The Distribution of Portfolio Payoffs and Increases in Risk Aversion

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(joint work with Mathias Beiglböck, Johannes Muhle-Karbe)

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- **? Relation between X_T^A and X_T^L ?**

In certain market models

X_T^L stochastically dominates X_T^A, i.e.,

$$X_T^A + \text{“risk premium”} + \text{“noise”} \overset{(d)}{=} X_T^L$$

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- Work in progress...
- Extension of Philip Dybvig, Yajun Wang, *Increases in Risk Aversion and the Distribution of Portfolio Payoffs*, 2009

Second Order Stochastic Dominance

Definition

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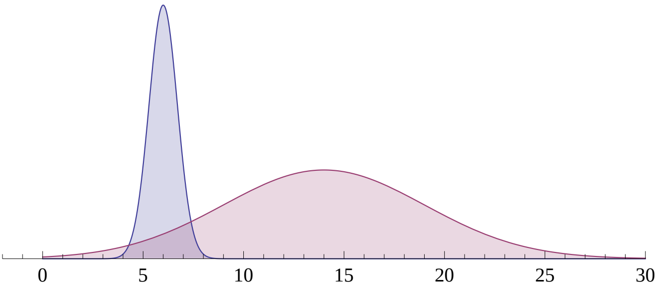
TFAE:

- $X \preceq_c Y$
- $Y \overset{(d)}{=} X + Z + \epsilon$, where
  $Z \geq 0$ ("risk premium"),
  $\mathbb{E}\left[\epsilon | X + Z\right] = 0$ ("noise")

A Visual Example

$$\forall K : \quad \mathbb{E}\left[(X-K)_+\right] \leq \mathbb{E}\left[(Y-K)_+\right]$$



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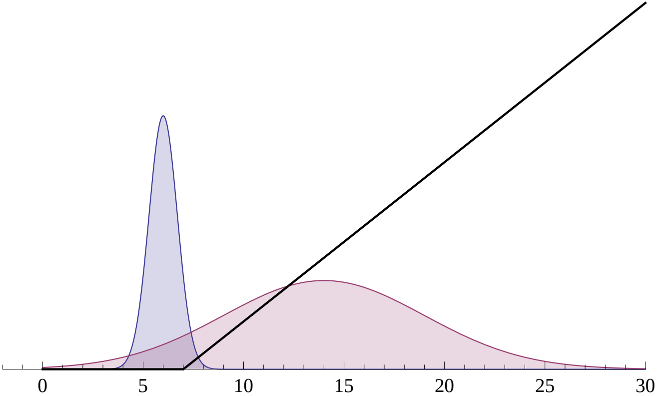

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$$U_A'(X_T^A) = y_A \frac{d\mathbb{Q}}{d\mathbb{P}}, \qquad U_L'(X_T^L) = y_L \frac{d\mathbb{Q}}{d\mathbb{P}}, \quad y_A, y_L > 0 \quad (1)$$

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Theorem (DW09)

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Corollary

In a complete market, $X_T^A \preceq_c X_T^L$, i.e., $X_T^L \stackrel{(d)}{=} X_T^A + Z + \epsilon$.

Main idea of proof: Extensive use of (1)

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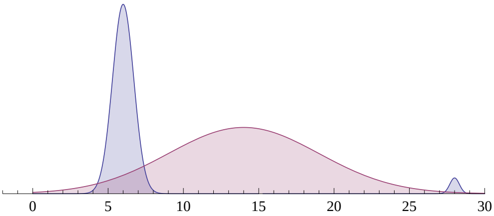
  where $\mathcal{Y}_A, \mathcal{Y}_L$ solve *dual* problem related to original optimization problem.
  BUT: no nice relation between $\mathcal{Y}_A$ and $\mathcal{Y}_L$ as in a complete market.

Counterexample: 1-Period Model, 2 stocks

- Incomplete market, agents A, L with power utility
- Two stocks:
  - "risky" red stock: should be bought by L
  - "secure" blue stock: should be bought by A

$$? \quad \forall K: \quad \mathbb{E}\left[(X^A - K)_+\right] \leq \mathbb{E}\left[(X^L - K)_+\right] \quad ?$$



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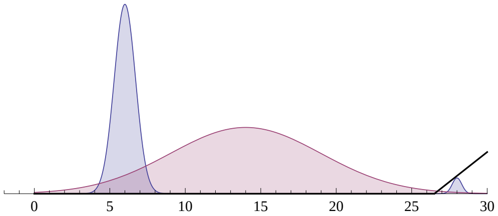

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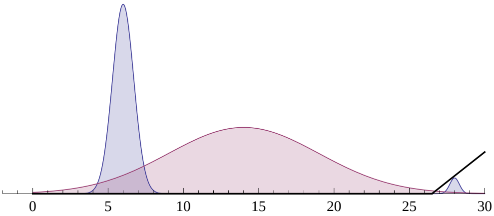

- Similar counterexample for 2-Period model with one risky stock & one risk-free stock and power utility

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- To check stochastic dominance in incomplete markets, we consider
  1. special market models (stochastic volatility model, exponential Lévy model)
  2. special utility functions (power utilities $U(x) = \frac{x^{1-p}}{1-p}$)

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  1. special market models (stochastic volatility model, exponential Lévy model)
  2. special utility functions (power utilities $U(x) = \frac{x^{1-p}}{1-p}$)
- $\rightarrow$ **exponential Lévy model** and **power utilities**
  1. N-period exponential Lévy model
  2. "essentially" 1-period model
  3. result for continuous time Lévy model by taking limit

N-Period Exponential Lévy Model

- Agents A, L with power utility functions U_A, U_L

N-Period Exponential Lévy Model

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- Risk-free stock $S^0 = 1$, ΔL_m i.i.d. random variables

$$S^1_n = \prod_{m=1}^{n} (1 + \Delta L_m) = \mathcal{E}(L)_n, \quad n = 1, \dots, N, \qquad \text{risky stock,}$$

N-Period Exponential Lévy Model

- Agents A, L with power utility functions U_A, U_L
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$$S_n^1 = \prod_{m=1}^{n} (1 + \Delta L_m) = \mathcal{E}(L)_n, \quad n = 1, \dots, N, \qquad \text{risky stock,}$$

- wealth process X_i evolves according to

$$X_0 = x \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_N} X_N$$

$$X_i = \pi_i X_{i-1}(1 + \Delta L_i) + (1 - \pi_i) X_{i-1}$$

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Proposition (Samuelson69)

Optimal strategy $(\pi_i)_{i=1}^N$ of N-period problem is given by $\pi_i = \pi^* \in \mathbb{R}$ (π^* optimal strategy for corr. 1-period problem).

Inheritance of Stochastic Dominance in 1-Period Models

- Agents A,L with power utility functions U_A, U_L
  stochastic initial capital-distributions μ_A, μ_L satisfying

$$\mu_A \preceq_c \mu_L.$$

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- One stock S, one risk-free stock normalized to 1, possibly incomplete market:

$$\max_{\pi} \mathbb{E}\left[U_i(\pi x + (1-\pi)x \cdot S)\right],$$

  where x is initial capital distributed according to μ_A, resp., μ_L.

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Proposition

$\mu_A \preceq_c \mu_L$ implies $X^A \preceq_c X^L$.

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$$X_0 = x \xrightarrow{\pi^*} X_1 \xrightarrow{\pi^*} \cdots \xrightarrow{\pi^*} X_N$$

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Corollary

The optimal terminal wealths X_N^A, X_N^L of the N-period Lévy problem satisfy $X_N^A \preceq_c X_N^L$.

Time-continuous Exponential Lévy Model: Idea of Proof

$$S_t^0 = 1, \qquad \text{risk-free stock}$$

$$\frac{dS_t^1}{S_t^1} = dL_t, \qquad \text{risky stock,}$$

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- Approximate $(S_t)_t$ by step function $t \mapsto \prod_{m=1}^{\lfloor t/N \rfloor}(1 + \Delta L_m)$

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2. Optimal payoffs X_N of N-period models converge in L^1 to optimal payoffs X_T in continuous time model

$$\mathbb{E}\left[|X_N^A - X_T^A|\right] \to 0 \qquad \mathbb{E}\left[|X_N^L - X_T^L|\right] \to 0$$

Exponential Lévy: Main Result & Future Plan

Theorem

In a time-continuous exponential Lévy model: $X_T^A \preceq_c X_T^L$, *i.e.,*

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- Stationarity of increments was not crucial.
- Attain results for models with conditionally independent increments, e.g.: BNS (Kallsen&Muhle-Karbe10)

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  - Result holds in exponential Lévy models with power utilities
  - Future plan: Result holds in models with conditionally independent increments and power utilities
  - "Strong" numerical evidence suggests that result holds in stochastic volatility models with correlation