Market models for the smile
Local volatility, local-stochastic volatility

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Outline

- Usable models?

  - The local volatility model
  - The carry P&L of the LV model
  - The delta – the delta of a vanilla option
  - Break-even levels for vols of implied vols / covariance of spot and implied volatilities
    - SSR and volatilities of volatilities in the local volatility model

- Local-stochastic volatility models
- A criterion for admissibility
- Examples
- Conclusion
Intro – a practically usable model?

- Imagine we have traded an option of maturity $T$ on an asset $S$, whose payoff is $f(S_T)$.
- The pricing library supplies a pricing function $P(t, S)$.
- We have no idea of what’s been implemented.
- How do we assess whether it’s OK to use $P(t, S)$?

- Sanity check 1
  - Set $t = T$; check that $P(t = T, S) = f(S), \forall S$.

- If OK, then sanity check 2
  - Compute delta: $\Delta = \frac{dP}{dS}$.
  - P&L of a short delta-hedged position during $[t, t + \delta t]$ is:
    $$ P&L = -\left( P(t + \delta t, S + \delta S) - (1 + r\delta t)P(t, S) \right) + \Delta \left( \delta S - (r - q)S\delta t \right) $$
  - Expand at order 2 in $\delta S$, 1 in $\delta t$:
    $$ P&L = - \left( -rP + \frac{dP}{dt} + (r - q)S\frac{dP}{dS} \right) \delta t - \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2 $$
Intro – a practically usable model? – 2

▶ P&L during $\delta t$ is:

$$P&L = -A(t, S)\delta t - B(t, S)\delta S^2$$

▶ if $A(t, S) \geq 0$, $B(t, S) \geq 0$ $\Rightarrow$ Always loosing money: no good.

▶ if $A(t, S) \leq 0$, $B(t, S) \leq 0$ $\Rightarrow$ Always making money: no good either.

▶ OK to use $P(t, S)$ only if signs of $A$ and $B$ different, $\forall S, \forall t$.

$$P&L = -BS^2 \left( \left( \frac{\delta S}{S} \right)^2 + \frac{A}{BS^2} \delta t \right)$$

▶ Reasonable ansatz, if $S$ is an equity: $\frac{A}{BS^2} = -cst = -\hat{\sigma}^2$. Using expressions of $A$ and $B$:

$$-rP + \frac{dP}{dt} + (r - q)S\frac{dP}{dS} = -\hat{\sigma}^2 \frac{1}{2} S^2 \frac{d^2 P}{dS^2}$$

▶ This is in fact the BS equation. Carry P&L acquires simple form:

$$P&L = -\frac{1}{2} S^2 \frac{d^2 P}{dS^2} \left( \left( \frac{\delta S}{S} \right)^2 - \hat{\sigma}^2 \delta t \right)$$
Intro – a practically usable model? – 3

- Simple form of $P&L \Rightarrow$ simple break-even criterion. Only reason why BS equation used in banks.
- No assumption that equities are lognormal – they are not.
- No assumption that volatility is constant – it is not.
- Not even the assumption of a process for $S$.
- Criterion for breakeven of $P&L$ at order 2 in $\delta S \Rightarrow P$ solves parabolic equation $\Rightarrow$ probabilistic interpretation & $P$ interpreted as an expectation.
- What if there are multiple hedge instruments? Carry $P&L$ reads:

$$P&L = -\frac{1}{2} S_i S_j \frac{d^2P}{dS_i dS_j} \left( \frac{\delta S_i}{S_i} \frac{\delta S_j}{S_j} - C_{ij} \delta t \right)$$

- Criterion for $P&L$ to be nonsensical: $C$ must be positive matrix.
- There is exist breakeven covariance levels $\forall S, \forall t$ that are payoff-independent.
- Important thing: only involves hedge instruments – not model’s state variables.
- $S_i$ underliers – or 1 underlying & associated vanilla options.
What’s left to do?

▶ Once option is delta-hedged, we are left with gamma/theta P&L. Total P&L incurred on \([0, T]\):

\[
P&L_T = - \sum_i e^{r(T-t_i)} S_i^2 \frac{d^2P}{dS^2} \bigg|_{t_i, S_i} \left( r_i^2 - \hat{\sigma}^2 \delta t \right), \quad r_i = \frac{\delta S_i}{S_i}
\]

▶ Is this P&L sizeable?

▶ If \(S\) follows a lognormal process with volatility \(\hat{\sigma}\) and \(\delta t \rightarrow 0\), then \(P&L = 0\).

▶ Returns of real undelyings (a) do not have exhibit volatility, (b) have non-Gaussian conditional distributions. Set:

\[
r_i = \sigma_i Z_i, \quad E[Z_i^2] = 1
\]

Then:

\[
P&L_T = - \sum_i e^{r(T-t_i)} S_i^2 \frac{d^2P}{dS^2} \bigg|_{t_i, S_i} \left( \sigma_i^2 Z_i^2 - \hat{\sigma}^2 \delta t \right)
\]

▶ \(Z_i\) non-Gaussian \(\Rightarrow\) impacts short-maturity options.

▶ \(\sigma_i\) random AND correlated \(\Rightarrow\) impacts longer-maturity options.
What’s left to do? – 2

- Use typical parameters. Stdev($P&L$) as fraction of price for an ATM option, as a function of maturity:

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Real case
without kurtosis term
Lognormal case
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- For 1y maturity: Black-Scholes: 5%, while $\approx 30\%$ in the real case.
- Delta hedging better than nothing – but remaining gamma/theta still too large.
- Gamma needs to be cancelled as well $\Rightarrow$ options are hedged with options.
What’s left to do? – conclusion

- $P$ becomes a function of $t$, $S$ and other derivative prices
  - For example vanilla options: $P(t, S, O_{KT})$.
  - This is called "calibration".

- Admissible models are such that the P&L of a delta/vega-hedged option reads:

\[
P&L = -\frac{1}{2} S_i S_j \frac{d^2 P}{dS_i dS_j} \left( \frac{\delta S_i}{S_i} \frac{\delta S_j}{S_j} - C_{ij} \delta t \right)
\]

with $C$ positive (implied) break-even covariance matrix of hedge instruments $S_i$.
  - $C$ is payoff-independent.
  - Ideally we would like to be able to choose the $C_{ij}$.

- We call "market models" models satisfying this condition.

- Usually not able to write down SDEs for hedge instruments directly, so condition needs to be checked \textit{a posteriori}.

- 2 examples:
  - Local volatility
  - Local-stochastic volatility
The local volatility model
Local volatility – intro: things heard on the street

- LV model used inconsistently: local vol surface is calibrated today; only to be recalibrated tomorrow.
  - violates model's assumption of fixed LV surface.

  - Rationale: so that vanilla options have BS delta.

- On a scale from dirty to downright ugly, where do we stand?
  - What is the carry P&L of an option position?
  - By the way, what's the delta of a vanilla option?
Local volatility – 1

- Local volatility: simplest model that is able to take as inputs vanilla option prices.

- Provided:
  - no time arbitrage: if zero int. rate: \( \frac{dC_{KT}}{dT} \geq 0 \) \( \Rightarrow \) \( T_1 \leq T_2 \) \( \Rightarrow \) \( T_1 \hat{\sigma}_{KT_1}^2 \leq T_2 \hat{\sigma}_{KT_2}^2 \)
  - no strike arbitrage: \( \frac{d^2 C_{KT}}{dK^2} \geq 0 \)

there exists a (single) local volatility function \( \sigma(t, S) \), given by the Dupire formula:

\[
\sigma(t, S)^2 = 2 \left( \frac{dC}{dT} + qC + (r - q) K \frac{dC}{dK} \right) \bigg|_{K=S,T=t} K^2 \frac{d^2 C}{dK^2} \]

such that, by using:

\[
dS_t = (r - q)S_t dt + \sigma(t, S_t)S_t dW_t
\]

vanilla option prices are recovered.

- Pricing function of LV model reads: \( P(t, S, O_{KT}) \) or \( P(t, S, \hat{\sigma}_{KT}) \) – no parameter beside time & values of hedge instruments.

- Model assumes fixed \( \sigma(t, S) \) while, in practice, local volatility function is recalibrated every day. Does this make any sense?

- What are the deltas (vegas)?
Local volatility – 2

Pricing equation of the local volatility model reads:

\[
\frac{dP^{LV}}{dt} + (r - q)S \frac{dP^{LV}}{dS} + \frac{1}{2} \sigma^2(t, S) S^2 \frac{d^2P^{LV}}{dS^2} = rP^{LV}
\]

Just like BS equation except \( \sigma(t, S) \) instead of cst volatility \( \hat{\sigma} \).

Solution of PDE is \( P^{LV}(t, S, \sigma) \)

In LV model all instruments have 1-d Markov representation as a function of \( t, S \):

\[
\hat{\sigma}_{KT}(t, S) \equiv \Sigma_{KT}^{LV}(t, S, \sigma)
\]

Imagine trading the LV delta:

\[
\Delta^{LV} = \left. \frac{dP^{LV}}{dS} \right|_{\sigma}
\]

P&L during \( \delta t \) of delta-hedged option is:

\[
P&L^{LV} = -\frac{1}{2} S^2 \frac{d^2P^{LV}}{dS^2} \left( \left( \frac{\delta S}{S} \right)^2 - \sigma^2(t, S) \delta t \right)
\]

\( P&L^{LV} \) actual P&L only if market implied vols move as prescribed by \( \Sigma_{KT}^{LV}(t, S, \sigma) \).

\( \Rightarrow \Delta^{LV} \) useless
Local volatility – carry P&L

- Let's compute the carry P&L in the LV model.
- Use (black-box) pricing function \( P(t, S, \hat{\sigma}_{KT}) \) given by:

\[
P(t, S, \hat{\sigma}_{KT}) \equiv P^{LV}(t, S, \sigma [t, S, \hat{\sigma}_{KT}])
\]

\[
P^{LV}(t, S, \sigma) = P\left(t, S, \Sigma_{KT}^{LV}(t, S, \sigma)\right)
\]

- Start with P&L of *naked* option position:

\[
P\&L = - \left[ P(t + \delta t, S + \delta S, \hat{\sigma}_{KT} + \delta \hat{\sigma}_{KT}) - (1 + r\delta t)P(t, S, \hat{\sigma}_{KT}) \right]
\]

- Expand at order 1 in \( \delta t \), 2 in \( \delta S \) and \( \delta \hat{\sigma}_{KT} \):

\[
P\&L = rP\delta t - \frac{dP}{dt}\delta t - \frac{dP}{dS}\delta S - \frac{dP}{d\hat{\sigma}_{KT}}\delta \hat{\sigma}_{KT}
\]

\[
- \left( \frac{1}{2} \frac{d^2 P}{dS^2} \delta S^2 + \frac{d^2 P}{dSd\hat{\sigma}_{KT}} \delta \hat{\sigma}_{KT} \delta S + \frac{1}{2} \frac{d^2 P}{d\hat{\sigma}_{KT}d\hat{\sigma}_{K'T'}} \delta \hat{\sigma}_{KT} \delta \hat{\sigma}_{K'T'} \right)
\]

- Notation \( \bullet \) stands for:

\[
\frac{df}{d\hat{\sigma}_{KT}} \bullet \delta \hat{\sigma}_{KT} \equiv \int dKdT \frac{\delta f}{\delta \hat{\sigma}_{KT}} \delta \hat{\sigma}_{KT} \equiv \Sigma_{ij} \frac{df}{d\hat{\sigma}_{K_i T_j}} \delta \hat{\sigma}_{K_i T_j}
\]
Local volatility – carry P&L – 2

- \( \frac{dP}{dS}, \frac{dP}{dt} \) are computed keeping the \( \hat{\sigma}_{KT} \) fixed – the LV function is not fixed.
  - Define sticky-strike delta \( \Delta^{SS} \):
    \[
    \Delta^{SS} = \left. \frac{dP}{dS} \right|_{\hat{\sigma}_{KT}}
    \]
  - \( P \) is not solution of the LV pricing PDE – \( P^{LV} \) is:
    \[
    P^{LV}(t, S, \sigma) = P \left( t, S, \hat{\sigma}_{KT} = \Sigma^{LV}_{KT}(t, S, \sigma) \right)
    \]
- Express derivatives of \( P^{LV} \) in terms of derivatives of \( P \):
  \[
  \frac{dP^{LV}}{dt} = \frac{dP}{dt} + \frac{dP}{d\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dt}
  \]
  \[
  \frac{dP^{LV}}{dS} = \frac{dP}{dS} + \frac{dP}{d\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS}
  \]
  \[
  \frac{d^2 P^{LV}}{dS^2} = \left( \frac{d^2 P}{dS^2} + 2 \frac{d^2 P}{dS d\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} + \frac{d^2 P}{d\hat{\sigma}_{KT} d\hat{\sigma}_{KT'}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} \frac{d\Sigma^{LV}_{KT'}}{dS} \right) + \frac{dP}{d\hat{\sigma}_{KT}} \cdot \frac{d^2 \Sigma^{LV}_{KT}}{dS^2}
  \]
- Now insert in LV pricing equation:
  \[
  \frac{dP^{LV}}{dt} + (r - q)S \frac{dP^{LV}}{dS} + \frac{1}{2} \sigma^2(t, S) S^2 \frac{d^2 P^{LV}}{dS^2} = rP^{LV}
  \]
  ... to generate relationship involving derivatives of \( P \).
Local volatility – carry P&L – 3

\[
\frac{dP}{dt} = rP - (r - q)S \frac{dP}{dS} - \frac{dP}{d\sigma_{KT}} \cdot \mu_{KT}
\]

\[
- \frac{1}{2} \sigma^2(t, S) S^2 \left( \frac{d^2P}{dS^2} + 2 \frac{d^2P}{dSd\sigma_{KT}} \cdot \frac{d\Sigma_{KT}^{LV}}{dS} + \frac{d^2P}{d\sigma_{KT}d\sigma_{K'T'}} \cdot \frac{d\Sigma_{KT}^{LV}}{dS} \frac{d\Sigma_{K'T'}^{LV}}{dS} \right)
\]

with \( \mu_{KT} \) given by:

\[
\mu_{KT} = \frac{d\Sigma_{KT}^{LV}}{dt} + \frac{1}{2} \sigma^2(t, S) S^2 \frac{d^2\Sigma_{KT}^{LV}}{dS^2} + (r - q)S \frac{d\Sigma_{KT}^{LV}}{dS}
\]

Now use this expression of \( \frac{dP}{dt} \) to rewrite P&L of *naked* option position:

\[
P&L = - \frac{dP}{dS} (\delta S - (r - q)S \delta t) - \frac{dP}{d\sigma_{KT}} \cdot (\delta \sigma_{KT} - \mu_{KT} \delta t)
\]

\[
+ \frac{1}{2} \sigma^2(t, S) S^2 \left( \frac{d^2P}{dS^2} + 2 \frac{d^2P}{dSd\sigma_{KT}} \cdot \frac{d\Sigma_{KT}^{LV}}{dS} + \frac{d^2P}{d\sigma_{KT}d\sigma_{K'T'}} \cdot \frac{d\Sigma_{KT}^{LV}}{dS} \frac{d\Sigma_{K'T'}^{LV}}{dS} \right) \delta t
\]

\[
- \left( \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2 + \frac{d^2P}{dSd\sigma_{KT}} \cdot \delta \sigma_{KT} \delta S + \frac{1}{2} \frac{d^2P}{d\sigma_{KT}d\sigma_{K'T'}} \cdot \delta \sigma_{KT} \delta \sigma_{K'T'} \right)
\]
Local volatility – carry P&L – 4

- Introduce implied (log-normal) vol of vol of $\hat{\sigma}_{KT}$:

$$\nu_{KT} = \frac{1}{\Sigma_{KT}^{LV}} \frac{d\Sigma_{KT}^{LV}}{dS} \sigma(t, S)$$

- Rewrite P&L as:

$$P&L = -\frac{dP}{dS} (\delta S - (r - q)S\delta t) - \frac{dP}{d\hat{\sigma}_{KT}} \bullet (\delta\hat{\sigma}_{KT} - \mu_{KT}\delta t)$$

$$- \frac{1}{2} S^2 \frac{d^2 P}{dS^2} \left[ \frac{\delta S^2}{S^2} - \sigma^2(t, S)\delta t \right]$$

$$- \frac{d^2 P}{dSd\hat{\sigma}_{KT}} \bullet S\hat{\sigma}_{KT} \left[ \frac{\delta S}{S} \frac{\delta\hat{\sigma}_{KT}}{\hat{\sigma}_{KT}} - \sigma(t, S)\nu_{KT}\delta t \right]$$

$$- \frac{1}{2} \frac{d^2 P}{d\hat{\sigma}_{KT}d\hat{\sigma}_{KT'}} \bullet \hat{\sigma}_{KT} \hat{\sigma}_{KT'} \left[ \frac{\delta\hat{\sigma}_{KT}}{\hat{\sigma}_{KT}} \frac{\delta\hat{\sigma}_{KT'}}{\hat{\sigma}_{KT'}} - \nu_{KT}\nu_{KT'}\delta t \right]$$

- Only uses market observables: $P(t, S, \hat{\sigma}_{KT})$ – no LV function involved.

- P&L expression is that of market model.
  - Variance/covariance breakeven levels are well-defined, payoff-independent, and make up a positive covariance matrix.
  - Delta is sticky-strike delta $\frac{dP}{dS}$, vegas simple vegas.
Local volatility – carry P&L – 5

- \( \hat{\sigma}_{KT} \equiv \) implied vol plays no special role. Use instead price \( O_{KT} \): \( P(t, S, O_{KT}) \).

\[
P(t, S, \hat{\sigma}_{KT}) = P(t, S, O_{KT} = P^{BS}_{KT}(t, S, \hat{\sigma}_{KT}))
\]

\[
P^{LV}(t, S, \sigma) = P(t, S, \Omega^{LV}_{KT}(t, S, \sigma))
\]

\( \Omega^{LV}_{KT}(t, S, \sigma) \) price in LV model with LV function \( \sigma \).

Everything same as before, except \( \hat{\sigma}_{KT} \rightarrow O_{KT}, \Sigma^{LV}_{KT} \rightarrow \Omega^{LV}_{KT} \).

- Drift \( \mu_{KT} \) simplifies:

\[
\mu_{KT} = \frac{d\Omega^{LV}_{KT}}{dt} + \frac{1}{2}\sigma^2(t, S) S^2 \frac{d^2\Omega^{LV}_{KT}}{dS^2} + (r - q)S \frac{d\Omega^{LV}_{KT}}{dS} = r\Omega^{LV}_{KT} = rO_{KT} \quad \text{OK}
\]

- P&L of naked option position – using only asset prices – no LV function involved:

\[
P&L = -\frac{dP}{dS} (\delta S - (r - q)S\delta t) - \frac{dP}{dO_{KT}} \cdot (\delta O_{KT} - rO_{KT}\delta t)
\]

\[
- \frac{1}{2} \frac{d^2P}{dS^2} [\delta S^2 - \sigma^2(t, S) S^2\delta t]
\]

\[
- \frac{d^2P}{dSdO_{KT}} \cdot \left[ \delta S\delta O_{KT} - \sigma^2(t, S) S^2 \frac{d\Omega^{LV}_{KT}}{dS} \delta t \right]
\]

\[
- \frac{1}{2} \frac{d^2P}{dO_{KT}dO_{K'T'}} \cdot \left[ \delta O_{KT}\delta O_{K'T'} - \sigma^2(t, S) S^2 \frac{d\Omega^{LV}_{KT}}{dS} \frac{d\Omega^{LV}_{K'T'}}{dS} \delta t \right]
\]
Local volatility – carry P&L – 6

▶ Expression of carry P&L – inclusive of recalibration of local volatility function – has typical form of market models.

▶ Hedge instruments all treated on equal footing.

▶ Implied break-even levels of cross-gammas are payoff-independent – are determined by market smile prevailing at time $t$.
  ▶ spot/vol correl = $-100\%$
  ▶ vol/vol correl = $100\%$
  ▶ vol of $\hat{\sigma}_{KT}$ is $\nu_{KT} = \frac{1}{\Sigma_{LV}^{KT}} \frac{d\Sigma_{LV}^{KT}}{dS} S\sigma(t, S)$

▶ Hedge ratios simply $\left. \frac{dP}{dS} \right|_{O_{KT}}$ and $\left. \frac{dP}{dO_{KT}} \right|_{S}$

▶ Delta of the local volatility model is – market model delta:
  $$\Delta^{MM} = \left. \frac{dP}{dS} \right|_{O_{KT}}$$

▶ Delta of vanilla option irrelevant notion.
  ▶ akin to asking model to generate a hedge ratio of one hedging instrument on another hedging instrument.

▶ Result seems $\approx$ natural; looks like any $P$ that’s the solution of a parabolic PDE will do the job – but see pathologies in local/stoch vol models.
Consistency of sticky-strike and market-model deltas

- Use $S, O_{KT} \Rightarrow \mathcal{P}(t, S, O_{KT})$. Hedge ratios $\Delta^{MM} = \left. \frac{dP}{dS} \right|_{O_{KT}}, \left. \frac{dP}{dO_{KT}} \right|_{S}$

- Use $S, \hat{\sigma}_{KT} \Rightarrow \mathcal{P}(t, S, \hat{\sigma}_{KT})$. Hedge ratios $\Delta^{SS} = \left. \frac{dP}{dS} \right|_{\hat{\sigma}_{KT}}, \left. \frac{dP}{d\hat{\sigma}_{KT}} \right|_{S}$
  - $\frac{dP}{d\hat{\sigma}_{KT}}$ offset by trading BS-delta-hedged vanilla options

- Hedge portfolio is:
  $$\Pi = \frac{dP}{dS} S + \frac{dP}{dO_{KT}} \cdot O_{KT}$$

- Rewrite in terms of delta-hedged vanillas:
  $$\Pi = \left[ \frac{dP}{dS} + \frac{dP}{dO_{KT}} \cdot \left. \frac{dP^{BS}_{KT}}{dS} \right|_{S} \right] S + \frac{dP}{dO_{KT}} \cdot \left[ O_{KT} - \left. \frac{dP^{BS}_{KT}}{dS} \right|_{S} S \right]$$

- Spot hedge ratio?
  - Move spot + move vanilla prices by their Black-Scholes deltas akin to: move vanilla prices keeping implied vols fixed $\Rightarrow$ sticky strike delta
  $$\Delta^{SS} = \left. \frac{dP}{dS} \right|_{\hat{\sigma}_{KT}}, \left. \frac{dP}{d\hat{\sigma}_{KT}} \right|_{S}$$

- Once hedge portfolio broken down into underlying + *naked* vanilla options, delta always equal to $\Delta^{MM} = \left. \frac{dP}{dS} \right|_{O_{KT}}$.

- Nothing fundamental about $\Delta^{SS}$ – tied to a particular representation of vanilla option prices.
So, what is the LV model?

- The LV model is a usable model. It is a market model for the underlying and vanilla options
  ... that happens to have a 1-d Markov representation in terms of \((t, S)\).

- This is a mathematical technicality – of which the LV function is a by-product – that facilitates pricing. Nothing fundamental.

- Daily recalibration of LV function is exactly how it has to be used.

- Consequences of 1-d Markov representation:
  - The break-even covariance matrix is of rank 1 – correls = 100%.
  - No control on break-even levels of volatilities of implied volatilities. They are set by the configuration of \(S, \hat{\sigma}_{KT}\) and will vary unpredictably.
  - Like them, use model – don’t like them, don’t use model.

- LV model completely specified by feeding in the values of the hedge instruments – no parameters whatsoever.

- This is how much we can get in a model with a 1-d Markov representation.
Using the LV model

- What's left before we can use LV model? Output the $\nu_{KT}$, see if we like them.
  
  - More practical to look at implied vols for floating strike – fixed moneyness.

- Look at vols of vols and spot/vol covariances.

- For ATMF vol $\hat{\sigma}_{FTT}$ equivalently look at SSR $R_T$

\[
R_T = \frac{1}{S_T} \frac{\langle d\hat{\sigma}_{FTT} \ d\ln S \rangle}{\langle (d\ln S)^2 \rangle} = \frac{1}{S_T} \frac{d\hat{\sigma}_{FTT}}{d\ln S}
\]

\[
S_T = \frac{d\hat{\sigma}_{KT}}{d\ln K} \bigg|_{FT}
\]

- Vol of vol:

\[
\frac{d\hat{\sigma}_{FTT}}{\hat{\sigma}_{FTT}} = \frac{1}{\hat{\sigma}_{FTT}} \frac{d\hat{\sigma}_{FTT}}{d\ln S} d\ln S_t = \frac{d\hat{\sigma}_{FTT}}{d\ln S} \frac{\sigma (t, S)}{\hat{\sigma}_{FTT}} dW_t
\]

Thus:

\[
\text{vol}(\hat{\sigma}_{FTT}) = R_T S_T \left( \frac{\hat{\sigma}_{F00}}{\hat{\sigma}_{FTT}} \right)
\]

- Assume following expression for LV function:

\[
\sigma(t, S) = \sigma(t) + \alpha(t)x + \frac{\beta(t)}{2} x^2, \quad x = \ln \left( \frac{S}{F_t} \right)
\]

and calculate $S_T, R_T$ at order 1 in $\alpha(t), \beta(t)$. 
Expansion of Implied volatilities

- Consider an LV model – Model 1: LV function $\sigma_1(t, S)$, pricing function $P_1(t, S)$.

$$\frac{dP_1}{dt} + (r - q)S\frac{dP_1}{dS} + \frac{1}{2}\sigma_1^2(t, S)S^2\frac{d^2P_1}{dS^2} = rP_1$$

- Now consider arbitrary diffusive model – Model 2: instantaneous volatility $\sigma_{2t}$.

$$dS_t = (r - q)S_t dt + \sigma_{2t}S_t dW_t$$

- Consider process $Q_t$ defined by:

$$Q_t = e^{-rt}P_1(t, S_t)$$

- At $t = 0$, $Q_{t=0} = P_1(0, S_0)$.

- At $t = T$, $Q_{t=T} = e^{-rT}P_1(T, S_{2T}) = e^{-rT}f(S_{2T})$, that is the final payoff.

$$dQ_t = e^{-rt}\left[\left(-rP_1 + \frac{dP_1}{dt}\right)dt + \frac{dP_1}{dS}dS_t + \frac{1}{2}\frac{d^2P_1}{dS^2}\langle dS_t^2 \rangle\right]$$

$$= e^{-rt}\left[\left(-rP_1 + \frac{dP_1}{dt}\right)dt + \frac{dP_1}{dS}dS_t + \frac{1}{2}S_t^2\frac{d^2P_1}{dS^2}\sigma_{2t}^2 dt\right]$$

$$= e^{-rt}\left[\frac{dP_1}{dS}(dS_t - (r - q)S_t dt) + \frac{1}{2}S_t^2\frac{d^2P_1}{dS^2}(\sigma_{2t}^2 - \sigma_1^2(t, S_t))dt\right]$$

$$E_2[dQ_t|t, S_t] = e^{-rt}\frac{S_t^2}{2}\frac{d^2P_1}{dS^2}(\sigma_{2t}^2 - \sigma_1(t, S_t)^2) dt$$
Expansion of Implied volatilities – 2

\[
E_2[Q_T] = Q_0 + \int_0^T E_2[dQ_t]
\]

\[
= P_1(0, S_0) + E_2 \left[ \int_0^T e^{-rt} \frac{S_t^2}{2} \frac{d^2P_1}{dS^2} \left( \sigma_{2t}^2 - \sigma_1(t, S_t)^2 \right) dt \right]
\]

▶ So, price in Model 2 given by:

\[
P_2(0, S_0, \bullet) = P_1(0, S_0) + E_2 \left[ \int_0^T e^{-rt} \frac{S_t^2}{2} \frac{d^2P_1}{dS^2} \left( \sigma_{2t}^2 - \sigma_1(t, S_t)^2 \right) dt \right]
\]

where \(\bullet\) other state variables of Model 2.

▶ Price(Model 2) = Price(Model 1) + gamma/theta P&L, incurred by hedging payoff using Model 1 with dynamics of \(S_t\) generated by Model 2.

揠 Efficient numerical algorithm for generating vanilla smiles of stochastic volatility models – see book.

▶ Imagine Model 1 is BS model with implied vol = \(\hat{\sigma}_{KT}\) \(\Rightarrow P_2(0, S_0, \bullet) = P_{\hat{\sigma}_{KT}}(0, S_0)\)

\[
0 = E_2 \left[ \int_0^T e^{-rt} \frac{S_t^2}{2} \frac{d^2P_{\hat{\sigma}_{KT}}}{dS^2} \left( \sigma_{2t}^2 - \hat{\sigma}_{KT}^2 \right) dt \right]
\]
Expansion of Implied volatilities – 3

▶ Thus:

\[
\hat{\sigma}_{KT}^2 = \frac{E_2 \left[ \int_0^T e^{-rt} S_t^2 \frac{d^2 P_{\sigma_{KT}}}{dS^2} \sigma_{2t}^2 dt \right]}{E_2 \left[ \int_0^T e^{-rt} S_t^2 \frac{d^2 P_{\sigma_{KT}}}{dS^2} dt \right]}
\]

▶ Work with variances \( u = \sigma^2 \). Set \( u_0 = \sigma^2_0 \) and \( \sigma_{2t}^2 = u_0 + \delta u(t, S) \).

▶ Expand at order 1 in \( \delta u \): \( \hat{\sigma}_{KT}^2 = \sigma^2_0 + \delta(\hat{\sigma}_{KT}) \).

\[
\hat{\sigma}_{KT}^2 = \sigma^2_0 + \delta(\hat{\sigma}_{KT}) = \frac{E_{u_0 + \delta u} [\bullet (u_0 + \delta u)]}{E_{u_0 + \delta u} [\bullet]} = u_0 + \frac{E_{u_0 + \delta u} [\bullet \delta u]}{E_{u_0 + \delta u} [\bullet]}
\]

\[
= u_0 + \frac{E_{u_0} [\bullet \delta u]}{E_{u_0} [\bullet]}
\]

\[
= \frac{E_{u_0} [\bullet (u_0 + \delta u)]}{E_{u_0} [\bullet]}
\]

▶ Thus:

\[
\hat{\sigma}_{KT}^2 = \frac{E_{\sigma_0} \left[ \int_0^T e^{-rt} \ u(t, S) S^2 \frac{d^2 P_{\sigma_0}}{dS^2} dt \right]}{E_{\sigma_0} \left[ \int_0^T e^{-rt} S^2 \frac{d^2 P_{\sigma_0}}{dS^2} dt \right]}
\]

▶ Density and gamma available in closed form in BS model.
Dynamics in LV model – 2

- Calculation can be done with deterministic \( u_0(t) = \sigma_0^2(t) \). At order 1 in \( \delta u \):

\[
\hat{\sigma}_{KT}^2 = \frac{1}{T} \int_0^T dt \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} u \left( t, F_t e^{\frac{\omega_t}{\omega_T} x_K} + \frac{\sqrt{(\omega_T - \omega_t) \omega_t}}{\sqrt{\omega_T}} y \right)
\]

where \( F_t \) forward for maturity \( t \), \( x_K = \ln \left( \frac{K}{F_T} \right) \) and \( \omega_t = \int_0^t \sigma_0^2(\tau) d\tau \).

- Expanding around a cst \( \sigma(t) = \sigma_0 \): \( u_0 = \sigma_0^2 \)

\[
\hat{\sigma}_{KT} = \frac{1}{T} \int_0^T dt \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \sigma \left( t, F_t e^{\frac{t}{\omega_T} x_K} + \sigma_0 \sqrt{\frac{t}{T}} y \right)
\]

\( \Rightarrow \) Implied vol \( \approx \) average of local vol around straight line – in \( \ln S \) – from \( S \) to \( K \).

\[
S_T = \frac{d\hat{\sigma}_{KT}}{d \ln K} \bigg|_{K=F_T} = \frac{1}{T} \int_0^T \frac{t}{T} \alpha(t) dt \quad \text{"skew averaging"} \quad \text{– see also V. Piterbarg}
\]

\[
 \frac{d^2\hat{\sigma}_{KT}}{d \ln K^2} \bigg|_{K=F_T} = \frac{1}{T} \int_0^T \left( \frac{t}{T} \right)^2 \beta(t) dt
\]

\[
 \frac{d\hat{\sigma}_{KT}}{d \ln S} \bigg|_{K=F_T} = \frac{1}{T} \int_0^T \left( 1 - \frac{t}{T} \right) \alpha(t) dt
\]

- Cst \( \alpha, \beta \): \( \frac{d\hat{\sigma}_{KT}}{d \ln K} \bigg|_{K=F_T} = \frac{\alpha}{2}, \quad \frac{d^2\hat{\sigma}_{KT}}{d \ln K^2} \bigg|_{K=F_T} = \frac{\beta}{3} \)
Dynamics in LV model – 3

- From 1st equation: \( \alpha(t) = \frac{d}{dt} (tS_t) + S_t \).

\[
\frac{d\sigma_{FT(S)}T}{d\ln S} = \left( \frac{d\sigma_{KT}}{d\ln K}_{K=F_T} + \frac{d\sigma_{KT}}{d\ln S}_{K=F_T} \right) = \frac{1}{T} \int_0^T \alpha(t)dt = S_T + \frac{1}{T} \int_0^T S_t dt
\]

- Get expression of SSR: \( R_T = \frac{1}{S_T} \frac{\langle d\sigma_{FT}T d\ln S \rangle}{\langle (d\ln S)^2 \rangle} = \frac{1}{S_T} \frac{d\sigma_{FT(S)}T}{d\ln S} : \\
R_T = 1 + \frac{1}{T} \int_0^T \frac{S_t}{S_T} dt
\]

- For typical equity smiles, \(|S_t|\) decreases with \( t \) \( \Rightarrow \) \( R_T \geq 2 \).

- Limiting behavior
  - Short maturities:
    \[
    \lim_{T \to 0} R_T = 2
    \]
    Lognormal vol of short ATMF vol = twice the skew.
  - Long maturities – take \( S_T \propto \frac{1}{T^\gamma} \):
    \[
    \lim_{T \to \infty} R_T = \frac{2 - \gamma}{1 - \gamma}
    \]
    For typical value \( \gamma = \frac{1}{2} \), \( \lim_{T \to \infty} R_T = 3 \).
Dynamics in LV model – 4

- Check approx of SSR on 2 smiles of Eurostoxx50

Figure: Top: smiles of the Eurostoxx50 index for a maturity \(\simeq 1\) year observed on October 4, 2010 (left) and May 16, 2013 (right). Bottom: term structures of ATMF skew and power-law fits with \(\gamma = 0.37\) (left), \(\gamma = 0.52\) (right), as a function of \(T\) (years).
Dynamics in LV model – 5

- Real versus approximate SSR

Figure: $R_T$ as a function of $T$ (years) computed: (a) in FD (actual), (b) using expression $R_T = 1 + \frac{1}{T} \int_0^T \frac{S_t}{S_T} dt$ (approx).

- What about smile with $S_T \propto \frac{1}{T}$? Approx formula gives $\lim_{T \to \infty} R_T = \infty$ (logarithmic divergence of $R_T$):

- Approx slightly overestimates SSR.
Conclusion

▶ LV model is a genuine market model for underlying + vanilla options
▶ The only diffusive market model that possesses a 1-d Markov representation in terms of \((t,S)\)
▶ Generates well-defined break-even levels for spot/vol and vol/vol covariances in the carry P&L.
▶ Daily recalibration of LV function – an ancillary object – is exactly how model should be used and deltas calculated.
  ▶ Spot/vol break-even correlations = \(-100\%\), vol/vol break-even correlations = \(100\%\).
  ▶ Volatilities of implied volatilities given by: \(\text{vol}(\hat{\sigma}_{KT}) = \frac{1}{\Sigma_{KT}^{LV}} \frac{d\Sigma_{KT}^{LV}}{dS} S \sigma(t, S)\).
▶ Delta is well-defined: \(\Delta^{MM} = \frac{dP}{dS}\bigg|_{O_{KT}}\). Delta of vanilla option irrelevant notion.
▶ When vega-hedging with (BS) delta-hedged vanilla options, sticky-strike delta should be used.
▶ Good approximate formulae for sizing up break-even vols of ATMF vols – or equivalently SSR:

\[
R_{\tau} = 1 + \frac{1}{T} \int_{0}^{T} \frac{S_t}{S_{\tau}} dt
\]

\[
\text{vol}(\hat{\sigma}_{F_{\tau} T}) = R_{\tau} S_{\tau}\left(\frac{\hat{\sigma}_{F_{\tau} 0}}{\hat{\sigma}_{F_{\tau} T}}\right)
\]
Local-stochastic volatility models – and non-models
Motivation

- In LV model, nothing to enter beside values of hedge instruments – zero parameter.
- Break-even covariances are set by prevailing smile. If smile is flat, implied vols of vols = 0.
- Can we regain some leverage on the model-implied dynamics of hedge instruments?

- Poor man’s fix:
  - Pick your favourite stochastic volatility model.
  - Decorate SV instantaneous volatility with local volatility component.

- Is it a (usable) model?

- Provided answer is positive
  - What is the delta? What are the vegas?
  - What kind of model is it?
SV models

- Which SV model should we use?

- Unlike LV model, SV models have parameters that we can use to drive the dynamics of the $\hat{\sigma}_{KT}$.

- First generation of SV models: based on instantaneous variance $V_t$, e.g. the Heston model:

  \[
  \begin{align*}
  dS_t &= (r - q)S_t dt + \sqrt{V_t} S_t dW_t \\
  dV_t &= -k(V_t - V^0)dt + \nu \sqrt{V_t}dZ_t
  \end{align*}
  \]

- Pbm: $V_t$ not an asset – no way to generate $P\&L \propto (V_{t_2} - V_{t_1}) \Rightarrow$ dynamics of $\hat{\sigma}_{KT}$ needs to be checked a posteriori.

- Better to model dynamics of hedge instruments directly, for example forward variances $\xi^T_t$:

  \[
  \xi^T_t = E_t \left[ \left( \frac{dS_T}{S_T} \right)^2 \right] = E_t[V_T]
  \]

  - Can be bought/sold by trading variance swaps (VS) – at no cost. VS volatility for maturity $T$ at time $t$, $\hat{\sigma}_T(t)$ given by:

    \[
    \hat{\sigma}_T^2(t) = \frac{1}{T-t} \int_t^T \xi^T_t dt
    \]

  - $\xi^T_t$ is driftless: $d\xi^T_t = \cdot dW^T_t$
Forward variance models

- Need to specify a dynamics for the curve $\xi^T_t$ such that:
  - Low-dimensional Markov representation
  - Able to generate flexible patterns for volatilities of VS volatilities $\hat{\sigma}_T$. Typically:
    $$\text{vol}(\hat{\sigma}_T) \propto \frac{1}{T^\alpha}, \quad \alpha \in [0.3, 0.6]$$

- In practice using two Brownian motions with exponential weightings is sufficient:
  $${d\xi^T_t \over \xi^T_t} = (2\nu)\mathcal{N} \left[ (1 - \theta)e^{-k_1(T-t)}dW^1_t + \theta e^{-k_2(T-t)}dW^2_t \right]$$
  with $\nu$: volatility of a volatility with vanishing maturity and $\mathcal{N}$ normalization factor.

  $$\xi^T_t = f^T(t, X^1_t, X^2_t)$$
  with $X^1_t, X^2_t$ two OU processes – easily simulated exactly.

- Process for $S_t$ is:
  $$dS_t = (r - q)S_t dt + \sqrt{\xi^t_t S_t} dW^S_t$$

- Also able to generate decay of ATMF skew $S_t \propto \frac{1}{T^{\gamma}}$ with $\gamma$ typically $\approx \frac{1}{2}$. see papers Smile Dynamics II, III, IV.
Models used as examples in presentation

- **Mixed Heston model**

\[
\begin{align*}
    dS_t &= (r - q)S_t dt + \sigma(t, S_t) \sqrt{V_t} S_t dW_t \\
    dV_t &= -k(V_t - V^0) dt + \nu \sqrt{V_t} dZ_t
\end{align*}
\]

- **Mixed two-factor model**

\[
\begin{align*}
    dS_t &= (r - q)S_t dt + \sigma(t, S_t) \sqrt{\zeta_t} S_t dW_t^S \\
    \frac{d\zeta_t}{\zeta_t} &= 2\nu \mathcal{N} \left[ ((1 - \theta)e^{-k_1(T-t)} dW_t^1 + \theta e^{-k_2(T-t)} dW_t^2 \right]
\end{align*}
\]

where \( \alpha_\theta = 1 / \sqrt{(1 - \theta)^2 + \theta^2 + 2\rho\theta (1 - \theta)} \); \( \nu \) vol of short vol.

- **LV component** \( \sigma(t, S) \) calibrated on vanilla smile.

- **Pricing function in mixed model:**
  - in Heston model \( P^M(t, S, \sigma, V) \), \( V \) number.
  - in two-factor model \( P^M(t, S, \sigma, \zeta^u) \), \( \zeta^u \) curve.
Usage of mixed models

▶ Choose model parameters & initial values of state variables:
  ▶ In Heston: \((k, \sigma, \rho, V^0), V\).
  ▶ In two-factor model: \((k_1, k_2, \theta, \nu, \rho_{12}, \rho_{S1}, \rho_{S2}), \zeta\).

▶ Calibrate local volatility function \(\sigma(t, S)\) to market smile.
  ▶ In 1-factor model like Heston: solve fwd PDE for density.

▶ Then Shift+F9 \(\Rightarrow\) produces a (real) number. Is it a price?

▶ What about deltas?
  ▶ Typically, move spot, recalibrate local vol and reprice.
  ▶ Is it right delta? What kind of carry P&L does this materialize?

▶ Let’s assume this is a model. Can we have an approximate way of sizing up:
  ▶ volatilities of implied vols
  ▶ covariances of spot and implied vols – equivalently SSR?
Two pricing functionals

- \( P^M(t, x) \): takes as inputs \( t, S \), \textit{LV function} + state variables \( \lambda \) of SV model:

\[
P^M(t, S, \sigma(, \lambda))
\]

- In Heston: \( \lambda = V \) - number
- In two-factor model: \( \lambda = \zeta^u \) - curve

- \( P(t, \hat{x}) \) takes as inputs \( t, S \), \textit{implied vols} + state variables of SV model:

\[
P(t, S, \hat{\sigma}_{KT}, \lambda)
\]

- Could include in \( x, \hat{x} \) model parameters as well (\( \equiv \) state variables with zero drift/vol).

- Will use \( P( ) \) – rather than \( P^M( ) \) – to do P&L accounting.
  - Could use prices rather than implied vols.
Carry P&L – $P(t, S, \hat{\sigma}_{KT}, \lambda)$

- In mixed model – for a set LV function – $\hat{\sigma}_{KT}$ is a function of $t, S, \sigma(,)$ + state variables: $\hat{x} = \hat{x}(t, x)$:

$$P^M(t, x) = P(t, \hat{x}(t, x))$$

- Implied vols given by:

$$\hat{\sigma}_{KT} \equiv \Sigma_{KT}(t, S, \sigma, \lambda)$$

$P^M, P$ related through:

$$P^M(t, S, \sigma, \lambda) = P(t, S, \Sigma_{KT}^M(t, S, \sigma, \lambda), \lambda)$$

- Pricing equation for $P^M$ – with set LV function – zero rates:

$$\frac{dP^M}{dt} + \left( \sum_k \mu_k \frac{d}{dx_k} + \frac{1}{2} \sum_{kl} a_{kl} \frac{d^2}{dx_k dx_l} \right) P^M = 0$$
Carry P&L – 2

Switch to variables $\hat{x}$:

$$\frac{dP}{dt} + \left( \sum_i \hat{\mu}_i \frac{d}{d\hat{x}_i} + \frac{1}{2} \sum_{ij} \hat{a}_{ij} \frac{d^2}{d\hat{x}_i d\hat{x}_j} \right) P = 0$$

with:

$$\begin{cases} 
\hat{\mu}_i = \frac{d\hat{x}_i}{dt} + \sum_k \mu_k \frac{d\hat{x}_i}{dx_k} + \frac{1}{2} \sum_{kl} a_{kl} \frac{d^2\hat{x}_i}{dx_k dx_l} \\
\hat{a}_{ij} = \sum_{kl} a_{kl} \frac{d\hat{x}_i}{dx_k} \frac{d\hat{x}_j}{dx_l}
\end{cases}$$

$\hat{\mu}_i$ drift of $\hat{x}_i$ and $\hat{a}_{ij}$ covariance matrix of $\hat{x}_i$ and $\hat{x}_j$

- as generated by mixed model with fixed LV function.
- $\frac{d\hat{x}_i}{dx_k}$ involve derivatives of functional $\Sigma^M_{KT}(t, S, \sigma, \lambda)$ with respect to $t, S, \lambda$.

Now consider P&L of short option position – unhedged for now – zero rates:

$$P&L = -P(t + \delta t, \hat{x} + \delta \hat{x}) + P(t, \hat{x})$$
Expand at order two in $\delta \hat{x}$, one in $\delta t$.

\[
P\&L = - \frac{dP}{dt} \delta t - \sum_i \frac{dP}{d\hat{x}_i} \delta \hat{x}_i - \frac{1}{2} \sum_{ij} \frac{d^2P}{d\hat{x}_i d\hat{x}_j} \delta \hat{x}_i \delta \hat{x}_j
\]

\[
= - \sum_i \frac{dP}{d\hat{x}_i} (\delta \hat{x}_i - \hat{\mu}_i \delta t) - \frac{1}{2} \sum_{ij} \frac{d^2P}{d\hat{x}_i d\hat{x}_j} (\delta \hat{x}_i \delta \hat{x}_j - \hat{a}_{ij} \delta t)
\]

Among components of $\hat{x}$:
- $O_i$ market observables: $S$, $\hat{\sigma}_{KT}$.
- $\lambda_k$ = state variables of SV model

Rewrite P&L:

\[
P\&L = - \sum_i \frac{dP}{dO_i} (\delta O_i - \hat{\mu}_i \delta t) - \frac{1}{2} \sum_{ij} \frac{d^2P}{dO_i dO_j} (\delta O_i \delta O_j - \hat{a}_{ij} \delta t)
\]

\[- \sum_k \frac{dP}{d\lambda_k} (\delta \lambda_k - \hat{\mu}_k \delta t)
\]

\[- \sum_{kl} \frac{d^2P}{d\lambda_k d\lambda_l} (\delta \lambda_k \delta \lambda_l - \hat{a}_{kl} \delta t) - \sum_{ik} \frac{d^2P}{dO_i d\lambda_k} (\delta O_i \delta \lambda_k - \hat{a}_{ik} \delta t)
\]
**P&L hedged portfolio**

- Portfolio: option + hedges that offset sensitivities $\frac{dP}{dO_i}$: $P_H = P + \sum_i \alpha_i f_i(t, S, O_i)$

- P&L equation also holds for hedge instruments $\rightarrow$ canceling $\delta O_i$ term cancels $\hat{\mu}_i \delta t$ contribution. P&L of hedged position:

$$P&L_H = - \frac{1}{2} \sum_{ij} d^2P_H \frac{d^2P_H}{dO_idO_j} (\delta O_i \delta O_j - \hat{\alpha}_{ij} \delta t)$$

$$- \sum_k \frac{dP_H}{d\lambda_k} (\delta \lambda_k - \hat{\mu}_k \delta t)$$

$$- \frac{1}{2} \sum_{kl} \frac{d^2P_H}{d\lambda_k d\lambda_l} (\delta \lambda_k \delta \lambda_l - \hat{\alpha}_{kl} \delta t) - \sum_{ik} \frac{d^2P_H}{dO_i d\lambda_k} (\delta O_i \delta \lambda_k - \hat{\alpha}_{ik} \delta t)$$

- 1st piece OK: thetas matching gammas on market instruments.

  - $\hat{\alpha}_{ij}$ positive covariance matrix: $\hat{\alpha}_{ij} = \sum_{kl} a_{kl} \frac{d\hat{x}_i}{d\hat{x}_k} \frac{d\hat{x}_j}{d\hat{x}_l}$

- 2nd / 3d pieces no good. P&L leakage from variation (or not) of SV state variables.

- By construction value of hedges indpdt on $\lambda_k$: $\frac{df_i}{d\lambda_k} = 0$, so:

$$\frac{dP_H}{d\lambda_k} = \frac{dP}{d\lambda_k}, \quad \frac{d^2P_H}{d\lambda_k d\lambda_l} = \frac{d^2P}{d\lambda_k d\lambda_l}, \quad \frac{d^2P_H}{dO_i d\lambda_k} = \frac{d^2P}{dO_i d\lambda_k}$$
P&L hedged portfolio – 2

- $\delta \lambda_k$ are not market values – are in our control. For example, take $\delta \lambda_k = \hat{\mu}_k \delta t$.

- Still leaves us with 3d piece in P&L:

$$P \& L_{H}^{\text{leak}} = -\frac{1}{2} \sum_{kl} \frac{d^2 P}{d\lambda_k d\lambda_l} (\delta \lambda_k \delta \lambda_l - \hat{a}_{kl} \delta t) - \sum_{ik} \frac{d^2 P}{dO_i d\lambda_k} (\delta O_i \delta \lambda_k - \hat{a}_{ik} \delta t)$$

- Is there a solution to P&L leakage?

- YES – need condition on $P(t, O, \lambda)$:

$$\frac{dP}{d\lambda_k} \bigg|_{S, \hat{\sigma}_{KT}} = 0, \ \forall k$$

- Pricing functional $P(t, S, \hat{\sigma}_{KT}, \lambda)$ must have zero sensitivity to SV state variables.
Conclusion: admissible (or gauge-invariant) models

▷ Criterion for models that can be used in trading: \( P(t, S, \hat{\sigma}_{KT}, \lambda) \):

\[
\left. \frac{dP}{d\lambda_k} \right|_{S, \hat{\sigma}_{KT}} = 0
\]

▷ \( P&L_H \) of delta-hedged/vega-hedged position then has typical form of market models:

\[
P&L_H = -\frac{1}{2} \sum_{ij} \frac{d^2 P}{dO_i dO_j} (\delta O_i \delta O_j - \hat{a}_{ij} \delta t)
\]

Break-even covariance levels are given by covariances in model \textit{with fixed LV function}: \( \hat{a}_{ij} = \Sigma_{kl} a_{kl} \left( \frac{dx_i}{dx_k} \right) \left( \frac{dx_j}{dx_l} \right) \).

▷ Pbm: condition \( \left. \frac{dP}{d\lambda} \right|_{S, \hat{\sigma}_{KT}} = 0 \) usually \textit{not} satisfied.

▷ Ex: not satisfied in local/stoch vol model built on Heston model:

\[
\frac{d}{dV} P(t, S, \hat{\sigma}_{KT}, V) \neq 0 \quad \Rightarrow \quad \text{P&L leakage}
\]

▷ Not usable in trading.

▷ Do admissible models exist at all?

▷ YES.
Admissible models – 2

Consider mixed two-factor model. Pricing function $P(t, S, \hat{\sigma}_{KT}, \zeta^u)$.

Model equivalently written as:

$$
\begin{align*}
    dS_t &= (r - q)S_t dt + \sqrt{\zeta t} \sqrt{f(t, X^1_t, X^2_t)} \sigma(t, S_t) S_t dW_t^S \\
    dX^1_t &= -k_1 X^1_t dt + dW^1_t \\
    dX^2_t &= -k_2 X^2_t dt + dW^2_t
\end{align*}
$$

with $X^1_0 = 0, X^2_0 = 0$ and:

$$
f(t, x_1, x_2) = e^{2\nu \alpha \theta [(1-\theta)x_1 + \theta x_2]} - \frac{(2\nu \alpha \theta)^2}{2} \chi(t)
$$

$$
\chi(t) = (1 - \theta)^2 \frac{1 - e^{-2k_1 t}}{2k_1} + \theta^2 \frac{1 - e^{-2k_2 t}}{2k_2} + 2\rho \theta (1 - \theta) \frac{1 - e^{-(k_1 + k_2) t}}{k_1 + k_2}
$$

Pick arbitrary $\varphi^u$, do following transformation:

$$
\begin{align*}
    \zeta^u &\rightarrow \varphi^u \zeta^u \\
    \sigma(u, S) &\rightarrow \sqrt{\frac{1}{\varphi^u}} \sigma(u, S)
\end{align*}
$$

SDEs for $S_t, X^1_t, X^2_t$ unchanged: $\frac{\delta P}{\delta \zeta^u} = 0 \Rightarrow$ mixed two-factor model admissible.
Admissible models – 3

- Other admissible models:
  - lognormal model for $V_t$ (SABR)
  - smiled version of two-factor model (see SD III)

- Significance of condition $\frac{dP}{d\lambda} \bigg|_{S,\hat{\sigma}_{KT}} = 0$

  - $\frac{dP}{d\lambda} \bigg|_{S,\hat{\sigma}_{KT}} \neq 0$: price depends on more state variables than hedge instruments. Ex. with Heston model: $P(t, S, \hat{\sigma}_{KT}, V)$.

  - State variables $\lambda$ are stochastic $\Rightarrow$ model allocates thetas proportional to $\frac{d^2P}{d\lambda^2}$, $\frac{d^2P}{d\lambda d\theta}$
  $\Rightarrow$ P&L leakage, even if $\delta\lambda = 0$.

  - Does not happen with model parameters $V^0, k, \nu, \rho$ – do not generate P&L leakage.

    - Model allocates no theta to gammas on model params.

    - Like making $P$ a function of a non-financial state variable – e.g. temperature.

- In admissible models, SV degrees of freedom do impact dynamics of assets, yet do not require extra hedges.
Now know which models are usable – what’s left to do?

▶ Size up break-even covariance levels for $S/\hat{\sigma}_{KT}, \hat{\sigma}_{KT}/\hat{\sigma}_{K'T'}$.
  ▶ Like them, use model; don’t like them, don’t use model.
  ▶ In practice, look at dynamics of implied vols with floating strikes – fixed moneyness, rather than fixed strikes.

▶ Approximate formulae for vols of vols and spot/vol covariances – for ATMF vols?

▶ Consider in particular SSR:

$$\mathcal{R}_T = \frac{1}{S_T} \frac{\langle d\hat{\sigma}_T d\ln S \rangle}{\langle (d\ln S)^2 \rangle}$$

▶ Expand at order one in vol of vol $\nu$ and local vol function.
Example

- Pick as mkt smile smile generated by two-factor model. Parameters typical of Eurostoxx50 smile. VS vols flat at 20%.
  - So that full SV situation attainable.

- Parameters so that \( \text{vol}(\hat{\sigma}_T) \propto \frac{1}{T^{0.6}} \).

- \( \rho_{SX_1}, \rho_{SX_2} \) (calibrated on actual smile) so that \( S_T \approx \frac{1}{T^{0.5}} \).

---

Model params

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>nu</td>
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<td>k2</td>
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<td>rho SY</td>
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95/105 vol pts

- Model
- Power-law exp = 0.5

---

Mat - years

95/105 vol pts

Model

Power-law exp = 0.5
Example – 2

Ϫ Test 1: use same parameters for underlying SV model – local vol flat = 1.
  ς MC: computed numerically – other curves: order-1 formulae
  ς Everything as function of maturity (years).

Ϫ Test 2: Now halve vol of vol of underlying SV model
Conclusion

- Characterization of local/stoch vol models that can be used for trading. Pricing function \( P(t, S, \hat{\sigma}_{KT}, \lambda) \) has to be such that:

\[
\frac{dP}{d\lambda} \bigg|_{S, \hat{\sigma}_{KT}} = 0
\]

Models not obeying this condition \( \Rightarrow \) P&L leakage.

- Models obeying condition are genuine market models: thetas matching asset/asset cross-gammas with positive break-even covariance matrix.

- Delta and vega given simply by \( \frac{dP}{dS} \bigg|_{\hat{\sigma}_{KT}} \) and \( \frac{dP}{d\hat{\sigma}_{KT}} \bigg|_{S} \) – the LV function is recalibrated.

- Good approximate expressions for break-even covariances for ATMF vols & spot.