Static / dynamic properties of stochastic volatility models:
a structural connection

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Outline

• Stochastic volatility models produce a smile AND generate a dynamics for implied volatilities

• Is there a connection of a structural nature?
• Enter the Skew Stickiness Ratio

  • Long-maturity smiles
    • an expression linking the decay of the ATM and the SSR
  • Historical dynamics of market smiles?

  • Short-maturity smiles
    • The realized skew

• Conclusion

Papers Smile Dynamics I, II, III, IV are available on the SSRN web site.
• Start with term-structure of Variance Swap Volatilities $\hat{\sigma}_{iT}$ observed at time $t = 0$ and define forward variances $\xi_t^T$ as:

$$\xi_t^T = \frac{d}{dT}((T-t)\hat{\sigma}_{iT}^2) \quad \hat{\sigma}_{iT}^2 = \frac{1}{T-t} \int_t^T \xi_t^\tau d\tau$$

• Forward variances can be traded – using Variance Swaps – at no cost: forward variances are driftless processes in the pricing measure.

• The general expression of a diffusive stochastic volatility model can be written as:

$$dS_t = (r-q)S_t \, dt + \sqrt{\xi_t^l} \, S_t \, dZ_t$$

$$d\xi_t^T = \bullet \, dW_t^T$$

• Any diffusive stochastic volatility model can be written this way

• The instantaneous variance of the spot process is $\xi_t^l$
The ATMF skew - 2

\[
\begin{align*}
\begin{cases}
   dS_t^\omega = (r - q)S_t^\omega dt + \sqrt{\xi_t} S_t^\omega dZ_t \\
   d\xi_t^T = \omega \xi_t^T \sum_k \lambda_{kt}^T (\xi_t) dW_t^k
\end{cases}
\end{align*}
\]

• Let us write the dynamics as:

• At order 1 in \( \omega \):

\[
\xi_t^T = \xi_0^T + \int_0^t \sum_k \lambda_{kt}^T dW_t^k
\]

– In particular, for the instantaneous variance:

\[
\delta \xi_t = \omega \xi_0^T \int_0^t \sum_k \lambda_{kt}^T dW_t^k
\]

– ATMF skew as a function of skewness \( S_T \) of \( x_T = \ln(S_T / F_T) \) is approximately given by:

\[
S_T = \left. \frac{d\hat{\sigma}_{KT}}{d \ln K} \right|_{F} \approx \frac{S_T}{6\sqrt{T}} \quad S_T = \frac{M_3^T}{(M_2^T)^{3/2}} \quad , \quad M_n^T = (x_T - <x_T>)^n
\]

(at order 1 in \( \omega \), exactly equivalent to 1st order perturbation of pricing equation in \( \omega \))

\[
\begin{align*}
&x_T - <x_T> = \int_0^T \sqrt{\xi_0^T} dZ_t + \frac{1}{2} \left( \int_0^T \delta \xi_t dZ_t - \int_0^T \delta \xi_t dt \right)
\end{align*}
\]

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The ATMF skew - 3

\[ M_2^T = \int_0^T \sqrt{\xi_0} dt \]

\[ M_3^T = \frac{3}{2} E \left[ \left( \int_0^T \sqrt{\xi_0} dZ_t \right)^2 \right] \left[ - \int_0^T \delta \xi_t dt + \int_0^T \frac{\delta \xi_t}{\sqrt{\xi_0}} dZ_t \right] \]

- Computing the expectation: \( M_3^T = 3 \omega \int_0^T dt \xi_0^t \int_0^t \sqrt{\xi_0^t} \sum_k \rho_{ij} (\lambda_{ik})_0 d\tau \]

\[ = 3 \int_0^T dt \left( \int_0^t E \left[ \frac{dS^0_t}{S^0_\tau} d\xi^t_\tau \right] \right) \]

- Intuition:

\[ M_3^T = \left( \Sigma_{ij} \right)^3 = \Sigma < r_i r_j r_k > = 3 \Sigma_{j>i} < r_i r_j^2 > \]

- Introduce spot/volatility covariance function:

\[ f(\tau, t) = \frac{1}{d \tau} E \left[ \frac{dS^0_t}{S^0_\tau} d\xi^t_\tau \right] \]

- At order 1 in volatility-of-volatility, ATMF skew is given by:

\[ S_T = \frac{1}{2\sqrt{T}} \left[ \int_0^T dt \int_0^t f(\tau, t) d\tau \right] \left( \int_0^T \xi_0^t d\tau \right)^{3/2} \]

95%-105% skew

- Actual
- Approximate
• How much does the ATMF volatility move when the spot moves – in units of the skew?

• Market-makers use following ratio:  
  \[ r_T = \frac{1}{d \hat{\sigma}_{KT}/d \ln K}_{F} \frac{\Delta \hat{\sigma}_{FT}}{\Delta \ln S} \]

  - \( r_T = 1 \): "sticky-strike" regime
  - \( r_T = 0 \): "sticky-delta" regime

• Introduce Skew Stickiness Ratio:

  \[ R_T = \frac{1}{d \hat{\sigma}_{KT}/d \ln K}_{F} \frac{E[d \hat{\sigma}_{FT} \, d\ln S]}{E[\ln^2 S]} \]

  - Jump / Lévy models :  \( R_T = 0 \)
  - Local vol, weak skew :  \( R_T = 2 \)
  - Stochastic volatility, short maturity & weak skew:  \( R_T = 2 \)
The SSR - 2

• At order 1 in volatility-of-volatility

– use VS volatility instead of ATMF volatility:

\[
E[d\ln S_t \, d\sigma_T] = \frac{1}{2\hat{\sigma}_T T} \int_0^T E[d\ln S \, d\xi^t] dt = \frac{1}{2\hat{\sigma}_T T} \int_0^T E[dS^0_0 / S^0_0 \, \delta\xi_t] dt = \frac{1}{2\hat{\sigma}_T T} \int_0^T f(0, t) dt
\]

– Final expression for the SSR, at order 1 in volatility-of-volatility:

\[
R_T = \int_0^T \xi^t_0 dt \int_0^T f(0, t) dt / \int_0^T dt \int_0^T f(\tau, t) d\tau
\]

\[
S_T = \frac{1}{2\sqrt{T}} \int_0^T dt \int_0^t f(\tau, t) d\tau / \left( \int_0^T \xi^t_0 dt \right)^{3/2}
\]

• Take short-maturity limit:

\[
R_0 = \lim_{T \to 0} \frac{T \int_0^T f(0, t) dt}{\int_0^T dt \int_0^T f(\tau, t) d\tau} = 2
\]

\[
S_0 = \lim_{T \to 0} \frac{1}{2\sqrt{T}} \int_0^T dt \int_0^t f(\tau, t) d\tau / \left( \int_0^T \xi^t_0 dt \right)^{3/2} = \frac{f(0, 0)}{4(\xi^0_0)^{3/2}}
\]

– Skew tends to a finite value which measures the covariance function at origin

– SSR tends to universal value: 2
Bounds for the SSR

- Let us make some additional assumptions:
  - VS curve is flat: \( \xi_0^T = \xi_0 \)
  - Stochastic volatility model is time-homogeneous: \( f(\tau, t) \equiv f(t - \tau) \)
  - ATMF skew and SSR take simpler forms:

\[
S_T = \frac{\int_0^T (T-t)f(t) dt}{2 \xi_0^{3/2} T^2} \quad R_T = \frac{\int_0^T f(t) dt}{\int_0^T (1 - \frac{t}{T}) f(t) dt}
\]

- Assume that \( |f(t)| \) decreases monotonically towards zero. Rewrite \( R_T \) as

\[
R_T = \frac{g(T)}{\frac{1}{T} \int_0^T g(t) dt}, \quad g(t) = \int_0^t f(\tau) d\tau
\]

- \( \frac{g(t)}{g(T)} \leq 1 \) implies \( R_T \geq 1 \)
- \( \frac{g(t)}{g(T)} \geq \frac{t}{T} \) implies \( R_T \leq 2 \)

⇒ Model-independent range for \( R_T \): \( R_T \in [1, 2] \)
Scaling of the SSR

• Assume that for large \( t \) \( f(t) \propto \frac{1}{t^\gamma} \)

• Then, for large \( T \) two types of behaviour, depending on value of \( \gamma \)

  – Type I: If \( \gamma > 1 \)
    
    \[
    S_T \propto \frac{1}{T} \quad \text{and} \quad \lim_{T \to \infty} R_T = 1
    \]

  – Type II: If \( \gamma < 1 \)
    
    \[
    S_T \propto \frac{1}{T^\gamma} \quad \text{and} \quad \lim_{T \to \infty} R_T = 2 - \gamma
    \]

  – exponential decay: Type I.

• Type I: scaling of \( S_T \) analogous to models with independent increments, however model becomes sticky-strike

• Both types of scaling can be summarized by:

  \[
  S_T \propto \frac{1}{T^{2-R_\infty}}
  \]

  where \( R_\infty = \lim_{T \to \infty} R_T \)
Type II scaling – in a model?

- Consider a model of the following type (Bergomi, 2005):

\[ d\xi_t^T = \omega\xi_0^T \sum w_i e^{-k_i(T-t)} dW_t \]

- expression for \( f \):

\[ f(\tau) = \omega\xi_0^{3/2} \sum w_i \rho_{Si} e^{-k_i\tau} \]

- expressions for skew and SSR:

\[
S_T = \frac{\omega}{2} \sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2} \\
R_T = \frac{\sum w_i \rho_{Si} \frac{1 - e^{-k_i T}}{k_i T}}{\sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2}}
\]

- for large \( \tau \) \( f(\tau) \propto e^{-(\min_i k_i)\tau} \) \( \Rightarrow \) for (really) large \( T \) \( S_T \propto \frac{1}{T} \), \( R_T \to 1 \)

- What about intermediate maturities?
Type II scaling – in a model?

• By suitably choosing parameter values, get non-trivial scaling over wide range of maturities:
  \[ k_1 = 8, \ k_2 = 0.35, \ \omega_1 = 72\%, \ \omega_2 = 28\%, \ \rho_{S1} = -70\%, \ \rho_{S2} = -35.7\%, \ \omega = 3.36 \]

• Look at maturity-dependence of ATMF skew:

⇒ ATMF skew decays like \( S_T \propto 1/\sqrt{T} \). What about the SSR?

⇒ We get a plateau around 1.5 (i.e. 2 minus exponent of skew decay) for an intermediate range of maturities.

⇒ It is possible to get in a model Type II behavior for a range of maturities that is practically relevant.
Type II scaling – in the market?

• Equity index smiles usually exhibit an ATMF skew that typically decays like $\frac{1}{\sqrt{T}}$
  
  – looks like Type II, but what about the SSR?
  
  – For the Eurostoxx, 3-month running estimates of the SSR for maturities 1 month, 6 months, 2 years:

  ⇒ SSR usually in interval [1, 2]

  ⇒ For longer maturities, average value of the SSR $\sim 1.5$ – compatible with a decay of the skew $\sim \frac{1}{\sqrt{T}}$

  ⇒ Equity volatility markets seem to be of Type II

• One puzzle left: short-maturity SSR always lower than 2 – can this be arbitraged?
  
  – is it possible materialize $2 - R_0$ as a P&L? (or are we just plain wrong?)
Arbing the realized SSR

- Introduce "realized" skew:

\[
\left. \frac{d\hat{\sigma}}{d \ln K} \right|_S^{\text{Realized}} = \frac{1}{2\sigma_0 \delta t} \begin{pmatrix} \delta S & \delta \sigma_0 \end{pmatrix}
\]

- implied versus "realized" 1-month 95 / 105 skew
Conclusion

- In stochastic volatility models, at order 1 in the vol-of-vol, the Skew Stickiness Ratio and the rate of decay of the ATMF skew are related through the spot/vol covariance function.

- For time-homogeneous model & flat VS term-structure:
  - SSR is bounded: \( R_T \in [1, 2] \)
  - 2 types of models
    - Type I: \( S_T \propto \frac{1}{T} \) and \( \lim_{T \to \infty} R_T = 1 \)
    - Type II: \( S_T \propto \frac{1}{T^\gamma} \) and \( \lim_{T \to \infty} R_T = 2 - \gamma \)

- Index volatility markets’ behavior consistent with Type II

- For short maturity smiles, SSR = 2
  - Markets display a realized SSR < 2
  - Introduce the notion of "realized skew": 
    - realized covariance of spot & implied vol
  - \( 2 - R_0 \) mismatch can be materialized as a P&L

\[ \frac{d\hat{\sigma}}{d\ln K}_{\text{Realized}} = \frac{1}{2\sigma_0 \delta t} \begin{pmatrix} \delta S \\ \delta \sigma_0 \\ S \\ \sigma_0 \end{pmatrix} \]