Contagion! The Spread of Systemic Risk in Financial Networks

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The Nature of the Minicourse

- FSR is an interdisciplinary topic: ideas from economics, social policy, finance, physics, computer science, other sciences, mathematics, probability and statistics.
- A new active field!
- This minicourse is based on the draft monograph “Contagion! The Spread of Systemic Risk in Financial Networks”, available for download at http://ms.mcmaster.ca/tom/tom.html
- Audience participation will be highly valued!

Systemic Risk: a Definition?

Quotation (Bank for International Settlements 1994)

The risk that the failure of a participant to meet its contractual obligations may in turn cause other participants to default with a chain reaction leading to broader financial difficulties.

Quotation (Kaufman 1995)

The probability that cumulative losses will accrue from an event that sets in motion a series of successive losses along a chain of institutions or markets comprising a system. . . . That is, systemic risk is the risk of a chain reaction of falling interconnected dominos.

Assessing These Definitions

- J. B. Taylor [2009] argued that these and others are not good definitions of SR.
- Any SR crisis also causes damage outside the network, through its failure to efficiently perform its key function of providing liquidity, credit and services.
- He says: "To some people, virtually everything is systemic. To others, it remains very rare."
- He also says, without a proper definition, public policy intending to identify "SIFIs" will fail: "we will make things worse by enshrining an inoperative concept. "

Systemic Risk: S. L Schwarcz' definition

Quotation

The risk that

1. an economic shock such as market or institutional failure triggers (through a panic or otherwise) either:
   - the failure of a chain of markets or institutions or;
   - a chain of significant losses to financial institutions,
2. resulting in increases in the cost of capital or decreases in its availability, often evidenced by substantial financial-market price volatility.

Cascades of shocks to banks plus general drop in liquidity
Correlation versus Contagion

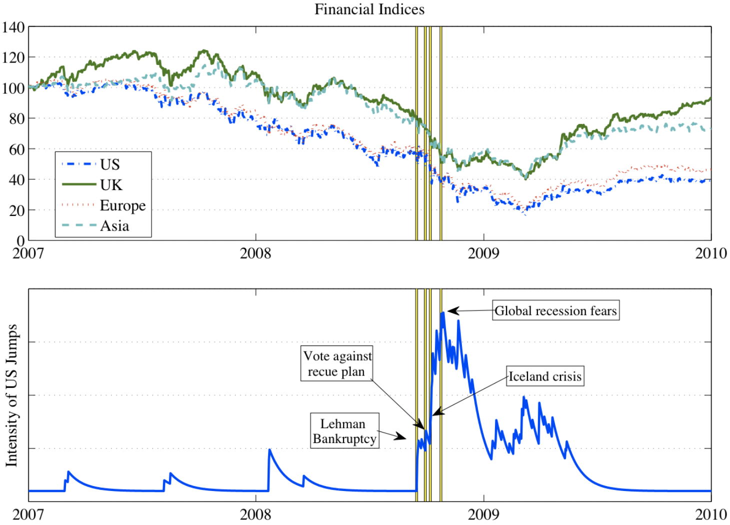
Systemic Network Risk: Overview

Andrew G Haldane's 2009 talk "Rethinking the Financial Network" is a brilliant summary of the nature of networks. He compares the 2002 SARS epidemic to the 2008 collapse of Lehman Bros. In both cases:

- an external event strikes;
- panic ensues and system seizes up;
- "collateral damage" is wide and deep;
- in hindsight, trigger event was modest;
- dynamics was chaotic.

Manifestation of a complex adaptive system

2007-2008 Crisis Schematic



Bank Failures: Sept 2008

September 7, 2008	Fannie Mae and Freddie Mac	Federal Housing Finance Agency	Subprime mortgage lender		[7][8][9]
September 8, 2008	Derbyshire Building Society	Nationwide Building Society	Building society	£7,100,000,000	[10]
September 8, 2008	Cheshire Building Society	Nationwide Building Society	Building society	£4,900,000,000	[11]
September 14, 2008	Merrill Lynch, New York City	Bank of America, Charlotte, North Carolina	Investment bank	$44,000,000,000	[12]
September 16, 2008 - *Presumed*	American International Group, New York City	United States federal government[A]	Insurance company	$182,000,000,000	[13]
September 17, 2008 -	Lehman Brothers, New York City [B]	Barclays plc	Investment bank	$1,300,000,000	[14]
September 18, 2008	HBOS	Lloyds TSB	Diversified financial services	$21,850,000,000	[15]
September 26, 2008	Washington Mutual, Seattle, Washington	JPMorgan Chase, New York City	Savings and loan association	$1,900,000,000	[16]
September 26, 2008	Lehman Brothers[C]	Nomura Holdings	Investment bank	$2	[17]
September 28, 2008	Bradford & Bingley[D]	Government of the United Kingdom (Mortgage Assets) Grupo Santander	Diversified financial services	£12,000,000	[18][19][20]

More Bank Failures: Sept-Oct 2008

September 28, 2008	Fortis	Government of the Netherlands (Dutch assets including ABN AMRO) BNP Paribas (Belgian and Luxembourg assets)	Diversified financial services	€11,200,000,000	[21]
September 30, 2008	Dexia	The Belgian, French and Luxembourg governments	Public finance and retail		[22]
October 3, 2008	Wachovia, Charlotte, North Carolina	Wells Fargo, San Francisco, California F	Retail and investment banking	$15,000,000,000	[23]
October 7, 2008	Landsbanki	Icelandic Financial Supervisory Authority	Commercial Bank		[24][25]
October 8, 2008	Glitnir	Icelandic Financial Supervisory Authority	Commercial bank		[26][27]
October 9, 2008	Kaupthing Bank	Icelandic Financial Supervisory Authority	Commercial bank		[28][29]
October 9, 2008	BankWest (subsidiary of HBOS)	Commonwealth Bank of Australia	Bank	£1,200,000,000	[30]
October 13, 2008	Sovereign Bank, Wyomissing, Pennsylvania	Banco Santander SA	Bank	$1,900,000,000	[31]
October 22, 2008	Barnsley Building Society	Yorkshire Building Society	Building society	£376,000,000	[32]

Haldane: Rethinking the Financial Network

Quotation (Haldane 2009, p. 3)

Both events [the failure of Lehman Brothers and the unfolding of the SARS epidemic] were manifestations of the behavior under stress of a complex, adaptive network. Complex because these networks were a cats-cradle of interconnections, financial and non-financial. Adaptive because behavior in these networks was driven by interactions between optimizing, but confused, agents. Seizures in the electricity grid, degradation of ecosystems, the spread of epidemics and the disintegration of the financial system: each is essentially a different branch of the same network family tree.

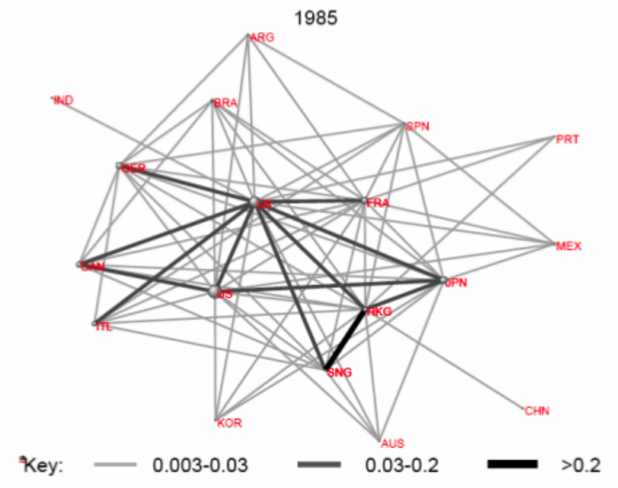
Complexity and Stability

What went wrong with the financial network?

- increasing complexity;
- decreasing diversity.

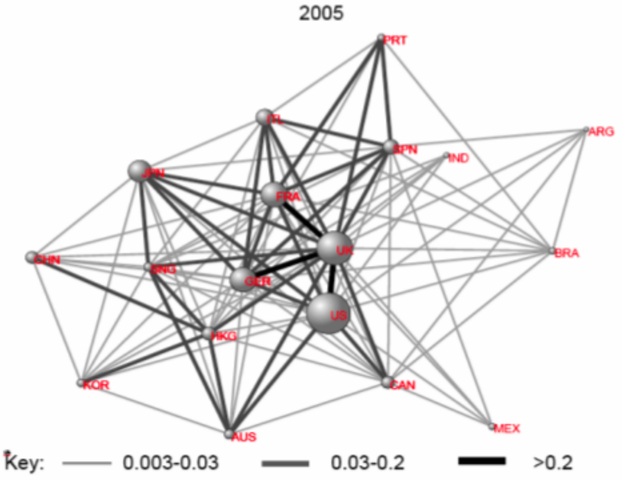
These two facts imply fragility and ring alarm bells for ecologists, engineers, geologists.

Global Financial Network 1985



(line denotes link strength as fraction of total GDP)

Global Financial Network 2005



Connectivity and Stability

Highly connected networks may be “robust yet fragile”:

- In a network, connections may be either shock absorbers or shock amplifiers;
- There may be a “tipping point” that separates these two regimes.
- A fat-tailed “degree distribution” (the number of links per node) implies robustness to random shocks but vulnerability to shocks that target highly connected nodes.

Feedback and Stability

How do agents respond to a crisis?

- Epidemics: "hide" vs "flight";
- Finance: "hoard liquidity" vs "sell assets".

In finance, both responses are rational, but make the systemic problem worse. Government intervention is important to provide liquidity when it is most needed!

Uncertainty and Stability

Networks generate chains of claims. At times of stress, these chains can amplify uncertainties about true counterparty exposures.

- In good times, counterparty risk is small, and thus "Knightian" uncertainty is small: stability improves with connectivity;
- In bad times, counterparty risk can be large and uncertain, due to the complicated web: stability declines with connectivity.

Innovation and Stability

Financial innovation, particularly "securitization", created instability.

- CDOs, MBSs, RMBSs and similar high dimensional products became pervasive internationally;
- The structure of these contracts was opaque, not transparent;
- They dramatically expanded the size and scope of the precrisis bubble (see Shin 2009, "Securitisation and Financial Stability");
- They dramatically increased the connectedness and complexity of the network;
- "Adverse selection" made them hard to evaluate.
- "With no time to read the small-print, the instruments were instead devoured whole. Food poisoning and a lengthy loss of appetite have been the predictable consequences. "

Diversity and Stability

- In ecosystems, biodiversity is known to improve stability;
- In "Great Moderation" period, financial diversity has been reduced;
- Pursuit of returns lead to many agents following similar strategies: portfolio correlations grew to $> 90\%$.
- Risk management regulation (a la Basel II) lead to similar risk management strategies for banks;
- As a result, bank balance sheet became increasingly homogeneous;

Finance became almost a "monoculture", and vulnerable to "viral infection".

Haldane: Summary

- Networks arising in ecology, engineering, the internet, finance, etc are complex and adaptive;
- They typically are "robust yet fragile";
- There is a role for intervention to create more stable networks;
- Key determinants for financial stability may be deduced by studying other types of networks.

What properties of the financial network most influence stability?

Aim of the Book

Main Aim

1. Inspired by Haldane's challenge and ideas from Network Science, to crystallize a basic modelling structure for systemic risk research.
2. Must enable mathematical tractability.
3. Must also be scalable and flexible to account for multiplicities of "bank" types, "interaction" types , and channels of contagion.

Network Science: Self-Organized Criticality

Four different types of cascading events that arise in nature or society:

- Floods caused by systems of dams and reservoirs or interconnected valleys.
- Snow avalanches in mountainous regions.
- Forest fires in areas susceptible to a lightning bolt or a lit match.
- Cascades of load shedding in power grids.

Is SR an example of this too?

Self-Organized Criticality (SOC)

- One of the mechanisms by which complexity arises in nature.
- Concept and name originated in Bak, Tang and Wiesenfeld's 1987 paper "Self-organized criticality: an explanation of $1/f$ noise", which introduces the "BTW sandpile model".
- See also: Per Bak (1996). "How Nature Works: The Science of Self-Organized Criticality." New York: Copernicus.

Quotation (Wikipedia)

Self-organized criticality (SOC) is a property of dynamical systems which have a critical point as an attractor. Their macroscopic behaviour thus displays the spatial and/or temporal scale-invariance characteristic of the critical point of a phase transition, but without the need to tune control parameters to precise values.

Self-Organized Criticality

Question

Does SOC Exist in Financial Markets?

Nature of Banking Balance Sheets

Long before the Crisis, Hyman Minsky and others argued that long periods of stable financial growth lead to evolving financial practises that make financial instability more likely.

Quotation (Minsky)

Stability–even of an expansion–is destabilizing in that more adventuresome financing of investment pays off to the leaders, and others follow.

From "Liquidity and Leverage" by Tobias Adrian and Hyun Song Shin 2009.

Quotation (Adrian and Shin)

In a financial system in which balance sheets are continuously marked to market, asset price changes appear immediately as changes in net worth, eliciting responses from financial intermediaries who adjust the size of their balance sheets. We document evidence that marked-to-market leverage is strongly procyclical.

Balance Sheet Arithmetic: a Household

- Suppose household is worth $A = 100$ (asset)...
- and mortgage value is $D = 90$ (debt):
- then net worth $E = A - D = 10$ (equity)
- and leverage $L = A/E = 10$.

Assets	Liabilities
100	10
	90

What happens to leverage as total assets A fluctuate?

Leverage for a Passive Investor

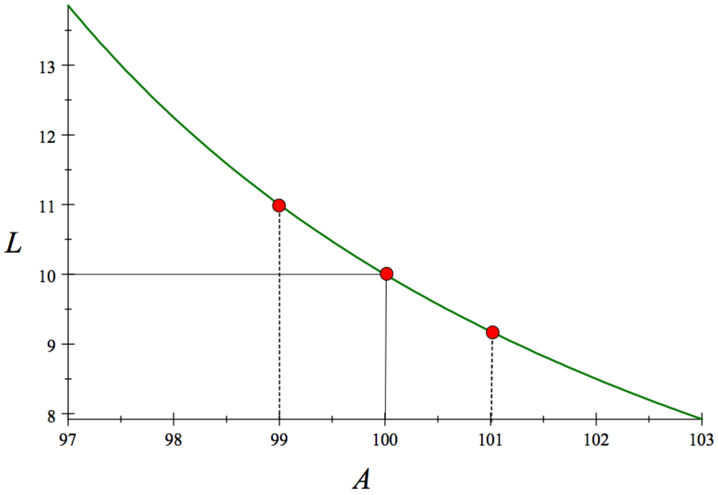


Figure: Leverage for a Passive Investor

Quarterly percentage changes in household leverage and asset value for period 1963-2006

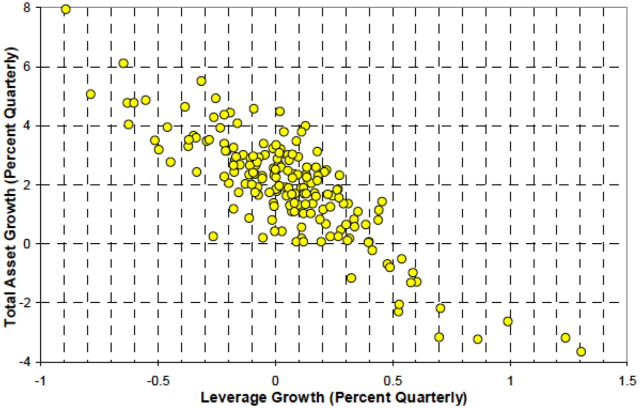


Figure 2.2: Total Assets and Leverage of Household

Investment Banks

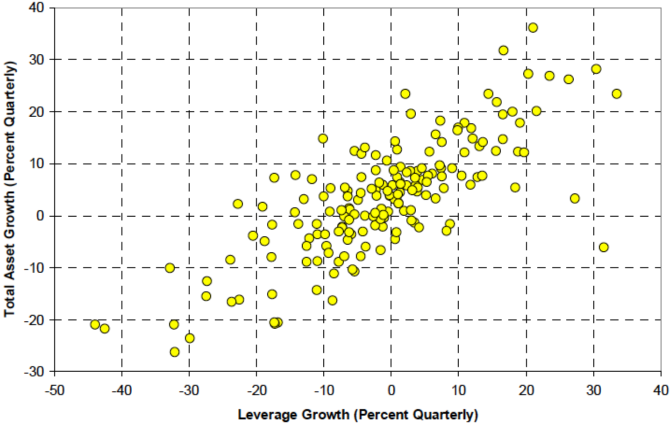


Figure 2.5: Total Assets and Leverage of Security Brokers and Dealers

Active Balance Sheets: Constant Leverage

Commercial bank that maintains $L = 10$:

Assets	Liabilities
securities 100	equity 10 debt 90

- Suppose asset value rises: $A \to 101$...
- new leverage: $L = 101/11 = 9.18$...
- raise debt by 9: $D \to 99$...
- buy 9 units of new assets: $A \to 110$...
- new leverage $L = 110/11 = 10$.

1% rise in security values leads to increase of 10% in assets:

demand curve is upward sloping!

Imperfectly liquid markets

If increase in demand leads to increase in security price:

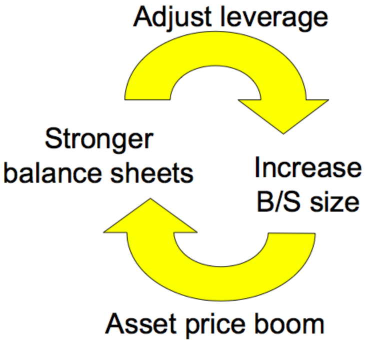


Figure: Leverage Spiral in an Upturn

Imperfectly liquid markets: (ctd)

If decrease in demand leads to decrease in security price:

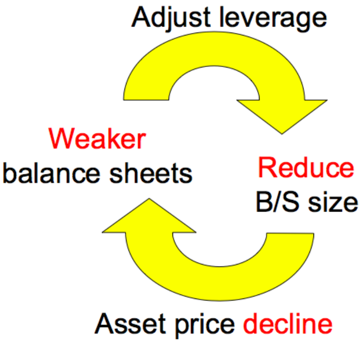
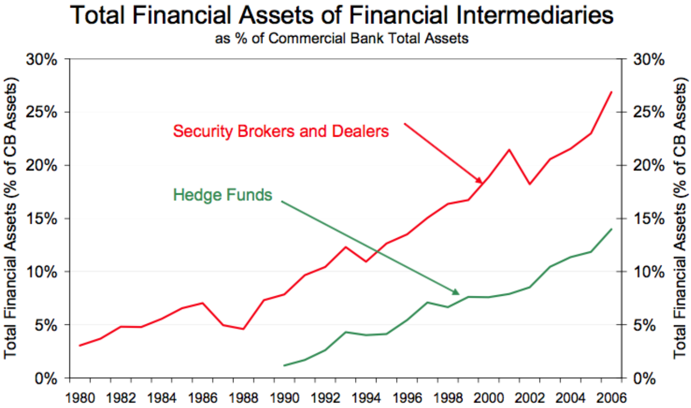


Figure: Leverage Spiral in a Downturn

Growth of the Investment Bank and Hedge Fund Sectors



Source:
Total financial assets of Security Brokers and Dealers are from table L.129 of the Flow of Funds, Board of Governors of the Federal Reserve.
Total financial assets of Bank Holding Companies are from table L.112 of the Flow of Funds, Board of Governors of the Federal Reserve.
Total Assets Under Management of Hedge Funds are from HFR.

My Interpretation of SOC

- Something like "sand piling" seems to happen in financial markets.
- Interpreting Minsky: a long period of stability allows network to build into a critical state.
- Critical systems exhibit power law statistics and universality.
- Eventually a dramatic "correction" hits, triggering a crisis.
- Scientists study SOC in very large or infinite systems using stochastic methods.

Stylized Bank Balance Sheet

Assets	external assets $\bar{Y}$	interbank assets $\bar{Z}$	
Liabilities	external debt $\bar{D}$	interbank debt $\bar{X}$	equity $\bar{E}$

Table: An over-simplified bank balance sheet.

Random Financial Network (RFN)...

- ...is a random object representing the possible states of the financial network at an instant in time.
- We think of it as three layers of mathematical structure.
- Base level, the *skeleton* is a random graph $(\mathcal{N}, \mathcal{E})$ whose nodes/vertices $v \in \mathcal{N}$ represent financial institutions or "banks".
- Directed edges/links $\ell = (wv) \in \mathcal{E}^{dir}$ may represent the presence of a non-negligible interbank exposure $\bar{\Omega}_{wv}$ between a debtor bank and its creditor bank.
- More generally, undirected edges $\ell \in \mathcal{E}^{un}$ might represent counterparty relationships, and Ω_ℓ will be some measure of its strength.

Random Financial Network (ctd)

- Conditioned on a realization of the skeleton, the second layer is a collection of random *balance sheets*, i.e. $(\bar{\mathrm{Y}}_v, \bar{\mathrm{Z}}_v, \bar{\mathrm{D}}_v, \bar{\mathrm{X}}_v)$ for each bank.
- Conditioned on a realization of the skeleton and balance sheets, the third level is a collection of random *exposures* $\bar{\Omega}_\ell$ for each link $\ell \in \mathcal{E}$.
- Constraints (directed case):

$$\bar{\mathrm{Z}}_v = \sum_w \bar{\Omega}_{wv}, \quad \bar{\mathrm{X}}_v = \sum_w \bar{\Omega}_{vw} \ .$$

Remark

$\bar{\mathrm{Y}}_v, \bar{\mathrm{Z}}_v, \bar{\mathrm{D}}_v, \bar{\mathrm{X}}_v, \bar{\Omega}_\ell$ can be multi-dimensional variables.

Assortative Configuration Skeletons

Given node-edge degree distribution pair (P, Q) and size N:

1. Draw a sequence of N node-degree pairs $X = ((j_1, k_1), \ldots, (j_N, k_N))$ independently from P, and accept draw if and only if $\sum_{n\in[N]} j_n = \sum_{n\in[N]} k_n$. Label the nth node with k_n *out-stubs* and j_n *in-stubs*. Number of out-stubs, in-stubs and edges are $e^+_k = \sum_n k\mathbf{1}(k_n = k), e^-_j = \sum_n j\mathbf{1}(j_n = j)$ and $E = \sum_k e^+_k = \sum_j e^-_j$.
2. Conditioned on X, the result of Step 1, choose an arbitrary ordering ℓ^- and ℓ^+ of the E in-stubs and E out-stubs. For each permutation $\sigma \in S[E]$, select the matching sequence or wiring W of "edges" $\ell = (\ell^- = \ell, \ell^+ = \sigma(\ell))$, labelled by $\ell \in [E]$, with probability weighted by the factor

$$\prod_{\ell\in[E]} Q_{k_{\sigma(\ell)} j_\ell} \ .$$

Studies of Specific Financial Systems

"Simulation methods to assess the danger of contagion in interbank markets" by Christian Upper (2011) reviews 15 precrisis studies of specific financial systems.

15 Precrisis Studies

	Country	Data source/estimation method	Simulation method	Shock	Extensions
Amundsen and Arnt (2005)	Denmark	Domestic overnight loans, computed from payments data	Sequential	IE	
Blavarg and Nimander (2002)	Sweden	Supervisory reports on 15 largest exposures of top 4 banks, incl. derivatives and FX settlement floats	Sequential	IE	Impact on liquidity
Degryse and Nguyen (2007)	Belgium	Domestic interbank loans and deposits of Belgian banks, ME. Foreign exposures from supervisory report on interbank exposures exceeding 10% of capital	Sequential		(i) Endogenise θ (ii) Relate contagion to concentration and internationalisation of interbank market
Elsinger et al. (2006a)	Austria	Domestic interbank loans and deposits, ME, complemented with equality constraints	EN	MC	(i) Risk management model for credit and market risk of individual banks (ii) Bankruptcy costs (iii) Bilateral netting
Elsinger et al. (2006b)	United Kingdom	Domestic interbank loans and deposits, ME (data as in Wells, 2004)	EN	MC	Asset correlations from multivariate Merton model
Frisell et al. (2007)	Sweden	Supervisory report on largest 15 exposures of top 4 Swedish banks (excl. repo)	Sequential	IE, AE	Probability of failure (from Moody's KMV, linked by Gaussian copula), and hence probability of contagion
Furfine (2003)	United States	Overnight loans in federal funds market, computed from payment data	Sequential	IE	Illiquidity due to inability to fund
Guerrero-Gómez and Lopez-Gallo (2004)	Mexico	Supervisory report on interbank exposures (incl. securities, derivatives and credit lines in payment system)	Sequential	IE	Illiquidity due to inability to fund
Lublóy (2005)	Hungary	Supervisory reports of all domestic interbank loans	Sequential	IE, AE	Topology of interbank market
Mistrulli (2007)	Italy	Supervisory reports of all domestic interbank loans	Sequential	IE	(i) Effect of intra-group guarantees (ii) Examines biases from ME
Müller (2006)	Switzerland	Supervisory reports covering 10 (20 for the two big banks) largest interbank exposures and liabilities (incl. off-balance sheet) of Swiss banks	EN	IE	(i) Topology of interbank market (ii) Illiquidity due to inability to fund
Sheldon and Maurer (1998)	Switzerland	Domestic interbank loans, ME aggregated by bank type	Sequential	AE	Probability of failure of trigger bank and hence probability of contagion
Toivanen (2009)	Finland	Supervisory reports on top ten interbank exposures of 6 largest banks, excl. repo; ME for remaining exposures	Sequential	IE	
Upper and Worms (2004)	Germany	Domestic interbank loans, ME performed separately for different maturity categories and bank types	Sequential	IE	(i) Bilateral netting (ii) Impact of safety net
Van Lelyveld and Liedorp (2006)	Netherlands	(i) Domestic interbank loans, ME (ii) Supervisory report on interbank exposures (incl. off balance sheet) larger than 3% of capital (iii) Survey of top 10 banks' for all domestic and top 15 foreign interbank exposures	Sequential	IE, AE	
Wells (2004)	United Kingdom	(i) Domestic interbank loans, ME (ii) Supervisory report on domestic and foreign exposures exceeding 10% of Tier 1 capital for some banks, ME for remaining banks	Sequential	IE, AE	(i) Estimate money-centred structure using ME and supervisory reports assuming that small banks hold all deposits with large banks

Channels for Contagion: Liability Side

Possible channels of contagion in the banking system.

Channel	References
Liability side	
Bank runs	
Multiple equilibria/fear of other withdrawals	Diamond and Dybvig (1983), Temzelides (1997), Goldstein and Pauzner (2004)
Common pool of liquidity	Aghion et al. (2000), Acharya and Yorulmazer (2008b), Diamond and Rajan (2005), Brunnermeier and Pedersen (2009)
Information about asset quality	Chen (1999), Acharya and Yorulmazer (2008a)
Portfolio rebalancing	Kodres and Pritsker (2002)
Fear of direct effects	Dasgupta (2004), Iyer and Peydró-Alcalde (2005), Lagunoff and Shreft (2001), Freixas et al. (2000)
Strategic behaviour by potential lenders	Acharya et al. (2008)

Channels for Contagion: Asset Side

Asset side	
Direct effects	
Interbank lending	Rochet and Tirole (1996), *studies reviewed in this paper*
Payment system	Humphrey (1986), Angelini et al. (1996), Bech and Garratt (2006)
Security settlement	Northcott (2002)
FX settlement	Blavarg and Nimander (2002)
Derivative exposures	Blavarg and Nimander (2002)
Equity cross-holdings	
Indirect effects	
Asset prices	Cifuentes et al. (2005), Fecht (2004)

Assumptions These Studies Make

Upper 2011 identifies the type of assumptions implicit in such studies.

1. Banks have limited liability.
   Virtually all banking systems feature institutions whose liabilities are either explicitly or implicitly guaranteed by the government or by other players.
2. Nonbank liabilities are senior to interbank liabilities.
   This is an open issue. Falsely assuming that all interbank claims are junior to claims by non-banks will overstate both the possibility and the severity of contagion.
3. Losses on interbank assets are shared equally across lenders.
   In fact, biases can go into either direction.
4. Nonbank assets can be sold at their book value.
   Failing banks liquidate their assets, which would tend to depress prices and thus increase the severity of contagion.

Further Assumptions

5. Banks spread their lending as evenly as possible given the assets and liabilities reported in the balance sheets of all other banks.
   This is far from true.
6. Contagion is only driven by domestic exposures.
   Assuming away contagion from abroad will lead to an underestimation of both the possibility and the severity of contagion.

Summary of Upper 2011

He identifies two major shortcomings:

- An exaggerated focus on scenarios involving idiosyncratic failure of a single bank, rather than a market shock;
- More important is the absence of "behavioural" foundations that preclude different channels for contagion. These studies assume "Banks sit tight as problems of their counterparties mount". We have seen that "asset hoarding" and "selling assets" are both rational responses that make systemic risk higher.

Most critics would also add: These papers focus too much on "insolvency" and underestimate the effects of "illiquidity".

Channels of Systemic Risk (SR)

Piecing things together, we identify four important channels of SR:

1. **Correlation** e.g. Subprime assets
2. **Default Contagion** e.g. default of Lehman
3. **Liquidity Contagion** e.g. the freezing of repo markets
4. **Firesales or Market Illiquidity** e.g. sales of ABS

In addition,

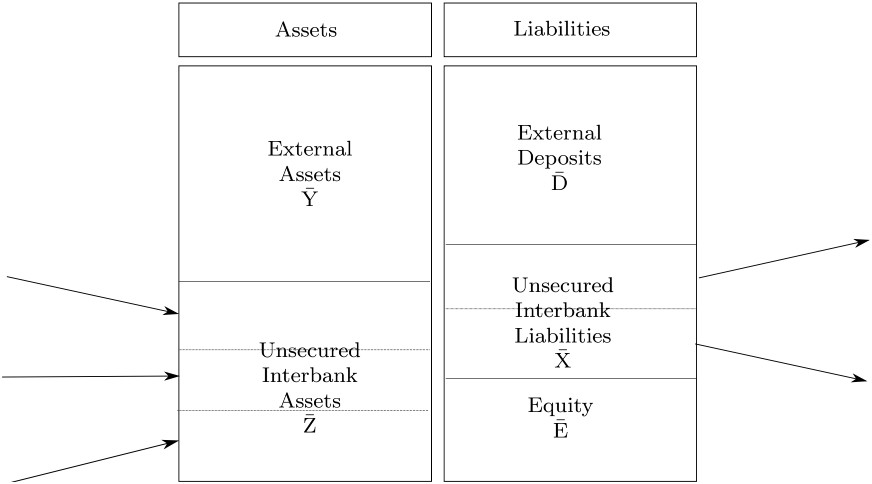
1. **Rising Haircuts**
2. **Confidence and Herd Behaviour**
3. **Rollover Risk**
4. **Central Clearing Party Failure**
5. ...

Eisenberg-Noe 2001 Network Model

Eisenberg-Noe 2001 model is a prototype:

1. Describes an abstract payment system with "entities" connected by a network of payment obligations.
2. It asks what should result in an ideal clearing system when some agents have insufficient assets to cover their obligations.
3. It has become a standard model of default cascades.

Balance Sheets



Balance Sheets

- The assets A_v of bank v
  1. *external assets* Y_v
  2. *internal (Interbank) assets* Z_v
- The liabilities of the bank v
  1. *external debts* D_v
  2. *internal (Interbank) debt* X_v
  3. *equity* or *net worth*, defined by $E_v = \mathrm{Y}_v + \mathrm{Z}_v - \mathrm{D}_v - \mathrm{X}_v \geq 0$
- *Promised payments*: Ω_ℓ, $\ell = (v, v')$, the amount v owes v'.
- Constraints

$$\mathrm{Z}_{v'} = \sum_v \Omega_{vv'}, \quad \mathrm{X}_v = \sum_{v'} \Omega_{vv'}, \quad \sum_{v'} \mathrm{Z}_{v'} = \sum_v \mathrm{X}_v$$

- *Debt ratios*: $\Pi_{vw} = \Omega_{vw}/\mathrm{X}_v$.

	1	2	$\cdots$	N	$\bar{X}$	$\bar{D}$	$\bar{E}$
1	0	$\bar{\Pi}_{12}\bar{X}_1$	$\cdots$	$\bar{\Pi}_{1N}\bar{X}_1$	$\bar{X}_1$	$\bar{D}_1$	$\bar{E}_1$
2	$\bar{\Pi}_{21}\bar{X}_2$	0	$\cdots$	$\bar{\Pi}_{2N}\bar{X}_2$	$\bar{X}_2$	$\bar{D}_2$	$\bar{E}_2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$\bar{\Pi}_{N1}\bar{X}_N$	$\bar{\Pi}_{N2}\bar{X}_N$	$\cdots$	0	$\bar{X}_N$	$\bar{D}_N$	$\bar{E}_N$
$\bar{Z}$	$\bar{Z}_1$	$\bar{Z}_2$	$\cdots$	$\bar{Z}_N$			
$\bar{Y}$	$\bar{Y}_1$	$\bar{Y}_2$	$\cdots$	$\bar{Y}_N$			

Table: The matrix of interbank exposures contains the values $\bar{\Omega}_{vw} = \bar{\Pi}_{vw}\bar{X}_v$. The first N rows of this table represent different banks' liabilities and the first N columns represent their assets.

Static default cascade assumptions

Definition

A *defaulted bank* is a bank with $\mathrm{E} \leq 0$. A *solvent bank* is a bank with $\mathrm{E} > 0$.

We describe the cascade as if it proceeds in daily steps:

Assumptions

1. Prior to the cascade, all banks are in the normal state, not insolvent.
2. The crisis commences on day 0 triggered by the default of one or more banks;
3. Balance sheets are recomputed daily on a mark-to-market basis;
4. Banks respond daily on the basis of their newly computed balance sheets;
5. All external cash flows, interest payments, and external asset

EN 2001 Liquidation Mechanism

Assumptions

1. External debt is senior to interbank debt and all interbank debt is of equal seniority; 2. No bankruptcy charges; and 3. $\bar{\mathrm{Y}}_v \geq \bar{\mathrm{D}}_v$.

- Let p_v be amount available to pay v's internal debt
- p_v is split amongst creditor banks in proportion to $\Pi_{vw} = \Omega_{vw}/\mathrm{X}_v$: bank w receives $\Pi_{vw} p_v$.
- Given $\mathbf{p} = [p_1, \dots, p_N]$, the clearing conditions are

$$
\begin{aligned}
p_v &= F_v^{(EN)}(\mathbf{p}) := \min(\mathrm{X}_v, \mathrm{Y}_v + \sum_w \Pi_{wv} p_w - \mathrm{D}_v) \\
\mathbf{p} &= \min(\mathrm{X}, \mathrm{Y} + \Pi^T * \mathbf{p} - \mathrm{D})
\end{aligned}
$$

Vector and matrix notation

For vectors $x = [x_v]_{v=1,\dots,N}, y = [y_v]_{v=1,\dots,N} \in \mathbb{R}^N$

$$
\begin{aligned}
x \le y & \quad \text{means} \quad \forall\, v,\ x_v \le y_v, \\
x < y & \quad \text{means} \quad x \le y,\ \exists\, v : x_v < y_v, \\
\min(x,y) = x \wedge y & \quad = \quad [\min(x_v, y_v)]_{v=1,\dots,N} \\
\max(x,y) = x \vee y & \quad = \quad [\max(x_v, y_v)]_{v=1,\dots,N} \\
(x)^+ & \quad = \quad \max(x, 0), \\
(x)^- & \quad = \quad \max(-x, 0)
\end{aligned}
$$

For $x \le y$, the hyperinterval $[x, y]$ is $\{z : x \le z \le y\}$. Any hyperinterval, with the above operations $\wedge, \vee$, is a complete lattice[1].

[1] A "lattice" (partially ordered set with "meet" $\vee$ and "join" $\wedge$) that is closed under sup and inf.

EN Main Theorem

Theorem

Corresponding to every financial system $(\bar{\mathrm{Y}}, \bar{\mathrm{Z}}, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\Omega})$ *satisfying Assumptions 2,*

1. *There exists a greatest and a least clearing vector* p^+ *and* p^-.
2. *Under all clearing vectors, the value of the equity at each node is the same, that is, if* p' *and* p'' *are any two clearing vectors,*

$$(\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}' - \bar{\mathrm{D}} - \bar{\mathrm{X}})^+ = (\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}'' - \bar{\mathrm{D}} - \bar{\mathrm{X}})^+$$

Proof of Part (1)

Knaster-Tarski Fixed Point Theorem states: "the fixed point set of a monotone mapping on a complete lattice is a complete lattice".

Note:

1. F^{EN} is monotonic: $x \leq y$ implies $F^{EN}(x) \leq F^{EN}(y)$.
2. Since also $F^{EN}(0) \geq 0$ and $F^{EN}(\bar{\mathrm{X}}) \leq \bar{\mathrm{X}}$, it maps the hyperinterval $[\mathbf{0}, \bar{\mathrm{X}}]$ into itself.
3. $[\mathbf{0}, \bar{\mathrm{X}}]$ is a complete lattice.

Conclude that the set of clearing vectors, being the fixed points of the mapping F^{EN}, is a complete lattice, hence nonempty, and with maximum and minimum elements p^{+} and p^{-}.

Proof of Part (2)

Show: for any clearing vector p′,

$$(\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}' - \bar{\mathrm{D}} - \bar{\mathrm{X}})^+ = (\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}^+ - \bar{\mathrm{D}} - \bar{\mathrm{X}})^+$$

1. By monotonicity, $\mathrm{p}' \le \mathrm{p}^+$ implies

$$(\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}' - \bar{\mathrm{D}} - \bar{\mathrm{X}})^+ \le (\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}^+ - \bar{\mathrm{D}} - \bar{\mathrm{X}})^+$$

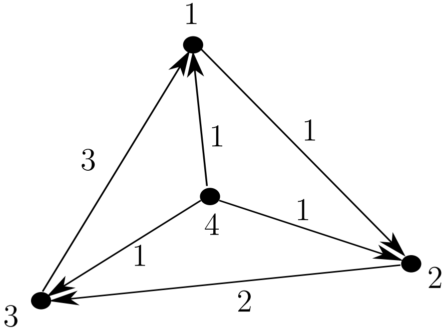
2. Because there are no bankruptcy charges,

$$\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}' - \bar{\mathrm{D}} - \mathrm{p}' \le \bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}^+ - \bar{\mathrm{D}} - \mathrm{p}^+$$

3. Inner product this equation with $\mathbf{1} = [1, \dots, 1]$ noting $\mathbf{1} * \bar{\Pi}^T = \mathbf{1}$:

$$\mathbf{1} * (\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}' - \bar{\mathrm{D}} - \mathrm{p}') = \mathbf{1} * (\bar{\mathrm{Y}} + \bar{\Pi}^T * \mathrm{p}^+ - \bar{\mathrm{D}} - \mathrm{p}^+)$$

Flaw: when $\bar{X}_v = 0$ we have defined $\bar{\Pi}^T_{wv} = 0$, invalidating the condition $\mathbf{1} * \bar{\Pi}^T = \mathbf{1}$. However for this v, $p_v = 0$, and hence it is still true that $\mathbf{1} * \bar{\Pi}^T * \mathrm{p} = \mathbf{1} * \mathrm{p}$.



Exercise: 1. Solve the EN clearing algorithm in Section 2.2.2 of my book in the case when $\bar{\mathrm{Y}} - \bar{\mathrm{D}} = (1/2, 1/2, 1/2, 1)$.
2. Show that only when $\bar{\mathrm{Y}} - \bar{\mathrm{D}} = 0$ can the system have multiple clearing vectors, of the form $\mathrm{p} = \lambda[1, 1, 1, 0]$ with $\lambda \in [0, 1]$.

Cases with $\bar{Y}_v < \bar{D}_v$

1. Write $\bar{Y} - \bar{D} = (\bar{Y} - \bar{D})^+ - (\bar{Y} - \bar{D})^-$.
2. Express the clearing condition in terms of $q = [q_1, \ldots, q_N]^T$ where q_v denotes the amount bank v has available to pay both the excess external debt $(\bar{Y}_v - \bar{D}_v)^-$ and the interbank debt $\bar{X}_v$:

$$
\begin{aligned}
q &= \min((\bar{Y} - \bar{D})^+ + \bar{\Pi}^T * p\ ,\ (\bar{Y} - \bar{D})^- + \bar{X}) \\
p &= (q - (\bar{Y} - \bar{D})^-)^+\ .
\end{aligned}
$$

Reduced Form Cascade Mapping

1. Different balance sheet specifications lead to identical cascades: find a reduced set of balance sheet data.
2. Initial default buffer $\Delta_v^{(0)} := \bar{\Delta}_v$ of bank v is its nominal equity:
$$\Delta_v^{(0)} := \bar{\mathrm{E}}_v = \bar{\mathrm{Y}}_v + \sum_w \Omega_{wv} - \bar{\mathrm{D}}_v - \bar{\mathrm{X}}_v \qquad (1)$$
3. $p_v^{(n)}$ is amount available to pay $\bar{\mathrm{X}}_v$ at cascade step n, $p_v^{(0)} = \bar{\mathrm{X}}_v$.
4. Threshold function $h(x) = (x+1)^+ - x^+$
5. nth step of E-N cascade is

$$\begin{cases} p_v^{(n)} &= \bar{\mathrm{X}}_v \; h(\Delta_v^{(n-1)}/\bar{\mathrm{X}}_v) \\ q_v^{(n)} &= ((\bar{\mathrm{Y}}_v - \bar{\mathrm{D}}_v)^- + \bar{\mathrm{X}}_v) \; h(\Delta_v^{(n-1)}/((\bar{\mathrm{Y}}_v - \bar{\mathrm{D}}_v)^- + \bar{\mathrm{X}}_v)) \\ \Delta_v^{(n)} &= \bar{\Delta}_v - \sum_w \Omega_{wv}(1 - p_w^{(n)}/\bar{\mathrm{X}}_w)) \\ &= \bar{\Delta}_v - \sum_w \Omega_{wv}(1 - h(\Delta_w^{(n-1)}/\bar{\mathrm{X}}_w)) \end{cases} \qquad (2)$$

1. The mark-to-market equity is the positive part of the default buffer, $\mathrm{E}_v^{(n)} = (\Delta_v^{(n)})^+$.
2. Default of bank v occurs at the first step that $\Delta_v^{(n)} \le 0$.
3. As $n \to \infty$, the monotone decreasing sequence $\mathrm{p}^{(n)}$ converges to the maximal fixed point p^+.
4. Cascade mapping: $\mathrm{p}^{(n-1)} \mapsto \mathrm{p}^{(n)} = F(\mathrm{p}^{(n-1)}|\bar{\Delta}, \Omega)$

$$F_v(\mathrm{p}) = \bar{\mathrm{X}}_v \; h\left(\bar{\Delta}_v/\bar{\mathrm{X}}_v - \sum_w \bar{\Pi}_{wv}(1 - p_w^{(n-1)}/\bar{\mathrm{X}}_w)\right)$$

5. It depends parametrically only on the initial equity buffers $\bar{\Delta}$ and the interbank exposures Ω.

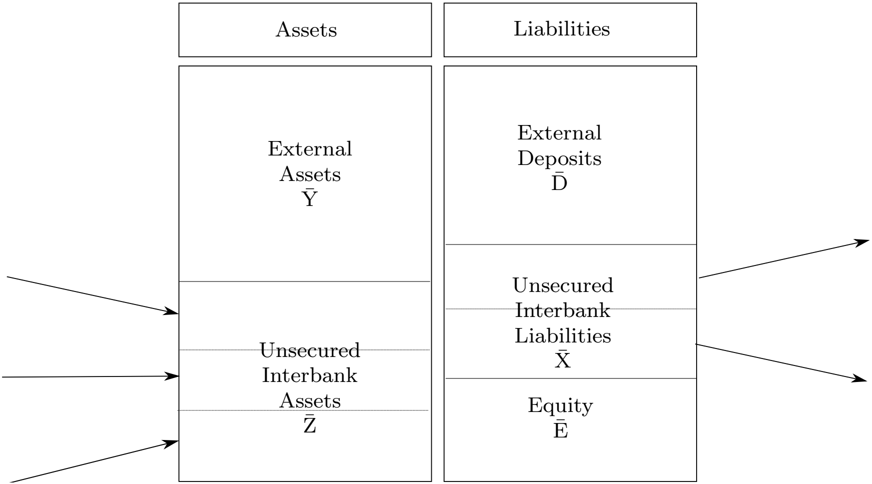
$$\mathrm{p}^+ = G^+(\bar{\Delta}, \Omega)$$

6. If instead of starting the cascade at the initial value $p_v^{(0)} = \bar{\mathrm{X}}_v$, we had begun with $p_v^{(0)} = 0$, we would obtain a monotone *increasing* sequence $\mathrm{p}^{(n)}$ that converges to the minimal fixed point $\mathrm{p}^- := G^-(\bar{\Delta}, \Omega)$.

1. The scaled variable $\Delta/\bar{\mathrm{X}}$ has the interpretation of a bank's "distance-to-default",
2. Threshold function h determines both the fractional loss on interbank debt and on total debt when Δ is negative.
3. Amount of external debt that bank v eventually repays requires $(\bar{\mathrm{Y}}_v - \bar{\mathrm{D}}_v)$:

$$\begin{aligned} q_v^+ &= ((\bar{\mathrm{Y}}_v - \bar{\mathrm{D}}_v)^- + \bar{\mathrm{X}}_v)\, h(\Delta_v^+/((\bar{\mathrm{Y}}_v - \bar{\mathrm{D}}_v)^- + \bar{\mathrm{X}}_v)); \\ \Delta_v^+ &= \bar{\Delta}_v - \sum_w \Omega_{wv}(1 - p_w^+/\bar{\mathrm{X}}_w)) \end{aligned}$$

Eisenberg-Noe 2001
Balance Sheet of a degree-type $(3, 2)$ Bank



Gai-Kapadia 2010 Default Cascade Assumptions

"Contagion in financial networks" aims to provide a stylized analytical model of default cascades.

1. Balance sheets as in EN 2001; $\bar{\Delta} = \bar{Y} + \bar{Z} - \bar{D} - \bar{Z}$.
2. Limited Liability: banks default the first time $\Delta \leq 0$.
3. Zero recovery of defaulted interbank loans: the worst case scenario, might be natural during a crisis, but not after.
4. Losses on external debt: not modelled.
5. Initial shocks cause one or more banks to have $\bar{\Delta} \leq 0$.

Gai-Kapadia Default Cascade Mapping

After n steps, $p_v^{(n)},\ \Delta_v^{(n)}$ have the same interpretation as before.

1. Begin with $p_v^{(0)} = \bar{X}_v,\ \Delta_v^{(0)} = \bar{\Delta}_v$.
2. Let $\mathcal{D}_n$ be the set of defaulted banks after step n.
3. Step n: like EN with new $\tilde{h}(x) = \mathbf{1}_{\{x\le 0\}}$:

$$
\begin{aligned}
\Delta_v^{(n)} &= \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}\left(1 - \tilde{h}(\Delta_w^{(n-1)}/\bar{X}_w)\right) \\
&= \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}\mathbf{1}_{\{w\in\mathcal{D}_{n-1}\}} \\
p_v^{(n)} &= \bar{X}_v\tilde{h}(\Delta_v^{(n)}/\bar{X}_v) \\
&= \bar{X}_v\tilde{h}\left(\bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}\left(1 - p_w^{(n-1)}/\bar{X}_w\right)\right)
\end{aligned}
$$

4. Clearing vector condition from $\Delta = \bar{\Delta} - \sum_w \bar{\Omega}_{w\cdot}(1-\tilde{h}(\Delta_w))$.

Watts 2002-style Random Cascade Model

1. Skeletons: $N \to \infty$ sequence of undirected Gilbert $G(N, z/(N-1))$ graphs with mean degree z.
2. Default buffers are integer random variables, $\bar{\Delta}_v \in \mathbb{Z}_+$. Conditioned on the skeleton they are independent with distributions that depend only on k_v:

$$\mathbb{P}[\bar{\Delta}_v \leq x | k_v = k] := D_k(x) := \sum_{0 \leq y \leq x} d_k(y)$$

3. We assume that v defaults, either initially if $\bar{\Delta}_v = 0$, or as soon as at least $\bar{\Delta}_v$ neighbours default.

Let $\mathcal{D}_n$ denote the set of defaulted nodes after n cascade steps. Initially defaulted nodes are

$$\mathcal{D}_0 = \{v | \bar{\Delta}_v = 0\}.$$

Then for $n > 0$, $v \in \mathcal{D}_n$ means $\bar{\Delta}_v \leq \sum_{w \in \mathcal{N}_v} \mathbf{1}(w \in \mathcal{D}_{n-1})$.

Percolation and Cascades

The right kind of connectivity turns out to be both necessary and sufficient for large scale cascades to propagate in a network.

1. First we outline percolation theory on random graphs and its relation to Galton-Watson branching processes.
2. Then we introduce bootstrap percolation.This proves to be the precise concept needed for unravelling and understanding the growth of simple network cascades.
3. These principles are illustrated by the Watts model of information cascades.

Extinction Theorem

Theorem

The extinction probability $\eta \in [0, 1]$ *is the smallest fixed point of* g.

1. *If* $\mathbb{E}X > 1$, *then* $\eta < 1$, *which says that with positive probability* $1 - \eta$ *the population will survive forever.*
2. *If* $\mathbb{E}X \leq 1$, *then apart from a trivial exception,* $\eta = 1$ *and the population becomes extinct almost surely.*

Case (1), when survival is possible, is called the *supercritical* case. The case of almost sure extinction subdivides: case $\mathbb{E}X < 1$ is called *subcritical*, and the case $\mathbb{E}X = 1$ and $g''(1) > 0$ is called *critical*.

Percolation Theorem (Molloy-Reed, Janson, ...

Theorem

Consider the random configuration multigraph sequence $G^*(N, \mathbf{d})$ *satisfying Assumptions. Let* $g(x)$ *be its asymptotic generating function and let* $\mathcal{C}$ *be the largest cluster. Then the following asymptotic properties hold:*

1. *If* $\sum_k k(k-2)\ P_k > 0$*, then there is a unique* $\xi \in (0,1)$ *such that* $g^*(\xi) = \xi$ *and*

$$\mathbb{P}[v \in \mathcal{C}] \xrightarrow{\mathrm{P}} 1 - g(\xi) > 0\ , \tag{3}$$

$$\mathbb{P}[v \in \mathcal{C} \cap \mathcal{N}_k] \xrightarrow{\mathrm{P}} P_k(1 - \xi^k),\ \textit{for every } k \geq 0\ , \tag{4}$$

$$\mathbb{P}[\ell \in \mathcal{C}] \xrightarrow{\mathrm{P}} (1 - \xi^2) > 0\ . \tag{5}$$

2. *If* $\sum_k k(k-2)\ P_k \leq 0$*, then unless* $P_2 = 1$*,* $\mathbb{P}[v \in \mathcal{C}] \xrightarrow{\mathrm{P}} 0.$

Cascade Dynamics = Bootstrap Percolation?

1. Bootstrap Percolation is a dynamic version of percolation introduced in 1979 by Chalupa, Leath and Reich for magnetic systems on regular lattices.
2. It follows the growth of connected clusters of nodes $v \in \mathcal{N}$ that become "activated" when the number of its active neighbours exceeds a threshold.
3. Exact analytic asymptotics are sometimes possible.
4. Watts' 2002 Information Cascade Model is a basic example of Bootstrap Percolation.

Watts 2002-style Toy Random Cascade Model

1. Skeletons: $N \to \infty$ sequence of undirected Gilbert $G(N, z/(N-1))$ graphs with mean degree z.
2. Default buffers are integer random variables, $\bar{\Delta}_v \in \mathbb{Z}_+$. Conditioned on the skeleton they are independent with distributions that depend only on k_v:

$$\mathbb{P}[\bar{\Delta}_v \le x | k_v = k] := D_k(x) := \sum_{0 \le y \le x} d_k(y)$$

3. We assume that v defaults, either initially if $\bar{\Delta}_v = 0$, or as soon as at least $\bar{\Delta}_v$ neighbours default.

Let $\mathcal{D}_n$ denote the set of defaulted nodes after n cascade steps. Initially defaulted nodes are

$$\mathcal{D}_0 = \{v | \bar{\Delta}_v = 0\}.$$

Then for $n > 0$, $v \in \mathcal{D}_n$ means $\bar{\Delta}_v \le \sum_{w \in \mathcal{N}_v} \mathbf{1}(w \in \mathcal{D}_{n-1})$.

The WOR property of the Watts model

Proposition

Let the Watts model be specified by $(\mathcal{N}, \mathcal{E}, \{\bar{\Delta}_v\})$ *and the sequences* $\{D_v^n, D_{v,w}^n\}_{n=-1,0,1,\dots}$ *defined by the recursive equations*

$$\begin{aligned} D_v^n &= \mathbf{1}(v \in \mathcal{D}_n) = \mathbf{1}\left(\bar{\Delta}_v \le \sum_{w' \in \mathcal{N}_v} D_{w'}^{n-1}\right) \\ D_{v,w}^n &= \mathbf{1}(v \in \mathcal{D}_n \text{ WOR } w) = \mathbf{1}\left(\bar{\Delta}_v \le \sum_{w' \in \mathcal{N}_v} D_{w',v}^{n-1} \mathbf{1}(w' \neq w)\right) . \end{aligned}$$

with $D_v^{-1}, D_{v,w}^{-1}, \tilde{D}_{v,w}^{-1} = 0$. Then *for all* $n \ge 0$ *and* $(v, w) \in \mathcal{E}$

$$D_v^n = \mathbf{1}\left(\bar{\Delta}_v \le \sum_{w' \in \mathcal{N}_v} D_{w',v}^{n-1}\right)$$

Watts model setup

Define for $n \geq 0$ and all k:

1. $p_k^{(n)} = \mathbb{E}[v \in \mathcal{D}_n | k_v = k]$;
2. $\hat{p}_k^{(n)} := \mathbb{P}[w \in \mathcal{D}_n \text{ WOR } v | w \in \mathcal{N}_k \cap \mathcal{N}_v]$.
3. $\hat{\pi}^{(n)} := \mathbb{P}[w \in \mathcal{D}_n \text{ WOR } v | v \in \cap \mathcal{N}_w \cap \mathcal{N}_k$ (which happens to be k independent)

Watts model: Main Result

Proposition

Consider the Watts model in the limit as $N \to \infty$ with initial adoption probabilities $p_k^{(0)} = \hat{p}_k^{(0)} = d_k(0)$. Then $(p_k^{(n)})$ and $(\hat{p}_k^{(n)})$ for $n \geq 1$ are given by the recursion formulas

$$p_k^{(n)} = G_k(\hat{p}^{(n-1)}) := \sum_{x=0}^{k} D_k(x)\ \mathrm{Bin}(k, \hat{\pi}^{(n-1)}, x) \quad (6)$$

$$\hat{p}_k^{(n)} = \hat{G}_k(\hat{p}^{(n-1)}) := \sum_{x=0}^{k-1} D_k(x)\ \mathrm{Bin}(k-1, \hat{\pi}^{(n-1)}, x) \quad (7)$$

where

$$\hat{\pi}^{(n-1)} = \sum_{k'} \hat{p}_k^{(n-1)} \frac{k' P_{k'}}{z} \quad (8)$$

Watts model: Formal proof

1. Requires two properties of the model: (i) LT property of the skeleton as long as N is sufficiently large, and (ii) conditional independence of the thresholds $\bar{\Delta}$, conditioned on the skeleton.
2. By the original definition of the set $\mathcal{D}_n$,
$$p_k^{(n)} = \mathbb{P}\Big[\bar{\Delta}_w \le \sum_{w' \in \mathcal{N}_w} \mathbf{1}(w' \in \mathcal{D}_{n-1}) | w \in \mathcal{N}_k\Big]$$
Problem: terms in the sum are not conditionally independent.
3. Instead use:
$$p_k^{(n)} = \mathbb{P}\Big[\bar{\Delta}_w \le \sum_{w' \in \mathcal{N}_w} \mathbf{1}(w' \in \mathcal{D}_{n-1} \text{ WOR } w) | w \in \mathcal{N}_k\Big]$$
where the terms are k independent $Bern(\hat{\pi}^{(n-1)})$ random variables.
4. This leads to equation (6).

Watts model: Formal proof

1. Similarly, for random links (v, w)

$$\begin{aligned}\hat{p}_k^{(n)} &:= \mathbb{P}[w \in \mathcal{D}_n \text{ WOR } v | w \in \mathcal{N}_v \cap \mathcal{N}_k] \\ &= \mathbb{P}\Big[\bar{\Delta}_w \leq \sum_{w' \in \mathcal{N}_w \setminus v} \mathbf{1}(w' \in \mathcal{D}_{n-1} \text{ WOR } w) | w \in \mathcal{N}_v \cap \mathcal{N}_k\Big]\end{aligned}$$

   where now there are $k-1$ independent Bern($\hat{\pi}^{(n-1)}$) random variables in the sum, leading to (7).

2. Finally, one can show (8) to finish off the formal proof.

Watts model: a Simple Rigorous proof

We need a bounded degree assumption: $K : P_k = 0$ for $k > K$.
Fix N, n and consider

$$p_k^{n,N} := \mathbb{E}[\frac{1}{N}\sum_{v\in[N]} \mathbf{1}(v \in \mathcal{D}_n, k_v = k)] = \mathbb{E}[\mathbb{E}[\mathbf{1}(1 \in \mathcal{D}_n, k_1 = k)|\mathcal{N}, \mathcal{E}]]$$

where 1 denotes the first node.

Configuration Probabilities for ACG Skeletons

Theorem

Consider the ACG sequence with (P, Q) (satisfying some weak conditions). For any fixed finite configuration g rooted to $v \in \mathcal{N}_{jk}$, with M added nodes and $L \geq M$ edges, labelled by the node bidegree sequence $X = (j_m, k_m)_{m\in[M]}$, the joint probability $p = \mathbb{P}[w_m \in \mathcal{N}_{j_m k_m}, m \in [M], g | v \in \mathcal{N}_{jk}, X]$ conditioned on X converges with high probability as $N \to \infty$:

$$p \xrightarrow{\mathrm{P}} \prod_{m\in[M],\ \textit{out-edge}} P_{k_m|j_m} Q_{j_m|k_{m'}} \prod_{m\in[M],\ \textit{in-edge}} P_{j_m|k_m} Q_{k_m|j_{m'}},$$

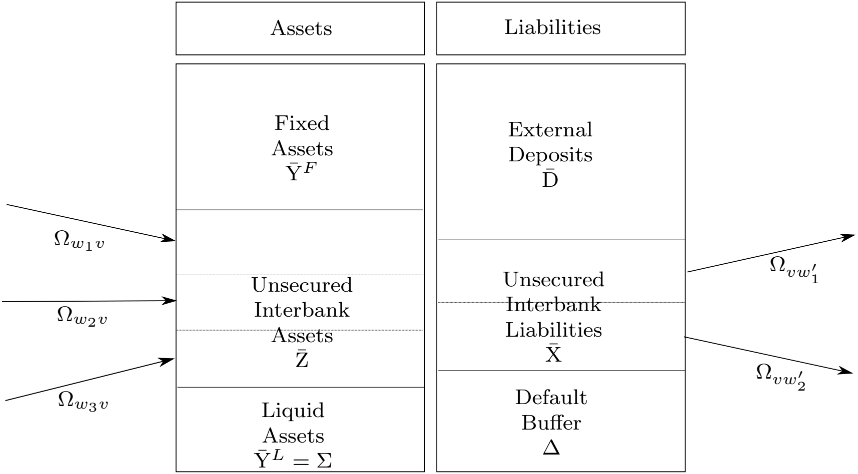
$$p = o(1), \qquad \textit{if } g \textit{ has cycles}.$$

Liquidity Cascade Models

Concerned with interbank liquidity rather than defaults.

1. Large fractions of bank liabilities are either insured deposits or uninsured wholesale funding (e.g. money markets).
2. Wholesale funding is prone to run ("rollover risk"); insured deposits tend to be free of rollover risk.
3. To guard against runs and other contingencies, banks keep reserves of liquid securities such as cash, treasury bills, Fed Reserve bonds, etc.
4. Liquid assets can be used in several ways to deal with liquidity demands, e.g. as collateral for repo borrowing.
5. Let $\bar{Y}^L$ denote such liquid assets.
6. Write $\bar{Y}^L = (\bar{\Sigma})^+$: as long as $\bar{\Sigma}$ is positive, it works as a liquidity buffer.
7. Let the remaining assets and liabilities be as before.

Illiquidity Cascades: Balance Sheets



Gai-Kapadia 2010 Liquidity Cascade Model

"Liquidity Hoarding, Network Externalities, and Interbank Market Collapse"

1. Designed to explain the dramatic shrinking of the interbank lending market in 2007/2008.
2. This occurred seemingly without regard to counterparty defaults.
3. They explain this event as precautionary hoarding of interbank lending by banks concerned about their own liquidity buffer, and the possibility of other banks' illiquidity.

Illiquidity Cascade: Gai-Kapadia 2010b

1. At time 0, some banks experience deposit withdrawals that deplete their liquidity buffer $\Sigma_v := \mathrm{Y}_v^L$ (allowing it to go negative).
2. Bank v with $\Sigma_v \leq 0$ reacts by hoarding liquidity; its debtor banks $w \in \mathcal{N}_v^+$ each receive a liquidity shock.
3. Under 100% hoarding, cascade mapping at step n is

$$\Sigma_v^{(n)} = \Sigma_v^{(0)} - \sum_{w \in \mathcal{N}_v^+} \bar{\Omega}_{vw}(1 - \tilde{h}(\Sigma_w^{(n-1)}/\bar{\mathrm{Z}}_w))$$

4. Formally identical to GK 2010 Default Cascade under interchange of assets and liabilities.

Generalized Liquidity Cascade

1. As in GK 2010b, each bank keeps "cash" (a "first line" reserve of liquid external assets) $\bar{\mathrm{Y}}^L = (\bar{\Sigma})^+$ to absorb liquidity shocks.
2. Stress buffer Σ is kept positive during normal business.
3. When the stress buffer becomes non-positive, i.e. bank is "stressed", bank meets further withdrawals by liquidating first interbank assets $\bar{\mathrm{Z}}$ (i.e. while bank is "stressed"), and finally the illiquid fixed assets $\bar{\mathrm{Y}}^F$ (when bank is "illiquid").
4. A fictitious sink bank 0 represents external agents that borrow amounts $\bar{\Omega}_{0v}$ with equal liquidation priority as interbank assets: $\bar{\mathrm{Z}}_v = \sum_{w=0}^{N} \bar{\Omega}_{wv}$.
5. Unlike GK 2010b, bank only liquidates interbank assets incrementally.

Generalized Liquidity Cascade

1. Banks' balance sheets are given by notional amounts $(\bar{\mathrm{Y}}^F, \bar{\mathrm{Z}}, \bar{\mathrm{Y}}^L, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\mathrm{E}}, \bar{\Omega})$.
2. At the onset of the liquidity crisis, all banks are hit by withdrawal shocks $\Delta \mathrm{D}_v$ that reduce the initial stress buffers $\Sigma_v^{(0)} = \bar{\mathrm{Y}}_v^L - \Delta \mathrm{D}_v$ of at least some banks to below zero, making them stressed.
3. Stressed banks then liquidate assets first from $\bar{\mathrm{Z}}$, inflicting additional liquidity shocks to their debtor banks' liabilities.
4. A stressed bank that has depleted all of $\bar{\mathrm{Z}}$ will be called "illiquid", and must sell external fixed assets $\bar{\mathrm{Y}}^F$ in order to survive.

Generalized Liquidity Cascade

1. Some thought reveals that this model is precisely equivalent to the extended EN 2001 model.
2. The role of assets and liabilities, and stress and default buffers, are interchanged: $\bar{\mathrm{Y}}^F \leftrightarrow \bar{\mathrm{D}}, \bar{\mathrm{Z}} \leftrightarrow \bar{\mathrm{X}}, \bar{\mathrm{Y}}^F \leftrightarrow \bar{\mathrm{E}}$, $\Delta \leftrightarrow \Sigma$.
3. GK 2010b model arises by replacing h by $\tilde{h}$.
4. S. K. Lee 2013 model arises by taking $\bar{\mathrm{Y}}^L_v = 0$, which also has the effect of making all the banks initially stressed since the initial stress buffers are $\Sigma^{(0)}_v = -\Delta \mathrm{D}_v \le 0$.

Asset Fire Sale Cascades

(c.f. Cifuentes et al 2005 and Caccioli et al 2012)

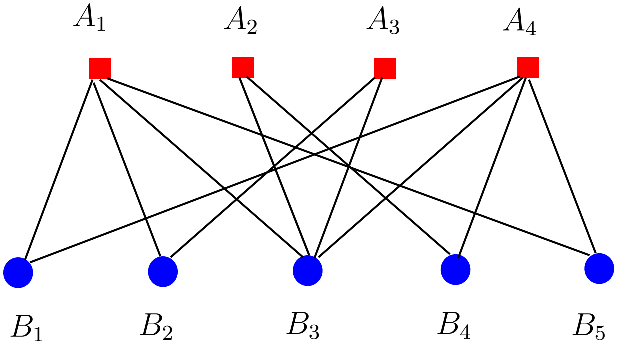


Figure: A bipartite graph with 5 banks (blue nodes) co-owning 4 assets (red nodes).

Asset Fire Sales

Banks $v \in \mathcal{N} = \{1, 2, \ldots, N\}$, Assets $a \in \mathcal{M} = \{1, 2, \ldots, M\}$. Let $\bar{s}_{av}$ be amount of asset a held by bank v.

On the nth cascade step:

1. When default buffer $\Delta_v^{(n)}$ hits a threshold, v begins to liquidate assets.
2. Amount $s_{av}^{(n)}$ of asset a held by bank v after n cascade steps is determined by $\Delta_v^{(n)}$.
3. The new mark-to-market price is determined by the total amount sold through an inverse demand function

$$p_a^{(n+1)} = d_a^{-1}\Big(\sum_v (\bar{s}_{av} - s_{av}^{(n)})\Big)$$

4. Banks mark-to-market to compute their new buffers $\Delta_v^{(n+1)}$.

Asset Fire Sale Cascades

1. Complex cascades result even with no interbank sector $\bar{\Omega} = 0$.
2. Each blue node v is governed by a buffer variable $\Delta_v^{(n)}$
3. Each red node a is governed its price $p_a^{(n)}$, which can be considered as a buffer variable.
4. One buffer per node!
5. Global cascades can start either in banks or in assets: once it starts it doesn't matter much where it started.

Extensions of the Watts/GK Model

1. **General degree distributions:** Poisson random graph model fails to capture most features of real world social networks. Easy to analyze Watts with arbitrary P_k.
2. **Mixtures of directed and undirected edges:** Percolation results can be proved in networks with both directed and undirected edges and arbitrary two-point correlations. Extending to Watts is easy.
3. **Assortative graphs:** For general assortativity, with non-independent edge-type distributions Q_{kj} limiting default probabilities are fixed points of a vector valued cascade mapping $G : \mathbb{R}^{\mathcal{K}} \to \mathbb{R}^{\mathcal{K}}$.
4. **Random edge weights:** Even purely deterministic dependence between link-strength and edge-degree requires analysis of random link weights. Then, p_k^∞ are fixed points of another vector-valued cascade mapping $G : \mathbb{R}^{\mathcal{K}} \to \mathbb{R}^{\mathcal{K}}$.

Random Financial Network (RFN)...

- ...is a random object representing the possible states of the financial network at an instant in time.
- Base level, the *skeleton* is a random directed graph $(\mathcal{N}, \mathcal{E})$ whose nodes $\mathcal{N}$ represent "banks" and whose edges represent the presence of a non-negligible "interbank exposure" between a debtor bank and its creditor bank.
- Conditioned on a realization of the skeleton, the second layer is a collection of random *balance sheets*, i.e. $(\bar{\mathrm{Y}}_v, \bar{\mathrm{Z}}_v, \bar{\mathrm{D}}_v, \bar{\mathrm{X}}_v)$ for each bank.
- Conditioned on a realization of the skeleton and balance sheets, the third level is a collection of random *exposures* $\bar{\Omega}_\ell$ for each link $\ell \in \mathcal{E}$.
- Constraints:

$$\bar{\mathrm{Z}}_v = \sum_w \bar{\Omega}_{wv}, \quad \bar{\mathrm{X}}_v = \sum_w \bar{\Omega}_{vw} \ .$$

The LTI Property

Our dependence hypothesis becomes the following definition.

Definition

[Locally Tree-like Independence] A random financial network (RFN) $(\mathcal{N}, \mathcal{E}, \bar{\Delta}, \bar{\Omega})$ is LTI when:

1. The skeleton graph is an infinite (directed, indirected or mixed) configuration graph $(\mathcal{N}, \mathcal{E})$, with arbitrary node and edge type distributions $\{P, Q\}$.
2. Conditioned on $(\mathcal{N}, \mathcal{E})$, the buffer random variables $\bar{\Delta}_v,\ v \in \mathcal{N}$ and exposure random variables $\bar{\Omega}_\ell,\ \ell \in \mathcal{E}$ form a mutually independent collection. Moreover, the buffer distribution of $\bar{\Delta}_v$ depends only on the type τ_v of v and the exposure distribution of $\bar{\Omega}_\ell$ depends only on the type τ_ℓ of ℓ.

LTI Dependence Structure

1. Probability space $(\Omega', \mathcal{F}, \mathbb{P})$, where

$$\mathcal{F} = \mathcal{G} \vee \mathcal{F}_\Delta \vee \mathcal{F}_\Omega.$$

2. Dependence of $\bar{\Delta}_v$ only on the type of the node v: it holds that there are Borel functions D_{jk} such that

$$\mathbb{E}[\bar{\Delta}_v \leq x | \mathcal{F} \setminus \sigma(\bar{\Delta}_v), v \in \mathcal{N}_{jk}] = D_{jk}(x) \tag{9}$$

3. In exactly the same way, for $\bar{\Omega}$ it holds that there are Borel functions W_{kj} such that

$$\mathbb{E}[\bar{\Omega}_\ell \leq x | \mathcal{F} \setminus \sigma(\bar{\Omega}_\ell), \ell \in \mathcal{E}_{kj}] = W_{kj}(x) \tag{10}$$

GK Assumptions

1. Banks have limited liability and receive a zero recovery of defaulted interbank liabilities.
2. The skeleton graph is a directed ACG $(\mathcal{N}, \mathcal{E})$ with $\{P_{jk}, Q_{kj}\}$ with a finite set of possible degrees $\mathcal{K} = \{0, 1, \ldots, K\}$ and mean degree $z = \sum_{jk} kP_{jk}$.
3. Conditionally on $(\mathcal{N}, \mathcal{E})$, banks' capital buffers $\bar{\Delta}_v$ are a collection of independent non-negative random variables with

$$\mathbb{P}[\bar{\Delta}_v \leq x | v \in \mathcal{N}_{jk}] = D_{jk}(x), \; x \geq 0 \; . \tag{11}$$

GK Assumptions (ctd)

4. Each interbank exposure $\bar{\Omega}_\ell$ depends randomly on its edge type (k_ℓ, j_ℓ). Conditionally on the skeleton, they form a collection of independent positive random variables, independent as well from the default buffers $\bar{\Delta}_v$. Their cumulative distribution functions (CDFs) and probability distribution functions (PDFs) are

$$
\begin{aligned}
W_{kj}(x) &= \mathbb{P}[\bar{\Omega}_\ell \le x | \ell \in \mathcal{E}_{kj}], \\
w_{kj}(x) &= W'_{kj}(x) ,
\end{aligned}
\tag{12}
$$

   with $W_{kj}(0) = 0$.

5. The remaining balance sheet quantities are freely specified.

GK Cascade Mapping Theorem

Proposition

Consider the LTI sequence of GK financial networks $(N, P, Q, \bar{\Delta}, \bar{\Omega})$*. Let* $p^{(0)}_{jk} = D_{jk}(0)$ *and* $\pi^{(0)}_{k'} = \mathbb{P}[w \in \mathcal{D}_0 | k_w = k']$*. Then the following formulas hold with high probability as* $N \to \infty$*:...*

Proposition

1. $\tilde{\pi}_j^{(0)} = \mathbb{P}[w \in \mathcal{D}_{n-1} | w \in \mathcal{N}_v^-,\ v \in \mathcal{N}_{jk}] = \sum_{k'} \pi_k^{(0)} Q_{k'|j}$
2. *For any* $n = 1, 2, \ldots,$ *the quantities* $\tilde{\pi}_j^{(n-1)}, p_{jk}^{(n)}, \pi_k^{(n)}$ *satisfy*

$$p_{jk}^{(n)} = \langle D_{jk}, (\widetilde{w}_j^{(n-1)})^{\circledast j} \rangle \ , \tag{13}$$

$$\pi_k^{(n)} = \sum_{j'} p_{j'k}^{(n)} \ P_{j'|k} \tag{14}$$

   where the PDFs $\widetilde{w}_j^{(n-1)}(x)$ *are given by* (**??**).
3. *The new probabilities* $\vec{\pi}^{(n)}$ *are a vector valued function* $G(\vec{\pi}^{(n-1)})$ *which is explicit in terms of the specification* $(N, P, Q, \bar{\Delta}, \bar{\Omega})$.
4. *The cascade mapping* G *maps* $[0,1]^{K+1}$ *onto itself, and is monotonic. Since* $\pi^{(0)} = G(0)$, *the sequence* $\pi^{(n)}$ *converges to the least fixed point* $\vec{\pi}^* \in [0,1]^{K+1}$, *that is*

Experiment 1: Benchmark Specification

1. The skeleton graph comprises $N = 1000$ banks taken from the Poisson random directed graph model with mean in and out degree z, and thus
   $P = \text{Bin}(N, z/(N-1)) \times \text{Bin}(N, z/(N-1))$ and $Q = Q^+Q^-$.
2. Capital buffers and assets are identical across banks, with $\Delta_v = 4\%$ and $Z_v = 20\%$.
3. Exposures are equal across the debtors of each bank, and so $\Omega_{wv} = \frac{20}{j_v}$.
4. Monte Carlo simulations were performed with $\text{Nsim} = 1000$.

Experiment 1 Results: Mean Cascade Size

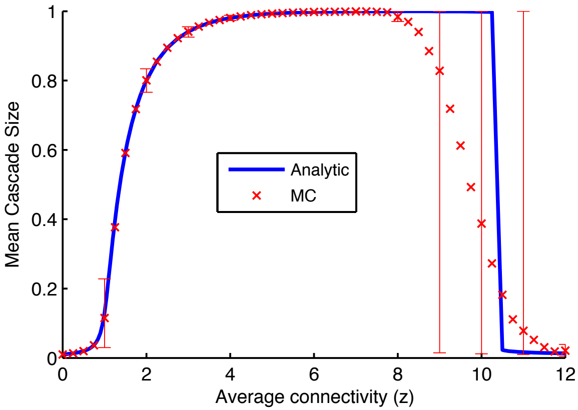


Figure: The mean cascade size in the benchmark GK model, as a function of z. The solid curve shows the analytic fixed point probability starting from a uniform seed density of $d_{k,0} = 10^{-2}$. Crosses show the Monte Carlo simulation mean with error bars, when the initial seed is a random set of 10 banks.

Experiment 1 Results: Cascade Frequency

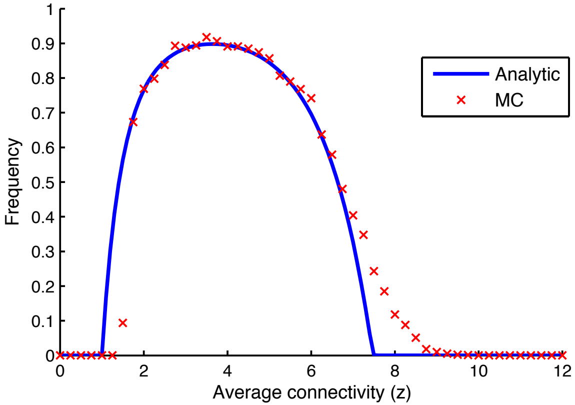


Figure: The frequency of global cascades in the benchmark GK model, as a function of z. The solid curve shows the analytic frequency starting from a uniform seed density of $d_{k,0} = 10^{-3}$. Crosses show the MC simulation results, with single initial seed.

Frequency and Size of Global Cascades

How the frequency of global cascades in large random networks is related to extended in-component of the giant vulnerable cluster. We define:

- *vulnerable* directed edge: edge whose weight is sufficient to exceed the default buffer of its downstream node.
- $\mathcal{E}_V \subset \mathcal{E}$, the set of vulnerable directed edges;
- $\mathcal{E}_s$, the largest strongly connected set of vulnerable edges (the *giant vulnerable cluster* of $\mathcal{E}_V$);
- $\mathcal{E}_i$ and $\mathcal{E}_o$, the *in-component* and *out-component* of the giant vulnerable cluster.
- $1 - b_k := \mathbb{P}[\ell \in \mathcal{E}_i | k_\ell = k]$, a conditional probability of an edge being in $\mathcal{E}_i$;
- $a_{k,jk'} = \mathbb{P}[\bar{\Delta}_v \leq \bar{\Omega}_{wv} | \ell \in \mathcal{E}_v^-, k_\ell = k, v \in \mathcal{N}_{jk'}]$, the conditional probability of an edge being vulnerable.

Zoology of Components of Directed Graphs

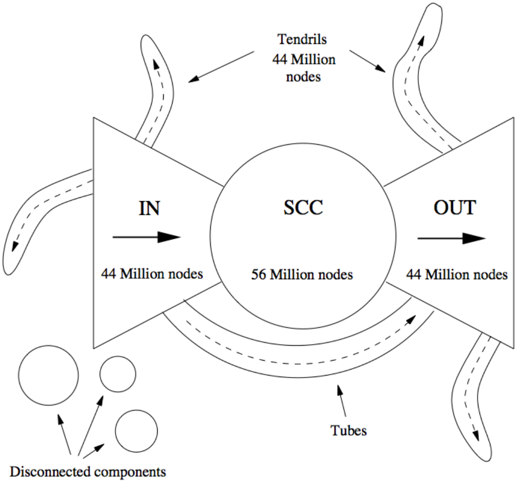


Figure: The connected components of the World Wide Web in 1999. (Source: Broder et al 2000.)

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