Model independent bounds for variance swaps

Martin Klimmek

with David Hobson
University of Warwick

August 23, 2010
Question:

What is the range of possible values for a security paying

$$\int_0^T \frac{(dX_t)^2}{X_t^2}$$

if \((X_t)_{t \geq 0}\) is a martingale started at a fixed point and it’s law at time \(T\) is \(\mu\)?
Answer: The continuous case

Suppose that $X$ is a continuous martingale. By Ito’s formula,

$$d \log(X_t) = \frac{dX_t}{X_T} - \frac{1}{2} \frac{(dX_t)^2}{X_t^2}.$$ 

Then

$$\int_0^T \frac{(dX_t)^2}{X_t^2} = -2 \log(X_T) + 2 \log(X_0) + \int_0^T \frac{2}{X_t} dX_t.$$ 

In the continuous case a model-independent price and hedge are trivial.
Continuity?

-2 log-contracts $\sim$ VIX.

Figure: 7th May 2010, Flash Crash - VIX
Intuition for an answer in the general case

Drop the continuity assumption and assume only right-continuity.

Itô for semimartingales:

\[
\int_0^T \frac{(dX_t)^2}{X_t^2} = -2 \log(X_T/X_0) + 2 \int_0^1 \frac{dX_t}{X_t} \quad - \sum_{0 \leq t \leq 1} 2 \left( \frac{\Delta X_t}{X_{t^-}} \right) - 2 \log \left( 1 + \frac{\Delta X_t}{X_{t^-}} \right) - \left( \frac{\Delta X_t}{X_{t^-}} \right)^2
\]

Let

\[ J(x) = -2x + 2 \log(1 + x) + x^2. \]

\[
\mathbb{E} \left[ \int_0^T \frac{(dX_t)^2}{X_t^2} \right] = -2 \mathbb{E} \left[ \log(X_T/X_0) \right] + \mathbb{E} \left[ \sum_{0 \leq t \leq T} J \left( \frac{\Delta X_t}{X_{t^-}} \right) \right].
\]
If jumps are positive:

\[ J \left( \sum_{t} \frac{\Delta X_t^J}{X_{t-}^J} \right) \geq \sum_{t} J \left( \frac{\Delta X_t^J}{X_{t-}^J} \right). \]

If jumps are negative:

\[ J \left( \sum_{t} \frac{\Delta X_t^J}{X_{t-}^J} \right) \leq \sum_{t} J \left( \frac{\Delta X_t^J}{X_{t-}^J} \right). \]

Intuition is to look for a one-jump martingales to maximise (up-jump), minimise (down-jump) the value of the variance swap.
Jump at the maximum for a lower bound?

There exists at time change $t \to A_t$ such that $X_t = B_{A_t}$.
If $A_t$ is discontinuous, so is $X_t$.

Define $R_t = \sup_{s \leq t} X_s$ and $S_t = \sup_{s \leq t} B_s$.
Note that $R_t \leq S_{A_t}$.

$$\int_0^T \frac{(dX_t)^2}{X_{t-}^2} \geq \int_0^T \frac{(dX_t)^2}{R_{t-}^2} \geq \int_0^T \frac{(dB_{A_t})^2}{S_{A_t-}^2} \geq \int_0^{A_T} \frac{du}{(S_u)^2}.$$  

Similarly, let $I_t = \inf_{s \leq t} B_s$ then:

$$\int_0^T \frac{(dX_t)^2}{X_{t-}^2} \leq \int_0^{A_T} \frac{du}{I_u^2}.$$
Simplifying $\int_{0}^{\tau} \frac{dt}{S_t^2}$

$$d\left(\frac{S_t - B_t}{S_t}\right)^2 = 2 \left(\frac{S_t - B_t}{S_t}\right) \frac{B_t}{S_t^2} dS_t - 2 \left(\frac{S_t - B_t}{S_t^2}\right) dB_t + \frac{dt}{(S_t)^2}$$

Then,

$$\int_{0}^{\tau} \frac{dt}{S_t^2} = \left(\frac{S_{\tau} - B_{\tau}}{S_{\tau}}\right)^2 + 2 \int_{0}^{\tau} \left(\frac{S_t - B_t}{S_t^2}\right) dB_t$$

The problem is to minimise $\left(\frac{S_{\tau} - B_{\tau}}{S_{\tau}}\right)^2$ over stopping times $\tau$, with the property $B_{\tau} \sim \mu$ which is:

**The Skorohod problem**

$$\min_{\tau} \mathbb{E} \left[ \left(\frac{S_{\tau} - B_{\tau}}{S_{\tau}}\right)^2 \mid B_{\tau} \sim \mu \right]$$
Solution to the Skorohod Problem

Given \( \mu \) with mean 1, there exists a decreasing function \( f : [0, \infty) \rightarrow [0, 1] \) with \( f(0) = 1 \) and a random variable \( Z \), \( \mathbb{P}(Z \geq x) = \exp(-R(x)) \) on \([1, \infty)\) such that if

\[
\tau_f = \inf\{ t \geq 0 | B_t \leq f(S_t) \},
\]
\[
\tau_G = \inf\{ t \geq 0 | S_t \geq G \},
\]

then \( \tau = \min(\tau_G, \tau_f) \) solves the embedding problem:

\[
\min_{\tau} \mathbb{E} \left[ \left( \frac{S_\tau - B_\tau}{S_\tau} \right)^2 \right| B_\tau \sim \mu]
\]
Properties of the solution

1. Let $\hat{\mu}$ be the law of $S_\tau$.
2. The embedding minimises $\hat{\mu}$ over embeddings i.e minimises $\mathbb{P}(S_\tau > x)$ for all $x$.
3. $\mathbb{P}(S_\tau \leq x) = \hat{\mu}(-\infty, x] = \mu(-\infty, x] - m(x)$.
4. $m(x) = \mathbb{P}(S_\tau \geq x, B_\tau < x) = \mathbb{P}(S_\tau \geq x) - \mathbb{P}(B_\tau \geq x)$
5. $R(x) = \int_0^x \frac{\mu(du)}{1-\hat{\mu}(-\infty,x]}$, (nice case - no atoms)
Construction of the martingale

Define the martingale

\[ N_t = B_{\min(H_{1+t}, \tau)}, \]

Note that \( N_\infty \sim \mu. \)

Let \( A(t) : [0, \infty) \to [0, T) \) be a deterministic time change.

\( M^A_t = N_{A(t)} \) is martingale with the requisite properties.
A martingale with the right properties on \([0, T]\)

Let \(F(x) = \mathbb{P}\{Z \leq x\}\) and \(h = F^{-1}\). Set \(M_t = N_{h(t/T)}\).

1. A right-continuous martingale
2. \(M_0 = 1, \ M_T \sim \mu\)
3. If \(M\) jumps at \(t\) then \(M_{t-} = \sup_{s \leq t} M_s\)
4. Carries the optimality properties of the Perkins solution and thus attains the lower bound.
Example: Target law is Uniform

\( M_T \sim U[1 - \epsilon, 1 + \epsilon], \ \epsilon \in [0, 1]. \)

The distribution function is \( F_\epsilon(x) = \frac{x-1+\epsilon}{2\epsilon} \)

\[
m(x) = \frac{1}{2\epsilon} (\epsilon - 1 + x - 2\sqrt{x\epsilon - \epsilon})
\]

\[
f(x) = F^{-1}(m(x)) = x - 2\sqrt{\epsilon(x - 1)}
\]

\[
\hat{\mu}((-\epsilon, x]) = F(x) - m(x) = 2\frac{\sqrt{\epsilon(x - 1)}}{\epsilon}
\]
Calculating the bounds for $\epsilon \in (0, 1)$

$$\mathbb{E} \left[ \int_{0}^{T} \frac{(dM_t)^2}{M_{t-}^2} \right] = \mathbb{E} \left[ \left( \frac{S_T - B_T}{S_T} \right)^2 \right]$$

$$= \int_{1}^{1+\epsilon} \frac{(x - f(x))^2}{x^2} \mathbb{P}(S_T \geq x, B_T < x) \, dx$$

$$= \int_{1}^{1+\epsilon} \frac{(x - f(x))^2}{x^2} \frac{\mathbb{P}(S_T \geq x)}{x - f(x)} \, dx$$

$$= 2 \int_{1}^{1+\epsilon} \sqrt{\epsilon(x - 1)} \times \frac{1 - \sqrt{\epsilon(x - 1)/\epsilon}}{x^2} \, dx$$

The target law is symmetric and so a reflection gives the upper bound which is:

$$\int_{1}^{1+\epsilon} \frac{\sqrt{\epsilon(x - 1)} \times 2(1 - \sqrt{\epsilon(x - 1)/\epsilon})}{(2 - x)^2} \, dx$$
Uniform Bounds - Perkins compared with Azema-Yor

Figure: Ratio of bound value to continuous log-contract value
Lognormal Example

\[ \mu_\epsilon \sim \text{lognormal}( -\frac{\epsilon^2}{2}, \epsilon ) \]
Suppose we know call prices with maturity $T$ for all strikes.

\[
C(K, T) = \mathbb{E}^{\mathbb{P}}[e^{-rT}(P_T - K)^+] \\
\mathbb{P}(P_T > K) = e^{rT} \left| \frac{\partial}{\partial K} C(K, T) \right| \\
\mathbb{P}(P_T \in K) = e^{rT} \frac{\partial^2}{\partial K^2} C(K, T)
\]

Set $X_t = e^{-rt} \mathbb{P}_t$ (martingale under a pricing measure).
$X_T \sim \mu$ is known.