Calibrating affine stochastic volatility models with jumps
An asymptotic approach

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Based on joint works with M. Forde, J. Gatheral, M. Keller-Ressel, R. Lee and A. Mijatović,

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in Financial Mathematics, Paris, August 2010
A short (non) fictitious story

I just finished my MSc (PhD) in Financial Mathematics from — and this is my first day as a bright junior quant in a large bank. First day, first assignment.

Boss: ‘Calibrate model $H(a)$ to market data.’
A short (non) fictitious story

Figure: Market implied volatilities for different strikes and maturities.
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Figure: Sum of squared errors: 4.53061E-05
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Me (10 minutes later): ‘Done.’
Boss: ‘Not good enough. Which initial point did you take?’
Me: ‘$a_1$.’
Boss: ‘Classic mistake!! You should take $a_2$ instead.’
A short (non) fictitious story

**Figure:** Sum of squared errors: 2.4856E-06
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Moral of the story:
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(ii) My boss is really good.
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"Start every day off with a smile and get it over with." (W.C. Fields)
So let us start off with a smile (one maturity slice)
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\[
\text{Solid blue: } x \mapsto g(x) := \mathcal{C}_{BS}^{-1} \left( \mathcal{F}^{-1} \mathcal{R} \left\{ f(x, z) \phi_a(z) \right\} \right)
\]
So let us start off with a smile (one maturity slice)

Solid blue: \( x \mapsto g(x) := C_{BS}^{-1}(\mathcal{F}^{-1}\mathbb{R}\{ f(x, z) \phi_{a}(z) \}) \)

Dashed black: \( x \mapsto \hat{g}(x) = \alpha x^2 + \beta x + \gamma \)
So let us start off with a smile (one maturity slice)

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Easier to calibrate \( \hat{g} \) than \( g \).
Motivation and goals

- Obtain closed-form formulae for the implied volatility under ASVM in the short and in the large-maturity limits.
- Propose an accurate starting point for calibration purposes.
- Discuss conditions on jumps for a model to be usable in practice.

**Definition:** The implied volatility is the unique parameter $\sigma \geq 0$ such that

$$C_{BS}(S_0, K, T, \sigma) = C_{obs}(S_0, K, T).$$
Lemma

The family of random variables $(Z_t)_{t \geq 1}$ satisfies the large deviations principle (LDP) with the good rate function $\Lambda^*$ if for every Borel measurable set $B$ in $\mathbb{R}$

$$\inf_{x \in B^c} \Lambda^*(x) \leq \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}(Z_t \in B) \leq \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}(Z_t \in B) \leq - \inf_{x \in \overline{B}} \Lambda^*(x),$$
The Gärtner-Ellis theorem

Assumption A.1: For all $u \in \mathbb{R}$, define the limiting cumulant generating function

$$\Lambda(u) := \lim_{t \to \infty} t^{-1} \log \mathbb{E} \left( e^{utX_t} \right) = \lim_{t \to \infty} t^{-1} \Lambda_t (ut)$$

as an extended real number. Denote $\mathcal{D}_\Lambda := \{ u \in \mathbb{R} : \Lambda(u) < \infty \}$. Assume further that

(i) the origin belongs to $\mathcal{D}_\Lambda^0$;

(ii) $\Lambda$ is essentially smooth.
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(i) the origin belongs to \( \mathcal{D}_\Lambda^0 \);
(ii) \( \Lambda \) is essentially smooth.

Theorem (Gärtner-Ellis) (special case of the general th. Dembo & Zeitouni)

Under Assumption A.1, the family of random variables \( (X_t)_{t \geq 0} \) satisfies the LDP with rate function \( \Lambda^* \), defined as the Fenchel-Legendre transform of \( \Lambda \),

\[
\Lambda^*(x) := \sup_{u \in \mathbb{R}} \{ ux - \Lambda(u) \}, \quad \text{for all } x \in \mathbb{R}.
\]
Methodology overview (large-time)

- Let \((S_t)_{t \geq 0}\) be a **martingale** share price process, and define \(X_t := \log (S_t/S_0)\).
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- Check the smoothness conditions for \(\Lambda\), in particular the set \(\mathcal{D}_\Lambda := \{u : \Lambda (u) < \infty\}\).
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- Conclude that \((X_t/t)_{t > 0}\) satisfies a full LDP with (good) rate function \(\Lambda^*\).
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• Translate the tail behaviour of \(X\) into an asymptotic behaviour of Call prices.
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- Translate the tail behaviour of \(X\) into an asymptotic behaviour of Call prices.
- Translate these Call price asymptotics into implied volatility asymptotics.
Option price and Share measure

Define the **Share** measure $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{P}} (A) := \mathbb{E} ((X_t - X_0) \mathbb{1}_A)$.

A European call option price reads

$$
\mathbb{E} \left( e^{X_t} - e^x \right)_+ = \mathbb{E} \left( \left( e^{X_t} - e^x \right) \mathbb{1}_{X_t \geq x} \right) = \mathbb{E} \left( e^{X_t} \mathbb{1}_{X_t \geq x} \right) - e^x \mathbb{P} (X_t \geq x) = \tilde{\mathbb{P}} (X_t \geq x) - e^x \mathbb{P} (X_t \geq x).
$$

Denote $\tilde{\Lambda}$ and $\tilde{\Lambda}^*$ the corresponding limiting cgf and Fenchel-Legendre transform under $\tilde{\mathbb{P}}$. They satisfy the following relations:

$$
\tilde{\Lambda} (u) = \Lambda (u + 1), \quad \text{if } (1 + u) \in \mathcal{D}_\Lambda, \quad \text{and} \quad \tilde{\Lambda}^* (x) = \Lambda^* (x) - x, \quad \text{for all } x \in \mathbb{R}.
$$
Let $x$ be a fixed real number.

(i) If $(X_t/t)_{t\geq 1}$ satisfies a full LDP under the measure $\mathbb{P}$ with the good rate function $\Lambda^*$, the asymptotic behaviour of a put option with strike $\exp(x t)$ reads

$$
\lim_{t \to \infty} t^{-1} \log \mathbb{E} \left[ (e^{x t} - e^{X_t})^+ \right] = \begin{cases} 
    x - \Lambda^*(x) & \text{if } x \leq \Lambda'(0), \\
    x & \text{if } x > \Lambda'(0).
\end{cases}
$$

(ii) If $(X_t/t)_{t\geq 1}$ satisfies a full LDP under the measure $\tilde{\mathbb{P}}$ with the good rate function $\tilde{\Lambda}^*$, the asymptotic behaviour of a call option struck at $e^{x t}$ is given by the formula

$$
\lim_{t \to \infty} t^{-1} \log \mathbb{E} \left[ (e^{X_t} - e^{x t})^+ \right] = \begin{cases} 
    x - \Lambda^*(x) & \text{if } x \geq \Lambda'(1), \\
    0 & \text{if } x < \Lambda'(1),
\end{cases}
$$

(iii) If $(X_t/t)_{t\geq 1}$ satisfies a full LDP under $\mathbb{P}$ and $\tilde{\mathbb{P}}$ with good rate functions $\Lambda^*$ and $\tilde{\Lambda}^*$, the covered call option with payoff $e^{X_t} - (e^{X_t} - e^{x t})^+$ satisfies

$$
\lim_{t \to \infty} t^{-1} \log \left( 1 - \mathbb{E} \left[ (e^{X_t} - e^{x t})^+ \right] \right) = x - \Lambda^*(x) \quad \text{if } x \in [\Lambda'(0), \Lambda'(1)].
$$
The following inequalities hold for all $t \geq 1$ and $\varepsilon > 0$:

$$e^{xt} (1 - e^{-\varepsilon}) \mathbb{I}\{X_t/t < x - \varepsilon\} \leq \left(e^{xt} - e^{x_t}\right)_+ \leq e^{xt} \mathbb{I}\{X_t/t < x\}.$$

Taking expectations, logarithms, dividing by $t$ and applying the LDP for $(X_t/t)_{t \geq 1}$

$$x - \inf_{y < x - \varepsilon} \Lambda^*(y) \leq \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[\left(e^{xt} - e^{x_t}\right)_+\right] \leq \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[\left(e^{xt} - e^{x_t}\right)_+\right] \leq x - \inf_{y \leq x} \Lambda^*(y).$$
Consider the Black-Scholes model: $dX_t = -\frac{\Sigma^2}{2} dt + \Sigma dW_t$, with $\Sigma > 0$. Then

$$\Lambda_{BS}(u) = u(u - 1) \frac{\Sigma^2}{2},$$
for all $u \in \mathbb{R}$,

$$\Lambda_{BS}^*(x, \Sigma) := \frac{(x + \frac{\Sigma^2}{2})^2}{(2\Sigma^2)},$$
for all $x \in \mathbb{R}$, $\Sigma \in \mathbb{R}^+_+$.

**Lemma**

Under the Black-Scholes model, we have the following option price asymptotics.

$$\lim_{t \to \infty} t^{-1} \log \mathbb{E}\left( e^{xt} - e^{X_t} \right)_+ = \begin{cases} 
  x - \Lambda_{BS}^*(x) & \text{if } x \leq -\frac{\Sigma^2}{2}, \\
  x & \text{if } x > -\frac{\Sigma^2}{2}, \\
  0 & \text{if } x < \frac{\Sigma^2}{2}, 
\end{cases}$$

$$\lim_{t \to \infty} t^{-1} \log \mathbb{E}\left( e^{X_t} - e^{xt} \right)_+ = \begin{cases} 
  x - \Lambda_{BS}^*(x) & \text{if } x \geq \frac{\Sigma^2}{2}, \\
  0 & \text{if } x < \frac{\Sigma^2}{2}, 
\end{cases}$$

$$\lim_{t \to \infty} t^{-1} \log \left( 1 - \mathbb{E}\left( e^{X_t} - e^{xt} \right)_+ \right) = x - \Lambda_{BS}^*(x) \quad \text{if } x \in \left[ -\frac{\Sigma^2}{2}, \frac{\Sigma^2}{2} \right].$$
Implied volatility asymptotics

Define the function $\hat{\sigma}_\infty^2 : \mathbb{R} \rightarrow \mathbb{R}_+$ by

$$\hat{\sigma}_\infty^2(x) := 2 \left( 2 \Lambda^*(x) - x + 2 \mathcal{I}(x) \sqrt{\Lambda^*(x)(\Lambda^*(x) - x)} \right) ,$$

where

$$\mathcal{I}(x) = \left( \mathbb{1}_{x \in (\Lambda'(0), \Lambda'(1))} - \mathbb{1}_{x \in \mathbb{R} \setminus (\Lambda'(0), \Lambda'(1))} \right) .$$

Note that $\Lambda^*(\Lambda'(0)) = 0$ and $\Lambda^*(\Lambda'(1)) = \Lambda'(1)$ (equivalently $\tilde{\Lambda}^*(\Lambda'(1)) = 0$).

Theorem

If the random variable $(X_t/t)_{t \geq 1}$ satisfies a full large deviations principle under $\mathbb{P}$ and $\tilde{\mathbb{P}}$, then the function $\hat{\sigma}_\infty$ is continuous on the whole real line and is the uniform limit of $\hat{\sigma}_t$ as $t$ tends to infinity.
Affine stochastic volatility models

Let \((S_t)_{t \geq 0}\) represent a share price process and a martingale. Define \(X_t := \log S_t\) and assume that \((X_t, V_t)_{t \geq 0}\) is a stochastically continuous, time-homogeneous Markov process satisfying

\[
\Phi_t(u, w) := \log \mathbb{E} \left( e^{uX_t + wV_t} \bigg| X_0, V_0 \right) = \phi(t, u, w) + \psi(t, u, w) V_0 + u X_0,
\]

for all \(t, u, w \in \mathbb{R}_+ \times \mathbb{C}^2\) such that the expectation exists.

Define 
\(F(u, w) := \partial_t \phi(t, u, w)\) \(\big|_{t=0^+}\), and 
\(R(u, w) := \partial_t \psi(t, u, w)\) \(\big|_{t=0^+}\). Then

\[
F(u, w) = \left\langle \frac{a}{2} \begin{pmatrix} u \\ w \end{pmatrix} + b, \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle + \int_{D \setminus \{0\}} \left( e^{xu+yw} - 1 - \left\langle \omega_F(x, y), \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle \right) m(dx, dy),
\]

\[
R(u, w) = \left\langle \frac{\alpha}{2} \begin{pmatrix} u \\ w \end{pmatrix} + \beta, \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle + \int_{D \setminus \{0\}} \left( e^{xu+yw} - 1 - \left\langle \omega_R(x, y), \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle \right) \mu(dx, dy),
\]

where \(D := \mathbb{R} \times \mathbb{R}_+\), and \(\omega_F\) and \(\omega_R\) are truncation functions.

Why this class of models?

- They feature most market characteristics: jumps, stochastic volatility, ...
- Their analytic properties are known (Duffie, Filipović & Schachermayer).
- They are tractable and pricing can be performed using Carr-Madan or Lewis inverse Fourier transform method.
- Most models used in practice fall into this category: Heston, Bates, exponential Lévy models (VG, CGMY), pure jump process (Merton, Kou), Barndorff-Nielsen & Shephard, ...
Continuous case

\[ dX_t = -\frac{1}{2} (a + V_t) \, dt + \rho \sqrt{V_t} \, dW_t + \sqrt{a + (1 - \rho^2)} V_t \, dZ_t, \quad X_0 = x \in \mathbb{R}, \]
\[ dV_t = (b + \beta V_t) \, dt + \sqrt{\alpha V_t} \, dW_t, \quad V_0 = v \in (0, \infty), \]

with \( a \geq 0, \ b \geq 0, \ \alpha > 0, \ \beta \in \mathbb{R}, \text{ and } \rho \in [-1, 1]. \)

In the Heston model: \( a = 0, \ b = \kappa \theta > 0, \ \beta = -\kappa < 0, \ \alpha = \sigma^2. \)

**Theorem**

\[ \Lambda(u) = -\frac{b}{\alpha} (\chi(u) + \gamma(u)) + \frac{a}{2} u (u - 1) \quad \text{for all } u \in \mathcal{D}_\Lambda, \]

where \( \chi(u) := \beta + u \rho \sqrt{\alpha} \) and \( \gamma(u) := (\chi(u)^2 + \alpha u (1 - u))^{1/2} \)

(i) If \( \chi(0) \leq 0, \)
    (a) if \( \chi(1) \leq 0 \) then \( \mathcal{D}_\Lambda = [u_-, u_+] \);  
    (b) if \( \chi(1) > 0 \) then \( \mathcal{D}_\Lambda = [u-, 1] \).

(ii) If \( \chi(0) > 0, \)
    (a) if \( \chi(1) \leq 0 \) then \( \mathcal{D}_\Lambda = [0, u_+] \);  
    (b) if \( \chi(1) > 0 \) then \( \mathcal{D}_\Lambda = [0, 1] \).

\( u_- \) and \( u_+ \) are explicit and \( u_- \leq 0 \) and \( u_+ \geq 1. \)
Antoine Jacquier
Calibrating affine stochastic volatility models with jumps
Implied volatility asymptotics

Case (i)(a): "Extended" \( a \neq 0 \) Heston model with \( \kappa - \rho \sigma > 0 \)
\( \Lambda \) is essentially smooth on \( \mathcal{D}_\Lambda \) hence the theorems apply and (after some rearrangements and changes of variables):

\[
\hat{\sigma}_\infty^2 (x) = \hat{\sigma}_{SVI}^2 (x) = \frac{\omega_1}{2} \left( 1 + \omega_2 \rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2} \right), \quad \text{for all } x \in \mathbb{R},
\]

i.e. Jim Gatheral’s SVI parameterisation is the genuine limit of the Heston smile.

Note that \( (X_t / t) \) converges weakly to a Normal Inverse Gaussian.

Case (i)(b): "Extended" \( a \neq 0 \) Heston model with \( \kappa - \rho \sigma \leq 0 \)
\cdot \( 0 \in \mathcal{D}_\Lambda^0 \) but \( 0 \in \mathcal{D}_\tilde{\Lambda}^0 \)
\cdot \( \Lambda \) is steep at \( u_- \) but not at 1.
The implied volatility formula holds for \( x \leq \Lambda' (0) \).

Other cases:
Implied volatility asymptotics

Case (i)(a): "Extended" ($a \neq 0$) Heston model with $\kappa - \rho \sigma > 0$

$\Lambda$ is essentially smooth on $D_\Lambda$ hence the theorems apply and (after some rearrangements and changes of variables):

$$\hat{\sigma}_\infty^2 (x) = \hat{\sigma}_{SVI}^2 (x) = \frac{\omega_1}{2} \left( 1 + \omega_2 \rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2} \right), \quad \text{for all } x \in \mathbb{R},$$

i.e. Jim Gatheral’s $SVI$ parameterisation is the genuine limit of the Heston smile. Note that $(X_t/t)$ converges weakly to a Normal Inverse Gaussian.

Case (i)(b): "Extended" ($a \neq 0$) Heston model with $\kappa - \rho \sigma \leq 0$

$0 \in D^o_\Lambda$ but $0 \notin D^o_{\tilde{\Lambda}}$

$\Lambda$ is steep at $u_-$ but not at 1.

The implied volatility formula holds for $x \leq \Lambda' (0)$.

Other cases: all the problems occur. Work in progress...
Recall that $\Lambda_t (u, w) := \phi (t, u, w) + \psi (t, u, w) V_0$. We are interested in the behaviour of $\lim_{t \to \infty} t^{-1} \Lambda_t (u, 0)$.

Define the function $\chi : \mathbb{R} \to \mathbb{R}$ by $\chi (u) := \partial_w R(u, w) |_{w=0}$, assume that $\chi (0) < 0$ and $\chi (1) < 0$.

**Lemma (Keller-Ressel, 2009)**

There exist an interval $\mathcal{I} \subset \mathbb{R}$ and a unique function $w \in C (\mathcal{I}) \cap C^1 (\mathcal{I}^c)$ such that $R (u, w (u)) = 0$, for all $u \in \mathcal{I}$ with $w (0) = w (1) = 0$. Define the set $\mathcal{J} := \{u \in \mathcal{I} : F (u, w (u)) < \infty\}$ and the function $\Lambda (u) := F (u, w (u))$ on $\mathcal{J}$, then

$$\lim_{t \to \infty} t^{-1} \Lambda_t (u, 0) = \lim_{t \to \infty} t^{-1} \phi (t, u, 0) = \Lambda (u), \quad \text{for all } u \in \mathcal{J},$$

$$\lim_{t \to \infty} \psi (t, u, 0) = w (u), \quad \text{for all } u \in \mathcal{I}.$$

For convenience, we shall write $\Lambda_t (u)$ in place of $\Lambda_t (u, 0)$. 
Properties and issues

- Can we have a limiting effective domain $\mathcal{D}_\Lambda = \mathcal{J}$ larger than $[0, 1]$? Yes.
Properties and issues

- Can we have a limiting effective domain $D_\Lambda = \mathcal{J}$ larger than $[0, 1]$? **Yes.**

- Is $\Lambda$ essentially smooth? **Not necessarily,** but we can find necessary and sufficient conditions.
Properties and issues

- Can we have a limiting effective domain $\mathcal{D}_\Lambda = \mathcal{I}$ larger than $[0, 1]$? Yes.

- Is $\Lambda$ essentially smooth? Not necessarily, but we can find necessary and sufficient conditions.

- What happens when the assumption $\chi(0) < 0$ and $\chi(1) < 0$ fails? Good question.
One-dimensional Lévy processes

Let \((X_t)_t \geq 0\) be a Lévy process with triplet \((\sigma, \eta, \nu)\). The standard Lévy assumptions as well as the martingale condition impose \(\nu(\{0\}) = 0\) and

\[
\int_{\mathbb{R}} (x^2 \land 1) \nu(dx) < \infty, \quad \int_{|x| \geq 1} e^x \nu(dx) < \infty, \quad \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x1_{|x| \leq 1}) \nu(dx) = -\eta.
\]

Now, \(\Phi_t(u, 0) = \exp(t\phi_X(u))\). Hence

\[
F(u, 0) = \phi_X(u) \quad \text{and} \quad R(u, 0) = 0.
\]

The conditions \(\chi(1) < 0\) and \(\chi(0) < 0\) fail. But clearly \(\Lambda \equiv \phi_X\) holds.
Let \((X_t)_t \geq 0\) be a Lévy process with triplet \((\sigma, \eta, \nu)\). The standard Lévy assumptions as well as the martingale condition impose \(\nu(\{0\}) = 0\) and

\[
\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty, \quad \int_{|x| \geq 1} e^x \nu(dx) < \infty, \quad \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x \mathbb{1}_{|x| \leq 1}) \nu(dx) = -\eta.
\]

Now, \(\Phi_t(u, 0) = \exp(t \phi_X(u))\). Hence

\[
F(u, 0) = \phi_X(u) \quad \text{and} \quad R(u, 0) = 0.
\]

The conditions \(\chi(1) < 0\) and \(\chi(0) < 0\) fail. But clearly \(\Lambda \equiv \phi_X\) holds.

- If \(\mathcal{D}_\Lambda\) is open and \(\{0, 1\} \in \mathcal{D}_\Lambda^\circ\) then \(\Lambda\) is essentially smooth.
One-dimensional Lévy processes

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- If \(D_\Lambda\) is open and \(\{0, 1\} \in D_\Lambda^o\) then \(\Lambda\) is essentially smooth.
- If \(D_\Lambda\) is not open then \(\Lambda\) is not necessarily essentially smooth.
One-dimensional Lévy processes

Let \((X_t)_{t \geq 0}\) be a Lévy process with triplet \((\sigma, \eta, \nu)\). The standard Lévy assumptions as well as the martingale condition impose \(\nu(\{0\}) = 0\) and

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- If \(\mathcal{D}_\Lambda\) is not open then \(\Lambda\) is not necessarily essentially smooth.

Example: \(\Lambda_{VG}(u) = \left(\frac{ab}{(a - u)(b + u)}\right)^c\), and \(\mathcal{D}_\Lambda = (-b, a)\).
Small-time asymptotics

We are interested in determining

\[
\lambda(u) := \lim_{t \to 0} t \Phi_t(u/t, 0) = \lim_{t \to 0} \left( t \phi(t, u/t, 0) + v_0 t \psi(t, u/t, 0) \right), \quad \text{for all } u \in D_\lambda.
\]

Let us define the Fenchel-Legendre transform \( \lambda^* : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\} \) of \( \lambda \) by

\[
\lambda^*(x) := \sup_{u \in \mathbb{R}} \{ux - \lambda(u)\}, \quad \text{for all } x \in \mathbb{R}.
\]
Small-time asymptotics

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\[ \lambda^*(x) := \sup_{u \in \mathbb{R}} \{ux - \lambda(u)\}, \quad \text{for all } x \in \mathbb{R}. \]

Proposition

If \( (X_t - X_0)_{t \geq 0} \) satisfies a full LDP with rate \( \lambda^* \) as \( t \) tends to zero. The small-time implied volatility reads

\[ \sigma_0(x) := \lim_{t \to 0} \sigma_t(x) = \frac{|x|}{\sqrt{2 \lambda^*(x)}}, \quad \text{for all } x \in \mathbb{R}^*, \]

and \( \sigma_0 \) is a continuous function on \( \mathbb{R} \).
Small-time for continuous affine SV models

Assume that the process has continuous paths, i.e. \( \mu \equiv 0 \) and \( \eta \equiv 0 \). Define

\[
\lambda_0 (u) := \lim_{t \to 0} t\psi(t, u/t, 0), \quad \text{for all } u \in \mathcal{D}_{\lambda_0}.
\]

Lemma

\[
\lambda_0 (u) = \alpha_{22}^{-1} \left( -\alpha_{12} u + \zeta u \tan \left( \frac{\zeta u}{2} + \arctan \left( \frac{\alpha_{12}}{\zeta} \right) \right) \right) \quad \text{and} \quad \mathcal{D}_{\lambda_0} = (u_-, u_+),
\]

where \( u_{\pm} := \zeta^{-1} \left( \pm \pi - 2 \arctan \left( \frac{\alpha_{12}}{\zeta} \right) \right) \in \mathbb{R}_{\pm} \) and \( \zeta := \det(\alpha)^{1/2} > 0 \). Therefore we obtain

\[
\lambda (u) = \lambda_0 (u) + a_{11} u^2 / 2.
\]
Small-time for continuous affine SV models

Assume that the process has continuous paths, i.e. $\mu \equiv 0$ and $m \equiv 0$. Define

$$\lambda_0(u) := \lim_{t \to 0} t \psi(t, u/t, 0), \quad \text{for all } u \in D_{\lambda_0}.$$ 

Lemma

$$\lambda_0(u) = \alpha_{22}^{-1} \left( -\alpha_{12} u + \zeta u \tan \left( \frac{\zeta u}{2} + \arctan \left( \frac{\alpha_{12}}{\zeta} \right) \right) \right) \quad \text{and} \quad D_{\lambda_0} = (u_-, u_+),$$

where $u_\pm := \zeta^{-1} (\pm \pi - 2 \arctan (\alpha_{12}/\zeta)) \in \mathbb{R}_\pm$ and $\zeta := \det (\alpha)^{1/2} > 0$. Therefore we obtain

$$\lambda(u) = \lambda_0(u) + a_{11} u^2 / 2.$$ 

• Everything works fine when there are no jumps, and $\lambda$ is known in closed-form.

Consider the Heston model

\[ dX_t = \left( \delta - \frac{1}{2} V_t \right) dt + \sqrt{V_t} dW_t + dJ_t, \quad X_0 = x_0 \in \mathbb{R}, \]

\[ dV_t = \kappa (\theta - V_t) dt + \xi \sqrt{V_t} dZ_t, \quad V_0 = v_0 > 0, \]

\[ d\langle W, Z \rangle_t = \rho dt, \]

where \( J := (J_t)_{t \geq 0} \) is a pure-jump Lévy process independent of \( (W_t)_{t \geq 0} \). Assume

\[ \chi (1) = \rho \sigma - \kappa < 0 \]

It is clear that

\[ \Lambda_t (u) := \log \mathbb{E} \left( e^{u(X_t - x_0)} \right) = \Lambda^h_t (u) + \overline{\Lambda}^J (u) t, \]

with \( \overline{\Lambda}^J (u) := \Lambda^J (u) - u \Lambda^J (1) \) (martingale condition). This means

\[ F (u, w) = \kappa \theta w + \overline{\Lambda}^J (u), \quad \text{and} \quad R (u, w) = \frac{u}{2} (u - 1) + \frac{\xi^2}{2} w^2 - \kappa w + \rho \xi uw. \]
Heston with jumps II

We know that, for all \( u \in [u_h^-, u_h^+] \)

\[
\Lambda^h(u) := \lim_{t \to \infty} t^{-1} \Lambda_t^h(u) = \frac{\kappa \theta}{\xi^2} \left( \kappa - \rho \xi u - \sqrt{(\kappa - \rho \xi u)^2 - \xi^2 u (u - 1)} \right),
\]

so that

\[
\Lambda(u) := \lim_{t \to \infty} t^{-1} \Lambda_t(u) = \Lambda^h(u) + \Lambda^J(u), \quad \text{for all } u \in [u_-^h \lor u_-^J, u_+^h \land u_+^J].
\]

and

\[
\Lambda^*(x) = \sup_{u \in [u_-^h \lor u_-^J, u_+^h \land u_+^J]} \{ux - \Lambda(u)\}, \quad \text{for all } x \in \mathbb{R}.
\]
Consider Normal Inverse Gaussian jumps, i.e. $J$ is an independent Normal Inverse Gaussian process with parameters $(\alpha, \beta, \mu, \delta)$ and Lévy exponent

$$\Lambda_{\text{NIG}}(u) = \mu u + \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right).$$

Then $u_{\pm}^{\text{NIG}} = -\beta \pm \alpha$. 
Numerical example: Heston without jumps

Heston (without jumps) calibrated on the Eurostoxx 50 on February, 15th, 2006, and then generated for $T = 9$ years. $\kappa = 1.7609$, $\theta = 0.0494$, $\sigma = 0.4086$, $v_0 = 0.0464$, $\rho = -0.5195$. 
Numerical example: Heston with NIG jumps

Same parameters as before for Heston and the following for NIG: $\alpha = 7.104$, $\beta = -3.3$, $\delta = 0.193$ and $\mu = 0.092$. Heston (with jumps) calibrated on the Eurostoxx 50.
Introduction and preliminary tools  
General results  
Applications to ASVM  
Examples  
Extensions

Barndorff-Nielsen & Shephard (2001) I

\[
\begin{align*}
\, & dX_t = -\left(\gamma k (\rho) + \frac{1}{2} V_t\right) dt + \sqrt{V_t} \, dW_t + \rho \, dJ_t, \quad X_0 = x_0 \in \mathbb{R}, \\
\, & dV_t = -\gamma V_t \, dt + dJ_t, \quad V_0 = v_0 > 0,
\end{align*}
\]

where \( \gamma > 0, \, \rho < 0 \) and \((J_t)_{t \geq 0}\) is a Lévy subordinator where the cgf of \( J_1 \) is given by \( \Lambda^J (u) = \log \mathbb{E} (e^{uJ_1}) \). \( D_{\Lambda^J} = (u_-, u_+) \), where

\[
\begin{align*}
u_\pm & := \frac{1}{2} - \rho \gamma \pm \sqrt{\frac{1}{4} - (2k^* - \rho) \gamma + \rho^2 \gamma^2}.
\end{align*}
\]

with \( k^* := \sup \{ u > 0 : k (u) < \infty \} \). We deduce the two functions \( F \) and \( R \),

\[
\begin{align*}
R (u, 0) & = \frac{1}{2} (u^2 - u) , \quad \text{and} \quad F (u, 0) = \gamma k (\rho u) - u \gamma k (\rho).
\end{align*}
\]

Consider the \( \Gamma \)-BNS model, where the subordinator is \( \Gamma (a, b) \)-distributed with \( a, b > 0 \). Hence \( k_\Gamma (u) = (b - u)^{-1} au \), and \( u_\Gamma^\pm := \frac{1}{2} - \rho \gamma \pm \sqrt{\left(\frac{1}{2} - \rho \gamma\right)^2 + 2b \gamma} \in \mathbb{R}_\pm \).
Barndorff-Nielsen & Shephard II

Γ-BNS model with $a = 1.4338$, $b = 11.6641$, $\nu_0 = 0.0145$, $\gamma = 0.5783$, (Schoutens)
Solid line: asymptotic smile. Dotted and dashed: 5, 10 and 20 years generated smile.
One step beyond

For more accurate results, it might be interesting to go one step beyond:

\[
\hat{\sigma}_t (x) = \hat{\sigma}_\infty (x) + \frac{1}{t} \hat{a} (x) + o \left( \frac{1}{t} \right), \quad \text{as} \; t \to \infty
\]

\[
\sigma_t (x) = \sigma_0 (x) + a (x) t + o (t), \quad \text{as} \; t \to 0.
\]

However large deviations do not provide the first-order term.
Complex saddlepoint methods (Heston)

From Lee (2004), we have, for any $\alpha \in \mathbb{R}$,

$$
\frac{1}{S_0} \mathbb{E}(S_t - K)^+ = \text{Residues} + \frac{1}{2\pi} \int_{-\infty-i\alpha}^{+\infty-i\alpha} e^{-izx} \phi_t(z - i) \frac{1}{iz - z^2} dz,
$$

where $x := \log (K/S_0)$ and $\phi_t$ is the Heston characteristic function. The methodology is the following (for the large-time):

- approximate $\phi_t(z) \sim e^{-\lambda(z)t} \phi(z)$. The integrand reads $\exp \{(izx - \lambda(z))t\} f(z)$. Find the saddlepoint of this function.
- Deform the integration contour through this saddlepoint using the steepest descent method.
- ‘Equate’ the Black-Scholes expansion with the model expansion.
- Back out the implied volatility.
Infinity is closer than what we think

**Figure:** Same parameters for the Heston model in the large-time regime, with $t = 5$ years.
Zero is even closer

Figure: Same parameters for the Heston model in the small-time regime, with $t = 0.2$ years.
Conclusion

Summary:

- Closed-form formulae for affine stochastic volatility models with jumps for large maturities.
- Closed-form formulae for continuous affine stochastic volatility models for small maturities.

Future research:

- What happens when 0 is not in the interior of $\mathcal{D}_\Lambda$?
- Remove the conditions $\chi(0) < 0$ and $\chi(1) < 0$.
- What happens precisely in the small-time when jumps are added?
- Determine the higher-order correction terms (in $t$ or $t^{-1}$).
- Statistical and numerical tests to assess the calibration efficiency.