Hedging under arbitrage

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Third SMAI European Summer School in Financial Mathematics
August 23, 2010
Motivation

- Given: a frictionless market of stocks with continuous Markovian dynamics.
- If there does not exist an equivalent local martingale measure can we have the concept of hedging?
- Answer: Yes, if a square-integrable “market price of risk” exists.
- If there exists an equivalent local martingale measure and a stock price process is a “strict local martingale” what is the cheapest way to hold this stock at time $T$?
- Answer: Delta-hedging.
- How can we compute hedging prices?
- Answer: PDE techniques, (non-)equivalent changes of measures
- Techniques: Itô’s formula, PDE techniques to prove smoothness of hedging prices, Föllmer measure
Two generic examples

- Reciprocal of the three-dimensional Bessel process (NFLVR):
  \[ d\tilde{S}(t) = -\tilde{S}^2(t)dW(t) \]

- Three-dimensional Bessel process:
  \[ dS(t) = \frac{1}{S(t)}dt + dW(t) \]
Strict local martingales

- A stochastic process $X(\cdot)$ is a *local martingale* if there exists a sequence of stopping times $(\tau_n)$ with $\lim_{n \to \infty} \tau_n = \infty$ such that $X^{\tau_n}(\cdot)$ is a martingale.
- Here, in our context, a local martingale is a nonnegative stochastic process $X(\cdot)$ which does not have a drift:
  
  $dX(t) = X(t) \text{something} dW(t)$.

- Strict local martingales (local martingales, which are not martingales) do only appear in continuous time.
- Nonnegative local martingales are supermartingales.
We assume a Markovian market model.

- Our time is finite: $T < \infty$. Interest rates are zero.
- The stocks $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))^T$ follow
  
  $$dS_i(t) = S_i(t) \left( \mu_i(t, S(t))dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_k(t) \right)$$

  with some measurability and integrability conditions.
- Markovian, but not necessarily complete ($K > d$ allowed).
- The covariance process is defined as
  
  $$a_{i,j}(t, S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))\sigma_{j,k}(t, S(t)).$$

- The underlying filtration is denoted by $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t \leq T}$. 
An important guy: the market price of risk.

- A market price of risk is an $\mathbb{R}^K$-valued process $\theta(\cdot)$ satisfying
  \[ \mu(t, S(t)) = \sigma(t, S(t))\theta(t). \]
- We assume it exists and
  \[ \int_0^T \|\theta(t)\|^2 dt < \infty. \]
- The market price of risk is not necessarily unique.
- We will always use a Markovian version of the form $\theta(t, S(t))$.
  (needs argument!)
Related is the stochastic discount factor.

- The *stochastic discount factor* corresponding to $\theta$ is denoted by

$$Z^\theta(t) := \exp \left( - \int_0^t \theta^T(u, S(u)) dW(u) - \frac{1}{2} \int_0^t \|\theta(u, S(u))\|^2 du \right).$$

- It has dynamics

$$dZ^\theta(t) = -\theta^T(t, S(t))Z^\theta(t)dW(t).$$

- If $Z^\theta(\cdot)$ is a martingale, that is, if $E[Z^\theta(T)] = 1$, then it defines a risk-neutral measure $Q$ with $dQ = Z^\theta(T)d\mathbb{P}$.

- Otherwise, $Z^\theta(\cdot)$ is a strict local martingale and classical arbitrage is possible.

- From Itô’s rule, we have

$$d \left( Z^\theta(t)S_i(t) \right) = Z^\theta(t)S_i(t) \sum_{k=1}^K (\sigma_{i,k}(t, S(t)) - \theta_k(t, S(t))) dW_k(t).$$
Everything an investor cares about: how and how much?

- We call *trading strategy* the number of shares held by an investor: \( \eta(t) = (\eta_1(t), \ldots, \eta_d(t))^T \)
- We assume that \( \eta(\cdot) \) is progressively measurable with respect to \( \mathbb{F} \) and self-financing.
- The corresponding wealth process \( V^{\nu,\eta}(\cdot) \) for an investor with initial wealth \( V^{\nu,\eta}(0) = \nu \) has dynamics
  \[
  dV^{\nu,\eta}(t) = \sum_{i=1}^{d} \eta_i(t) dS_i(t).
  \]
- We restrict ourselves to trading strategies which satisfy \( V^{1,\eta}(t) \geq 0 \)
The terminal payoff

- Let $p : \mathbb{R}_+^d \rightarrow [0, \infty)$ denote a measurable function.
- The investor wants to have the payoff $p(S(T))$ at time $T$.
- For example,
  - market portfolio: $\tilde{p}(s) = \sum_{i=1}^{d} s_i$
  - money market: $p^0(s) = 1$
  - stock: $p^1(s) = s_1$
  - call: $p^C(s) = (s_1 - L)^+$ for some $L \in \mathbb{R}$.
- We define a candidate for the hedging price as
  $$h^p(t, s) := \mathbb{E}_{t}^{s, t} \left[ \tilde{Z}^\theta(T)p(S(T)) \right],$$
  where $\tilde{Z}^\theta(T) = Z^\theta(T)/Z^\theta(t)$ and $S(t) = s$ under the expectation operator $\mathbb{E}_{t}^{s, t}$.
Prerequisites

- We shall call \((t, s) \in [0, T] \times \mathbb{R}^d\) a point of support for \(S(\cdot)\) if there exists some \(\omega \in \Omega\) such that \(S(t, \omega) = s\).
- We have assumed Markovian stock price dynamics such that \(S(t)\) is \(\mathbb{R}^d\)-valued, unique and stays in the positive orthant and a square-integrable Markovian market price of risk \(\theta(t, S(t))\).
- We have defined
  \[
  h^p(t, s) := \mathbb{E}^{t,s} \left[ \tilde{Z}^{\theta}(T)p(S(T)) \right],
  \]
  where \(\tilde{Z}^{\theta}(T) = Z^{\theta}(T)/Z^{\theta}(t)\) and \(S(t) = s\) under the expectation operator \(\mathbb{E}^{t,s}\).
- In particular,
  \[
  h^p(T, s) := p(s).
  \]
A first result: non path-dependent European claims

Assume that we have a contingent claim of the form \( p(S(T)) \geq 0 \) and that for all points of support \((t, s)\) for \( S(\cdot)\) with \( t \in [0, T)\) we have \( h^p \in C^{1,2}(U_{t,s})\) for some neighborhood \( U_{t,s}\) of \((t, s)\). Then, with \( \eta^p_i(t, s) := D_i h^p(t, s) \) and \( v^p := h^p(0, S(0))\), we get

\[
V^{v^p, \eta^p}(t) = h^p(t, S(t)).
\]

The strategy \( \eta^p \) is optimal in the sense that for any \( \tilde{v} > 0 \) and for any strategy \( \tilde{\eta} \) whose associated wealth process is nonnegative and satisfies \( V^{\tilde{v}, \tilde{\eta}}(T) \geq p(S(T)) \), we have \( \tilde{v} \geq v^p \). Furthermore, \( h^p \) solves the PDE

\[
\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 h^p(t, s) = 0
\]

at all points of support \((t, s)\) for \( S(\cdot)\) with \( t \in [0, T)\).
The proof relies on Itô’s formula.

- Define the martingale $N^p(\cdot)$ as
  \[
  N^p(t) := \mathbb{E}[Z^\theta(T)p(S(T))|\mathcal{F}(t)] = Z^\theta(t)h^p(t, S(t)).
  \]

- Use a localized version of Itô’s formula to get the dynamics of $N^p(\cdot)$. Since it is a martingale, its $dt$ term must disappear which yields the PDE.

- Then, another application of Itô’s formula yields
  \[
  dh^p(t, S(t)) = \sum_{i=1}^{d} D_i h^p(t, S(t)) dS_i(t) = dV^{v^p,\eta^p}(t).
  \]

- This yields directly $V^{v^p,\eta^p}(\cdot) \equiv h^p(\cdot, S(\cdot))$. 
• Next, we prove optimality.

• Assume we have some initial wealth \( \tilde{v} > 0 \) and some strategy \( \tilde{\eta} \) with nonnegative associated wealth process such that \( V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T)) \) is satisfied.

• Then, \( Z^\theta(\cdot)V^{\tilde{v},\tilde{\eta}}(\cdot) \) is a supermartingale.

• This implies

\[
\tilde{v} \geq \mathbb{E}[Z^\theta(T)V^{\tilde{v},\tilde{\eta}}(T)] \geq \mathbb{E}[Z^\theta(T)p(S(T))] \\
= \mathbb{E}[Z^\theta(T)V^{v^p,\eta^p}(T)] = v^p
\]
Non-uniqueness of PDE

- Usually, 

\[ \frac{\partial}{\partial t} v(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 v(t, s) = 0 \]

does not have a unique solution.

- However, if $h^p$ is sufficiently differentiable, it can be characterized as the minimal nonnegative solution of the PDE.

- This follows as in the proof of optimality. If $\tilde{h}$ is another nonnegative solution of the PDE with $\tilde{h}(T, s) = p(s)$, then $Z^\theta(\cdot)\tilde{h}(\cdot, S(\cdot))$ is a supermartingale.
Corollary: Modified put-call parity

For any $L \in \mathbb{R}$ we have the modified put-call parity for the call- and put-options $(S_1(T) - L)^+$ and $(L - S_1(T))^+$, respectively, with strike price $L$:

$$
\mathbb{E}^{t,s} \left[ \tilde{Z}^\theta(T)(L - S_1(T))^+ \right] + h^p_1(t,s) \\
= \mathbb{E}^{t,s} \left[ \tilde{Z}^\theta(T)(S_1(T) - L)^+ \right] + Lh^p_0(t,s),
$$

where $p^0(\cdot) \equiv 1$ denotes the payoff of one monetary unit and $p^1(s) = s_1$ the price of the first stock for all $s \in \mathbb{R}_+^d$. 
We shall call a function $f : [0, T] \times \mathbb{R}^d_+ \rightarrow \mathbb{R}$ locally Lipschitz and bounded on $\mathbb{R}^d_+$ if for all $s \in \mathbb{R}^d_+$ the function $t \rightarrow f(t, s)$ is right-continuous with left limits and for all $M > 0$ there exists some $C(M) < \infty$ such that for all $t \in [0, T]$.

$$\sup_{\frac{1}{M} \leq \|y\|, \|z\| \leq M \atop y \neq z} \left| \frac{f(t, y) - f(t, z)}{\|y - z\|} \right| + \sup_{\frac{1}{M} \leq \|y\| \leq M} |f(t, y)| \leq C(M).$$
Sufficient conditions for the differentiability of $h^p$.

(A1) The functions $\theta_k$ and $\sigma_{i,k}$ are for all $i = 1, \ldots, d$ and $k = 1, \ldots, K$ locally Lipschitz and bounded.

(A2) For all points of support $(t, s)$ for $S(\cdot)$ with $t \in [0, T)$ there exist some $C > 0$ and some neighborhood $U$ of $(t, s)$ such that

$$
\sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(u, y)\xi_i \xi_j \geq C \|\xi\|^2
$$

for all $\xi \in \mathbb{R}^d$ and $(u, y) \in U$.

(A3) The payoff function $p$ is chosen so that for all points of support $(t, s)$ for $S(\cdot)$ there exist some $C > 0$ and some neighborhood $U$ of $(t, s)$ such that $h^p(u, y) \leq C$ for all $(u, y) \in U$.

We will proceed in three steps to show that these conditions imply smoothness of $h^p$. 
Step 1: Stochastic flows

We define $X^{t,s,z}(\cdot) := (S^{t,s}_{\cdot}, z\tilde{Z}^{\phi,t,s}(\cdot))^T$.

Take $(t, s) \in [0, T] \times \mathbb{R}^d$ a point of support for $S(\cdot)$. Then under Assumption (A1) [locally Lipschitz and bounded] we have for all sequences $(t_k, s_k)_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} (t_k, s_k) = (t, s)$ that

$$\lim_{k \to \infty} \sup_{u \in [t, T]} \|X^{t_k,s_k,1}(u) - X^{t,s,1}(u)\| = 0$$

almost surely.

In particular, for $K(\omega)$ sufficiently large we have that $X^{t_k,s_k,1}(u, \omega)$ is strictly positive and $\mathbb{R}^{d+1}_+$-valued for all $k > K(\omega)$ and $u \in [t, T]$. 
Step 2: Schauder estimates

Fix a point \((t, s) \in [0, T) \times \mathbb{R}^d_+\) and a neighborhood \(\mathcal{U}\) of \((t, s)\). Suppose Assumptions (A1) and (A2) [locally Lipschitz and bounded, non-degenerate] hold.

Let \((f_k)_{k \in \mathbb{N}}\) denote a sequence of solutions of the Black-Scholes PDE on \(\mathcal{U}\), uniformly bounded under the supremum norm on \(\mathcal{U}\). If \(\lim_{k \to \infty} f_k(t, s) = f(t, s)\) on \(\mathcal{U}\) for some function \(f : \mathcal{U} \to \mathbb{R}\), then \(f\) solves also the PDE on some neighborhood \(\tilde{\mathcal{U}}\) of \((t, s)\). In particular, \(f \in C^{1,2}(\tilde{\mathcal{U}})\).

- Janson and Tysk (2006), Tysk and Ekström (2009)
- Interior Schauder estimates by Knerr (1980) together with Arzelà-Ascoli type of arguments
Under Assumptions (A1)-(A3) [locally Lipschitz and bounded, non-degenerate \(a\), locally boundedness of \(h^p\)] there exists for all points of support \((t, s)\) for \(S(\cdot)\) with \(t \in [0, T]\) some neighborhood \(U\) of \((t, s)\) such that the function \(h^p\) is in \(C^{1,2}(U)\).

- Define \(\tilde{p}(s_1, \ldots, s_d, z) := zp(s_1, \ldots, s_d)\).
- Define \(\tilde{p}^M(\cdot) := \tilde{p}(\cdot)1_{\{\tilde{p}(\cdot) \leq M\}}\) for some \(M > 0\)
- Approximate by sequence of continuous functions \(\tilde{p}^M,m\) such that \(\tilde{p}^M,m \leq 2M\) for all \(m \in \mathbb{N}\).
Proof (continuation)

- The corresponding expectations are defined as

\[ \tilde{h}^{p,M}(u, y) := \mathbb{E}^{u,y}[\check{p}^{M}(S_1(T), \ldots, S_d(T), \check{Z}^{\theta}(T))] \]

for all \((u, y) \in \tilde{U}\) for some neighborhood \(\tilde{U}\) of \((t, s)\) and equivalently \(\tilde{h}^{p,M,m}\).

- We have continuity of \(\tilde{h}^{p,M,m}\) for large \(m\) due to the bounded convergence theorem.

- A result from Jansen and Tysk (2006) yields that under Assumption (A2) [non-degenerate a] \(\tilde{h}^{p,M,m}\) is a solution of the PDE.

- Then, by Step 2 firstly, \(\tilde{h}^{p,M}\) and secondly, \(h^p\) also solve the PDE.
We can change the measure to compute $h^p$

- There exists not always an equivalent local martingale measure.
- However, after making some technical assumptions on the probability space and the filtration we can construct a new measure $\mathbb{Q}$ which corresponds to a “removal of the stock price drift”.
- Based on the work of Föllmer and Meyer and along the lines of Delbaen and Schachermayer.
Theorem: Under a new measure $\mathbb{Q}$ the drifts disappear.

There exists a measure $\mathbb{Q}$ such that $\mathbb{P} \ll \mathbb{Q}$. More precisely, for all nonnegative $\mathcal{F}(T)$-measurable random variables $Y$ we have

$$
\mathbb{E}^{\mathbb{P}}[Z^\theta(T)Y] = \mathbb{E}^{\mathbb{Q}}\left[Y \mathbf{1}_{\{ \frac{1}{Z^\theta(T)}>0 \}} \right].
$$

Under this measure $\mathbb{Q}$, the stock price processes follow

$$
dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) d\widetilde{W}_k(t)
$$

up to time $\tau^\theta := \inf\{t \in [0, T] : 1/Z^\theta(t) = 0\}$. Here,

$$
\widetilde{W}_k(t \wedge \tau^\theta) := W_k(t \wedge \tau^\theta) + \int_0^{t \wedge \tau^\theta} \theta_k(u, S(u))du
$$

is a $K$-dimensional $\mathbb{Q}$-Brownian motion stopped at time $\tau^\theta$. 
What happens in between time 0 and time $T$: Bayes’ rule.

For all nonnegative $\mathcal{F}(T)$-measurable random variables $Y$ the representation

$$
E^Q \left[ Y 1\{1/Z^{\theta}(T) > 0\} \mid \mathcal{F}(t) \right] = E^P \left[ Z^{\theta}(T) Y \mid \mathcal{F}(t) \right] \frac{1}{Z^{\theta}(t)} 1\{1/Z^{\theta}(t) > 0\}
$$

holds $Q$-almost surely (and thus $P$-almost surely) for all $t \in [0, T]$. 
The class of Bessel processes with drift provides interesting arbitrage opportunities.

- We begin with defining an auxiliary stochastic process $X(\cdot)$ as

$$dX(t) = \left(\frac{1}{X(t)} - c\right) dt + dW(t)$$

with $W(\cdot)$ denoting a Brownian motion and $c \geq 0$ a constant.

- $X(t)$ is for all $t \geq 0$ strictly positive since $X(\cdot)$ is a Bessel process under an equivalent measure.

- The stock price process is now defined via

$$dS(t) = \frac{1}{X(t)} dt + dW(t) = S(t) \left(\frac{1}{S^2(t) - S(t)ct} dt + \frac{1}{S(t)} dW(t)\right)$$

with $S(0) = X(0) > 0$. 
After a change of measure, the Bessel process becomes Brownian motion.

- As a reminder:
  \[ dS(t) = \frac{1}{S(t) - ct} \, dt + dW(t). \]

- We have \( S(t) \geq X(t) > 0 \) for all \( t \geq 0 \).
- The market price of risk is \( \theta(t, s) = 1/(s - ct) \).
- Thus, the inverse stochastic discount factor \( 1/Z^\theta \) becomes zero exactly when \( S(t) \) hits \( ct \).
- Removing the drift with a change of measure as before makes \( S(\cdot) \) a Brownian motion (up to the first hitting time of zero by \( 1/Z^\theta(\cdot) \)) under \( Q \).
The optimal strategy for getting one dollar at time $T$ can be explicitly computed.

- For $p(s) \equiv p^0(s) \equiv 1$ we get
  \[ h^{p^0}(t, s) = \mathbb{E}^P \left[ \frac{Z^\theta(T)}{Z^\theta(t)} \cdot 1 \middle| \mathcal{F}_t \right] \bigg|_{S(t)=s} = \mathbb{E}^Q[1\{1/Z^\theta(T)>0\}\mid \mathcal{F}_t] \bigg|_{S(t)=s} 
  = \Phi \left( \frac{s - cT}{\sqrt{T-t}} \right) - \exp(2c s - 2c^2 t) \Phi \left( \frac{-s - cT + 2ct}{\sqrt{T-t}} \right). \]
- This yields the optimal strategy
  \[ \eta^0(t, s) = \frac{2}{\sqrt{T-t}} \Phi \left( \frac{s - cT}{\sqrt{T-t}} \right) - 2c \exp(2cs - 2c^2 t) \Phi \left( \frac{-s - cT + 2ct}{\sqrt{T-t}} \right). \]
- The hedging price $h^p$ satisfies on all points $\{s > ct\}$ the PDE
  \[ \frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} D^2 h^p(t, s) = 0. \]
Conclusion

- No equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge.
- Sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration.
- The dynamics of stochastic processes under a non-equivalent measure and a generalized Bayes' rule might be of interest themselves.
- We have computed some optimal trading strategies in standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.
Thank you!
Strict local martingales II

Assume \( X(\cdot) \) is a nonnegative local martingale:

\[
dX(t) = X(t) \text{something} dW(t).
\]

- We always have \( \mathbb{E}[X(T)] \leq X(0) \).
- If \( \mathbb{E}[X(T)] = X(0) \) then \( X(\cdot) \) is a (true) martingale.
- If “something” behaves nice (for example is bounded) then \( X(\cdot) \) is a martingale.
- If \( \mathbb{E}[X(T)] < X(0) \) then \( X(\cdot) \) is a strict local martingale.
Let $M \geq 0$ be a random variable measurable with respect to $\mathcal{F}^S(T)$. Let $\nu(\cdot)$ denote any MPR and $\theta(\cdot, \cdot)$ a Markovian MPR. Then, with

$$M^\nu(t) := \mathbb{E} \left[ \frac{Z^\nu(T)}{Z^\nu(t)} M \bigg| \mathcal{F}_t \right]$$

and

$$M^\theta(t) := \mathbb{E} \left[ \frac{Z^\theta(T)}{Z^\theta(t)} M \bigg| \mathcal{F}_t \right]$$

for $t \in [0, T]$, we have $M^\nu(\cdot) \leq M^\theta(\cdot)$ almost surely.
Proof

• We define \( c(\cdot) := \nu(\cdot) - \theta(\cdot, S(\cdot)) \) and \( c^n(\cdot) := c(\cdot)1_{\{\|c(\cdot)\| \leq n\}} \)

• Then,

\[
\frac{Z^\nu(T)}{Z^\nu(t)} = \lim_{n \to \infty} \frac{Z^{c^n}(T)}{Z^{c^n}(t)}
\]

\[
\cdot \exp \left( - \int_t^T \theta^T (dW(u) + c^n(u)du) - \frac{1}{2} \int_t^T \|\theta\|^2 du \right).
\]

• Since \( c^n(\cdot) \) is bounded, \( Z^{c^n}(\cdot) \) is a martingale.

• Fatou’s lemma, Girsanov’s theorem and Bayes’ rule yield

\[
M^\nu(t) \leq \lim inf_{n \to \infty} \mathbb{E}^{Q^n} \left[ \exp \left( - \int_t^T \theta^T dW^n(u) - \frac{1}{2} \int_t^T \|\theta\|^2 du \right) M \bigg| \mathcal{F}_t \right].
\]

• Since \( \sigma(\cdot, S(\cdot))c^n(\cdot) \equiv 0 \) the process \( S(\cdot) \) has the same dynamics under \( Q^n \) as under \( P \).
Open problem

The last result might be related to the “Markovian selection results”, as in Krylov (1973) and Ethier and Kurtz (1986). There, the existence of a Markovian solution for a martingale problem is studied. It is observed that a supremum over a set of expectations indexed by a family of distributions is attained and the maximizing distribution is a Markovian solution of the martingale problem.
Open problem

\( h^p \) can be characterized as the minimal nonnegative solution of the Cauchy problem

\[
\frac{\partial}{\partial t} v(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D^2 v(t, s) = 0
\]

\[
v(T, s) = p(s)
\]

Can an iterative method be constructed, which converges to the minimal solution of this PDE?
• Reminder: \( dZ^\theta(t) = -\theta^T(t, S(t))Z^\theta(t)dW(t) \), where \( \theta \) denotes the market price of risk.

• Assume: \( Z^\theta(\cdot) \) is a true martingale.

• Then, there exists a risk-neutral measure \( \mathbb{Q} \), under which \( S(\cdot) \) has dynamics

\[
dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW^\mathbb{Q}_k(t).
\]

• Then,

\[
h^p(t, s) = \mathbb{E}^{t,s}[\tilde{Z}^\theta(T)p(S(T))] = \mathbb{E}^{\mathbb{Q},t,s}[p(S(T))].
\]

• Below: Generalization to the situation where \( Z^\theta(\cdot) \) is a strict local martingale and risk-neutral measure \( \mathbb{Q} \) does not exist.
If we assume that the number of stocks $d$ and the number of driving Brownian motions $K$ is equal, that is, $d = K$, and $\sigma$ has full rank, then the market is called **complete**.

Then, by the Martingale Representation Theorem, there exists some strategy $\eta$ such that

$$V^{v,\eta}(T) = p(S(T))$$

for initial capital $v = h^P(0, S(0))$.

That is, the contingent claim / payoff can be **hedged**.

Often, one can use Itô’s rule to compute

$$\eta_i(t) = D_i h^P(t, S(t)),$$

which is called **delta hedge**.
“Classical” Mathematical Finance III

- Often, the hedging price $h^p$ needs to be computed numerically.
- Theory behind it: *Feynman-Kac Theorem*
- It states that under some continuity and growth conditions on $a$ and $p$, any solution $v : [0, T] \times \mathbb{R}^d_+ \rightarrow \mathbb{R}$ of the Cauchy-Problem (*Black-Scholes PDE*)

$$\frac{\partial}{\partial t} v(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 v(t, s) = 0$$

$$v(T, s) = p(s)$$

with polynomial growth can be represented as

$$v(t, s) = \mathbb{E}^{Q_{t,s}} [p(S(T))] = h^p(t, s),$$

where $a(\cdot, \cdot) = \sigma(\cdot, \cdot) \sigma^T(\cdot, \cdot)$ and $S(\cdot)$ has $Q$-dynamics

$$dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) dW_{k}^{Q}(t).$$
Feynman-Kac does not always work.

- We have seen, as long as
  - some growth and continuity conditions on $\sigma$ and $p$ are satisfied,
  - the risk-neutral measure $Q$ exists,
  - $h^p$ is of polynomial growth,
  - the Black-Scholes equation has a solution

we know that the hedging price $h^p$ is a solution.

- Growth conditions are often not satisfied, for example

$$d\tilde{S}(t) = -\tilde{S}^2(t)dW(t)$$

with corresponding PDE

$$\frac{\partial}{\partial t}v(t, s) + \frac{1}{2}s^4D^2v(t, s) = 0.$$

- Then, $v_1(t, s) = s$ and $v_2(t, s) = 2s\Phi\left(\frac{1}{s\sqrt{T-t}}\right) - s$ are solutions of polynomial growth, satisfying $v(T, s) = s$ and $v(t, 0) = 0.$
“Classical” Mathematical Finance IV

• Remember: We have assumed that there exists some $\theta$ which maps the volatility into the drift, that is $\sigma(\cdot, \cdot)\theta(\cdot, \cdot) = \mu(\cdot, \cdot)$.

• It can be shown that this assumption excludes “unbounded profit with bounded risk”.

• Thus “making (a considerable) something out of almost nothing” is not possible.

• However, it is still possible to “certainly make something more out of something”.

• The reason that the arbitrage is not scalable is due to the credit constraint (admissibility) $V^{1,\eta}(\cdot) \geq 0$. 
Digression: Problems of the no-arbitrage assumption.

- A typical market participant can statistically detect whether a market price of risk $\theta$ exists or does not exist.
- However, there exists no statistical test to decide whether $Z^\theta(\cdot)$ is a true martingale or not (whether arbitrage exists or does not exist).
- Instead of starting from the normative assumption of no arbitrage, *Stochastic Portfolio Theory* takes a descriptive approach.
- One goal is to find models which provide realistic dynamics of the market weights $S_i(\cdot)/(S_i(\cdot) + \ldots S_d(\cdot))$.
- These models tend to violate the no-arbitrage assumption.
Stationarity of the market weights.

Figure: Market weights against ranks on logarithmic scale, 1929 - 1999, from Fernholz, *Stochastic Portfolio Theory*, page 95.