Limit theory for heavy-tailed models on a lattice

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State space $(E, \mathcal{E})$

- $M_p(E)$ - space of point measures on E.
- $\mathcal{M}_p(E)$ - smallest σ-algebra making the evaluation maps $m \to m(F)$ measurable, $m \in M_p(E)$, $F \in \mathcal{E}$.
- $C_K^+ := \{f : E \to \mathbb{R}_+ : \ f \text{ continuous with compact support}\}$.

Vague convergence of measures $\mu_n \in M_p(E)$, $n > 0$

$$\mu_n \xrightarrow{v} \mu_0 \iff \mu_n(f) \to \mu_0(f) \text{ for all } f \in C_K^+.$$

Poisson point process ξ on $(E, \mathcal{E})$

- $P(\xi(F) = k) = \begin{cases} e^{-\mu(F)}(\mu(F))^k/k! & \text{if} \quad \mu(F) < \infty \\ 0 & \text{if} \quad \mu(F) = \infty, \end{cases}$ for all $F \in \mathcal{E}$.
- $F_1, \ldots, F_n \in \mathcal{E}$ mutually disjoint $\Rightarrow \xi(F_1), \ldots, \xi(F_n)$ independent.

Regular variation in $\mathbb{R}^d$

- $X \in \mathbb{R}^d$ and its distribution are **regularly varying with index** $\alpha > 0$:

$$\frac{P(x^{-1}X \in \cdot)}{P(|X| > x)} \xrightarrow{v} \mu(\cdot)$$

for a non-null Radon measure μ on $\overline{\mathbb{R}}^d \setminus \{0\}$ with $\mu(tA) = t^{-\alpha}\mu(A), t > 0$.

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- Equivalently, there exists $\Theta \in \mathbb{S}^{d-1}$ such that for any $t > 0$, $S \subset \mathbb{S}^{d-1}$ with $P(\Theta \in \partial S) = 0$,

$$\lim_{x \to \infty} \frac{P(|X| > tx, \overline{X} \in S)}{P(|X| > x)} = t^{-\alpha} P(\Theta \in S),$$

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- Equivalently, there exist $a_n \to \infty$ such that

$$nP(a_n^{-1}X \in \cdot) \xrightarrow{v} \mu(\cdot).$$

Regular variation in $\mathbb{R}^d$

$\{X_i\}$ i.i.d. and (a_n) such that $nP(|X| > a_n) \sim 1$. The following are equivalent:

- X_1 regularly varying with index $\alpha > 0$.
- Point process convergence with limiting Poisson process on $\overline{\mathbb{R}}^d \setminus \{0\}$ with mean measure μ:

$$\sum_{i=1}^{n} \delta_{a_n^{-1} X_i} \Rightarrow \sum_{i=1}^{+\infty} \delta_{\pi_i}.$$

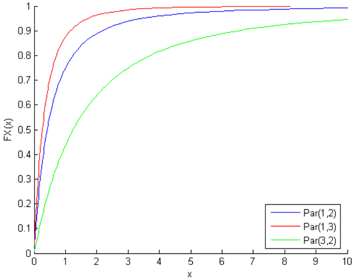
- Convergence of partial sums with α-stable limit S_α (for $\alpha < 2$)

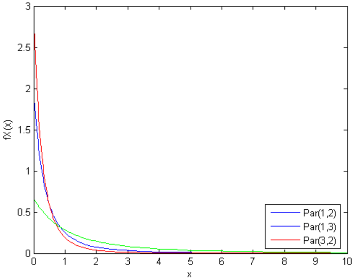
$$a_n^{-1}(X_1 + \ldots X_n - b_n) \Rightarrow S_\alpha.$$

Example: Pareto distribution

$X \sim Par(c, \alpha)$

- Density function $f_X(x) = \alpha \frac{c^\alpha}{(c+x)^{\alpha+1}} \mathbf{1}_{\{x \geq 0\}}$
- Cumulative distribution function $F_X(x) = 1 - (\frac{c}{c+x})^\alpha \mathbf{1}_{\{x \geq 0\}}$
- Tail function $\overline{F}_X(x) = (\frac{c}{c+x})^\alpha \mathbf{1}_{\{x \geq 0\}}$





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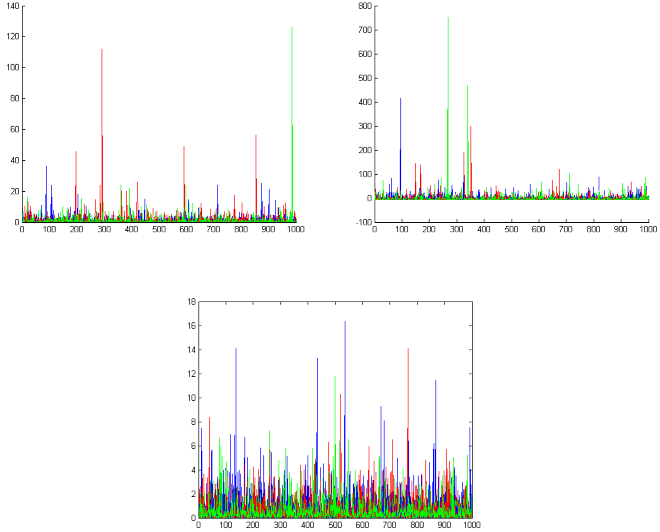

Left: $X_1, \ldots, X_{1000} \sim Par(1,2)$, Center: $X_1, \ldots, X_{1000} \sim Par(1,3)$, Right: $X_1, \ldots, X_{1000} \sim Par(3,2)$

- $\{Z_{i,j} : i,j \in \mathbb{Z}\}$ real valued iid random variables such that:

$$P(|Z_{i,j}| > x) = x^{-\alpha}L(x)\ ,\ L \text{ slowly varying at } \infty, \alpha > 0 \tag{1}$$

$$\frac{P(Z_{i,j} > x)}{P(|Z_{i,j}| > x)} \to p \text{ and } \frac{P(Z_{i,j} \leq -x)}{P(|Z_{i,j}| > x)} \to q \tag{2}$$

as $x \to \infty, 0 \leq p \leq 1, q = 1 - p$.

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- $\{a_n\}$ sequence of positive constants such that

$$n^2 P(|Z_{1,1}| > a_n x) \to x^{-\alpha} \text{ for all } x > 0. \tag{3}$$

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- $\lambda(dx) = \alpha p x^{-\alpha-1} \mathbf{1}_{(0,\infty)}(x)dx + \alpha q(-x)^{-\alpha-1} \mathbf{1}_{(-\infty,0)}(x)dx$ measure on $\mathbb{R} \setminus \{0\}$.
- $\mu = Leb \times Leb \times \lambda$ measure on $\mathbb{R}^2 \times \mathbb{R} \setminus \{0\}$.

Convergence to a Poisson process

Theorem 1

For each n suppose $\{X_{n,i,j} : i,j \in \mathbb{Z}\}$ are iid random elements of $(E, \mathcal{E})$ and let λ be a Radon measure on $(E, \mathcal{E})$.
Define $\xi_n := \sum_{i,j\in\mathbb{Z}} \delta_{(\frac{i}{n}, \frac{j}{n}, X_{n,i,j})}$ and suppose ξ is PRM on $\mathbb{R}^2 \times E$ with mean measure $\mu = Leb \times Leb \times \lambda$. Then

$$\xi_n \Rightarrow \xi \text{ in } M_p(\mathbb{R}^2 \times E)$$

iff

$$n^2 P(X_{n,1,1} \in \cdot) \xrightarrow{v} \lambda(\cdot) \text{ on } E. \quad (4)$$

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$X_{n,i,j} := a_n^{-1} Z_{i,j}$ satisfies (4) on $\mathbb{R} \setminus \{0\}$, so

$$\sum_{i,j\in\mathbb{Z}} \delta_{(\frac{i}{n}, \frac{j}{n}, a_n^{-1} Z_{i,j})} \Rightarrow \sum_h \delta_{(t_h^{(1)}, t_h^{(2)}, w_h)},$$

where $t_h^{(1)}, t_h^{(2)}, w_h$ are such that the sum on the right is a Poisson random measure with mean measure $\mu = Leb \times Leb \times \lambda$ on $\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\})$.

For a fixed $m \in \mathbb{N}$, $Z_{i,j}^{(m)} := (Z_{i-m,j-m}, Z_{i-m+1,j-m}, \ldots, Z_{i+m,j+m}) \in \mathbb{R}^{(2m+1)^2}$.

Theorem 2

Let $\{Z_{i,j}\}$ be i.i.d. satisfying (1) and (2) with $\{a_n\}$ satisfying (3). Then for each fixed positive integer m

$$\sum_{i,j\in\mathbb{Z}} \delta_{\left(\frac{i}{n},\frac{j}{n},Z_{i,j}^{(m)}\right)} \Rightarrow \sum_{h} \sum_{k=1}^{(2m+1)^2} \delta_{\left(t_h^{(1)},t_h^{(2)},w_h \mathbf{e}_k\right)}$$

in $M_p(\mathbb{R}^2 \times (\mathbb{R}^{(2m+1)^2} \setminus \{0\}))$ as $n \to \infty$, where $\mathbf{e}_k \in \mathbb{R}^{(2m+1)^2}$ is the basis element with kth component equal to one and the rest zero, and $t_h^{(1)}$, $t_h^{(2)}$, w_h are defined as above.

- $\{c_{k,l}\}$ array of real numbers such that

$$\sum_{k,l\in\mathbb{Z}} |c_{k,l}|^{\delta} < \infty \text{ for some } \delta < \alpha,\ \delta \le 1. \tag{5}$$

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$$X_{i,j} := \sum_{k,l\in\mathbb{Z}} c_{k,l} Z_{i+k,j+l}. \tag{6}$$

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Theorem 3

Suppose that $\{Z_{i,j}\}, \{a_n\}, \{c_{k,l}\}$ satisfy (1), (2), (3) and (7), and $\{X_{i,j}\}$ is given by (6). Let $\{(t_h^{(1)}, t_h^{(2)}, w_h)\}$ be the points of PRM(μ) on $\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\})$. Then

$$\sum_{i,j\in\mathbb{Z}} \delta_{(\frac{i}{n},\frac{j}{n},a_n^{-1}X_{i,j})} \Rightarrow \sum_{k,l\in\mathbb{Z}} \sum_{h} \delta_{(t_h^{(1)},t_h^{(2)},w_h c_{k,l})} \text{ in } M_p(\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})) \text{ as } n \to \infty.$$

Applications

- weak limiting behaviour of extremes
- joint limiting distribution of upper and lower extremes
- exceedances

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Extremal index

If $P(M_n \leq u_n) \to e^{-\theta\tau}$ for each τ, with u_n satisfying $n\overline{F}(u_n) \to \tau$, we say that the stationary sequence $\{X_n\}$ has extremal index θ.

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Example (extremal index)

Denote $M_n = \max_{0 \leq i,j \leq n} X_{i,j}$, $c_+ = \max_{k,l}\{c_{k,l}, 0\}$, $c_- = \max_{k,l}\{-c_{k,l}, 0\}$.

$$P(a_n^{-1} M_n \leq x) = e^{-(pc_+^\alpha + qc_-^\alpha)x^{-\alpha}},$$

i.e. the extremal index of the array $\{X_{i,j}\}$ is $pc_+^\alpha + qc_-^\alpha$.

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Example 2

Denote $M_{n,m}^r$ - rth largest among $\{X_{(-n,-m)}, \dots, X_{(n,m)}\}$. For $0 < y < x$ we have

$$P(a_n^{-1} M_{n,n} \leq x, a_n^{-1} M_{n,n}^2 \leq y) \to P(N(\langle x, \infty\rangle) = 0, N([y, x]) \leq 1).$$

Convergence of moving maxima

- $\{c_{k,l}\}$ array of real numbers such that

$$\sum_{k,l\in\mathbb{Z}} |c_{k,l}|^{\delta} < \infty \text{ for some } \delta < \alpha, \ \delta \leq 1. \tag{7}$$

- $$Y_{i,j} := \bigvee_{k,l\in\mathbb{Z}} c_{k,l} Z_{i+k,j+l}. \tag{8}$$

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Theorem 4

Suppose that $\{Z_{i,j}\}, \{a_n\}, \{c_{k,l}\}$ satisfy (1), (2), (3) and (7), and $\{Y_{i,j}\}$ is given by (8). Let $\{(t_h^{(1)}, t_h^{(2)}, w_h)\}$ be the points of PRM(μ) on $\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})$. Then

$$\sum_{i,j\in\mathbb{Z}} \delta_{(\frac{i}{n},\frac{j}{n},a_n^{-1}Y_{i,j})} \Rightarrow \bigvee_{k,l\in\mathbb{Z}} \sum_{h} \delta_{(t_h^{(1)},t_h^{(2)},w_h c_{k,l})} \text{ in } M_p(\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})) \text{ as } n\to\infty.$$

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Example (extremal index)

Denote $M_n = \max_{0 \leq i,j \leq n} Y_{i,j}$, $c_+ = \max_{k,l}\{c_{k,l}, 0\}$, $c_- = \max_{k,l}\{-c_{k,l}, 0\}$.

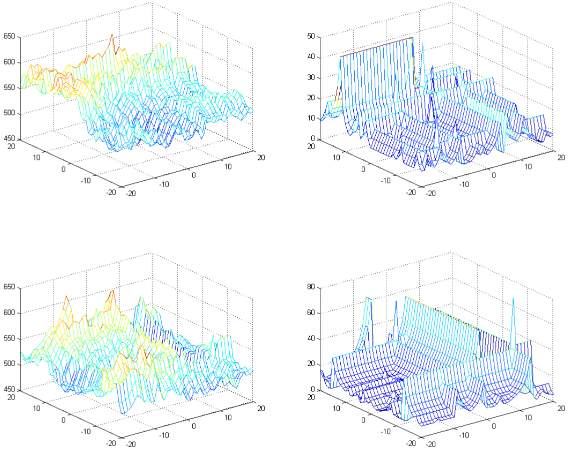
$$P(a_n^{-1} M_n \leq y) = e^{-(p c_+^{\alpha} + q c_-^{\alpha}) y^{-\alpha}},$$

i.e. the extremal index of the array $\{Y_{i,j}\}$ is also $p c_+^{\alpha} + q c_-^{\alpha}$.

Example

- $Z_{i,j} \sim Par(1,2)$
- $c_{k,l} = \begin{cases} \max\{\frac{1}{|k|}, \frac{1}{|l|}\} & , |k|, |l| \leq 20 \\ 0 & \text{, otherwise .} \end{cases}$

left $X_{i,j}$, right $Y_{i,j}$, $|i|, |j| \leq 20$:



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