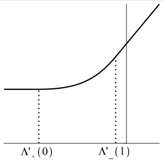
(e) Case (i)(a)



(f) Case (i)(b)



(g) Case (ii)(a)



(h) Case (ii)(b)

## Implied volatility asymptotics

**Case (i)(a): "Extended" ($a \neq 0$) Heston model with $\kappa - \rho\sigma > 0$**
$\Lambda$ is essentially smooth on $\mathcal{D}_\Lambda$ hence the theorems apply and (after some rearrangements and changes of variables):

$$\hat{\sigma}_\infty^2(x) = \hat{\sigma}_{\mathrm{SVI}}^2(x) = \frac{\omega_1}{2}\left(1 + \omega_2\rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2}\right), \quad \text{for all } x \in \mathbb{R},$$

i.e. Jim Gatheral's *SVI* parameterisation is the genuine limit of the Heston smile.
Note that $(X_t/t)$ converges weakly to a Normal Inverse Gaussian.

**Case (i)(b): "Extended" ($a \neq 0$) Heston model with $\kappa - \rho\sigma \leq 0$**
· $0 \in \mathcal{D}_\Lambda^o$ but $0 \in \mathcal{D}_{\tilde{\Lambda}}^o$
· $\Lambda$ is steep at $u_-$ but not at 1.
The implied volatility formula holds for $x \leq \Lambda'(0)$.

**Other cases:**

# Implied volatility asymptotics

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**Case (i)(b): "Extended" ($a \neq 0$) Heston model with $\kappa - \rho\sigma \leq 0$**
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· $\Lambda$ is steep at $u_-$ but not at 1.
The implied volatility formula holds for $x \leq \Lambda'(0)$.

**Other cases:** all the problems occur. Work in progress...

## Jump case

Recall that $\Lambda_t(u,w) := \phi(t,u,w) + \psi(t,u,w)V_0$. We are interested in the behaviour of $\lim_{t\to\infty} t^{-1}\Lambda_t(u,0)$.

Define the function $\chi : \mathbb{R} \to \mathbb{R}$ by $\chi(u) := \partial_w R(u,w)|_{w=0}$, assume that

$$\chi(\mathbf{0}) < \mathbf{0} \quad \text{and} \quad \chi(\mathbf{1}) < \mathbf{0}.$$

**Lemma (Keller-Ressel, 2009)**

There exist an interval $\mathcal{I} \subset \mathbb{R}$ and a unique function $w \in C(\mathcal{I}) \cap C^1(\mathcal{I}^\circ)$ such that $R(u, w(u)) = 0$, for all $u \in \mathcal{I}$ with $w(0) = w(1) = 0$. Define the set $\mathcal{J} := \{u \in \mathcal{I} : F(u, w(u)) < \infty\}$ and the function $\Lambda(u) := F(u, w(u))$ on $\mathcal{J}$, then

$$\lim_{t\to\infty} t^{-1}\Lambda_t(u,0) = \lim_{t\to\infty} t^{-1}\phi(t,u,0) = \Lambda(u), \quad \text{for all } u \in \mathcal{J},$$

$$\lim_{t\to\infty} \psi(t,u,0) = w(u), \quad \text{for all } u \in \mathcal{I}.$$

For convenience, we shall write $\Lambda_t(u)$ in place of $\Lambda_t(u,0)$.

# Properties and issues

- Can we have a limiting effective domain $\mathcal{D}_\Lambda = \mathcal{J}$ larger than $[0, 1]$? **Yes**.

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- Can we have a limiting effective domain $\mathcal{D}_\Lambda = \mathcal{J}$ larger than $[0, 1]$? **Yes**.
- Is $\Lambda$ essentially smooth? **Not necessarily**, but we can find necessary and sufficient conditions.
- What happens when the assumption $\chi(0) < 0$ and $\chi(1) < 0$ fails? **Good question**.

# One-dimensional Lévy processes

Let $(X_t)_t \geq 0$ be a Lévy process with triplet $(\sigma, \eta, \nu)$. The standard Lévy assumptions as well as the martingale condition impose $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \left(x^2 \wedge 1\right) \nu(\mathrm{d}x) < \infty, \quad \int_{|x| \geq 1} \mathrm{e}^x \nu(\mathrm{d}x) < \infty, \quad \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - x \mathbb{1}_{|x| \leq 1}\right) \nu(\mathrm{d}x) = -\eta.$$

Now, $\Phi_t(u, 0) = \exp(t\phi_X(u))$. Hence

$$F(u, 0) = \phi_X(u) \quad \text{and} \quad R(u, 0) = 0.$$

The conditions $\chi(1) < 0$ and $\chi(0) < 0$ fail. But clearly $\Lambda \equiv \phi_X$ holds.

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$$\text{Example: } \Lambda_{\mathrm{VG}}(u) = \left(\frac{ab}{(a-u)(b+u)}\right)^c, \quad \text{and} \quad \mathcal{D}_\Lambda = (-b, a).$$

## Small-time asymptotics

We are interested in determining

$$\lambda(u) := \lim_{t\to 0} t\Phi_t(u/t, 0) = \lim_{t\to 0}\Big(t\phi(t, u/t, 0) + v_0 t\psi(t, u/t, 0)\Big), \quad \text{for all } u \in \mathcal{D}_\lambda.$$

Let us define the Fenchel-Legendre transform $\lambda^* : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ of $\lambda$ by

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**Proposition**

If $(X_t - X_0)_{t \geq 0}$ satisfies a full LDP with rate $\lambda^*$ as $t$ tends to zero. The small-time implied volatility reads

$$\sigma_0(x) := \lim_{t \to 0} \sigma_t(x) = \frac{|x|}{\sqrt{2\lambda^*(x)}}, \quad \text{for all } x \in \mathbb{R}^*,$$

and $\sigma_0$ is a continuous function on $\mathbb{R}$.

# Small-time for continuous affine SV models

Assume that the process has continuous paths, i.e. $\mu \equiv 0$ and $\mathrm{m} \equiv 0$. Define

$$\lambda_0(u) := \lim_{t \to 0} t\psi(t, u/t, 0), \quad \text{for all } u \in \mathcal{D}_{\lambda_0}.$$

**Lemma**

$$\lambda_0(u) = \alpha_{22}^{-1}\left(-\alpha_{12}u + \zeta u \tan\left(\zeta u/2 + \arctan(\alpha_{12}/\zeta)\right)\right) \quad \text{and} \quad \mathcal{D}_{\lambda_0} = (u_-, u_+),$$

where $u_\pm := \zeta^{-1}(\pm\pi - 2\arctan(\alpha_{12}/\zeta)) \in \mathbb{R}_\pm$ and $\zeta := \det(\alpha)^{1/2} > 0$. Therefore we obtain

$$\lambda(u) = \lambda_0(u) + a_{11}u^2/2.$$

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- Everything works fine when there are no jumps, and $\lambda$ is known in closed-form.
- Jump case: proper scaling needed: Nutz & Muhle-Karbe (2010), Rosenbaum & Tankov (2010): in progress.

# Heston with jumps I

Consider the Heston model

$$
\begin{aligned}
\mathrm{d}X_t &= \left(\delta - \frac{1}{2}V_t\right)\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \quad X_0 = x_0 \in \mathbb{R}, \\
\mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\,\mathrm{d}Z_t, \quad V_0 = v_0 > 0, \\
\mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\mathrm{d}t,
\end{aligned}
$$

where $J := (J_t)_{t\geq 0}$ is a pure-jump Lévy process independent of $(W_t)_{t\geq 0}$. Assume

$$\chi\left(1\right) = \rho\sigma - \kappa < 0$$

It is clear that

$$\Lambda_t\left(u\right) := \log\mathbb{E}\left(\mathrm{e}^{u(X_t - x_0)}\right) = \Lambda_t^h\left(u\right) + \overline{\Lambda}^J\left(u\right)t,$$

with $\overline{\Lambda}^J\left(u\right) := \Lambda^J\left(u\right) - u\Lambda^J\left(1\right)$ (martingale condition). This means

$$F\left(u, w\right) = \kappa\theta w + \overline{\Lambda}^J\left(u\right), \quad \text{and} \quad R\left(u, w\right) = \frac{u}{2}\left(u-1\right) + \frac{\xi^2}{2}w^2 - \kappa w + \rho\xi u w.$$

# Heston with jumps II

We know that, for all $u \in \left[u_-^h, u_+^h\right]$

$$\Lambda^h(u) := \lim_{t\to\infty} t^{-1}\Lambda_t^h(u) = \frac{\kappa\theta}{\xi^2}\left(\kappa - \rho\xi u - \sqrt{(\kappa-\rho\xi u)^2 - \xi^2 u(u-1)}\right),$$

so that

$$\Lambda(u) := \lim_{t\to\infty} t^{-1}\Lambda_t(u) = \Lambda^h(u) + \overline{\Lambda}^J(u), \quad \text{for all } u \in \left[u_-^h \vee u_-^J, u_+^h \wedge u_+^J\right].$$

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$$\Lambda^*(x) = \sup_{u\in\left[u_-^h \vee u_-^J, u_+^h \wedge u_+^J\right]} \{ux - \Lambda(u)\}, \quad \text{for all } x \in \mathbb{R}.$$

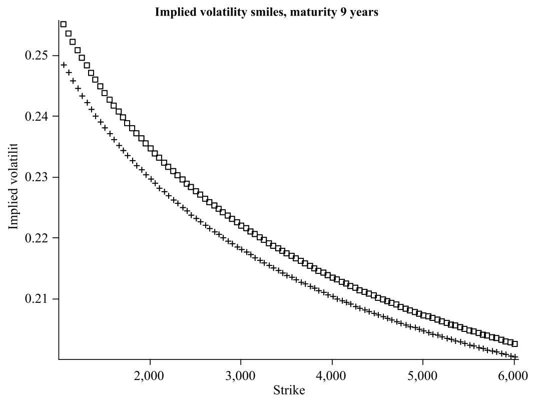
# Heston with jumps III

Consider Normal Inverse Gaussian jumps, i.e.
$J$ is an independent Normal Inverse Gaussian process with parameters $(\alpha, \beta, \mu, \delta)$ and Lévy exponent

$$\Lambda^{\mathrm{NIG}}(u) = \mu u + \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right).$$

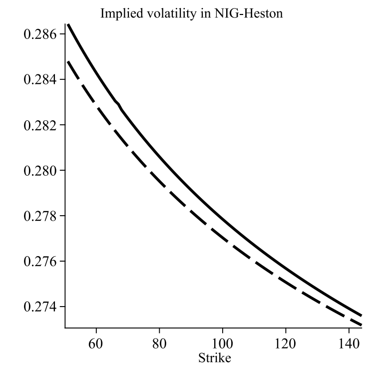
Then $u_{\pm}^{NIG} = -\beta \pm \alpha$.

# Numerical example: Heston without jumps




Heston (without jumps) calibrated on the Eurostoxx 50 on February, 15th, 2006, and then generated for $T = 9$ years. $\kappa = 1.7609$, $\theta = 0.0494$, $\sigma = 0.4086$, $v_0 = 0.0464$, $\rho = -0.5195$.

# Numerical example: Heston with NIG jumps




Same parameters as before for Heston and the following for NIG: $\alpha = 7.104$, $\beta = -3.3$, $\delta = 0.193$ and $\mu = 0.092$. Heston (with jumps) calibrated on the Eurostoxx 50.

# Barndorff-Nielsen & Shephard (2001) I

$$\mathrm{d}X_t = -\left(\gamma k(\rho) + \frac{1}{2}V_t\right)\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t + \rho\,\mathrm{d}J_{\gamma t}, \quad X_0 = x_0 \in \mathbb{R},$$
$$\mathrm{d}V_t = -\gamma V_t \mathrm{d}t + \mathrm{d}J_{\gamma t}, \quad V_0 = v_0 > 0,$$

where $\gamma > 0$, $\rho < 0$ and $(J_t)_{t\geq 0}$ is a Lévy subordinator where the cgf of $J_1$ is given by $\Lambda^J(u) = \log \mathbb{E}\left(\mathrm{e}^{uJ_1}\right)$. $\mathcal{D}_\Lambda = (u_-, u_+)$, where

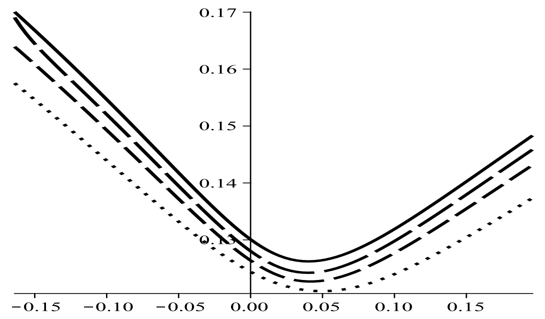
$$u_\pm := \frac{1}{2} - \rho\gamma \pm \sqrt{\frac{1}{4} - (2k^* - \rho)\gamma + \rho^2\gamma^2}.$$

with $k^* := \sup\{u > 0 : k(u) < \infty\}$. We deduce the two functions $F$ and $R$,

$$R(u,0) = \frac{1}{2}\left(u^2 - u\right), \quad \text{and} \quad F(u,0) = \gamma k(\rho u) - u\gamma k(\rho).$$

Consider the $\Gamma$-BNS model, where the subordinator is $\Gamma(a,b)$-distributed with $a$, $b > 0$. Hence $k_\Gamma(u) = (b-u)^{-1} au$, and $u_\pm^\Gamma := \frac{1}{2} - \rho\gamma \pm \sqrt{\left(\frac{1}{2} - \rho\gamma\right)^2 + 2b\gamma} \in \mathbb{R}_\pm$.

# Barndorff-Nielsen & Shephard II




Γ-BNS model with $a = 1.4338$, $b = 11.6641$, $v_0 = 0.0145$, $\gamma = 0.5783$, (Schoutens)
Solid line: asymptotic smile. Dotted and dashed: 5, 10 and 20 years generated smile.

## One step beyond

For more accurate results, it might be interesting to go one step beyond:

$$\hat{\sigma}_t(x) = \hat{\sigma}_\infty(x) + \frac{1}{t}\hat{a}(x) + o(1/t), \qquad \text{as } t \to \infty$$

$$\sigma_t(x) = \sigma_0(x) + a(x)\,t + o(t), \qquad \text{as } t \to 0.$$

However large deviations do not provide the first-order term.

# Complex saddlepoint methods (Heston)

From Lee (2004), we have, for any $\alpha \in \mathbb{R}$,

$$\frac{1}{S_0}\mathbb{E}(S_t - K)^+ = \text{Residues} + \frac{1}{2\pi}\int_{-\infty-\mathrm{i}\alpha}^{+\infty-\mathrm{i}\alpha} \mathrm{e}^{-izx}\frac{\phi_t(z-\mathrm{i})}{\mathrm{i}z - z^2}\mathrm{d}z,$$

where $x := \log(K/S_0)$ and $\phi_t$ is the Heston characteristic function. The methodology is the following (for the large-time):

- approximate $\phi_t(z) \sim \mathrm{e}^{-\lambda(z)t}\phi(z)$. The integrand reads $\exp\{(-\mathrm{i}zx - \lambda(z))\,t\}\,f(z)$. Find the saddlepoint of this function.
- Deform the integration contour through this saddlepoint using the steepest descent method.
- 'Equate' the Black-Scholes expansion with the model expansion.
- Back out the implied volatility.

# Infinity is closer than what we think

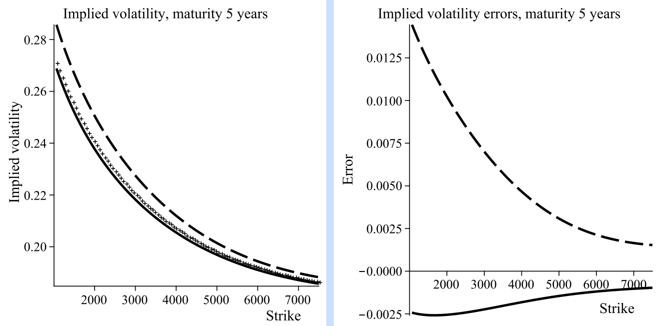



Figure: Same parameters for the Heston model in the large-time regime, with $t = 5$ years.

# Zero is even closer

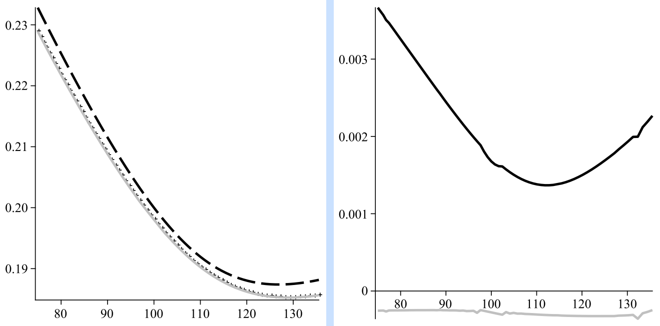



Figure: Same parameters for the Heston model in the small-time regime, with $t = 0.2$ years.

# Conclusion

Summary:

- Closed-form formulae for affine stochastic volatility models with jumps for large maturities.
- Closed-form formulae for continuous affine stochastic volatility models for small maturities.

Future research:

- What happens when 0 is not in the interior of $\mathcal{D}_\Lambda$?
- Remove the conditions $\chi(0) < 0$ and $\chi(1) < 0$.
- What happens precisely in the small-time when jumps are added?
- Determine the higher-order correction terms (in $t$ or $t^{-1}$).
- Statistical and numerical tests to assess the calibration efficiency.