# Model independent bounds for variance swaps

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# Question:

What is the range of possible values for a security paying

$$\int_0^T \frac{(dX_t)^2}{X_{t-}^2}$$

if $(X_t)_{t\geq 0}$ is a martingale started at a fixed point and it's law at time T is $\mu$ ?

# Answer: The continuous case

Suppose that $X$ is a continuous martingale. By Ito's formula,

$$d\log(X_t) = \frac{dX_t}{X_T} - \frac{1}{2}\frac{(dX_t)^2}{X_t^2}.$$

Then

$$\int_0^T \frac{(dX_t)^2}{X_{t-}^2} = -2\log(X_T) + 2\log(X_0) + \int_0^T \frac{2}{X_t} dX_t.$$

In the continuous case a model-independent price and hedge are trivial.

# Continuity?

-2 log-contracts $\sim$ VIX.


Figure: 7th May 2010, Flash Crash - VIX

# Intuition for an answer in the general case

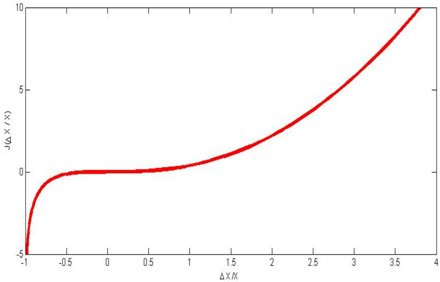
Drop the continuity assumption and assume only right-continuity.
Itô for semimartingales:

$$\int_0^T \frac{(dX_t)^2}{X_{t-}^2} = -2\log(X_T/X_0) + 2\int_0^1 \frac{dX_t}{X_{t-}} - \sum_{0\le t\le 1} 2\left(\frac{\Delta X_t}{X_{t-}}\right) - 2\log\left(1+\frac{\Delta X_t}{X_{t-}}\right) - \left(\frac{\Delta X_t}{X_{t-}}\right)^2$$

Let

$$J(x) = -2x + 2\log(1+x) + x^2.$$

$$\mathbb{E}\left[\int_0^T \frac{(dX_t)^2}{X_{t-}^2}\right] = -2\,\mathbb{E}\left[\log(X_T/X_0)\right] + \mathbb{E}\left[\sum_{0\le t\le T} J\left(\frac{\Delta X_t}{X_{t-}}\right)\right].$$




If jumps are positive:

$$J\left(\sum_t \Delta X_t^J / X_{t-}^J\right) \geq \sum_t J\left(\Delta X_t^J / X_{t-}^J\right).$$

If jumps are negative:

$$J\left(\sum_t \Delta X_t^J / X_{t-}^J\right) \leq \sum_t J\left(\Delta X_t^J / X_{t-}^J\right).$$

Intuition is to look for a one-jump martingales to maximise (up-jump), minimise (down-jump) the value of the variance swap.

# Jump at the maximum for a lower bound?

There exists at time change $t \to A_t$ such that $X_t = B_{A_t}$.
If $A_t$ is discontinuous, so is $X_t$.

Define $R_t = \sup_{s \leq t} X_s$ and $S_t = \sup_{s \leq t} B_s$.
Note that $R_t \leq S_{A_t}$.

$$\int_0^T \frac{(dX_t)^2}{X_{t-}^2} \geq \int_0^T \frac{(dX_t)^2}{R_{t-}^2} \geq \int_0^T \frac{(dB_{A_t})^2}{S_{A_{t-}}^2} \geq \int_0^{A_T} \frac{du}{(S_u)^2}.$$

Similarly, let $I_t = \inf_{s \leq t} B_s$ then:

$$\int_0^T \frac{(dX_t)^2}{X_{t-}^2} \leq \int_0^{A_T} \frac{du}{I_u^2}.$$

## Simplifying $\int_0^\tau \frac{dt}{S_t^2}$

$$d\left(\frac{S_t - B_t}{S_t}\right)^2 = 2\left(\frac{S_t - B_t}{S_t}\right)\frac{B_t}{S_t^2}dS_t - 2\left(\frac{S_t - B_t}{S_t^2}\right)dB_t + \frac{dt}{(S_t)^2}$$

Then,

$$\int_0^\tau \frac{dt}{S_t^2} = \left(\frac{S_\tau - B_\tau}{S_\tau}\right)^2 + 2\int_0^\tau \left(\frac{S_t - B_t}{S_t^2}\right)dB_t$$

The problem is to minimise $\left(\frac{S_\tau - B_\tau}{S_\tau}\right)^2$ over stopping times $\tau$, with the property $B_\tau \sim \mu$ which is:

**The Skorohod problem**

$$\min_\tau \mathbb{E}\left[\left(\frac{S_\tau - B_\tau}{S_\tau}\right)^2 | B_\tau \sim \mu]\right]$$

# Solution to the Skorohod Problem

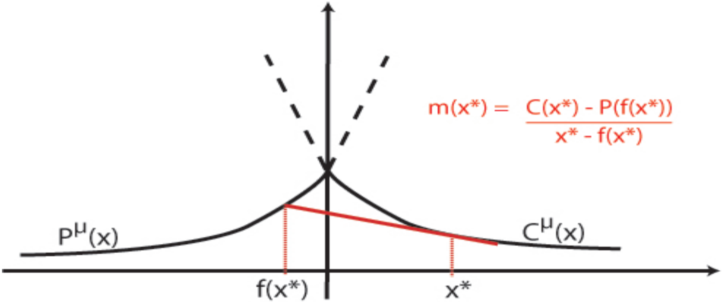
Given $\mu$ with mean 1, there exists a decreasing function $f : [0, \infty) \to [0, 1]$ with $f(0) = 1$ and a random variable $Z$, $\mathbb{P}(Z \geq x) = \exp(-R(x))$ on $[1, \infty)$ such that if

$$\begin{aligned} \tau_f &= \inf\{t \geq 0 | B_t \leq f(S_t)\} \\ \tau_G &= \inf\{t \geq 0 | S_t \geq G\} \end{aligned}$$

then $\tau = \min(\tau_G, \tau_f)$ solves the embedding problem:

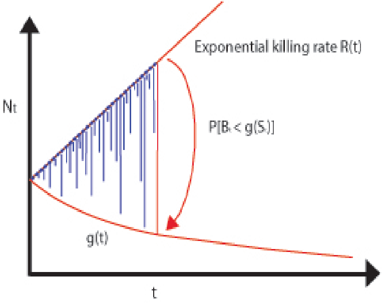
$$\min_\tau \mathbb{E}\left[\left(\frac{S_\tau - B_\tau}{S_\tau}\right)^2 |B_\tau \sim \mu]\right]$$

# Properties of the solution




1. Let $\hat{\mu}$ be the law of $S_\tau$.
2. The embedding minimises $\hat{\mu}$ over embeddings i.e minimises $\mathbb{P}(S_\tau > x)$ for all $x$.
3. $\mathbb{P}(S_\tau \leq x) = \hat{\mu}(-\infty, x] = \mu(-\infty, x] - m(x)$.
4. $m(x) = \mathbb{P}(S_\tau \geq x, B_\tau < x) = \mathbb{P}(S_\tau \geq x) - \mathbb{P}(B_\tau \geq x)$
5. $R(x) = \int_0^x \frac{\mu(du)}{1-\hat{\mu}(-\infty,x]}$, (nice case - no atoms)

# Construction of the martingale




Define the martingale

$$N_t = B_{\min(H_{1+t},\tau)},$$

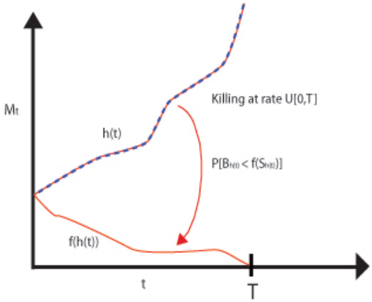
Note that $N_\infty \sim \mu$.

Let $A(t) : [0, \infty) \to [0, T)$ be a deterministic time change .

$M_t^A = N_{A(t)}$ is martingale with the requisite properties.

# A martingale with the right properties on $[0, T]$

Let $F(x) = \mathbb{P}\{Z \leq x\}$ and $h = F^{-1}$. Set $M_t = N_{h(t/T)}$.

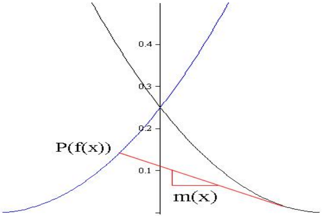



1. A right-continuous martingale
2. $M_0 = 1$, $M_T \sim \mu$
3. If $M$ jumps at $t$ then $M_{t-} = \sup_{s \leq t} M_s$
4. Carries the optimality properties of the Perkins solution and thus attains the lower bound.

# Example: Target law is Uniform

$M_T \sim U[1-\epsilon, 1+\epsilon]$, $\epsilon \in [0,1]$.

The distribution function is $F_\epsilon(x) = \frac{x-1+\epsilon}{2\epsilon}$




$$
\begin{aligned}
m(x) &= \frac{1}{2\epsilon}(\epsilon - 1 + x - 2\sqrt{x\epsilon - \epsilon}) \\
f(x) &= F^{-1}(m(x)) = x - 2\sqrt{\epsilon(x-1)} \\
\hat{\mu}((-\epsilon, x]) &= F(x) - m(x) = 2\frac{\sqrt{\epsilon(x-1)}}{\epsilon}
\end{aligned}
$$

# Calculating the bounds for $\epsilon \in (0, 1)$

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \frac{(dM_t)^2}{M_{t-}^2}\right] &= \mathbb{E}\left[\left(\frac{S_\tau - B_\tau}{S_\tau}\right)^2\right] \\
&= \int_1^{1+\epsilon} \frac{(x - f(x))^2}{x^2} \mathbb{P}(S_\tau \geq x, B_\tau < x) dx \\
&= \int_1^{1+\epsilon} \frac{(x - f(x))^2}{x^2} \frac{\mathbb{P}(S_\tau \geq x)}{x - f(x)} dx \\
&= 2\int_1^{1+\epsilon} \frac{\sqrt{\epsilon(x-1)} \times (1 - \sqrt{\epsilon(x-1)}/\epsilon)}{x^2} dx
\end{aligned}$$

The target law is symmetric and so a reflection gives the upper bound which is:

$$\int_1^{1+\epsilon} \frac{\sqrt{\epsilon(x-1)} \times 2(1 - \sqrt{\epsilon(x-1)}/\epsilon)}{(2-x)^2} dx$$

# Uniform Bounds - Perkins compared with Azema-Yor

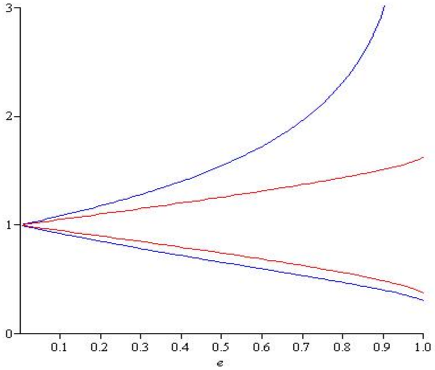
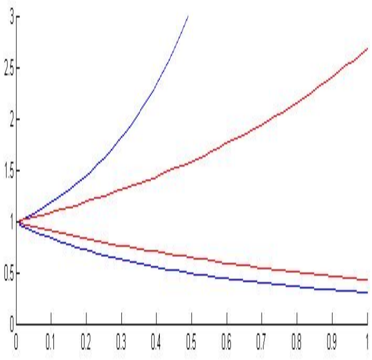



Figure: Ratio of bound value to continuous log-contract value

# Lognormal Example

$$\mu_\epsilon \sim lognormal(-\frac{\epsilon^2}{2}, \epsilon)$$



# 'Model-independence'

Suppose we know call prices with maturity $T$ for all strikes.

$$\begin{aligned} C(K,T) &= \mathbb{E}^{\mathbb{P}}[e^{-rT}(P_T - K)^+] \\ \mathbb{P}(P_T > K) &= e^{rT}\left|\frac{\partial}{\partial K}C(K,T)\right| \\ \mathbb{P}(P_T \in K) &= e^{rT}\frac{\partial^2}{\partial K^2}C(K,T) \end{aligned}$$

Set $X_t = e^{-rt}P_t$ (martingale under a pricing measure).
$X_T \sim \mu$ is known.