Martingale Optimal Transport: A Nice Ride in Quantitative Finance

Pierre Henry-Labordère[1]

[1]Global markets Quantitative Research, SOCIÉTÉ GÉNÉRALE

Contents

- Optimal transport versus Martingale optimal transport.
- Applications in mathematical finance:
  - Model-independent bounds for exotic options: Numerical methods.
  - Particle's methods for non-linear McKean SDEs: Calibration of LSVMs.
  - Skorokhod embedding problem [see Nizar's talk]

Optimal Transport in Mathematics

- Optimal transport, first introduced by G. Monge in his work “Théorie des déblais et des remblais" (1781).
- Has recently spread out in various mathematical domains as highlighted by the last Fields medallist C. Villani. Let us cite
  - Analysis of non-linear (kinetic) partial differential equations arising in statistical physics such as McKean-Vlasov PDE.
  - Mean-field limits, convergence of particle’s methods.
  - Optimal fundamental inequalities (Poincaré, (Log)-Sobolev, Talagrand...)
  - Study of Ricci flows in differential geometry.

Optimal Transport in Quantitative Finance

- Despite these large ramifications with analysis and probability, optimal transport has not yet attracted the attention of practitioners in financial mathematics.
- However, various long-standing problems in quantitative finance can be tackled using the framework of optimal transport. In particular,
  - Calibration of (hybrid) models on market smiles using particle's method.
  - Computation of efficient model-independent bounds for exotic options.

$\Rightarrow$ Leads to a nice modification of optimal transport ["Martingale version" of MK]

Optimal Transport in a Nutshell (1)

- Payoff c depending on two assets S_1, S_2.
- The distributions of S_1 and S_2 are known from Vanilla options

$$\mathbb{P}^i(K) = \partial_K^2 \mathcal{C}^i(T, K)$$

- Monge-Kantorovich[1]

$$\mathrm{MK}_c = \inf_{\mathbb{P}, S_1 \sim \mathbb{P}^1, S_2 \sim \mathbb{P}^2} \mathbb{E}^{\mathbb{P}}[c(S_1, S_2)]$$

[1] $S_1 \sim \mathbb{P}^1$ means $\mathrm{Law}(S_1) = \mathbb{P}^1$

Optimal Transport in a Nutshell (2): Kantorovich duality

- (Linear) duality (Minimax):

$$\mathrm{MK}_c = \sup_{u_1(\cdot),u_2(\cdot)} \quad \mathbb{E}^{\mathbb{P}^1}[u_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[u_2(S_2)]$$

$$u_1(S_1) + u_2(S_2) \leq c(S_1, S_2)\ ,\ \mathbb{P}^1 \times \mathbb{P}^2 \ \mathrm{a.s}$$

- The dual bound can be statically replicated by holding European options with payoffs $u_1(S_1)$ and $u_2(S_2)$ with market prices $\mathbb{E}^{\mathbb{P}^1}[u_1(S_1)]$ and $\mathbb{E}^{\mathbb{P}^2}[u_2(S_2)]$. The intrinsic value of the portfolio $u_1(S_1) + u_2(S_2)$ is lower than the payoff $c(S_1, S_2)$.

Martingale Optimal Transport (1)

- Payoff $c(S_{t_1}, \ldots, S_{t_n})$ depending on one asset evaluated at $t_1 < \ldots < t_n$.
- No-arbitrage condition: S_t is required to be a (local) positive martingale [2].
- The distribution of S_{t_i} is known from Vanilla options at t_i.
- Primal (Lower bound):

$$P = \inf_{S_{t_i} \sim \mathbb{P}^i, \mathbb{E}^{\mathbb{P}}_{t_{i-1}}[S_{t_i}] = S_{t_{i-1}}} \mathbb{E}^{\mathbb{P}}[c(S_{t_1}, \ldots, S_{t_n})]$$

[2] We take zero interest rate, no dividends for the sake of simplicity. This can be easily relaxed.

Martingale Optimal Transport (2)

- Feasibility of $\{\mathbb{P} : S_{t_i} \sim \mathbb{P}^i, \mathbb{E}^{\mathbb{P}}_{t_{i-1}}[S_{t_i}] = S_{t_{i-1}}]\}$: Convex order [Kellerer].
- Convex order: $\mathbb{P}^1 \leq \mathbb{P}^2$ if $\mathbb{E}^{\mathbb{P}^1}[(S_{t_1} - K)^+] \leq \mathbb{E}^{\mathbb{P}^2}[(S_{t_2} - K)^+]$.
- Dual[3] :

$$D = \inf_{(u_i(\cdot))_{1\leq i\leq n},(\Delta_i(\cdot))_{1\leq i\leq n}} \sum_{i=1}^{n} \mathbb{E}^{\mathbb{P}^i}[u_i(S_i)]$$

$$\sum_{i=1}^{n} u_i(S_i) + \sum_{i=1}^{n} \Delta_i(S_1, \ldots, S_{i-1})(S_i - S_{i-1}) \leq c(S_1, \ldots, S_n)$$

$$, \ \mathbb{P}^1 \times \ldots \times \mathbb{P}^n \text{ a.s.}$$

- Financial interpretation: sub-hedging strategy $\bigoplus$ static portfolio of Vanillas.

[3] $\bigoplus$ Markov assumption: $\Delta_i(S_1, \ldots, S_{i-1}) = \Delta_i(S_{i-1})$

“Martingale version" of MK duality

Theorem (Beiglböck, PHL, Penkner)

Assume that $\mathbb{P}^1, \ldots, \mathbb{P}^n$ are Borel probability measures on $\mathbb{R}_+$ such that $\mathbb{P}^1 \leq \ldots \leq \mathbb{P}^n$. Let $c : \mathbb{R}_+^n \to (-\infty, \infty]$ be a lower semi-continuous function such that

$$c(S_1, \ldots, S_n) \geq -K \cdot (1 + |S_1| + \ldots + |S_n|) \tag{1}$$

on $\mathbb{R}_+^n$ for some constant K. Then there is no duality gap, i.e. $P = D \equiv \widetilde{\mathrm{MK}}_c$. Moreover, the primal value P is attained, i.e. there exists a martingale measure $\mathbb{P}$ with marginals $(\mathbb{P}^1, \ldots, \mathbb{P}^n)$ such that $P = \mathbb{E}^{\mathbb{P}}[c]$. The dual supremum is in general not attained.

MK_c versus $\widetilde{\mathrm{MK}}_c$

- $\widetilde{\mathrm{MK}}_c > \mathrm{MK}_c \Longrightarrow$ tight bounds.

MK_c	$\widetilde{\mathrm{MK}}_c$
$\inf_{\mathbb{P}, S_1 \sim \mathbb{P}^1, S_2 \sim \mathbb{P}^2} \mathbb{E}^{\mathbb{P}}[c(S_1, S_2)]$	$\inf_{\mathbb{P}, S_1 \sim \mathbb{P}^1, S_2 \sim \mathbb{P}^2, \mathbb{E}[S_2 \mid S_1] = S_1} \mathbb{E}^{\mathbb{P}}[c(S_1, S_2)]$
$\sup_{u_1, u_2} \mathbb{E}^{\mathbb{P}^1}[u_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[u_2(S_2)]$ $u_1(S_1) + u_2(S_2) \leq c(S_1, S_2)$	$\sup_{u_1, u_2, \Delta} \mathbb{E}^{\mathbb{P}^1}[u_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[u_2(S_2)]$ $u_1(S_1) + u_2(S_2) + \Delta(S_1)(S_2 - S_1) \leq c(S_1, S_2)$
$\sup_u \mathbb{E}^{\mathbb{P}^2}[u(S_2)] + \mathbb{E}^{\mathbb{P}^1}[u^c(S_1)]$[4]	$\sup_u \mathbb{E}^{\mathbb{P}^2}[u(S_2)] + \mathbb{E}^{\mathbb{P}^1}[(c(S_1, \cdot) - u(\cdot))^{\mathrm{conv}}(S_1)]$[5]

- Important results in optimal transport are derived for the quadratic cost $c(S_1, S_2) = |S_2 - S_1|^2$ [see Brenier's Theorem].
- In the Martingale version, the quadratic cost is degenerate:

$$\mathbb{E}^{\mathbb{P}}[(S_2 - S_1)^2] = \mathbb{E}^{\mathbb{P}^2}[S_2^2] - \mathbb{E}^{\mathbb{P}^1}[S_1^2] \; \forall \, \mathbb{P} \text{ mart. } \oplus S_i \sim \mathbb{P}^i$$

$\Longrightarrow$ Important results in MK need to be rewritten for $\widetilde{\mathrm{MK}}$!

[4] $u^c(S_1) \equiv \inf_{S_2} c(S_1, S_2) - u(S_2)$

[5] f^{conv}: largest convex function smaller than or equal to f

Optimal Transport on the real line

We note F_1 the cumulative distribution associated to $\mathbb{P}_1$. Let $c(S_1, S_2) = c(S_2 - S_1)$ be a C^1 strictly concave.

Proposition

The upper bound is given by [Fréchet copula]

$$\mathrm{MK}_c = \int_0^1 c(F_1^{-1}(u), F_2^{-1}(u))du$$

The (optimal) upper bound is reached for

$$\hat{u}_2(y) = \int_0^y c'(F_1^{-1} F_2(z), z)dz$$

$$\hat{u}_1(x) = c(x, F_2^{-1} F_1(x)) - \hat{u}_2(F_2^{-1} F_1(x))$$

Brenier's theorem

Let $c(S_1, S_2) = c(S_2 - S_1)$ be a C^1 strictly convex.

Theorem (Brenier)

There exists a unique optimal transference plan for the MK_c *transportation problem and it has the form*

$$\mathbb{P}^*(S_1, S_2) = \delta\left(S_2 - T(S_1)\right)\mathbb{P}^1(S_1),\ T_{\#}\mathbb{P}_1 = \mathbb{P}_2$$

and $T(x) = x - \nabla c^{-1}(\nabla\psi)$ *for some c-concave function* ψ*. The optimal lower bound is given by*

$$\mathrm{MK}_c = \int_0^\infty c(x, T(x))\mathbb{P}^1(x)dx$$

On the real line, $T(x) = F_2^{-1} F_1(x)$: monotone rearrangement map.

Martingale version of Brenier's theorem (1) [Hobson-Neuberger], [Beiglböck-Juillet]

Let $c(S_1, S_2) = c(S_2 - S_1)$ be a C^1 function such that c' is strictly concave. Suppose $\mathbb{P}^1 \leq \mathbb{P}^2$.

Theorem (Beiglböck-Juillet)

There exists a unique optimal transference plan for $\widetilde{\mathrm{MK}}_c$*:*

$$\mathbb{P}^*(S_1, S_2) = \Big(\delta(S_2 - T_1(S_1))\frac{T_2(S_1) - S_1}{T_2(S_1) - T_1(S_1)} + \delta(S_2 - T_2(S_1))\frac{S_1 - T_1(S_1)}{T_2(S_1) - T_1(S_1)}\Big)\mathbb{P}^1(S_1)$$

The optimal upper bound is given by

$$\int_0^\infty \frac{(T_2(x) - x)\, c(x, T_1(x)) + (x - T_1(x))\, c(x, T_2(x))}{T_2(x) - T_1(x)}\mathbb{P}^1(x)dx$$

Explicit characterization of T_1, T_2 [PHL]

- The maps (T_1, T_2) are solutions of the equations ($T_1(x) \leq x \leq T_2(x)$, T_1, T_2 C^1 functions)

$$c_2(T_1^{-1}(x), x) - c_2(T_2^{-1}(x), x) = \int_{T_2^{-1}(x)}^{T_1^{-1}(x)} \frac{c_1(y, T_2(y)) - c_1(y, T_1(y))}{T_2(y) - T_1(y)} dy$$

$$\mathbb{P}^2(x) = \frac{T_2 T_1^{-1}(x) - T_1^{-1}(x)}{T_2 T_1^{-1}(x) - x} \mathbb{P}^1(T_1^{-1}(x)) |T_1'^{-1}(x)| + \frac{T_2^{-1}(x) - T_1 T_2^{-1}(x)}{x - T_1 T_2^{-1}(x)} \mathbb{P}^1(T_2^{-1}(x)) |T_2'^{-1}(x)|$$

- Semi-static superreplication:

$$\frac{du_2(x)}{dx} = c_2(T_1^{-1}(y), x) - \int_0^{T_1^{-1}(x)} \frac{c_1(y, T_2(y)) - c_1(y, T_1(y))}{T_2(y) - T_1(y)} dy$$

$$u_1(x) = \frac{(c(x, T_1(x)) - u_2(T_1(x)))(x - T_2(x)) - (c(x, T_2(x)) - u_2(T_2(x)))(x - T_1(x))}{T_1(x) - T_2(x)}$$

$$\Delta(x) = \frac{(c(x, T_1(x)) - u_2(T_1(x))) - (c(x, T_2(x)) - u_2(T_2(x)))}{T_1(x) - T_2(x)}$$

Examples

- Spread option $(S_2 - S_1)^+$ [Fréchet]:

$$\mathrm{MK}_c = \int_0^\infty (T(x) - x)^+ \mathbb{P}^1(x)dx \ , \ T(x) = F_2^{-1} F_1(x)$$

- Forward-start options [Hobson-Neuberger] $(S_{t_2} - S_{t_1})^+$:

$$\widetilde{\mathrm{MK}}_2 \quad = \quad \int_0^\infty \frac{(T_2(x) - x)\,(x - T_1(x))}{T_2(x) - T_1(x)} \mathbb{P}^1(x)dx$$

- Variance swap $c(S_{t_2}, S_{t_1}) = \ln^2 \frac{S_{t_2}}{S_{t_1}}$ [PHL]:

$$\int_0^\infty \frac{(T_2(x) - x) \ln^2 \frac{T_1(x)}{x} + (x - T_1(x)) \ \ln^2 \frac{T_2(x)}{x}}{T_2(x) - T_1(x)} \mathbb{P}^1(x)dx$$

Optimal Transport and Hamilton-Jacobi (1)

Here $c(S_1, S_2) := c(S_2 - S_1)$, c is strictly concave.

Theorem (see Villani, *Topics in Optimal Transport*, AMS)

$$\mathrm{MK}_c = \sup -\mathbb{E}^{\mathbb{P}^1}[u(0, S_1)] + \mathbb{E}^{\mathbb{P}^2}[u(1, S_2)]$$

where the supremum is taken over all continuous viscosity solutions u to the following HJ equation:

$$\partial_t u(t,x) + c^*(\nabla u) = 0\ ,\ \ c^*(p) := \sup_q \{pq - c(q)\}$$

Proof uses Hopf-Lax's formula:

$$-u(0,x) = \inf_y c(y-x) - u(1,y)$$

Guess: Martingale optimal transport $\Longrightarrow$ HJB.

See Nizar's talk: Generalization of Mikani-Thiellen approach.

Hopf-Lax's formula: Reminder

1. Dynamic programming:

$$u(t,x) = \sup_{\dot\zeta} u(1, x + \int_t^1 \dot\zeta(s)ds) - \int_t^1 c(\dot\zeta(s))ds$$

2. Maximization over $\dot\zeta$: $\dot\zeta$ is a constant q.

$$u(t,x) = \sup_q u(1, x + q(1-t)) - c(q)(1-t)$$

3. Set $y = x + q(1-t)$. Get the Hopf-Lax solution:

$$u(t,x) = \sup_y u(1,y) - c(\frac{y-x}{1-t})(1-t)$$

4. For $t = 0$, $-u(0,\cdot)$ is the c-transform of $u(1,\cdot)$:

$$-u(0,x) = \inf_y c(y-x) - u(1,y)$$

Time-continuous limit

- Robust super-hedging price of a payoff given vanilla options ($S_{t_i} \sim \mu_i$, $\mu(\lambda) := \mathbb{E}^{\mu}[\lambda]$):

$$U_n^{\mu}(\xi) := \inf\{U_0 \ : \ \exists \Delta, \exists \lambda \ : \ U_0 + \int_0^T \Delta_s dS_s + \sum_{i=1}^n \lambda_i(S_{t_i}) - \sum_{i=1}^n \mu_i(\lambda_i) \geq \xi \ , \ \forall \, \mathbb{P} \text{ Mart.}\}$$

Measures are singular: Quasi-sure analysis (see Nizar's talk)

Duality in continuous-time

Theorem (Galichon, PHL, Touzi)

Let $\xi \in UC(\Omega_{S_0})$ be such that $\xi^+ \in \mathbb{L}^1(\mathbb{P})$ for all $\mathbb{P}$ Mart.. Then, for all $\mu := (\mu_i)_i \in M(\mathbb{R}_+)$ in convex order:

$$U_n^\mu(\xi) \quad = \quad \inf_{\lambda_i \in \Lambda^\mu_{\mathrm{UC}}} \sup_{\mathbb{P}\ \mathrm{Mart.}} \Big\{ \sum_{i=1}^n \mu_i(\lambda_i) + \mathbb{E}^{\mathbb{P}}\big[\xi - \sum_{i=1}^n \lambda_i(S_{t_i})\big] \Big\}.$$

Robust version of [Kramkov, Schachermayer] duality.
If we can apply formally a min-max duality,

$$U_n^\mu(\xi) = \sup_{\mathbb{P} \in \mathrm{Mart.}\ ,\ S_{t_i} \sim \mu_i} \mathbb{E}^{\mathbb{P}}[\xi]$$

$\Rightarrow$ Martingale optimal transport problem.
$\Rightarrow$ Give models calibrated to vanilla options.

Models calibrated to Vanillas: Some examples

- Local volatility model [Dupire]:

$$
\begin{aligned}
df_t &= \sigma_{\mathrm{loc}}(t, f_t) dW_t \\
\sigma_{\mathrm{loc}}(t, f)^2 &= 2 \frac{\partial_t \mathcal{C}(t, f)}{\partial_f^2 \mathcal{C}(t, f)}
\end{aligned}
$$

- Local stochastic volatility models:

$$
\begin{aligned}
df_t &= \sigma(t, f_t) a_t dW_t \\
\sigma_{\mathrm{loc}}(t, f)^2 &= \sigma(t, f)^2 \mathbb{E}[a_t^2 | f_t = f]
\end{aligned}
$$

Equivalent to

$$
df_t = \sigma_{\mathrm{loc}}(t, f) \frac{a_t}{\sqrt{\mathbb{E}[a_t^2 | f_t]}} dW_t
$$

$\Longrightarrow$ Non-linear McKean SDEs for which optimal transport shows up again! [see Tanaka's approach for Boltzmann equation]

McKean SDEs

Definition

$$dX_t = b(t, X_t, \mathbb{P}_t)dt + \sigma(t, X_t, \mathbb{P}_t) \cdot dW_t$$

with W_t a d-dimensional Brownian motion and $\mathbb{P}_t = \mathrm{Law}(X_t)$.

- Example: McKean-Vlasov SDEs:

$$\begin{aligned} b \equiv \left(b^i(t, x, \mathbb{P}_t)\right)_{i=1,\ldots,n} &= \int b^i(t, x, y) p(t, y|X_0) dy \\ \sigma \equiv \{\sigma^i_j(t, x, \mathbb{P}_t)\}_{i=1,\ldots,n; j=1,\ldots d} &= \int \sigma^i_j(t, x, y) p(t, y|X_0) dy \end{aligned}$$

Existence result

Theorem (Sznitman)

Let $b : \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{P}_2 \to \mathbb{R}^{n \times d}$ be Lipschitz continuous functions for the sum of canonical metric on $\mathbb{R}^n$ and the MK metric d on the set $\mathcal{P}_2$ of probability measures with finite second order moments. Then the non-linear SDE

$$dX_t = b(t, X_t, \mathbb{P}_t) + \sigma(t, X_t, \mathbb{P}_t) dW_t \ , \ X_0 \in \mathbb{R}^n$$

where $\mathbb{P}_s$ denotes the probability distribution of X_s admits an unique solution such that $\mathbb{E}(\sup_{t \leq T} |X_t|^p) < \infty$ for all $p \geq 2$.

Open problem: Existence of LSVMs?

Proof: fixed point.

Monte-Carlo simulation: interacting particle system

- Replace $\mathbb{P}_t$ by its empirical measure: Let $X_t^1, \ldots X_t^N$ be i.i.e. with law $\mathbb{P}_t$: $\mathbb{P}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$. Note that $\mathbb{P}_t^N$ is a random probability measure.
- N interacting **bosons** (i.e. symmetric):

$$df_t^i = f_t^i \sigma_{\mathrm{loc}}(t, f_t^i)\sqrt{\frac{\sum_{j=1}^N \delta(\ln f_t^j - \ln f_t^i)}{\sum_{j=1}^N (a_t^j)^2 \delta(\ln f_t^j - \ln f_t^i)}} a_t^i dW_t^i$$

  $\rightarrow$ Needs to be replaced $\delta(\cdot)$ by a regularizing kernel.
- *Propagation of chaos* for McKean-Vlasov SDEs: If at $t = 0$, $X_0^{i,N}$ are independent particles then as $N \rightarrow \infty$, for any fixed $t > 0$, the $X_t^{i,N}$ are asymptotically independent and their empirical measure $\mathbb{P}_t^N$ converges in distribution towards the true measure $\mathbb{P}_t$.

Algorithm [Guyon-PHL]

1. Initialize $k = 0$ and set $\sigma(t, f) = \frac{\sigma_{\mathrm{Dup}}(0,f)}{a_0}$ for all $t \in [k\Delta, (k+1)\Delta]$.
2. Simulate the N processes $\{f_t^i, a_t^i\}_{i=1,\ldots,N}$ from $t = k\Delta$ to $(k+1)\Delta$ using a discretization scheme such as Euler.
3. Compute the local volatility $\sigma((k+1)\Delta, f)$ on a space-grid $f \in [f_{k\Delta}^{\mathrm{min}}, f_{k\Delta}^{\mathrm{max}}]$ using

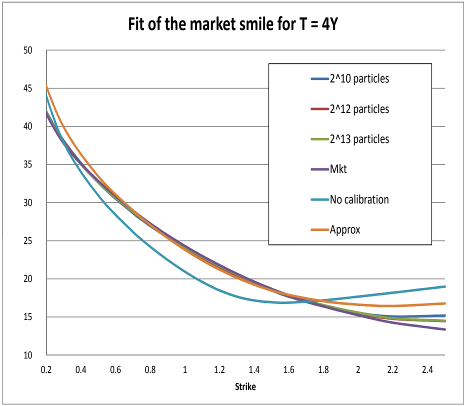
$$\sigma(t, f) = \frac{\sigma_{\mathrm{Dup}}(t, f)}{\sqrt{\frac{\sum_{j=1}^{N} (a_t^{(j)})^2 \delta_{t,N}(f_t^{(j)} - f)}{\sum_{j=1}^{N} \delta_{t,N}(f_t^{(j)} - f)}}}$$

   Set $\sigma(t, f) \equiv \sigma((k+1)\Delta, f)$ for all $t \in [(k+1)\Delta, (k+2)\Delta]$.
4. $k := k + 1$. Iterate step 2 and 3 up to the maturity date T.

Convergence issue: prove the propagation of chaos for LSVMs?

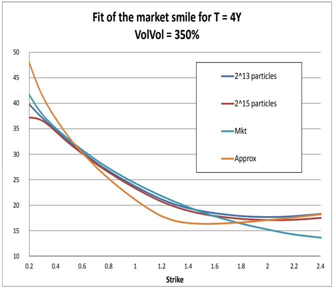
Local Bergomi model

▷ DAX market smiles (30-May-11):



Local stochastic volatility model: Existence under question

- The existence of LSV models for a given market smile is not at all obvious although this seems to be a common belief in the quant community.
- Checked our algorithm with a volatility-of-volatility $\sigma = 350\%$. Our algorithm converge with $N = 2^{13}$ particles but the market smile is not properly calibrated:



Digression: Talagrand-like inequality

- Talagrand inequality:

$$\mathrm{T}(\lambda) \; : \; \forall \mathbb{P}^1 \; W_2(\mathbb{P}^1, \mathbb{P}^2)^2 \leq \frac{2}{\lambda} H(\mathbb{P}^1 | \mathbb{P}^2)$$

  Relative entropy:

$$\begin{aligned} H(\mathbb{P}|\mathbb{P}^0) &= \mathbb{E}^{\mathbb{P}}[\ln \frac{d\mathbb{P}}{d\mathbb{P}^0}] \; , \; \mathbb{P} \text{ is absolutely continuous w.r.t. } \mathbb{P}^0 \\ &= +\infty \; , \; \text{otherwise} \end{aligned}$$

- Villani-Otto, Ledoux-al: $\mathrm{LSI}(\lambda) \longrightarrow \mathrm{T}(\lambda)$ [Proof: dual expression for the Talagrand inequality + contraction of HJ]
- A similar dual expression appears in mathematical finance $\Rightarrow$ (Martingale) Weighted Monte-Carlo.

(Martingale) Weighted Monte-Carlo [Avellaneda-al], [PHL]

- Consider instruments c_a , $a = 1, \ldots, N$, with bid/ask market prices $\underline{c}_a/\overline{c}_a$:

$$\underline{c}_a \leq \mathbb{E}^{\mathbb{P}}[c_a] \leq \overline{c}_a$$

- $\mathcal{M}(\mathbb{P}_1, \ldots, \mathbb{P}_n | c_1, \ldots, c_N)$: the set of all martingale measures $\mathbb{P}$ on $(\mathbb{R}_+^d)^n$ with prescribed marginals $\{\mathbb{P}_i\}_{i=1,\ldots,n}$ and satisfying (2).
- Primal:

$$\mathrm{P}_\lambda \equiv \sup\{\mathbb{E}^{\mathbb{P}}[c] : \mathbb{P} \in \mathcal{M}(\mathbb{P}_1, \ldots, \mathbb{P}_n | c_1, \ldots, c_N) \ , \ H(\mathbb{P}, \mathbb{P}^0) \leq \lambda\}$$

- Some particular limits:

$$\begin{aligned} \mathrm{P}_\infty &= \mathrm{MK}_c \\ \mathrm{P}^0 &= \inf\{H(\mathbb{P}, \mathbb{P}^0) : \mathbb{P} \in \mathcal{M}(\mathbb{P}_1, \ldots, \mathbb{P}_n | c_1, \ldots, c_N)\} \end{aligned}$$

(Martingale) Weighted Monte-Carlo [PHL]

- Dual:

$$\mathrm{D}_\lambda \equiv \inf_{(u_i(\cdot))_{1\leq i\leq n},(\Delta_i(\cdot))_{1\leq i\leq n},\underline{\Lambda}_a\in\mathbb{R}^+,\overline{\Lambda}_a\in\mathbb{R}^+,\zeta\in\mathbb{R}^+}$$

$$\sum_{i=1}^{n}\mathbb{E}^{\mathbb{P}^i}[u_i] + \sum_{a=1}^{N}\left(\overline{\Lambda}_a\overline{c}_a - \underline{\Lambda}_a\underline{c}_a\right)$$

$$+\zeta\left(\lambda + \ln\mathbb{E}^{\mathbb{P}^0}[e^{\zeta^{-1}\left(c-\sum_{a=1}^{N}\left(\overline{\Lambda}_a-\underline{\Lambda}_a\right)c_a-\sum_{i=1}^{n}u_i-\sum_{i=1}^{n}\Delta_i(S_i-S_{i-1})\right)}]\right)$$

(Martingale) Weighted Monte-Carlo [PHL]

Theorem

There is no duality gap $D_\lambda = P_\lambda$. The supremum is attained by the optimal measure $\mathbb{P}^*$ given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}^0} = \frac{e^{(\zeta^*)^{-1}\left(c - \sum_{a=1}^N \left(\overline{\Lambda}_a^* - \underline{\Lambda}_a^*\right) c_a - \sum_{i=1}^n u_i^* - \sum_{i=1}^n \Delta_i^*(S_i - S_{i-1})\right)}}{\mathbb{E}^{\mathbb{P}^0}\left[e^{(\zeta^*)^{-1}\left(c - \sum_{a=1}^N \left(\overline{\Lambda}_a^* - \underline{\Lambda}_a^*\right) c_a - \sum_{i=1}^n u_i^* - \sum_{i=1}^n \Delta_i^*(S_i - S_{i-1})\right)}\right]}$$

where $\left((u_i^*(\cdot))_{1 \le i \le n}, (\Delta_i^*(\cdot))_{1 \le i \le n}, \underline{\Lambda}_a^*, \overline{\Lambda}_a^*, \zeta^*\right)$ achieves the infimum in D_λ.

P_∞: Semi-Infinite Linear Programming Approach

- Dual:

$$\inf_{(u_i(\cdot))_{1\le i\le n},(\Delta_i(\cdot))_{2\le i\le n},\overline{\Lambda}_a\ge 0,\underline{\Lambda}_a\ge 0} \sum_{i=1}^{n} \mathbb{E}^{\mathbb{P}^i}[u_i] + \sum_a \left(\overline{\Lambda}_a \overline{c}_a - \underline{\Lambda}_a \underline{c}_a\right)$$

subject to the constraints

$$\sum_{i=1}^{n} u_i + \sum_{i=2}^{n} \Delta_i (S_i - S_{i-1}) + \sum_a \left(\overline{\Lambda}_a - \underline{\Lambda}_a\right) c_a \ge c$$

- Deltas Δ_i are decomposed over a finite-dimensional basis:

$$\Delta_i(S_0, \cdots, S_{i-1}) = \sum_b [\Delta_i]^b e_b(S_0, \cdots, S_{i-1})$$

- Similarly, European options with payoffs u_i are decomposed over a finite set of call options:

$$u_i(S_i) = \sum_b [u_i]^b (S_i - K^b)^+$$

P_∞: Semi-Infinite Linear Programming Approach

- Leads to a semi-infinite linear program:

$$U = \min_{x \in \mathbb{R}^n} c^\dagger x \;\; A(S)x \geq B(S) \; \forall \; S \in (\mathbb{R}_+)^d$$

- Our algorithm will produce an upper bound

$$\mathrm{D}_{\mathrm{basis}} \geq \mathrm{D} = \mathrm{P}$$

Dealing with ∞ constraints: Cutting-plane method

Let $G \subset (\mathbb{R}_+)^d$, $|G| < \infty$ be a given initial grid and (ϵ_k) a sequence of non-negative numbers converging to 0. Let $\mathrm{TOL} > 0$ be a suitable convergence tolerance and set $k = 0$.

1. Solve the relaxed finite-dimensional LP: optimal solution $x = x^*$

$$
\begin{aligned}
U \geq \min_{x \in \mathbb{R}^n} c^\dagger x \\
A(S)x \geq B(S) \; \forall \; S \in G
\end{aligned}
$$

2. Determine the constraint violation: $\delta = \min_{S \in \mathbb{R}_+^d} A(S)x^* - B(S)$
3. If $\delta > -\mathrm{TOL}$ then stop. Otherwise add the constraints $A(S)x^* - B(S) < \delta + \epsilon_k$
4. Go to step 1.

Algorithm for the risk-neutral WMC: calibration and pricing

1. Simulate Monte-Carlo paths under the measure $\mathbb{P}^0$.
2. Solve the non-linear programming problem (2) using for example a gradient-based optimization routine (Note that this problem is strictly convex and admits an unique solution.
3. The exotic option price with payoff c is given by D_λ. Note that the optimal measure $\mathbb{P}^*$ as given by Equation (2) can be used to value any exotic options depending on $(S_1, \cdots, S_n)$.

Pricing variance swap on an illiquid stock (1)

- Assumption: Diffusion $\to \mathrm{VS} = -\frac{2}{T}\mathbb{E}[\ln S_T]$.
- Input: finite set of strikes (with $K = 0$).
- Dual:

$$\min_{(\omega_i)_{i=1,\dots,n},\nu} \nu + \sum_{i=1}^{n} \omega_i \mathcal{C}(K_i) \; ; \; \nu + \sum_{i=1}^{n} \omega_i (S - K_i)^+ \geq -\frac{2}{T} \ln S \, , \; \forall \, S \in$$

- Input: Smile DAX $5/07/2011$ $T = 1.5\mathrm{Y}$, static replication: 34.06.

Strike range	Lower	Upper	Mid
$[0.15 - 2.50]$	33.32	34.74	34.06
$[0.50 - 1.50]$	30.99	40.35	35.67
$[0.80 - 1.20]$	27.20	54.94	41.07
$[0.90 - 1.10]$	24.89	62.87	43.88

Illiquid Fx smile

- Input: Smile 1 & 2 , ATM smile 3, call spread on 3.
- Dual:

$$\min_{(\omega_i^j)_{j=1,2;i=1,\ldots,n},\nu,\omega_3,\Delta} \nu + \sum_{j=1}^{2}\sum_{i=1}^{n}\omega_i^j \mathcal{C}^j(K_i) + \omega_3 \mathcal{C}^3(S_0^3) + \Delta \mathcal{CS}^3$$

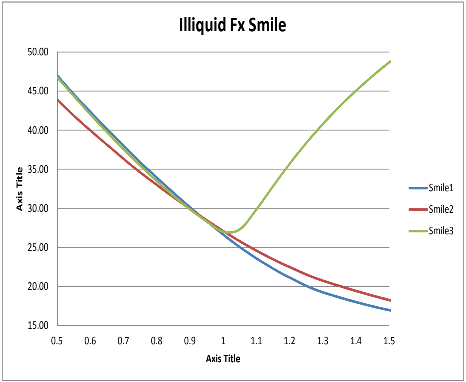
$$\nu + \sum_{j=1}^{2}\sum_{i=1}^{n}\omega_i^j (S^j - K_i^j)^+ + \omega_3 (S^2 - S_0^3 S^1)^+$$

$$+\Delta\left((S^2 - 0.95 S_0^3 S^1)^+ - (S^2 - 1.05 S_0^3 S^1)^+\right) \geq (S^2 - K S^1)^+$$

- Fact: constraints are piecewise linear w.r.t. S_1, S_2:
  Extremal points: prob. with a discrete support.

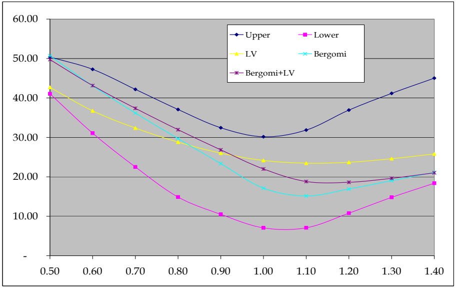
Illiquid Fx smile (1)

- $\sigma_{\text{ATM}} = 27$
- Call Spread: $\sigma(0.95) = 25.5, \sigma(1.05) = 28.5$



Cliquet: $\left(\frac{S_2}{S_1} - K\right)^+$

- Eurostock implied volatilities(2-Feb-2010). $t_1 = 1$ year and $t_2 = 1.5$ years.
- Parameters for the Bergomi model: $\sigma = 2.0$, $\theta = 22.65\%$, $k_1 = 4$, $k_2 = 0.125$, $\rho = 34.55\%$, $\rho_{\text{SX}} = -76.84\%$, $\rho_{\text{SX}} = -86.40\%$.



Asian option with monthly returns 1Y (1)

- Input: DAX 5/09/2011.
- Parameters for the Bergomi model: $\theta = 25\%$, $k_1 = 8$, $k_2 = 0.3$, $\rho = 0\%$, $\rho_{\mathrm{SX}} = -80\%$, $\rho_{\mathrm{SX}} = -48\%$.

LV	8.36%
Bergomi [6]	9.23%
Bergomi+LV	8.71%
Upper/Lower	9.42%/5.47%
Minimal entropy martingale	8.32%
WMC	8.84%/7.51%

[6] calibrated on VS

Some references (1)

- Beiglböck, M., PHL and Penkner, F. : *Model independent Bounds for Option Prices: A Mass Transport Approach*. arXiv:1106.5929, submitted (2011).
- Galichon, A., PHL, Touzi, N. : *A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options*, submitted.
- Guyon, J., Henry-Labordère, P. : *Being particular about calibration*, Risk magazine (2012).
- Guyon, J., Henry-Labordère, P. : *Non-linear Methods in Quantitative Finance*, CRC Chapman-Hall, to appear in 2013.
- PHL : *Automated Option Pricing: Numerical methods*, submitted (2012).

Some references (2)

- Villani, C. : *Topics in Optimal Transportation*, Graduate Studies in Mathematics (58), AMS(2003).
- Villani, C. : *Limite de Champ Moyen*, Cours DEA (2002), http://www.umpa.ens-lyon.fr/ cvillani/Cours/moyen.pdf