Joint Dynamics using Asymptotic Methods
August 2010

Equity Markets Cross Smile Prediction Adil REGHAI (2010) adil.reghai@natixis.com
Outline of Presentation

1. Problem

2. Model Description
   - Local Volatility Component
   - Stochastic Volatility Component

3. Cross Smile Estimation
   - Spread Pricing
   - Smile Calculation

4. Back testing

5. Summary & Conclusions
1 Problem
Financial Problem

USDJPY and AUDUSD are two liquid currency pairs (ATM vols known, Smile known)

AUDJPY is less liquid (ATM vols known)

What is the smile of AUDJPY?
Mathematical Problem

$S_{1,t}, S_{2,t}$ Marginal Laws known

What is the law of $\frac{S_{1,t}}{S_{2,t}}$ ?
Approach

Use **Risk Neutral** Approach to take on board all observed data (photography)

Use **Historical Data** to incorporate evolution information (dynamic)

Base analysis on **rich** models (Multi LSVLC)

Use **efficient numerical** techniques to perform all calculations (asymptotics)
2.1 Model Description
**Model**

Each currency pair is a mixture of local volatility and stochastic volatility.

Stochastic volatility introduces correlation between volatilities.

Local volatility introduces level dependency to the cross.
Literature I – on local correlation


C-Bruno Dupire « Basket Skew Asymptotics » working paper 2004


F-B. Jourdain, Mohamed Sbai “Coupling Index and stocks” 2009
Literature II – some comments

A-It gives the framework for calibrating baskets and numerical algorithms for short term asymptotics for pricing

B-C-It provides a good grasp of the phenomenology with model free approach
Good for the phenomenology

D-Simple idea to expand the dimension and obtain stochastic correlation at a cheap cost (is used for the local correlation model)

E-Simplest local volatility extension plus direct calibration formulae and model risk illustration through the chewing gum effect

F- Nice numerical method – particle method, specific to baskets
However,

As will be shown in the sequel, we need:

- local volatility and local correlation
- Fast Calibration: flow business
- Precise formulae for pricing
Local Volatility Component

\[
\frac{dS_{1,t}}{S_{1,t}} = \sigma_1 (S_{1,t}, t) dW_{1,t} \\
\frac{dS_{2,t}}{S_{2,t}} = \sigma_2 (S_{2,t}, t) dW_{2,t} \\
< dW_{1,t}, dW_{2,t} > = \rho_{12} \left( \frac{S_{1,t}}{S_{2,t}} \right) dt
\]

Equation (1)

Historical Data confirm link between correlation and cross level
Local Volatility Component (2)

Calibrated to atm vol of the cross

\[
\ln \left( \frac{1 + \rho_{12} \left( \frac{S_{1,t}}{S_{2,t}} \right)}{1 - \rho_{12} \left( \frac{S_{1,t}}{S_{2,t}} \right)} \right) = a + b \ln \left( \frac{S_{1,t}}{S_{2,t}} \right) + c \ln^2 \left( \frac{S_{1,t}}{S_{2,t}} \right)
\]

Eq. (2)

Classical Fisher Transform: avoid boundary problems

Slope: trader’s input

Curvature: trader’s input
Stoch. Volatility Component

\[
\frac{dS_{1,t}}{S_{1,t}} = \sigma_1 e^{\alpha_1 \tilde{W}_{1,t} - \frac{1}{2} \alpha_1^2 t} dW_{1,t}
\]

\[
\frac{dS_{2,t}}{S_{2,t}} = \sigma_2 e^{\alpha_2 \tilde{W}_{2,t} - \frac{1}{2} \alpha_2^2 t} dW_{2,t}
\]

\[
< dW_{1,t}, dW_{2,t} >= \rho_{12}^S dt
\]

\[
< d\tilde{W}_{1,t}, d\tilde{W}_{2,t} >= \rho_{12}^\sigma dt
\]

\[
< dW_{1,t}, d\tilde{W}_{2,t} >= \rho_{12}^{S, \sigma} dt
\]

\[
< d\tilde{W}_{1,t}, dW_{2,t} >= \rho_{12}^{\sigma, S} dt
\]

Eq. (3)

Spot correlation is calibrated to the atm of the cross

Volvol correlation is a trader’s input that can be estimated through historical data

Spot vol correlation has a very small impact
2.2 Mathematical Results
Results

A- Short Term Asymptotic for LSVLC

B- Multi Stochastic VoL Perturbation approach

C- Multi Local Volatility Using most likely path combined with gradient conditioning

D- LSVLC combination result
A-Ito and Short Term Asymptotic

Pricing European Options under the general Local Stochastic Volatility and Local Correlation

\[
\frac{dS_{i,t}}{S_{i,t}} = \sigma_i dW_{i,t}^S, \quad i = 1, \ldots, n
\]

\[
\frac{d\sigma_i}{\sigma_i} = \alpha_i dW_{i,t}^\sigma
\]

\[
< dW_{i,t}^u, dW_{j,t}^v >= \rho_{i,j}^{u,v} (S_{1,t}, \ldots, S_{n,t}) dt, \quad u, v \in \{S, \sigma\} \quad \text{Eq. (4)}
\]

We want to price the Basket European options “linear”

\[
B_T = \sum_{i=1}^{n} S_i(T) \quad \text{Eq. (5)}
\]
A-Ito and Short Term Asymptotic

Notations

\[ \sigma_B^2 = \sum_{i,j=1}^{n} \rho_{i,j} \sigma_i \sigma_j \omega_i \omega_j \]

\[ \omega_i = \frac{S_i}{\sum_{j=1}^{n} S_j} \]

\[ \beta_{i,j} = \frac{\sum_{i,j=1}^{n} \rho_{i,j} \sigma_i \sigma_j \omega_i \omega_j}{\sum_{i,j=1}^{n} \rho_{i,j} \sigma_i \sigma_j \omega_i \omega_j} \]

\[ \beta_i = \sum_{j=1}^{n} \beta_{i,j} = \sum_{j=1}^{n} \beta_{j,i} \]

Approach

- Use Ito on special variable
- Take limit when time goes to zero
A-Ito and Short Term Asymptotic

Result

\[ X_t = B_t \frac{1}{\sigma_B} \]

\[ dX_t = \sum_{i=1}^n \frac{\sigma_i \omega_i dW_{i,t}}{\sigma_B} - \frac{1}{2} \ln X_t \frac{d \sigma_B^2}{\sigma_B^2} + \theta_i dt \]

\[ \frac{d \sigma_B^2}{\sigma_B^2} = \sum_{i,j=1}^n \beta_{i,j} \frac{d \rho_{i,j}}{\rho_{i,j}} + 2 \sum_{i=1}^n \beta_i \frac{d \sigma_i}{\sigma_i} + 2 \sum_{i=1}^n \beta_i \frac{d \omega_i}{\omega_i} \]

Three terms contributing to the distortion from a log normal

- (1) Weights variability
- (2) Each underlying own distortion
- (3) Correlation skew
A-Case I : Pat Hagan formula recovered

We look at a one stoch vol model - keep one underlying :

\[ X_t = B_T \frac{1}{\sigma_B} \]

\[ \frac{dX_t}{X_t} = \sigma(X_t)dz_t \]

\[ \sigma^2(X_t) = 1 + \alpha^2 \ln^2(X_t) - 2 \rho_{S,\sigma} \alpha \ln(X_t) \]

This becomes a local volatility model for which the implied volatility is given by the classical BBF formula in [6]

Recover easily the Pat Hagan formula cf [7]
A-Case II : Sum of log-normals is not a log normal

We keep one term coming from the weights variability:

\[
X_t = B_T \frac{1}{\sigma_B}
\]
\[
\frac{dX_t}{X_t} = \sigma(X_t) dZ_t
\]

\[
\sigma^2(X_t) = \sum_{i, j=1}^{n} \frac{w_i \sigma_i \sigma_j \rho_{i,j}^{s,s}}{\sigma_B^2}
\]

\[
-2 \ln(X_t) \sum_{i, j=1}^{n} (\beta_i - w_i) \frac{\sigma_i \sigma_j \rho_{i,j}^{s,s}}{\sigma_B}
\]
\[
+ \ln^2(X_t) \sum_{i, j=1}^{n} (\beta_i - w_i)(\beta_j - w_j) \sigma_i \sigma_j \rho_{i,j}^{s,s}
\]

- We have a skew
- We have a curvature
- The distribution that is generated is not a log-normal
A-Case III : Multi Stoch vol and no local correlation nor local volatility

We keep contribution from each underlying smile

We neglect the variability of the weights (in practice it is negligible)

\[
X_t = B_T \frac{1}{\sigma_B} \\
\frac{dX_t}{X_t} = \sigma(X_t) dZ_t \\
\sigma^2(X_t) = 1 - 2 \ln(X_t) \sum_{i,j=1}^n \frac{\sigma_i}{\sigma_B} w_i \beta_j \alpha_j \rho_{i,j}^{\sigma,\sigma} + \ln^2(X_t) \sum_{i,j=1}^n \beta_i \alpha_i \beta_j \alpha_j \rho_{i,j}^{\sigma,\sigma}
\]
A-Case IV: Multi Stoch vol asymptotic implied volatility calculation

Moment match the two coefficients of the log(X) expansions

Use the Pat Hagan formula

\[ \tilde{\alpha}^2 = \sum_{i,j=1}^{n} \beta_i \alpha_i \beta_j \alpha_j \rho_{i,j}^{\sigma,\sigma} \]

\[ \tilde{\rho} \tilde{\alpha} = \sum_{i,j=1}^{n} \frac{\sigma_i}{\sigma_B} w_i \beta_i \alpha_i \rho_{i,j}^{\sigma,\sigma} \]

\[ \Sigma_{BS} (T, K) = \Sigma (T, B_0) f \left( \frac{\ln \left( \frac{B_0}{K} \right)}{\Sigma (T, B_0)} \right) \]

\[ f(x) = \frac{\tilde{\alpha} x}{\ln \left( \frac{\sqrt{\tilde{\alpha}^2 x^2 - 2 \tilde{\alpha} \tilde{\rho} x + 1 - \tilde{\rho} + \tilde{\alpha} x}}{1 - \tilde{\rho}} \right)} \]

Moment Match

Pat Hagan Formula cf[7]
A-Case V: Local Correlation Model

No stochastic volatility

We assume that the local correlation is given by the following formula

\[
\frac{d \rho_{i,j}}{\rho_{i,j}} = - (1 - \delta_{i,j}) \lambda_{i,j} \ln \left( X_t \right) dZ_t
\]

We obtain the following dynamic

\[
X_t = B_T \frac{1}{\sigma_B}
\]

\[
\frac{dX_t}{X_t} = \sum_{i=1}^{n} \frac{\sigma_i \omega_i dW_{i,t}}{\sigma_B} - \frac{1}{2} \ln^2 X_t \left( \sum_{i,j=1}^{n} \beta_{i,j} \lambda_{i,j} dZ_t \right) + \theta \, dt
\]

\[
\frac{dX_t}{X_t} = \sigma \left( X_t \right) dB_t
\]

\[
\sigma^2 \left( X_t \right) = 1 + \frac{1}{4} \ln^4 X_t \left( \sum_{i,j=1}^{n} \beta_{i,j} \lambda_{i,j} \right)
\]
Pricing European Options under multi asset Stochastic Volatility can be performed using perturbation techniques

\[
\begin{align*}
\frac{dS_i}{S_i} &= \mu_i dt + \sigma_i dW_i^s \\
\sigma_i &= \varepsilon \alpha_i \sigma_i dW_i^\sigma
\end{align*}
\]

Eq. (6)

\[< dW_i^s, dW_j^\sigma > = \sqrt{\varepsilon} \rho_i^{s,\sigma} dt, < dW_i^s, dW_j^s > = \rho_i^s dt, < dW_i^\sigma, dW_j^\sigma > = \rho_i^\sigma dt\]

We want to price the European payoff \( f \)

\[u = E(f(S_1(T),...,S_n(T))| S_1(t)=s_1,...,S_n(t)=s_n, \sigma_1(t)=\sigma_1,...,\sigma_n(t)=\sigma_n)\]
B-Multi Stoch. Volatility (2)

It is based on the Black Scholes price (eps=0) and its greeks

Result

\[
\begin{aligned}
\partial_t u_0 + \sum_i \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) \partial_{x_i} u_0 + \frac{1}{2} \sum_{i,j} \rho_{i,j} \sigma_i \sigma_j \partial_{x_i x_j} u_0 &= 0 \\
u_0(T) &= f
\end{aligned}
\]

Eq. (7)

\[
u = u_0
\]

\[
\begin{aligned}
&T - t \left\{ \sum_{i,j} \frac{1}{12} \alpha_i \alpha_j \sigma_i(t) \sigma_j(t) \rho_{i,j}^{\sigma,\sigma} \frac{\partial u_0}{\partial (\sigma_i \sigma_j)} \right\} \\
&T - t \left\{ \sum_{i,j} \frac{1}{6} \alpha_i \alpha_j \sigma_i(t) \sigma_j(t) \rho_{i,j}^{\sigma,\sigma} \frac{\partial^2 u_0}{\partial \sigma_i \partial \sigma_j} \right\} \\
&T - t \left\{ \sum_{i,j} \frac{1}{2} \sigma_i(t) \alpha_j \sigma_j(t) \rho_{i,j}^{\sigma,\sigma} S_i \frac{\partial^2 u_0}{\partial S_i \partial \sigma_j} \right\}
\end{aligned}
\]

\[
\begin{aligned}
&\text{Black & Scholes} \\
&\text{cross Varga} \\
&\text{cross Vomma} \\
&\text{cross Vanna}
\end{aligned}
\]

Eq. (8)
B-Proof in 1D

Dynamic in 1D is given by:

\[
\frac{dS^\varepsilon}{S^\varepsilon} = \mu dt + \sigma dW^1_t \\
d\sigma_t = \varepsilon \eta(t, \sigma_t) dt + \sqrt{\varepsilon} \alpha(\sigma_t, t) dW^2_t
\]

Option’s price satisfies

\[
C(t, s, \varepsilon)
\]

\[
C_t + \mu_s C_s + \frac{1}{2} \sigma^2 s^2 C_{ss} + \varepsilon \left( \eta(t, \sigma) C_{\sigma} + \frac{\alpha^2(t, \sigma)}{2} C_{\sigma\sigma} + \sigma \rho_s \sigma(t, \sigma) \right) = 0 
\]

\[
C(T, s) = \phi(s)
\]

Eq. (9)  
Eq. (10)
B-Proof in 1 D : order 0

Equation order 0 is Black & Scholes:

\[ C_{t}^0 + \mu_s s C_{s}^0 + \frac{1}{2} \sigma_{0}^2 s^2 C_{ss}^0 = 0 \]
\[ C^0(T,s) = \phi(s) \]

Eq. (11)

Change of variables

\[ \phi_{\exp}(x) = \phi(S_0 e^x) \]

\[ C_t + (\mu - \frac{\sigma^2}{2})C_s + \frac{1}{2} \sigma^2 C_{ss} + \epsilon \left( \eta(t,\sigma)C_{\sigma} + \frac{\alpha^2(t,\sigma)}{2} C_{\sigma\sigma} + \sigma \rho \alpha(t,\sigma) C_{x\sigma} \right) = 0 \]
\[ C(T,s) = \phi_{\exp}(s) \]

Eq. (12)
B-Proof in 1 D : order 1

Compute derivative w.r.t to epsilon:
\[ v^\varepsilon (t, x) = \partial_\varepsilon C(t, x, \varepsilon) \]

Equation becomes:
\[
v^\varepsilon_i + (\mu_i - \frac{\sigma^2}{2})v^\varepsilon_i + \frac{1}{2} \sigma^2 v^\varepsilon_{xx} + \varepsilon \left( \eta(t, \sigma)v^\varepsilon_{\sigma} + \frac{\alpha^2(t, \sigma)}{2} v^\varepsilon_{\sigma\sigma} + \sigma \rho \alpha(t, \sigma) v^\varepsilon_{x\sigma} \right) = \\
- \left( \eta(t, \sigma)C_{\sigma} + \frac{\alpha^2(t, \sigma)}{2} C_{\sigma\sigma} + \sigma \rho \alpha(t, \sigma) C_{x\sigma} \right)
\]
\[ v^\varepsilon(T, x) = 0 \]
B-Proof in 1 D : Math Toolbox

(1) Lemma (magical lemma):

- Let $X_t$ be a martingale and let $P(t, X_t)$ a pricing function then:

Eq. (14)

$$\frac{\partial^n P}{\partial x^n}(t, X_t) = E\left[\frac{\partial^n P}{\partial x^n}(T, X_T)\big| X_t\right] \quad \forall n$$

Feynmann-Kac

If $X_t$ satisfies the following SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

Then the following value function

$$u(t, x) = E_{X_t=x}\left(f(X_T) + \int_t^T g(X_s)ds\right)$$

Satisfies The Feymann - Kac equation

$$u_t + au_x + \frac{1}{2}bu_{xx} = g$$

$$u_T = f$$

Eq. (15)
B-Proof in 1 D : order 1

Keep only order 1 in eps:

\[ v_t + \left( \mu - \frac{\sigma^2}{2} \right) v_x + \frac{1}{2} \sigma^2 v_{xx} = -\left( \eta(t, \sigma_0) C_0^0 + \frac{\alpha^2(t, \sigma_0)}{2} C_0^0 \sigma + \sigma_0 \rho \alpha(t, \sigma_0) C_0^0 \right) \]

\[ v(T, x) = 0 \]

Eq. (16)

Use Feymann-Kac:

\[ v(t, x) = E \left[ \int_t^T (\theta, \sigma_0) C_0^0 (\theta, X_\theta) + \frac{1}{2} \alpha^2(\theta, \sigma_0) C_0^0 (\theta, X_\theta) \right. \]

\[ + \sigma_0 \rho \theta \alpha^2(\theta, \sigma_0) C_0^0 (\theta, X_\theta))d\theta | X_t = x \]

Eq. (17)

To use lemma we need to transform vol derivatives into \( x \) derivatives
B-Proof in 1 D : order 1

Following Black&Scholes relations hold:

\[ C^0_x(t, x) = \sigma_0 (T - t)(C^0_{xx}(t, x) - C^0_x(t, x)) \] \hspace{1cm} \text{Eq. (18)}

\[ C^0_{x\sigma}(t, x) = \sigma_0 (T - t)(C^0_{xxx}(t, x) - C^0_{xx}(t, x)) \]

\[ C^0_{\sigma\sigma}(t, x) = (T - t)(C^0_{xx}(t, x) - C^0_x(t, x)) + \sigma_0^2 (T - t)^2 (C^0_{xxxx}(t, x) - 2C^0_{xxx}(t, x) + C^0_{xx}(t, x)) \]
Therefore:

\[
E\left[ \int_t^T \eta(\theta, \sigma_0)C_0^0(\theta, X_\theta) d\theta \right] = \sigma_0 \left( \int_t^T (T-\theta)\eta(\theta, \sigma_0) d\theta \right) (C_{xx}^0(t,x) - C_x^0(t,x)) = \frac{1}{(T-t)} \left( \int_t^T (T-\theta)\eta(\theta, \sigma_0) d\theta \right) C_\sigma(t,x)
\]

\[
E\left[ \int_t^T \rho_\sigma \alpha(\theta, \sigma_0)C_{x\sigma}^0(\theta, X_\theta) d\theta \right] = \sigma_0 \left( \int_t^T (T-\theta)\eta(\theta, \sigma_0) d\theta \right) (C_{xxx}^0(t,x) - C_{xx}^0(t,x)) = \frac{1}{(T-t)} \left( \int_t^T (T-\theta)\rho_\sigma \alpha(\theta, \sigma_0) d\theta \right) C_{\sigma x}^0(t,x)
\]

\[
E\left[ \int_t^T \alpha^2(\theta, \sigma_0)C_{\sigma\sigma}^0(\theta, X_\theta) d\theta \right] = \left( \int_t^T (T-\theta)\alpha^2(\theta, \sigma_0) d\theta \right) (C_{xxx}^0(t,x) - C_x^0(t,x)) + \sigma_0^2 \left( \int_t^T (T-\theta)^2\alpha^2(\theta, \sigma_0) d\theta \right) (C_{xxxx}^0(t,x) - 2C_{xxx}^0(t,x) + C_{xx}^0(t,x))
\]
finally:

\[ C(t, x, \varepsilon) = C^0(t, x) + \varepsilon(VegaFactor \times C^0_{\sigma}(t, x) + VannaFacto rC^0_{\chi\sigma}(t, x) + Volg aFactorC^0_{\sigma\sigma}(t, x)) \]

\[ VegaFactor = \frac{1}{T-t} \left( \int_t^T (T-\theta)\eta(\theta, \sigma_0)d\theta \right) + \frac{1}{2(T-t)\sigma_0} \left( \int_t^T (T-\theta)^2\psi(\theta, \sigma_0)d\theta \right) \]

\[ Volg aFactor = \frac{1}{2(T-t)^2} \int_t^T (T-\theta)^2\alpha^2(\theta, \sigma_0)d\theta \]

\[ VannaFactor = \frac{s}{T-t} \int_t^T (T-\theta)\sigma_0\rho_0\alpha(\theta, \sigma_0)d\theta \]

Eq. (20)
Pricing European Options under multi local volatility local correlation model can be performed using perturbation techniques

\[
\frac{dS_i}{S_i} = \mu_i dt + \sigma_i(t, S_i) dW_i \\
<dW_i, dW_j> = \rho_{i,j}(t, S_1, \ldots, S_n) dt
\]

Eq. (21)

We want to price the European payoff \( f \)

\[
u = E(f(S_1(T), \ldots, S_n(T)) | S_1(t) = s_1, \ldots, S_n(t) = s_n)
\]
Using a simple perturbation analysis and Feymann Kac, we obtain the following result – just like before in the stochastic volatility case:

\[
\begin{align*}
    u_{LVLC} &= u_{BS} + E_{BS} \left( \frac{1}{2} \sum_{i,j}^{n} \int_{0}^{T} \left( \rho_{i,j}(t, S_1, \ldots, S_n) \sigma_i(t, S_i) \sigma_j(t, S_j) - \rho_{i,j}^{BS} \sigma_i^{BS} \sigma_j^{BS} \frac{\partial^2 u_{BS}}{\partial S_i \partial S_j} \right) dt \right) + O(\varepsilon^2)
\end{align*}
\]

Eq. (22)

Yet formulae are not practical – many integrals to be computed

We do not have the magical lemma 😞
C- Specific work for linear payoffs (3)

Consider a linear payoff

- Basket with positive weights
- Spread options

Under the general dynamic

\[
\frac{dS_i}{S_i} = \mu_i dt + \sigma_i(t, S_i) dW_i \\
< dW_i , dW_j > = \rho_{i,j}(t, S_1, \ldots, S_n) dt
\]

\[
\psi = \left( \sum_{i=1}^{n} w_i S_i(T) - k \right)^+ 
\]

Eq. (23)
C-Most Likely Path Pricing under Multi Local Volatility (4)

Model Reduction – using Gradient Conditionning (Curran)

\[ k_i = E\left(S_i(T) \mid B(T) = k\right) \]
\[ \cong E\left(S_i(T) \mid Z = z^*\right) \]
\[ E\left(B(T) \mid Z = z^*\right) = k \]

Eq. (24)

<table>
<thead>
<tr>
<th>Asset</th>
<th>Forward</th>
<th>Vol. ATM</th>
<th>Slope</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.</td>
<td>0.20</td>
<td>-0.30</td>
<td>0.333</td>
</tr>
<tr>
<td>2</td>
<td>1.</td>
<td>0.25</td>
<td>-0.30</td>
<td>0.333</td>
</tr>
<tr>
<td>3</td>
<td>1.</td>
<td>0.30</td>
<td>-0.30</td>
<td>0.333</td>
</tr>
</tbody>
</table>
Model Reduction – works as well for spreads

<table>
<thead>
<tr>
<th>Asset/Value</th>
<th>Forward</th>
<th>Vol. ATM</th>
<th>Slope</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.20</td>
<td>-0.30</td>
<td>0.50</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.25</td>
<td>-0.30</td>
<td>0.50</td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.30</td>
<td>-0.30</td>
<td>-1.00</td>
</tr>
</tbody>
</table>
C-Most Likely Path Pricing under Multi Local Volatility (6)

Local Volatility Model becomes like when pricing $\psi$

A simpler model – A multi Black Scholes Model:

Same methodology as in [3] and [4]:

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i * dW_i$$

Eq. (25)

$$< dW_i, dW_j >= \rho_{i,j}(t, k_1, ..., k_n) dt$$
Differentiating twice and integrating with \( \exp(ik) \) we obtain the moment generating function for the joint distribution - trick in [6]

\[
\int_{-\infty}^{\infty} \frac{d^2 \psi}{dk^2} \exp(ik) dk = E \left( \exp \left( i \sum_{j=1}^{n} w_j S_j(T) \right) \right)
\]

Eq. (26)

We can recover the joint density (Fourier inversion) and be able to price all European payoffs ☺

Numerically tractable in low dimensions 3 to 4 😞
D-Multi Local Vol & Stoch. Volatility

Using Perturbation techniques under the general model

\[
\frac{dS_i}{S_i} = \mu_i dt + \sigma_i f_i(t, S_i) dW_i^s \\
d\sigma_i = \varepsilon\sigma_i dW_i^\sigma
\]

Eq. (27)

We have the following Pricing approximating results

Price = Price Local Vol + \varepsilon^2(Price Stoch Vol - Price Local Vol Local Correl)

Eq. (28)
Summary

65% from Barrier Business
Relies on fast approximation
 Allows to Price Correlation Products
Determines Cross Smile
3 Cross Smile Estimation
Pricing Calls on the cross

Pricing a call on the cross

\[ \left( \frac{S_{2,t} - k}{S_{1,t}} \right)^+ \]

Is equivalent to pricing a spread option on the two currency pairs

\[ (S_{2,t} - kS_{1,t})^+ \]

Warning: only true after change of numeraire
Pricing Example: AUDJPY

Pricing a call on the cross

\[
\left( \frac{\text{JPY}}{\text{AUD}} - k \right)^+ \quad \text{under JPY measure}
\]

Is equivalent to pricing a spread option on the two currency pairs

\[
\left( \frac{\text{USD}}{\text{AUD}} - k \frac{\text{USD}}{\text{JPY}} \right)^+ \quad \text{under USD measure}
\]
Calculating Smile

From call prices we back out implied volatilities

From Implied Volatilities we back out smile characteristics
4 Back Testing
Method

Collect complete data of:
- AUDUSD, USDJPY and AUDJPY

Apply Model with different inputs

Compare Predicted smiles with observed ones of AUDJPY
# Historical Data

## From 18/11/02 to 17/11/04

### Data for 1y cross smile

<table>
<thead>
<tr>
<th></th>
<th>AUDUSD</th>
<th>USDJPY</th>
<th>AUDJPY</th>
<th>AUDUSD</th>
<th>USDJPY</th>
<th>AUDJPY</th>
<th>USD</th>
<th>AUD</th>
<th>JPY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>spot</strong></td>
<td>0.56235</td>
<td>121.02</td>
<td>68.04</td>
<td>0.54498</td>
<td>119.11</td>
<td>64.91257</td>
<td>0.984225738</td>
<td>0.953849</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.55895</td>
<td>122.19</td>
<td>68.28</td>
<td>0.54165</td>
<td>120.29</td>
<td>65.15508</td>
<td>0.98412732</td>
<td>0.953849</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.56055</td>
<td>122.67</td>
<td>68.78</td>
<td>0.54345</td>
<td>120.74</td>
<td>65.61615</td>
<td>0.983438672</td>
<td>0.953801</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.56285</td>
<td>122.65</td>
<td>69.04</td>
<td>0.54563</td>
<td>120.645</td>
<td>65.82753</td>
<td>0.983242004</td>
<td>0.953324</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.56375</td>
<td>122.835</td>
<td>69.25</td>
<td>0.5465</td>
<td>120.805</td>
<td>66.01993</td>
<td>0.983340333</td>
<td>0.9528</td>
<td>0.99935</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>USD</th>
<th>AUD</th>
<th>JPY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>fwd</strong></td>
<td>0.984225738</td>
<td>0.953849</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.98412732</td>
<td>0.953849</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.983438672</td>
<td>0.953801</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.983242004</td>
<td>0.953324</td>
<td>0.99935</td>
</tr>
<tr>
<td></td>
<td>0.983340333</td>
<td>0.9528</td>
<td>0.99935</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>AUDUSD</th>
<th>USDJPY</th>
<th>AUDJPY</th>
<th>AUDUSD</th>
<th>USDJPY</th>
<th>AUDJPY</th>
<th>USD</th>
<th>AUD</th>
<th>JPY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>vols</strong></td>
<td>9.60%</td>
<td>9.35%</td>
<td>11.10%</td>
<td>-0.10%</td>
<td>-0.50%</td>
<td>0.55%</td>
<td>0.31%</td>
<td>0.37%</td>
<td>0.37%</td>
</tr>
<tr>
<td></td>
<td>9.60%</td>
<td>9.35%</td>
<td>10.90%</td>
<td>-0.10%</td>
<td>-0.40%</td>
<td>0.55%</td>
<td>0.31%</td>
<td>0.37%</td>
<td>0.37%</td>
</tr>
<tr>
<td></td>
<td>9.55%</td>
<td>9.30%</td>
<td>10.90%</td>
<td>-0.12%</td>
<td>-0.35%</td>
<td>0.55%</td>
<td>0.31%</td>
<td>0.37%</td>
<td>0.37%</td>
</tr>
<tr>
<td></td>
<td>9.60%</td>
<td>9.30%</td>
<td>11.05%</td>
<td>-0.12%</td>
<td>-0.30%</td>
<td>0.55%</td>
<td>0.31%</td>
<td>0.37%</td>
<td>0.37%</td>
</tr>
<tr>
<td></td>
<td>9.60%</td>
<td>9.25%</td>
<td>11.00%</td>
<td>-0.10%</td>
<td>-0.30%</td>
<td>0.55%</td>
<td>0.31%</td>
<td>0.37%</td>
<td>0.37%</td>
</tr>
</tbody>
</table>

---

*Not liquid data*
Tests

We consider 4 different cases:

- Slope corr=0, curve corr=0, volvolcorr=0
- Slope corr=2, curve corr=0, volvolcorr=0
- Slope corr=2, curve corr=0, volvolcorr=0.5
- Slope corr=1.8, curve corr=-28, volvolcorr=0.5
Test 1

**Strangle**

Local correlation with slope 0
volvol correlation = 0
AUDJPY Strangle

<table>
<thead>
<tr>
<th>Date</th>
<th>0.00%</th>
<th>0.10%</th>
<th>0.20%</th>
<th>0.30%</th>
<th>0.40%</th>
<th>0.50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>01-Sep-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10-Dec-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20-Mar-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28-Jun-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>06-Oct-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14-Jan-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23-Apr-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01-Aug-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>09-Nov-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17-Feb-05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Risk Reversal**

Local Correlation with slope 0
volvol correlation = 0
AUDJPY Risk Reversal

<table>
<thead>
<tr>
<th>Date</th>
<th>-2.50%</th>
<th>-2.00%</th>
<th>-1.50%</th>
<th>-1.00%</th>
<th>-0.50%</th>
<th>0.00%</th>
<th>0.50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>01-Sep-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10-Dec-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20-Mar-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28-Jun-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>06-Oct-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14-Jan-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23-Apr-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>01-Aug-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>09-Nov-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17-Feb-05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Smile Strangle given by Model
Historical Smile Strangle

Risk Reversal given by model
Historical Risk Reversal
Test 2

Strangle

Local Correlation Slope = 2
volvol correlation = 0
AUDJPY Strangle

Risk Reversal

Local Correlation Slope = 2
volvol correlation = 0
AUDJPY Risk Reversal
Test 3

Strangle

Risk Reversal

Local Correlation Slope = 2
volvol correlation = 0.5
AUDJPY Strangle

Local Correlation Slope = 2
volvol correlation = 0.5
AUDJPY Risk Reversal
Test 4

Local Correlation Slope = 2
Curvature = -28
volvol correlation = 0.5
AUDJPY Strangle

Local Correlation Slope = 1.8
Curvature = -28
volvol correlation = 0.5
AUDJPY Risk Reversal

Smile Strangle given by the model
Historical Smile Strangle

Risk Reversal given by the model
Historical Risk Reversal

Aug-10
Summary

A new model for cross smile estimation is produced

Uses a mixture of local vol and stoch. Vol

Introduces volvol correlation and local correlation

Playing on the parameters offers flexibility to predict market levels

It is based on efficient numerical techniques
Questions

Thank you for your attention
References

- [1] www.sciencedirect.com
Reference prices are based on closing prices.

The information contained in these publications is exclusively intended for a client base consisting of professionals or qualified investors. It is sent to you by way of information and cannot be disclosed to a third party without the prior consent of Natixis. It cannot be considered under any circumstances as an offer to sell, or a solicitation of any offer to buy financial instruments. While all reasonable effort has been made to ensure that the information contained is not untrue or misleading at the time of publication, no representation is made as to its accuracy or completeness and it should not be relied upon as such. Past and simulated performances offer no guarantee as to future performances. Any opinions offered herein reflect our current judgement and may change without notice. Natixis cannot be held responsible for the consequences of any decision made with regard to the information contained in those documents. Natixis has set up due procedures for the separation of activities, notably in order to prevent conflicts of interest between the research activities and its other activities. Details of these ‘information barriers’ are available on request from the head of compliance. On the date of those reports, Natixis and/or one of its subsidiaries may be in a conflict of interest with the issuer mentioned herein. In particular, it may be that Natixis or any person or company linked thereto, their respective directors and/or representatives and/or employees, has/have invested on their own account in, or act or intend to act, in the next twelve months, as an advisor, provider of underwriting transactions, placements or connected transactions, for a company discussed in this report.

This research may be disseminated from the United Kingdom by Natixis, London Branch, which is authorised by the ACP and subject to limited regulation by the Financial Services Authority. Details about the extent of regulation by the Financial Services Authority are available from the London Branch on request.

The transfer / distribution of this document in Germany is done by / under the responsibility of Natixis Zweigniederlassung Deutschland. NATIXIS is authorized by the ACP and regulated by BaFin (Bundesanstalt für Finanzdienstleistungsaufsicht) for the conduct of its business in Germany.

Natixis is authorized by the ACP and regulated by Bank of Spain and the CNMV (Comisión Nacional para la Sociedad de Bolsa) for the conduct of its business in Spain.

Natixis is authorized by the ACP and regulated by Bank of Italy and the CONSOB (Commissione Nazionale per le Società e la Borsa) for the conduct of its business in Italy.

Natixis, a foreign bank and broker-dealer, makes this research report available solely for distribution in the United States to major U.S. institutional investors as defined in Rule 15a-6 under the U.S. Securities Act of 1934. This document shall not be distributed to any other persons in the United States. All major U.S. institutional investors receiving this document shall not distribute the original nor a copy thereof to any other person in the United States. Natixis Bleichroeder LLC a U.S. registered broker-dealer and member of FINRA is a subsidiary of Natixis. Natixis has no officers or employees in common with Natixis Bleichroeder LLC. Natixis Bleichroeder LLC did not participate in the preparation of this research report and as such assumes no responsibility for its content. This research report has been prepared and reviewed by research analysts employed by Natixis, who are not associated persons of Natixis Bleichroeder LLC and are not registered or qualified as research analysts with FINRA, and may change without notice. Natixis Bleichroeder LLC did not participate in the preparation of this research report and as such assumes no responsibility for its content. If you do not wish to receive any additional information about or to effect a transaction in any security or financial instrument mentioned herein, please contact a registered representative Natixis Bleichroeder LLC 1345 Avenue of the Americas, New York, NY 10019.