Trading Strategies Generated by Lyapunov Functions *

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March 30, 2016

Dedicated to Dr. E. Robert Fernholz on the occasion of his 75th Birthday

Abstract

Functional portfolio generation, initiated by E.R. Fernholz almost twenty years ago, is a methodology for constructing trading strategies with controlled behavior. It is based on very weak and descriptive assumptions on the covariation structure of the underlying market model, and needs no estimation of model parameters. In this paper, the corresponding generating functions G are interpreted as Lyapunov functions for the vector process $\mu(\cdot)$ of market weights; that is, via the property that $G(\mu(\cdot))$ is a supermartingale under an appropriate change of measure. This point of view unifies, generalizes, and simplifies several existing results, and allows the formulation of conditions under which it is possible to outperform the market portfolio over appropriate time-horizons. From a probabilistic point of view, the present paper yields results concerning the interplay of stochastic discount factors and concave transformations of semimartingales on compact domains.

Keywords and Phrases: Trading strategies, functional generation, outperformance, relative arbitrage, regular and Lyapunov functions, concavity, semimartingale property, deflators.

AMS 2000 Subject Classifications: 60G44, 60H05, 60H30, 91G10, 93D30.

1 Introduction

Back in 1999, E.R. Fernholz introduced a construction that was both remarkable and remarkably easy to establish. He showed that for a certain class of so-called "functionally-generated" portfolios, it is possible to express the wealth these portfolios generate, discounted by (that is, denominated in terms of) the total market capitalization, solely in terms of the individual companies' *market weights* – and to do so in a pathwise manner, that *does not involve stochastic integration.* This fact can be proved by a somewhat determined application of Itô's rule. Once the result is known, its proof becomes a moderate exercise in stochastic calculus.

*We are grateful to Robert Fernholz for initiating this line of research and for encouraging us to think about the issues studied here. Many discussions with Kostas Kardaras helped us sharpen our thoughts. We are also deeply indebted to Adrian Banner, Christa Cuchiero, Freddy Delbaen, David Hobson, Tomoyuki Ichiba, Philip Protter, Mathieu Rosenbaum, Walter Schachermayer, Konrad Swanepoel, Kangjia'Nan Xie, and Hao Xing for helpful comments, and Alexander Vervuurt and Minghan Yan for their detailed reading and suggestions on successive versions of this paper. I.K. acknowledges the support of the National Science Foundation under grant NSF-DMS-14-05210. J.R. acknowledges generous support from the Oxford-Man Institute of Quantitative Finance, University of Oxford.

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The discovery paved the way for finding simple and very general structural conditions on *large* equity markets – that involve more than one stock, and typically thousands – under which it is possible strictly to outperform the market portfolio. Put a little differently: conditions under which strong relative arbitrage with respect to the market portfolio is possible, at least over sufficiently long time-horizons. Fernholz (1999, 2001, 2002) showed also how to implement this strong relative arbitrage, or "outperformance," using portfolios that can be constructed solely in terms of observable quantities, and without any need for estimation or optimization. Pal and Wong (2015) related functional generation to optimal transport in discrete time.

Although well-known, celebrated, and quite easy to prove, Fernholz's construction has been viewed over the past 15 years as somewhat "mysterious." In this paper we hope to help make the result a bit more celebrated and a bit less mysterious, via an interpretation of portfolio-generating functions G as Lyapunov functions for the vector process $\mu(\cdot)$ of relative market weights. Namely, via the property that $G(\mu(\cdot))$ is a supermartingale under an appropriate change of measure; see Remark 3.3 for elaboration. We generalize this functional generation from portfolios to trading strategies, as well as to situations where some, but not all, of the market weights can vanish; along the way we simplify the underlying arguments considerably, and answer an old question of Fernholz (2002), Problem 4.2.3. Conditions for strong outperformance of the market over appropriate time horizons become extremely simple via this interpretation, as do the strategies that implement such outperformance and the accompanying proofs that establish such results; see Theorems 5.1 and 5.2.

We have cast all our results in the framework of continuous semimartingales for the market weights; this seems to us a very good compromise between generality on the one hand, and conciseness, unity and readability on the other. The reader will easily decide which of the results can be extended to general semimartingales, and which cannot.

Here is an outline of the paper. Section 2 presents the market model and recalls the financial concepts of trading strategies, outperformance, and deflators. Section 3 then introduces the notions of regular and Lyapunov functions. Section 4 discusses how such functions generate trading strategies, and Section 5 uses these observations to formulate conditions that guarantee trading strategies which outperform the market over sufficiently long time horizons. Section 6 contains several relevant examples for regular and Lyapunov functions and the corresponding generated strategies. Section 7 proves that concave functions satisfying certain additional assumptions are indeed Lyapunov and provides counterexamples if those additional assumptions are not satisfied. Finally, Section 8 concludes.

2 The setup

2.1 Market model

On a given probability space $(\Omega, \mathscr{F}, \mathsf{P})$ endowed with a right-continuous filtration $\mathfrak{F} = (\mathscr{F}(t))_{t\geq 0}$ that satisfies $\mathscr{F}(0) = \{\emptyset, \Omega\}$ mod. P, we consider a vector process $S(\cdot) = (S_1(\cdot), \cdots, S_d(\cdot))'$ of continuous, non-negative semimartingales with $S_1(0) > 0, \cdots, S_d(0) > 0$ and

$$\Sigma(t) := S_1(t) + \cdots + S_d(t) > 0, \qquad t \geq 0. \tag{2.1}$$

We interpret these processes as the capitalizations of a fixed number $d \geq 2$ of companies in an equity market. An individual company's capitalization $S_i(\cdot)$ is allowed to vanish; but the total capitalization $\Sigma(\cdot)$ of the equity market is not. Throughout this paper we study trading strategies that only invest in these d assets, and abstain from introducing a money market explicitly: the financial market of available investment opportunities is represented here by the d–dimensional continuous semimartingale $S(\cdot)$.

Having introduced these quantities, we now define the vector process $\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_d(\cdot))'$ that consists of the various companies' relative *market weights*

$$\mu_i(t) := \frac{S_i(t)}{\Sigma(t)} = \frac{S_i(t)}{S_1(t) + \cdots + S_d(t)}, \qquad t \geq 0 \tag{2.2}$$

for each $i = 1, \cdots, d$. These processes are continuous, non-negative semimartingales in their own right; each of them takes values in the unit interval [0,1] and they satisfy $\mu_1(\cdot) + \cdots + \mu_d(\cdot) \equiv 1$. In other words, the vector process $\mu(\cdot)$ takes values in the lateral face $\mathbf{\Delta}^d$ of the unit simplex in $\mathbb{R}^d$. We are using throughout the notation

$$\mathbf{\Delta}^d := \Big\{ \big(x_1, \cdots, x_d\big)' \in [0,1]^d : \sum_{i=1}^d x_i = 1 \Big\}, \qquad \mathbf{\Delta}^d_+ := \mathbf{\Delta}^d \cap (0,1)^d \tag{2.3}$$

and note that, by assumption, $\mu(0) \in \mathbf{\Delta}^d_+$.

An important special case of the above setup arises, when each semimartingale $S_i(\cdot)$ is strictly positive; equivalently, when the process $\mu(\cdot)$ takes values in $\mathbf{\Delta}^d_+$, that is,

$$\mathsf{P}\big(\mu(t) \in \mathbf{\Delta}^d_+, \ \forall \ t \geq 0\big) = 1. \tag{2.4}$$

2.2 Trading strategies

Let $X(\cdot) = \big(X_1(\cdot), \cdots, X_d(\cdot)\big)'$ denote a generic $[0,\infty)^d$–valued continuous semimartingale. For the purposes of this section, $X(\cdot)$ will stand for either the vector process $S(\cdot)$ of capitalizations, or for the vector process $\mu(\cdot)$ of market weights. We consider a predictable process $\vartheta(\cdot) = \big(\vartheta_1(\cdot), \cdots, \vartheta_d(\cdot)\big)'$ with values in $\mathbb{R}^d$, and interpret $\vartheta_i(t)$ as the number of shares held at time $t \geq 0$ in the stock of company $i = 1, \cdots, d$. Then the total *value*, or "wealth," of this investment in a market whose price processes are given by the vector process $X(\cdot)$, is

$$V^\vartheta(\cdot\,; X) := \sum_{i=1}^d \vartheta_i(\cdot) X_i(\cdot). \tag{2.5}$$

Definition 2.1 (Trading strategies)**.** Suppose that the $\mathbb{R}^d$–valued, predictable process $\vartheta(\cdot)$ is integrable with respect to the continuous semimartingale $X(\cdot)$; and write $\vartheta(\cdot) \in \mathscr{L}(X)$ to express this. We shall say that such $\vartheta(\cdot) \in \mathscr{L}(X)$ is a *trading strategy* with respect to $X(\cdot)$, if the so-called "self-financibility" condition

$$V^\vartheta(\cdot\,; X) - V^\vartheta(0; X) = \int_0^\cdot \big\langle \vartheta(t), \mathrm{d}X(t) \big\rangle \tag{2.6}$$

is satisfied. We shall denote by $\mathscr{T}(X)$ the collection of all such trading strategies. □

Remark 2.2 (On notation and interpretation). Here and in what follows, we use for any fixed $T \geq 0$ the notation on the right-hand side of (2.6), namely

$$\int_0^T \big\langle \vartheta(t), \mathrm{d}X(t) \big\rangle = \int_0^T \sum_{i=1}^d \vartheta_i(t) \mathrm{d}X_i(t),$$

as a short-hand for vector stochastic integration. This quantity gives the "gains-from-trade" realized over the interval $[0,T]$ (gains, if it is positive; losses, if it is negative). The self-financibility requirement of (2.6) posits that these "gains" account for the entire change in the value generated by the trading strategy $\vartheta(\cdot)$ between the start $t = 0$ and the end $t = T$ of the time-interval $[0,T]$: there is no infusion of funds, and neither are there transaction or other fees. □

The following result can be proved via a somewhat determined application of Itô's rule. It formalizes the intuitive idea that the concept of trading strategy should not depend on the manner in which prices or capitalizations are quoted. We refer to Proposition 1 in Geman et al. (1995) for a proof.

Proposition 2.3 (Change of numéraire)**.** *An $\mathbb{R}^d$–valued process $\vartheta(\cdot) = \big(\vartheta_1(\cdot), \cdots, \vartheta_d(\cdot)\big)'$ is a trading strategy with respect to the $\mathbb{R}^d$–valued semimartingale $S(\cdot)$, if and only if it is a trading strategy with respect to the $\mathbb{R}^d$–valued semimartingale $\mu(\cdot)$ given in* (2.2)*. In particular, $\mathscr{T}(\mathcal{S}) = \mathscr{T}(\mu)$; and in this case, we have $V^\vartheta(\cdot\,; S) = \Sigma(\cdot)V^\vartheta(\cdot\,; \mu)$.*

Suppose we are given an element $\vartheta(\cdot) = (\vartheta_1(\cdot), \cdots, \vartheta_d(\cdot))'$ in the space $\mathscr{L}(\mu)$ of predictable processes which are integrable with respect to the continuous vector semimartingale $\mu(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_d(\cdot)\big)'$ of (2.2). Let us consider the quantity

$$Q^\vartheta(T;\mu) \,:=\, V^\vartheta(T;\mu) - V^\vartheta(0;\mu) - \int_0^T \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle, \qquad T \geq 0, \tag{2.7}$$

which measures the "defect of self-financibility" of this process $\vartheta(\cdot)$ relative to $\mu(\cdot)$ over the time-horizon $[0,T]$. If $Q^\vartheta(\cdot\,;\mu) \equiv 0$ fails, the process $\vartheta(\cdot) \in \mathscr{L}(\mu)$ is not a trading strategy with respect to $\mu(\cdot)$. How do we modify it then, in order to turn it into a trading strategy? Our next result describes a way, which essentially adjusts each component of $\vartheta(\cdot)$ by the defect of self-financibility.

Proposition 2.4 (From integrands to trading strategies)**.** *For a given process $\vartheta(\cdot) \in \mathscr{L}(\mu)$, a given real constant $\boldsymbol{C} \in \mathbb{R}$, and with the notation of* (2.7)*, we introduce the processes*

$$\varphi_i(t) \,:=\, \vartheta_i(t) - Q^\vartheta(t\,;\mu) + \boldsymbol{C}, \qquad i = 1, \cdots, d\,, \quad t \geq 0\,. \tag{2.8}$$

The resulting $\mathbb{R}^n$–valued, predictable process $\varphi(\cdot) = \big(\varphi_1(\cdot), \cdots, \varphi_d(\cdot)\big)'$ is then a trading strategy with respect to the vector process $\mu(\cdot)$ of market weights; to wit, $\varphi(\cdot) \in \mathscr{T}(\mu)$. Moreover, the value process $V^\varphi(\cdot\,;\mu) = \sum_{i=1}^d \varphi_i(\cdot)\mu_i(\cdot)$ of this trading strategy satisfies

$$V^\varphi(\cdot\,;\mu) = V^\vartheta(0\,;\mu) + \boldsymbol{C} + \int_0^\cdot \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle = V^\varphi(0\,;\mu) + \int_0^\cdot \big\langle \varphi(t), \mathrm{d}\mu(t) \big\rangle. \tag{2.9}$$

Proof. Consider the vector process $\widetilde{\vartheta}(\cdot) = \big(\widetilde{\vartheta}_1(\cdot), \cdots, \widetilde{\vartheta}_d(\cdot)\big)'$ with components $\widetilde{\vartheta}_i(\cdot) = \boldsymbol{C} - Q^\vartheta(\cdot\,;\mu)$ for each $i = 1, \cdots, d$. Then $\widetilde{\vartheta}(\cdot)$ is predictable, since $V^\vartheta(\cdot\,;\mu)$ and $\int_0^\cdot \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle$ are. Moreover, Lemma 4.13 in Shiryaev and Cherny (2002) yields $\widetilde{\vartheta}(\cdot) \in \mathscr{L}(\mu)$; thus, we have also $\varphi(\cdot) = \vartheta(\cdot) + \widetilde{\vartheta}(\cdot) \in \mathscr{L}(\mu)$. Furthermore,

$$\int_0^\cdot \big\langle \widetilde{\vartheta}(t), \mathrm{d}\mu(t) \big\rangle \equiv 0$$

holds thanks to $\sum_{i=1}^d \mu_i(\cdot) \equiv 1$, and therefore so does

$$\int_0^\cdot \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle = \int_0^\cdot \big\langle \varphi(t), \mathrm{d}\mu(t) \big\rangle.$$

Since we also have $\varphi_i(0) = \vartheta_i(0) + \boldsymbol{C}$ for each $i = 1, \cdots, d$, hence $V^\varphi(0;\mu) = V^\vartheta(0;\mu) + \boldsymbol{C}$, we obtain (2.9). This yields that $\varphi(\cdot)$ is indeed a trading strategy, and concludes the proof. $\square$

2.3 Outperforming the market

Let us fix a real number $T > 0$. We say that a given trading strategy $\varphi(\cdot) \in \mathscr{T}(S)$ *outperforms the market over the time-horizon* $[0,T]$, if we have

$$V^\varphi(t;S) \geq 0, \ \ \forall \ t \in [0,T]; \qquad V^\varphi(0;S) = \Sigma(0) \tag{2.10}$$

in the notation of (2.1), along with

$$\mathsf{P}\left(V^\varphi(T;S) \geq \Sigma(T)\right) = 1; \qquad \mathsf{P}\left(V^\varphi(T;S) > \Sigma(T)\right) > 0. \tag{2.11}$$

Whenever a given $\varphi(\cdot) \in \mathscr{T}(S)$ satisfies these conditions, and if in fact the second probability in (2.11) is not just positive but actually equal to 1, that is, if

$$\mathsf{P}\left(V^\varphi(T;S) > \Sigma(T)\right) = 1 \tag{2.12}$$

holds, then we say that this $\varphi(\cdot)$ *outperforms the market strongly* over $[0,T]$.

Remark 2.5 (Change of numéraire). It follows from Proposition 2.3 that the above requirements (2.10)–(2.12) can be cast, respectively, as

$$\begin{gathered} V^\varphi(t;\mu) \geq 0, \ \ \forall \ t \in [0,T]; \qquad V^\varphi(0;\mu) = 1; \\ \mathsf{P}\left(V^\varphi(T;\mu) \geq 1\right) = 1; \qquad \mathsf{P}\left(V^\varphi(T;\mu) > 1\right) > 0 \end{gathered} \tag{2.13}$$

and $\mathsf{P}(V^\varphi(T;\mu) > 1) = 1$. □

We remark that the concept of (strong) outperformance is often also called (strong) relative arbitrage in the literature; see, for example, Fernholz et al. (2005) and Fernholz and Karatzas (2009).

2.4 Deflators

For some of our results we shall need the notion of *deflator* for the vector process $\mu(\cdot)$ of market weights in (2.2). This is any continuous, strictly positive and adapted process $Z(\cdot)$ with $Z(0) = 1$ for which

$$\text{all products} \quad Z(\cdot)\,\mu_i(\cdot), \quad i = 1, \cdots, d \quad \text{are local martingales;} \tag{2.14}$$

thus $Z(\cdot)$ is also a local martingale itself. An apparently stronger condition is that the product

$$Z(\cdot) \int_0^\cdot \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle \quad \text{is a local martingale, for every } \ \vartheta(\cdot) \in \mathscr{L}(\mu). \tag{2.15}$$

Proposition 2.6 (Equivalence of conditions)**.** *The conditions in* (2.14) *and* (2.15) *are equivalent.*

Proof. Let us suppose (2.14) holds; then $Z(\cdot)$ is a local martingale, so there exists a nondecreasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with $\lim_{n\uparrow\infty} \tau_n = \infty$ and the property that $Z(\cdot \wedge \tau_n)$ is a uniformly integrable martingale for each $n \in \mathbb{N}$. The recipe $\mathsf{Q}_n(A) = \mathsf{E}^\mathsf{P}[Z(\tau_n)\mathbf{1}_A]$ for each $A \in \mathscr{F}(\tau_n)$ defines a probability measure on $\mathscr{F}(\tau_n)$, under which $\mu_i(\cdot \wedge \tau_n)$ is a martingale, for each $i = 1, \cdots, d$ and $n \in \mathbb{N}$. However, the "stopped" version $\int_0^{\cdot\wedge\tau_n} \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle$ of the stochastic integral as in (2.15) is then also a Q_n–local martingale for each $n \in \mathbb{N}$; therefore each product $Z(\cdot \wedge \tau_n) \int_0^{\cdot\wedge\tau_n} \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle$ is a P–local martingale, and the property of (2.15) follows. The reverse implication is trivial. □

Remark 2.7 (Equivalent martingale measure for market weights). If a deflator $Z(\cdot)$ exists and is a martingale, then for any real number $T > 0$ we can define a probability measure on $\mathscr{F}(T)$ by $\mathsf{Q}_T(A) = \mathsf{E}^\mathsf{P}[Z(T)\mathbf{1}_A]$, $A \in \mathscr{F}(T)$. Under this measure the market weights $\mu_i(\cdot \wedge T)$, $i = 1, \cdots, d$ are local martingales; thus actual martingales, as they take values in [0,1]. □

Now let us introduce the stopping times

$$\mathscr{D} := \mathscr{D}_1 \wedge \cdots \wedge \mathscr{D}_d, \qquad \mathscr{D}_i := \inf\big\{ t \geq 0 : \mu_i(t) = 0 \big\}. \tag{2.16}$$

Whenever a deflator for the vector $\mu(\cdot)$ of market weights exists, each continuous process $Z(\cdot)\,\mu_i(\cdot)$, being non-negative and a local martingale, is a supermartingale. From this, and from the strict positivity of $Z(\cdot)$, we see that then

$$\mu_i\big(\mathscr{D}_i + u\big) = 0 \text{ holds for all } u \geq 0, \text{ on the event } \big\{\mathscr{D}_i < \infty\big\}.$$

Thus, the vector process $\mu(\cdot)$ of market weights starts life at a point $\mu(0) \in \mathbf{\Delta}_+^d$. It may then – that is, when a deflator exists – begin a "descent" into simplices of successively lower dimensions, possibly all the way up until the time the entire market capitalization concentrates in just one company, that is

$$\mathscr{D}_\star := \inf\big\{ t \geq 0 : \mu_i(t) = 1 \quad \text{for some } i = 1, \cdots, d \big\}. \tag{2.17}$$

3 Regular and Lyapunov functions

For a generic d-dimensional semimartingale $X(\cdot)$ we write $\mathbf{supp}\,(X)$ to denote the support of $X(\cdot)$, that is, the smallest closed set $\mathfrak{S} \subset \mathbb{R}^d$ such that

$$\mathsf{P}(X(t) \in \mathfrak{S},\ \forall\, t \geq 0) = 1.$$

In case of the vector process $\mu(\cdot)$ of market weights in (2.2) we always have $\mathbf{supp}\,(\mu) \subset \mathbf{\Delta}^d$ for the lateral face $\mathbf{\Delta}^d$ of the unit simplex, defined in (2.3).

Definition 3.1 (Regular functions)**.** We say that a continuous function $G : \mathbf{supp}\,(X) \to \mathbb{R}$ is *regular* for the d-dimensional semimartingale $X(\cdot)$ if

(i) there exists a measurable function $DG = \big(D_1 G, \cdots, D_d G\big)' : \mathbf{supp}\,(X) \to \mathbb{R}^d$ such that the process $\boldsymbol{\vartheta}(\cdot) = \big(\boldsymbol{\vartheta}_1(\cdot), \cdots, \boldsymbol{\vartheta}_d(\cdot)\big)'$ with components

$$\boldsymbol{\vartheta}_i(t) := D_i G\big(X(t)\big), \qquad i = 1, \cdots, d\,, \quad t \geq 0 \tag{3.1}$$

is in $\mathscr{L}(X)$; and

(ii) the continuous, adapted process

$$\Gamma^G(T) := G\big(X(0)\big) - G\big(X(T)\big) + \int_0^T \big\langle \boldsymbol{\vartheta}(t), \mathrm{d}X(t) \big\rangle\,, \qquad T \geq 0 \tag{3.2}$$

has finite variation on compact intervals. □

Definition 3.2 (Lyapunov functions)**.** We say that a regular function G as in Definition 3.1 is *a Lyapunov function* for the d-dimensional semimartingale $X(\cdot)$ if, for some function DG as in Definition 3.1, the finite-variation process $\Gamma^G(\cdot)$ of (3.2) is actually non-decreasing. □

Remark 3.3 (Supermartingale properties). Let us suppose that a probability measure Q exists, under which the market weights $\mu_1(\cdot), \cdots, \mu_d(\cdot)$ are (local) martingales. Then, for any given regular function $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$, it is seen from (3.2) that the continuous process

$$G\big(\mu(\cdot)\big) + \Gamma^G(\cdot) = G\big(\mu(0)\big) + \int_0^{\cdot} \sum_{i=1}^d D_i G\big(\mu(t)\big) \mathrm{d}\mu_i(t) \tag{3.3}$$

is a Q–local martingale, provided that G is regular for $\mu(\cdot)$. If, furthermore, this G is actually a Lyapunov function for $\mu(\cdot)$, then it follows that the process $G\big(\mu(\cdot)\big)$ is a Q–local supermartingale – thus in fact a Q–supermartingale, as it is bounded from below due to the continuity of G.

A bit more generally, let us assume now that there exists a deflator $Z(\cdot)$ for the market weight process $\mu(\cdot)$. Then Proposition 2.6 yields that the product $Z(\cdot)\int_0^\cdot \sum_{i=1}^d D_iG\big(\mu(t)\big)\mathrm{d}\mu_i(t)$ is a P–local martingale. If now G is a Lyapunov function for $\mu(\cdot)$, integration by parts shows that the process

$$Z(\cdot)G\big(\mu(\cdot)\big) = G(\mu(0)) + Z(\cdot)\int_0^\cdot \sum_{i=1}^d D_iG\big(\mu(t)\big)\mathrm{d}\mu_i(t) - \int_0^\cdot \Gamma^G(t)\mathrm{d}Z(t) - \int_0^\cdot Z(t)\mathrm{d}\Gamma^G(t)$$

is a P–local supermartingale, thus also a P–supermartingale as it is bounded from below. □

The process $\Gamma^G(\cdot)$ in (3.2) might depend on the choice of DG. For example, consider the situation when each component of $\mu(\cdot)$ is of first finite variation but not constant. Then it is easy to see that different choices of DG lead to different processes $\Gamma^G(\cdot)$ in (3.2). However, if a deflator for $\mu(\cdot)$ exists then we get the following uniqueness result.

Proposition 3.4 (Uniqueness in (3.2))**.** *If a function $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$ is regular for the vector process $\mu(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_d(\cdot)\big)'$ of market weights, and if a deflator for the process $\mu(\cdot)$ exists, then the continuous, adapted, finite-variation process $\Gamma^G(\cdot)$ of* (3.2) *does not depend on the choice of DG.*

Proof. Suppose that there exist a deflator $Z(\cdot)$ for the vector process $\mu(\cdot)$ of market weights; as well as two functions DG, $\widetilde{DG}$ as in Definition 3.1, with corresponding processes $\boldsymbol{\vartheta}(\cdot)$, $\widetilde{\boldsymbol{\vartheta}}(\cdot)$ in (3.1) and $\Gamma^G(\cdot)$, $\widetilde{\Gamma}^G(\cdot)$ in (3.2). We need to show $\Gamma^G(\cdot) = \widetilde{\Gamma}^G(\cdot)$, or equivalently

$$\Upsilon(\cdot) := \int_0^\cdot \big\langle \boldsymbol{\phi}(t), \mathrm{d}\mu(t)\big\rangle \equiv 0\,, \qquad \text{where} \quad \boldsymbol{\phi}(\cdot) := \boldsymbol{\vartheta}(\cdot) - \widetilde{\boldsymbol{\vartheta}}(\cdot)\,.$$

Now, on the strength of (3.2), this continuous process $\Upsilon(\cdot)$ is of finite variation on compact intervals, so the product rule gives

$$\int_0^\cdot Z(t)\,\mathrm{d}\Upsilon(t) = Z(\cdot)\,\Upsilon(\cdot) - \int_0^\cdot \Upsilon(t)\,\mathrm{d}Z(t) \; = Z(\cdot)\int_0^\cdot \big\langle \boldsymbol{\phi}(t), \mathrm{d}\mu(t)\big\rangle - \int_0^\cdot \Upsilon(t)\,\mathrm{d}Z(t).$$

As a consequence of Proposition 2.6, the process on the right-hand side is a local martingale; on the other hand, the process $\int_0^\cdot Z(t)\,\mathrm{d}\Upsilon(t)$ is continuous and of finite variation on compact intervals, and thus identically equal to zero. The strict positivity of $Z(\cdot)$ gives now $\Upsilon(\cdot) \equiv 0$. □

3.1 Sufficient conditions for a function to be regular or Lyapunov

Example 3.5 (The smooth case). Suppose that a given continuous function $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$ can be extended to a twice continuously differentiable function on some open set $\mathcal{U} \subset \mathbb{R}^d$ with

$$\mathsf{P}\big(\mu(t) \in \mathcal{U},\ \forall\ t \geq 0\big) = 1.$$

Elementary stochastic calculus expresses then the process of (3.2) as

$$\Gamma^G(\cdot) = -\frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \int_0^\cdot D^2_{ij}G\big(\mu(t)\big)\,\mathrm{d}\big\langle \mu_i, \mu_j\big\rangle(t) \tag{3.4}$$

with the notation $D_iG = \partial G/\partial x_i$, $D^2_{ij}G = \partial^2 G/(\partial x_i \partial x_j)$. (See Propositions 4 and 6 in Bouleau (1984) for slight generalizations of this result.) Therefore, such a function G is regular; if it is also concave, then the process $\Gamma^G(\cdot)$ in (3.4) is non-decreasing, and G becomes a Lyapunov function. □

Quite a bit more generally, we have the following results.

Theorem 3.6 (The concave case)**.** *A given continuous function $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$ is a Lyapunov function for the vector process $\mu(\cdot)$ of market weights if one of the following conditions holds:*

(i) *G can be extended to a continuous, concave function on $\mathbf{\Delta}^d_+$, and* (2.4) *holds.*

(ii) *G can be extended to a continuous, concave function on the set*

$$\mathbf{\Delta}^d_e := \Big\{ \big(x_1, \cdots, x_d\big)' \in \mathbb{R}^d : \sum_{i=1}^{d} x_i = 1 \Big\}. \tag{3.5}$$

(iii) *G can be extended to a continuous, concave function on the set $\mathbf{\Delta}^d$ of* (2.3)*, and there exists a deflator for the vector process $\mu(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_d(\cdot)\big)'$ of market weights.*

We refer to Section 7 for a review of some basic notions from convexity, and for the proof of Theorem 3.6. The existence of a deflator is essential for the sufficiency in Theorem 3.6(iii) (that is, whenever the market-weight process $\mu(\cdot)$ is "allowed to hit a boundary"), as illustrated by Example 7.3 below.

3.2 Rank-based regular and Lyapunov functions

Let us introduce the "rank operator" $\mathfrak{R}$, namely, the mapping $\mathbf{\Delta}^d \ni (x_1, \cdots, x_d) \mapsto \mathfrak{R}(x_1, \cdots, x_d) = (x_{(1)}, \cdots, x_{(d)}) \in \mathbb{W}^d$, where

$$\mathbb{W}^d := \Big\{ \big(x_1, \cdots, x_d\big)' \in \mathbf{\Delta}^d : 1 \ge x_1 \ge x_2 \ge \cdots \ge x_{d-1} \ge x_d \ge 0 \Big\}. \tag{3.6}$$

We denote by

$$\max_{i=1,\cdots,d} x_i = x_{(1)} \ge x_{(2)} \ge \cdots \ge x_{(d-1)} \ge x_{(d)} = \min_{i=1,\cdots,d} x_i$$

the descending order statistics of the components of the vector $x = (x_1, \cdots, x_d)'$, constructed with a clear rule for breaking ties (say, the lexicographic rule that always favors the smallest "index" $i = 1, \cdots, d$); moreover, for each $x \in \mathbf{\Delta}^d$, we denote by

$$N_\ell(x) := \sum_{i=1}^{d} \mathbf{1}_{x_{(\ell)} = x_i} \tag{3.7}$$

the number of components of the vector $x = (x_1, \cdots, x_d)'$, that coalesce in the given rank $\ell = 1, \cdots, d$.

Finally, we introduce the process of market weights ranked in descending order, namely

$$\boldsymbol{\mu}(t) = \mathfrak{R}(\mu(t)) = \big(\mu_{(1)}(t), \cdots, \mu_{(d)}(t)\big)', \qquad t \ge 0\,. \tag{3.8}$$

We note that $\boldsymbol{\mu}(\cdot)$ can be interpreted again as a market model. However, this rank-based model may fail to admit a deflator, even when the original vector process of market weights $\mu(\cdot)$ admits one. This is due to the appearance, in the dynamics for $\boldsymbol{\mu}(\cdot)$, of local time terms, which correspond to the reflections whenever two or more components of the original process $\mu(\cdot)$ collide with each other; see, for instance, (3.9) below.

Theorem 3.7 (The concave case, continued)**.** *Consider a function $\boldsymbol{G} : \mathbf{supp}\,(\boldsymbol{\mu}) \to \mathbb{R}$. Then $\boldsymbol{G}$ is a Lyapunov function for the vector process $\boldsymbol{\mu}(\cdot)$ in* (3.8)*, if one of the following two conditions holds.*

(i) $\boldsymbol{G}$ *can be extended to a continuous, concave function on* $\boldsymbol{\Delta}^d_+$ *, and* (2.4) *holds; or*

(ii) $\boldsymbol{G}$ *can be extended to a continuous, concave function on* $\boldsymbol{\Delta}^d_e$ *, given in* (3.5).

Under any of these two conditions, the composition $G = \boldsymbol{G} \circ \mathfrak{R}$ *is a regular function for the vector process* $\mu(\cdot)$*. More generally, if* $\boldsymbol{G}$ *is a regular function for* $\boldsymbol{\mu}(\cdot)$*, then* $G = \boldsymbol{G} \circ \mathfrak{R}$ *is a regular function for* $\mu(\cdot)$*.*

We refer again to Section 7 for the proof of Theorem 3.7. A simple modification of Example 7.3 illustrates that a function $\boldsymbol{G}$ can be concave and continuous on $\mathbb{W}^d$ without being regular for $\mu(\cdot)$. Indeed, this can happen even when a deflator for $\mu(\cdot)$ exists, as Example 7.4 illustrates.

Example 3.8 (The smooth case, continued). Example 3.5 has an equivalent formulation for the rank-based case. Assume again that the function $\boldsymbol{G} : \mathbf{supp}\,(\boldsymbol{\mu}) \to \mathbb{R}$ can be extended to a twice continuously differentiable function on some open set $\mathcal{U} \subset \mathbb{R}^d$ with

$$\mathsf{P}\big(\boldsymbol{\mu}(t) \in \mathcal{U},\ \forall\, t \geq 0\big) = 1.$$

Then $\boldsymbol{G}$ is regular for $\boldsymbol{\mu}(\cdot)$. Indeed, as in Example 3.5, applying Itô's formula yields

$$\boldsymbol{G}(\boldsymbol{\mu}(\cdot)) = \boldsymbol{G}(\boldsymbol{\mu}(0)) + \int_0^\cdot \sum_{\ell=1}^d D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\boldsymbol{\mu}_\ell(t) + \frac{1}{2} \sum_{k=1}^d \sum_{\ell=1}^d \int_0^\cdot D^2_{k\ell} \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\langle \boldsymbol{\mu}_k, \boldsymbol{\mu}_\ell \rangle(t)$$

with $D_\ell \boldsymbol{G} = \partial \boldsymbol{G}/\partial x_\ell$, $D^2_{k\ell}\boldsymbol{G} = \partial^2 \boldsymbol{G}/(\partial x_k \partial x_\ell)$, and the regularity of $\boldsymbol{G}$ for $\boldsymbol{\mu}(\cdot)$ follows.

Next, let $\Lambda^{(k,\ell)}(\cdot)$ denote the local time process of the continuous semimartingale $\mu_{(k)}(\cdot) - \mu_{(\ell)}(\cdot) \geq 0$ at the origin, for $1 \leq k < \ell \leq d$. Then with the notation of (3.7), Theorem 2.3 in Banner and Ghomrasni (2008) yields the semimartingale representation for the ranked market weights

$$\begin{aligned} \boldsymbol{\mu}_\ell(\cdot) = \boldsymbol{\mu}_\ell(0) &+ \int_0^\cdot \sum_{i=1}^d \frac{1}{N_\ell(\mu(t))} \mathbf{1}_{\{\mu_{(\ell)}(t) = \mu_i(t)\}} \mathrm{d}\mu_i(t) + \sum_{k=\ell+1}^d \int_0^\cdot \frac{1}{N_\ell(\mu(t))} \mathrm{d}\Lambda^{(\ell,k)}(t) \\ &- \sum_{k=1}^{\ell-1} \int_0^\cdot \frac{1}{N_\ell(\mu(t))} \mathrm{d}\Lambda^{(k,\ell)}(t)\,, \qquad \ell = 1, \cdots, d\,. \end{aligned} \tag{3.9}$$

Thus, we obtain for the function $G = \boldsymbol{G} \circ \mathfrak{R}$ the representations of (3.1)–(3.3), with

$$D_i G(x) = \sum_{\ell=1}^d \frac{1}{N_\ell(x)} D_\ell \boldsymbol{G}\big(\mathfrak{R}(x)\big) \mathbf{1}_{x_{(\ell)} = x_i}\,, \qquad x \in \mathcal{U}\,,\ \ i = 1, \cdots, d\,; \tag{3.10}$$

$$\begin{aligned} \Gamma^G(\cdot) = &-\frac{1}{2} \sum_{k=1}^d \sum_{\ell=1}^d \int_0^\cdot D^2_{k\ell}\boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\langle \boldsymbol{\mu}_k, \boldsymbol{\mu}_\ell \rangle(t) - \sum_{\ell=1}^{d-1} \sum_{k=\ell+1}^d \int_0^\cdot \frac{1}{N_\ell(\mu(t))} D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\Lambda^{(\ell,k)}(t) \\ &+ \sum_{\ell=2}^d \sum_{k=1}^{\ell-1} \int_0^\cdot \frac{1}{N_\ell(\mu(t))} D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\Lambda^{(k,\ell)}(t). \end{aligned} \tag{3.11}$$

In particular, G is indeed regular for $\mu(\cdot)$; this confirms the last statement of Theorem 3.6 in this case.

Let us consider now the special case when the collision local times of order 3 or higher vanish:

$$\Lambda^{(k,\ell)}(\cdot) \equiv 0\,; \qquad 1 \leq k < \ell \leq d\,, \quad \ell \geq k+2\,. \tag{3.12}$$

This will happen, of course, when actual triple collisions never occur. It will also happen when triple- or higher-order collisions *do* occur but are sufficiently "weak," so as not to lead to the accumulation of collision local time; see Ichiba et al. (2011) and Ichiba et al. (2013) for examples of this situation. Under (3.12), only the term corresponding to $k = \ell + 1$ appears in the second summation on the right-hand side of (3.11), and only the term corresponding to $k = \ell - 1$ appears in the third summation. $\square$

Example 3.9 (Regular, but not Lyapunov). Let us consider the function $\boldsymbol{G} : \mathbb{W}^d \to [0,1]$ defined by $\boldsymbol{G}(x) := x_1$. This $\boldsymbol{G}$ is twice continuously differentiable and concave. In particular, as in Example 3.5, $\boldsymbol{G}$ is a Lyapunov function for the process $\boldsymbol{\mu}(\cdot)$ in (3.8).

However, the function $G = \boldsymbol{G} \circ \mathfrak{R}$, which has the representation $G(x) = \max_{i=1,\cdots,d} x_i$ for all $x \in \boldsymbol{\Delta}^d$, is regular for $\mu(\cdot)$, but *typically not Lyapunov.* Indeed, in the notation of Example 3.8, we have $D_1\boldsymbol{G} = 1$, $D_\ell \boldsymbol{G} = 0$ for all $\ell = 2, \cdots, d$, and $D^2_{k\ell}\boldsymbol{G} = 0$ for all $1 \le k, \ell \le d$. Thus, in the notation of Example 3.8, we have

$$D_i G(x) \;=\; \frac{1}{\sum_{j=1}^d \mathbf{1}_{\,x_{(1)}=x_j}}\, \mathbf{1}_{\,x_{(1)}=x_i}\,, \qquad x \in \boldsymbol{\Delta}^d, \quad i = 1, \cdots, d$$

as follows directly from (3.10), and the expression in (3.11) simplifies to

$$\Gamma^G(\cdot) = -\sum_{k=2}^{d} \int_0^{\cdot} \frac{1}{\sum_{i=1}^d \mathbf{1}_{\{\mu_{(1)}(t)=\mu_i(t)\}}}\, \mathrm{d}\Lambda^{(1,k)}(t).$$

Unless the nondecreasing process $\Lambda^{(1,2)}(\cdot)$ is identically equal to zero, the process $\Gamma^G(\cdot)$ is non-increasing. If we now additionally assume the existence of a deflator, then, by Proposition 3.4, the process $\Gamma^G(\cdot)$ does not depend on the choice of DG; thus, $\Gamma^G(\cdot)$ is determined uniquely by the above expression, so G cannot be a Lyapunov function for $\mu(\cdot)$. Example 6.2 below generalizes this setup. $\square$

4 Functionally generated trading strategies

To simplify notation, and when it is clear from the context, we shall write from now on $V^\vartheta(\cdot)$ to denote the value process $V^\vartheta(\cdot\,;\mu)$ given in (2.5) for $X(\cdot) = \mu(\cdot)$. Proposition 2.3 allows us to interpret $V^\vartheta(\cdot) = V^\vartheta(\cdot\,;\mu) = V^\vartheta(\cdot\,;S)/\Sigma(\cdot)$ as the "relative value" of the trading strategy $\vartheta(\cdot) \in \mathscr{T}(S)$ with respect to the market portfolio.

4.1 Additive generation

For any given function $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$ which is regular for the vector process $\mu(\cdot)$ of market weights as in Definition 3.1, we consider the vector $\boldsymbol{\vartheta}(\cdot) = \big(\boldsymbol{\vartheta}_1(\cdot), \cdots, \boldsymbol{\vartheta}_d(\cdot)\big)'$ of processes $\boldsymbol{\vartheta}_i(\cdot) := D_i G\big(X(\cdot)\big)$ in (3.1), as well as the trading strategy $\boldsymbol{\varphi}(\cdot) = \big(\boldsymbol{\varphi}_1(\cdot), \cdots, \boldsymbol{\varphi}_d(\cdot)\big)'$ with components

$$\boldsymbol{\varphi}_i(\cdot) := \boldsymbol{\vartheta}_i(\cdot) - Q^{\boldsymbol{\vartheta}}(\cdot) + \boldsymbol{C}, \qquad i = 1, \cdots, d \tag{4.1}$$

in the manner of (2.8) and (2.7), and with the real constant

$$\boldsymbol{C} := G\big(\mu(0)\big) - \sum_{j=1}^{d} \mu_j(0) D_j G\big(\mu(0)\big). \tag{4.2}$$

Definition 4.1 (Additive functional generation (AFG))**.** We say that the trading strategy $\boldsymbol{\varphi}(\cdot) = \big(\boldsymbol{\varphi}_1(\cdot), \cdots, \boldsymbol{\varphi}_d(\cdot)\big)' \in \mathscr{T}(\mu)$ of (4.1) is *additively generated* by the regular function $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$. $\square$

Remark 4.2 (Non-uniqueness of trading strategies). There might be two different trading strategies $\boldsymbol{\varphi}(\cdot) \neq \widetilde{\boldsymbol{\varphi}}(\cdot)$, both generated additively by the same regular function G. This is because the function DG in Definition 3.1 need not be unique. However, if there exists a deflator for $\mu(\cdot)$, then the process $\Gamma^G(\cdot)$ is uniquely determined by Proposition 3.4, and (4.3) below yields $V^{\boldsymbol{\varphi}}(\cdot) = V^{\widetilde{\boldsymbol{\varphi}}}(\cdot)$. □

Proposition 4.3 (Representation and value of AFG strategies)**.** *The trading strategy* $\boldsymbol{\varphi}(\cdot) = \big(\boldsymbol{\varphi}_1(\cdot), \cdots, \boldsymbol{\varphi}_d(\cdot)\big)'$*, generated additively as in* (4.1) *by a regular function* $G : \mathbf{supp}\,(\mu) \to \mathbb{R}$*, has relative value process*

$$V^{\boldsymbol{\varphi}}(\cdot) = G\big(\mu(\cdot)\big) + \Gamma^G(\cdot), \tag{4.3}$$

and can be represented in the form

$$\boldsymbol{\varphi}_i(\cdot) \;=\; D_i G\big(\mu(\cdot)\big) + \Gamma^G(\cdot) + G\big(\mu(\cdot)\big) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G\big(\mu(\cdot)\big)\,, \qquad i = 1, \cdots, d\,. \tag{4.4}$$

Proof. We substitute from (4.1) and (3.1) into (2.9), and recall (3.2) and (4.2), to obtain

$$V^{\boldsymbol{\varphi}}(\cdot) = \sum_{j=1}^{d} \mu_j(0)\, D_j G\big(\mu(0)\big) + \boldsymbol{C} + \int_0^{\cdot} \big\langle \boldsymbol{\vartheta}(t), \mathrm{d}\mu(t) \big\rangle = G\big(\mu(\cdot)\big) + \Gamma^G(\cdot),$$

that is, (4.3). Using (4.1), (2.7), and (2.9) we also obtain

$$\begin{aligned}
\boldsymbol{\varphi}_i(\cdot) &= D_i G(\mu(\cdot)) - V^{\boldsymbol{\vartheta}}(\cdot) + V^{\boldsymbol{\vartheta}}(0) + \int_0^{\cdot} \big\langle \boldsymbol{\vartheta}(t), \mathrm{d}\mu(t) \big\rangle + \boldsymbol{C} \\
&= D_i G(\mu(\cdot)) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G\big(\mu(\cdot)\big) + V^{\boldsymbol{\varphi}}(\cdot), \qquad i = 1, \cdots, d
\end{aligned}$$

leading to (4.4). □

The expression for $\boldsymbol{\varphi}(\cdot)$ in (4.4) motivates the interpretation of $\boldsymbol{\varphi}(\cdot)$ as "delta hedge" for a given "generating function" G. Indeed, if we interpret DG as the gradient of G, then for each $i = 1, \cdots, d$ and $t \geq 0$ the quantity $\boldsymbol{\varphi}_i(t)$ is exactly the "derivative" $D_i G\big(\mu(t)\big)$ in the i-th direction, plus the global correction term

$$w(t) := V^{\boldsymbol{\varphi}}(t) - \sum_{j=1}^{d} \mu_i(t) D_j G(\mu(t)) = \Gamma^G(t) + G\big(\mu(t)\big) - \sum_{j=1}^{d} \mu_i(t) D_j G(\mu(t))\,,$$

the same for all i, which ensures the self-financibility of the trading strategy $\boldsymbol{\varphi}(\cdot)$.

To implement the trading strategy $\boldsymbol{\varphi}(\cdot)$ in (4.4) at some time $t \geq 0$, assume it has been implemented up to the present time t. It now suffices to compute $D_i G(\mu(t))$ for each $i = 1, \cdots, d$, and to buy exactly $D_i G(\mu(t))$ shares of the i-th asset. If not all wealth gets invested, that is, if the quantity $w(t)$ is positive, then one buys exactly $w(t)$ shares of each asset, costing exactly $\sum_{i=1}^{d} w(t)\mu_i(t) = w(t)$. If $w(t)$ is negative, one sells those $|w(t)|$ shares instead of buying them. Thus, an *implementation of the functionally generated strategy does not require the computations of any stochastic integral.*

If the function G is nonnegative and concave, the following result guarantees that the strategy it generates holds a nonnegative amount of each asset, even if $D_i G(\mu(t))$ is negative for some $i = 1, \cdots, d$.

Proposition 4.4 (Long-only trading strategies)**.** *Assume that one of the three conditions in Theorem 3.6 holds for some continuous function* $G : \mathbf{supp}\,(\mu) \to [0, \infty)$*. Then there exists a trading strategy* $\boldsymbol{\varphi}(\cdot)$*, additively generated by* G*, which satisfies* $\boldsymbol{\varphi}_i(\cdot) \geq 0$ *for each* $i = 1, \cdots, d$*; in other words, the trading strategy* $\boldsymbol{\varphi}(\cdot)$ *is then "long-only."*

The proof of Proposition 4.4 requires some convex analysis and is contained in Subsection 7.1 below.

Remark 4.5 (Associated portfolios). Let G be a regular function for the vector process $\mu(\cdot)$, generating the trading strategy $\boldsymbol{\varphi}(\cdot)$ as in (4.1) and (4.4). Whenever $V^{\boldsymbol{\varphi}}(\cdot) > 0$ holds (for example, if G is a Lyapunov function taking values in $(0, \infty)$), the portfolio weights

$$\boldsymbol{\pi}_i(\cdot) := \frac{\mu_i(\cdot)\boldsymbol{\varphi}_i(\cdot)}{V^{\boldsymbol{\varphi}}(\cdot)} = \frac{\mu_i(\cdot)\boldsymbol{\varphi}_i(\cdot)}{\sum_{j=1}^d \mu_j(\cdot)\boldsymbol{\varphi}_j(\cdot)}, \qquad i = 1, \cdots, d \tag{4.5}$$

of the trading strategy $\boldsymbol{\varphi}(\cdot)$ can be cast as

$$\boldsymbol{\pi}_i(\cdot) = \mu_i(\cdot)\left(1 + \frac{1}{G(\mu(\cdot)) + \Gamma^G(\cdot)}\left(D_i G\big(\mu(\cdot)\big) - \sum_{j=1}^d \mu_j(\cdot) D_j G\big(\mu(\cdot)\big)\right)\right) \tag{4.6}$$

for each $i = 1, \cdots, d$. $\square$

4.2 Multiplicative generation

Let us recall the functionally-generated portfolio introduced by Fernholz (1999, 2001, 2002). Suppose that the function $G : \mathbf{supp}\,(\mu) \to [0, \infty)$ is regular for the vector process $\mu(\cdot)$ of market weights in (2.2), and that $1/G(\mu(\cdot))$ is locally bounded. This holds if G is bounded away from zero, or if (2.4) is satisfied and G is strictly positive on $\boldsymbol{\Delta}^d_+$. We introduce now the predictable portfolio-weights

$$\boldsymbol{\Pi}_i(\cdot) := \mu_i(\cdot)\left(1 + \frac{1}{G\big(\mu(\cdot)\big)}\left(D_i G\big(\mu(\cdot)\big) - \sum_{j=1}^d D_j G\big(\mu(\cdot)\big)\, \mu_j(\cdot)\right)\right), \qquad i = 1, \cdots, d. \tag{4.7}$$

These processes satisfy $\sum_{i=1}^d \boldsymbol{\Pi}_i(\cdot) \equiv 1$ rather trivially; and it is shown as in Proposition 4.4 that they are non-negative, if one of the three conditions in Theorem 3.6 holds.

In order to relate these portfolio weights to a trading strategy, let us consider the vector process $\widetilde{\boldsymbol{\vartheta}}(\cdot) = \big(\widetilde{\boldsymbol{\vartheta}}_1(\cdot), \cdots, \widetilde{\boldsymbol{\vartheta}}_d(\cdot)\big)'$ given in the notation of (3.1) by

$$\widetilde{\boldsymbol{\vartheta}}_i(\cdot) := \boldsymbol{\vartheta}_i(\cdot) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)}\right) = D_i G(\mu(\cdot)) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)}\right), \qquad i = 1, \cdots, d\,.$$

Note that the integral is well-defined, as $1/G(\mu(\cdot))$ is locally bounded by assumption. Moreover, we have $\widetilde{\boldsymbol{\vartheta}}(\cdot) \in \mathscr{L}(\mu)$ since $\boldsymbol{\vartheta}(\cdot) \in \mathscr{L}(\mu)$ and the exponential function is locally bounded. We can turn the predictable process $\widetilde{\boldsymbol{\vartheta}}(\cdot)$ into a trading strategy $\boldsymbol{\psi}(\cdot) = \big(\boldsymbol{\psi}_1(\cdot), \cdots, \boldsymbol{\psi}_d(\cdot)\big)'$ by setting

$$\boldsymbol{\psi}_i(\cdot) := \widetilde{\boldsymbol{\vartheta}}_i(\cdot) - Q^{\widetilde{\boldsymbol{\vartheta}}}(\cdot) + \boldsymbol{C}, \qquad i = 1, \cdots, d \tag{4.8}$$

in the manner of (2.8) and (2.7), and with $\boldsymbol{C}$ given by (4.2).

Definition 4.6 (Multiplicative functional generation (MFG))**.** We say that the trading strategy $\boldsymbol{\psi}(\cdot) = \big(\boldsymbol{\psi}_1(\cdot), \cdots, \boldsymbol{\psi}_d(\cdot)\big)' \in \mathscr{T}(\mu)$ of (4.8) is *multiplicatively generated* by the function $G : \mathbf{supp}\,(\mu) \to [0, \infty)$. $\square$

Proposition 4.3 has now an equivalent formulation.

Proposition 4.7 (Representation and value of MFG strategies)**.** *The trading strategy* $\boldsymbol{\psi}(\cdot) = \big(\psi_1(\cdot), \cdots, \psi_d(\cdot)\big)'$*, generated as in* (4.8) *by a function* $G : \mathbf{supp}\,(\mu) \to [0,\infty)$ *which is regular for the process* $\mu(\cdot)$ *of market weights and such that* $1/G(\mu(\cdot))$ *is locally bounded, has relative value process*

$$V^{\boldsymbol{\psi}}(\cdot) \;=\; G\big(\mu(\cdot)\big)\exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)}\right) \;>\; 0 \tag{4.9}$$

and can be represented in the form

$$\psi_i(\cdot) = V^{\boldsymbol{\psi}}(\cdot)\left(1 + \frac{1}{G\big(\mu(\cdot)\big)}\left(D_iG\big(\mu(\cdot)\big) - \sum_{j=1}^d D_jG\big(\mu(\cdot)\big)\mu_j(\cdot)\right)\right), \qquad i = 1, \cdots, d. \tag{4.10}$$

Proof. With $K(\cdot) := \exp\big(\int_0^{\cdot}\big(1/G(\mu(t))\big)\,\mathrm{d}\Gamma^G(t)\big)$, the product rule yields

$$\begin{aligned}
\mathrm{d}\big(G(\mu(t))K(t)\big) &= K(t)\mathrm{d}G\big(\mu(t)\big) + K(t)\mathrm{d}\Gamma^G(t) = K(t)\sum_{i=1}^d \boldsymbol{\vartheta}_i(t)\mathrm{d}\mu_i(t) \\
&= \sum_{i=1}^d \widetilde{\boldsymbol{\vartheta}}_i(t)\mathrm{d}\mu_i(t) = \sum_{i=1}^d \psi_i(t)\mathrm{d}\mu_i(t) = \mathrm{d}V^{\boldsymbol{\psi}}(t), \qquad t \geq 0,
\end{aligned}$$

where the second equality uses (3.2), and the second-to-last relies on (2.9). Since (4.9) holds at time zero, namely $V^{\boldsymbol{\psi}}(0) = \sum_{i=1}^d \psi_i(0)\mu_i(0) = \sum_{i=1}^d \big(\boldsymbol{\vartheta}_i(0) + \boldsymbol{C}\big)\mu_i(0) = G\big(\mu(0)\big)$ on the strength of (2.5), (4.8), (2.7) and (4.2), it follows from the above display that (4.9) holds in general.

On the other hand, starting with (4.8) we obtain

$$\begin{aligned}
\psi_i(\cdot) = \;& \widetilde{\boldsymbol{\vartheta}}_i(\cdot) - Q^{\widetilde{\boldsymbol{\vartheta}}}(\cdot) + \boldsymbol{C} = K(\cdot)D_iG(\mu(\cdot)) - V^{\widetilde{\boldsymbol{\vartheta}}}(\cdot) + V^{\widetilde{\boldsymbol{\vartheta}}}(0) + \int_0^{\cdot}\big\langle\widetilde{\boldsymbol{\vartheta}}(t), \mathrm{d}\mu(t)\big\rangle + \boldsymbol{C} \\
= \;& K(\cdot)D_iG(\mu(\cdot)) - K(\cdot)\sum_{j=1}^d D_jG\big(\mu(\cdot)\big)\mu_j(\cdot) + V^{\boldsymbol{\psi}}(\cdot), \qquad i = 1, \cdots, d,
\end{aligned}$$

using (2.9) and the definition of $Q^{\widetilde{\boldsymbol{\vartheta}}}(\cdot)$ in (2.7). This yields the representation (4.10). □

It is now easy to see how the portfolio process $\boldsymbol{\Pi}(\cdot)$ in (4.7) is obtained from (4.10) in the same manner as (4.5), as $V^{\boldsymbol{\psi}}(\cdot)$ is strictly positive. The representation in (4.9) is a "generalized master equation" in the spirit of Theorem 3.1.5 in Fernholz (2002).

4.3 Comparison of additive and multiplicative functional generation

It is instructive at this point to compare additive and multiplicative functional generation. On a purely formal level, the multiplicative generation of Definition 4.6 requires a regular function G such that $1/G(\mu(\cdot))$ is locally bounded. On the other side, additive functional generation requires only the regularity of the function G.

At time $t = 0$, the additively-generated strategy agrees with the multiplicatively-generated one; that is, we have $\boldsymbol{\varphi}(0) = \boldsymbol{\psi}(0)$ in the notation of (4.4) and (4.10). However, at any time $t > 0$ with $\Gamma^G(t) \neq 0$, these two strategies usually differ; this is seen most easily by looking at their corresponding portfolios (4.6) and (4.7). More precisely, the two strategies differ in the way they allocate the proportion of their wealth captured by $\Gamma^G(\cdot)$. The additively-generated strategy tries to allocate this proportion uniformly across all assets in the market; whereas the multiplicatively-generated strategy tends to correct for this amount by proportionally adjusting the asset holdings.

To see this, consider again (4.10) and assume that $\sum_{j=1}^{d} x_j D_j G(x) = G(x)$ for all $x \in \mathbf{\Delta}^d$, a case which occurs often in examples; we have then

$$\boldsymbol{\psi}_i(\cdot) \,=\, D_i G\big(\mu(\cdot)\big)\, \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big)}\right), \qquad i = 1, \cdots, d.$$

Thus, in this situation, the multiplicatively-generated $\boldsymbol{\psi}(t)$ does not invest in assets for which $D_i G(\mu(t)) = 0$, for each $t \geq 0$, but instead adjusts the holdings proportionally. By contrast, the additively-generated $\boldsymbol{\varphi}(\cdot)$ buys shares

$$\boldsymbol{\varphi}_i(t) \,=\, D_i G\big(\mu(t)\big) + \Gamma^G(t), \qquad i = 1, \cdots, d$$

of the different assets in this case, and does *not* shun stocks for which $D_i G(\mu(t)) = 0$, at time t.

Ramifications: This difference in the two strategies leads to two observations.

First, if one is interested in a trading strategy that invests through time only in a subset of the market, such as for example the set of "small-capitalization stocks", then strategies generated multiplicatively by functions G that satisfy $\sum_{j=1}^{d} x_j D_j G(x) = G(x)$ for all $x \in \mathbf{\Delta}^d$, are appropriate. If, on the other hand, one wants to invest the trading strategy's earnings in a proportion of the whole market, additive generation is better suited. This is illustrated by Examples 6.2 and 6.3.

Secondly, the trading strategy which holds equal weights across all assets, can be generated multiplicatively, by the "geometric mean" function $\mathbf{\Delta}^d \ni x \mapsto G(x) = (x_1 \times \cdots \times x_d)^{1/d} \in (0,1)$ as long as (2.4) holds; indeed, the portfolio weights in (4.7) become now $\mathbf{\Pi}_i(\cdot) = 1/d$ for all $i = 1, \cdots, d$. But such a trading strategy cannot be additively generated; for instance, the portfolio in (4.6), namely

$$\boldsymbol{\pi}_i(t) \,=\, \frac{(1/d) + \mu_i(t) R^G(t)}{1 + R^G(t)}\,, \qquad i = 1, \cdots, d\,, \quad t \geq 0 \qquad \text{with} \qquad R^G(t) := \frac{\Gamma^G(t)}{G(\mu(t))}\,,$$

that corresponds to the strategy generated additively by this geometric-mean function G, distributes the gains described by $\Gamma^G(\cdot)$ uniformly across stocks, and this destroys equal weighting.

Comparison of portfolios: Let us compare the two portfolios in (4.6) and (4.7) more closely. These differ only in the denominators that appear inside the brackets on their right-hand sides. Computing the quantities of (4.7) needs, at any given time $t \geq 0$, knowledge of the configuration of market weights $\mu_1(t), \cdots, \mu_d(t)$ prevalent at that time – *and nothing else*. By contrast, the quantities of (4.6) need the entire history of these market weights during the interval $[0, t]$, in order to compute the integral in (3.2). When these portfolios are expressed as trading strategies, as is done in (4.4) and (4.10), then in both cases only the wealth process and the market weights $\mu_1(t), \cdots, \mu_d(t)$ are necessary.

5 Sufficient conditions for outperformance

We have developed by now the necessary machinery in order to present sufficient conditions for the possibility of outperforming the market, as introduced in Subsection 2.3 – at least over sufficiently long time horizons.

In this section, $G : \mathbf{supp}\,(\mu) \to [0, \infty)$ denotes a nonnegative regular function for the market-weight process $\mu(\cdot)$ with $G(\mu(0)) = 1$. This normalization ensures that the initial wealth of a functionally generated strategy starts with one dollar, as required by (2.13); see (4.3) and (4.9). Such a normalization can always be achieved upon replacing G by $G+1$ if $G(\mu(0)) = 0$, or by $G/G(\mu(0))$ if $G(\mu(0)) > 0$.

Theorem 5.1 (Additively generated outperformance)**.** *Fix a Lyapunov function* $G : \mathbf{supp}\,(\mu) \to [0, \infty)$ *satisfying* $G(\mu(0)) = 1$*, and suppose that for some real number* $T_* > 0$ *we have*

$$\mathsf{P}\big(\Gamma^G(T_*) > 1\big) = 1. \tag{5.1}$$

Then the additively generated strategy $\boldsymbol{\varphi}(\cdot) = (\boldsymbol{\varphi}_1(\cdot), \cdots, \boldsymbol{\varphi}_d(\cdot))'$ *of Definition 4.1 strongly outperforms the market over every time-horizon* $[0, T]$ *with* $T \geq T_*$.

Proof. We recall the observations in Remark 2.5 and note that (4.3) yields $V^{\boldsymbol{\varphi}}(0) = 1$, $V^{\boldsymbol{\varphi}}(\cdot) \geq 0$, and $V^{\boldsymbol{\varphi}}(T) = G\big(\mu(T)\big) + \Gamma^G(T) \geq \Gamma^G(T_*) > 1$ for all $T \geq T_*$. □

The following result complements Theorem 5.1.

Theorem 5.2 (Multiplicatively generated outperformance)**.** *Fix a regular function* $G : \mathbf{supp}\,(\mu) \to [0, \infty)$ *satisfying* $G(\mu(0)) = 1$*, and suppose that for some real numbers* $T_* > 0$ *and* $\varepsilon > 0$ *we have*

$$\mathsf{P}\big(\Gamma^G(T_*) > 1 + \varepsilon\big) = 1\,.$$

Then there exists a constant $c > 0$ *such that the trading strategy* $\boldsymbol{\psi}^{(c)}(\cdot) = \big(\boldsymbol{\psi}_1^{(c)}(\cdot), \cdots, \boldsymbol{\psi}_d^{(c)}(\cdot)\big)'$*, multiplicatively generated by the regular function* $G^{(c)} := (G + c)/(1 + c)$ *as in Definition 4.6, strongly outperforms the market over the time-horizon* $[0, T_*]$*; and, if G is a Lyapunov function, also over every time-horizon* $[0, T]$ *with* $T \geq T_*$.

Proof. For $c > 0$, the representation (4.9) yields the comparisons $V^{\boldsymbol{\psi}^{(c)}}(0) = 1$, $V^{\boldsymbol{\psi}^{(c)}}(\cdot) > 0$, and

$$V^{\boldsymbol{\psi}^{(c)}}(T_*) \geq \frac{c}{1+c} \times \exp\left(\int_0^{T_*} \frac{\mathrm{d}\Gamma^G(t)}{G\big(\mu(t)\big) + c}\right) > \frac{c}{1+c} \times \exp\left(\frac{1+\varepsilon}{\boldsymbol{\kappa} + c}\right), \tag{5.2}$$

where $\boldsymbol{\kappa}$ is an upper bound on G, which is assumed to be continuous on the compact set $\mathbf{supp}\,(\mu)$. Here, we used in the first inequality the bound $G \geq 0$ and the identity $\Gamma^{G^{(c)}}(\cdot) = \Gamma^G(\cdot)/(1 + c)$. Now, with the help of Remark 2.5 we may conclude again, as soon as we have argued the existence of a constant $c > 0$ such that the last term in (5.2) is greater than one. Taking logarithms yields

$$-\log\left(1 + \frac{1}{c}\right) + \frac{1+\varepsilon}{\boldsymbol{\kappa} + c} > \frac{\varepsilon - \boldsymbol{\kappa}\log\left(1 + 1/c\right)}{\boldsymbol{\kappa} + c} \tag{5.3}$$

for all $c > 0$, since $1 > c\log(1 + 1/c)$. However, the right-hand side of (5.3) is positive for sufficiently large c, and this concludes the proof.

If G is a Lyapunov function, $\mathsf{P}\big(\Gamma^G(T) > 1 + \varepsilon\big) = 1$ and the inequalities in (5.2) are valid for all $T \geq T_*$, and the same reasoning as above works once again. □

We illustrate now the previous two theorems with two examples.

Example 5.3 (Entropy function and excess growth). Consider the Gibbs *entropy function*

$$H(x) = \sum_{i=1}^d x_i \log\left(\frac{1}{x_i}\right), \qquad x \in \boldsymbol{\Delta}^d$$

with values in $[0, \log(d)]$ and the understanding $0 \times \log(\infty) = 0$. This H is concave and continuous on $\boldsymbol{\Delta}^d$ and strictly positive on $\boldsymbol{\Delta}^n_+$. It is a Lyapunov function for $\mu(\cdot)$ provided that, as we assume from now on in this example, either a deflator for $\mu(\cdot)$ exists, or (2.4) holds; cf. Theorem 3.6(i)&(iii).

Elementary computations then show that the process of (3.4) takes now the form

$$\Gamma^H(\cdot) = \frac{1}{2}\sum_{i=1}^d \int_0^{\cdot} \mathbf{1}_{\{\mu_i(t)>0\}} \frac{\mathrm{d}\langle \mu_i\rangle(t)}{\mu_i(t)} = \frac{1}{2}\sum_{i=1}^d \int_0^{\cdot} \mu_i(t)\mathrm{d}\big\langle \log(\mu_i)\big\rangle(t).$$

This is the *cumulative excess growth of the market,* a trace-like quantity which plays a very important role in Stochastic Portfolio Theory. It measures the market's cumulative "relative variation" – stock-by-stock, then averaged according to each stock's market weight. It is easy to see that $\Gamma^H(\cdot)$ is clearly non-decreasing, which confirms that the Gibbs entropy is indeed a Lyapunov function for any market $\mu(\cdot)$ that allows for a deflator or satisfies (2.4).

The additively-generated strategy $\boldsymbol{\varphi}(\cdot)$ of (4.4) invests a number

$$\boldsymbol{\varphi}_i(\cdot) = \left(\log\left(\frac{1}{\mu_i(\cdot)}\right) + \Gamma^H(\cdot)\right)\mathbf{1}_{\{\mu_i(\cdot)>0\}}\,, \qquad i = 1, \cdots, d$$

of shares in each of the various assets, and generates strictly positive value

$$V^{\boldsymbol{\varphi}}(\cdot) = H(\mu(\cdot)) + \Gamma^H(\cdot) > 0\,.$$

This strict positivity is obvious if (2.4) holds; on the other hand, to see this assuming the existence of a deflator, consider the stopping time $\tau := \inf\{t \geq 0 : V^{\boldsymbol{\varphi}}(t) = 0\} > 0$ on the strength of $\mu(0) \in \boldsymbol{\Delta}^d_+$. On the event $\{\tau < \infty\}$ we have both $H(\mu(\tau)) = 0$ and $\Gamma^H(\tau) = 0$. From the properties of the entropy function, the first of these requirements implies that, at time τ, the process of market weights is at one of the vertices of the simplex: $\tau \geq \mathscr{D}_*$ in the notation of (2.17). The second requirement gives $\Gamma^H(\mathscr{D}_*) = 0$, thus $\Gamma^H(\mathscr{D}) = 0$ in the notation of (2.16). But then

$$2\,\Gamma^H(\mathscr{D}) \;=\; \sum_{i=1}^d \int_0^{\mathscr{D}} \frac{d\langle \mu_i\rangle(t)}{\mu_i(t)} \;\geq\; \sum_{i=1}^d \langle \mu_i\rangle(\mathscr{D})$$

implies that, for each $i = 1, \cdots, d$, we have $\langle \mu_i\rangle(\mathscr{D}) = 0$ on the event $\{\tau < \infty\}$; the existence of a deflator leads to $\mu_i(t) = \mu_i(0)$ for all $0 \leq t \leq \mathscr{D}$, and this to $\mathbb{P}(\tau < \infty) = 0$.

Multiplicative generation needs a regular function that is bounded away from zero, so let us consider $H^{(c)} = H + c$ for some $c > 0$. According to (4.10), the multiplicatively generated strategy invests a number

$$\boldsymbol{\psi}_i^{(c)}(\cdot) = \left(\log\left(\frac{1}{\mu_i(\cdot)}\right) + c\right) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^H(t)}{H\big(\mu(t)\big) + c}\right)\mathbf{1}_{\{\mu_i(\cdot)>0\}}\,, \qquad i = 1, \cdots, d$$

of shares in each of the various assets.

We can compute now the portfolio weights corresponding to these two strategies from (4.6) and (4.7), respectively, as

$$\boldsymbol{\pi}_i(\cdot) = \frac{\mu_i(\cdot)}{H(\mu(\cdot)) + \Gamma^H(\cdot)}\left(\log\left(\frac{1}{\mu_i(\cdot)}\right) + \Gamma^H(\cdot)\right), \qquad i = 1, \cdots, d\,,$$

$$\boldsymbol{\Pi}_i^{(c)}(\cdot) = \frac{\mu_i(\cdot)}{H(\mu(\cdot)) + c}\left(\log\left(\frac{1}{\mu_i(\cdot)}\right) + c\right), \qquad i = 1, \cdots, d\,,$$

with the previous understanding $0 \times \log(\infty) = 0$. The process $\boldsymbol{\Pi}^{(c)}(\cdot)$ has been termed "entropy-weighted portfolio" in the literature; see Fernholz (2002), and Fernholz and Karatzas (2005).

Let us now consider the question of outperformance. By definition, a trading strategy that strongly outperforms the market starts with wealth of one dollar; cf. (2.13). Hence we shall consider the Lyapunov function $G = H/H(\mu(0))$ along with its nondecreasing process $\Gamma^G(\cdot) = \Gamma^H(\cdot)/H(\mu(0))$. Then Theorems 5.1 and 5.2 yield the existence of such a strategy over the time horizon $[0, T]$, as long as we have, respectively,

$$\mathsf{P}\big(\Gamma^H(T) > H(\mu(0))\big) = 1\,,$$

or

$$\mathsf{P}\big(\Gamma^H(T) > H(\mu(0)) + \varepsilon\big) = 1$$

for some $\varepsilon > 0$. In the first case, this strong outperformance is additively generated through the trading strategy $\boldsymbol{\varphi}(\cdot)/H(\mu(0))$; in the second, it is multiplicatively generated through the trading strategy $\boldsymbol{\psi}^{(c)}(\cdot)/(H(\mu(0)) + c)$ for some sufficiently large $c > 0$.

For example, if $\mathsf{P}\big(\Gamma^H(t) \geq \eta\, t,\ \forall\, t \geq 0\big) = 1$ holds for some real constant $\eta > 0$, strong outperformance of the market can be implemented over any time-horizon $[0, T]$ with $T > H(\mu(0))/\eta$. □

Remark 5.4 (An old question). It has been a long-standing open problem, dating to Fernholz and Karatzas (2005), whether the validity of $\mathsf{P}\big(\Gamma^G(t) \geq \eta\, t,\ \forall\, t \geq 0\big) = 1$ for some real constant $\eta > 0$, can guarantee the existence of a strategy that outperforms the market over *any* time-horizon $[0, T]$, of *arbitrary* length $T \in (0, \infty)$. For explicit examples showing that this is not possible in general, see our companion paper Fernholz et al. (2016). □

Example 5.5 (Quadratic function and sum of variations). Fix, for the moment, a constant $c \in \mathbb{R}$ and consider, in the manner of Example 3.3.3 in Fernholz (2002), the quadratic function

$$H^{(c)}(x) \;:=\; c - \sum_{i=1}^{d} x_i^2\,, \qquad x \in \boldsymbol{\Delta}^d,$$

with values in $[c-1, c-1/d]$. The term $\sum_{i=1}^{d} \mu_i^2(\cdot)$ is the weighted average capitalization of the market and may be used to quantify the concentration of capital in a market.

Clearly, $H^{(c)}$ is concave and Theorem 3.6(ii), or alternatively, Example 3.5, yields that $H^{(c)}$ is a Lyapunov function for $\mu(\cdot)$, without any additional assumption. The nondecreasing process of (3.2) is then given by

$$\Gamma^{H^{(c)}}(\cdot) = \sum_{i=1}^{d} \langle \mu_i \rangle(\cdot)$$

and the additively generated strategy $\boldsymbol{\varphi}^{(c)}(\cdot)$ of (4.4) is given by

$$\boldsymbol{\varphi}_i^{(c)}(\cdot) = c - 2\mu_i(\cdot) + \sum_{j=1}^{d} \Big(\langle \mu_j \rangle(\cdot) + \big(\mu_j(\cdot)\big)^2 \Big), \qquad i = 1, \cdots, d.$$

If $c > 1$, the multiplicatively generated strategy $\boldsymbol{\psi}^{(c)}(\cdot)$ of (4.10) is well-defined and is given as

$$\boldsymbol{\psi}_i^{(c)}(\cdot) = K^{(c)}(\cdot) \left(-2\mu_i(\cdot) + \sum_{j=1}^{d} \big(\mu_j(\cdot)\big)^2 + c \right), \qquad i = 1, \cdots, d,$$

where

$$K^{(c)}(\cdot) = \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^{H^{(c)}}(t)}{H^{(c)}\big(\mu(t)\big)} \right) = \exp\left(\sum_{i=1}^{d} \int_0^{\cdot} \frac{\mathrm{d}\langle \mu_i \rangle(t)}{c - \sum_{j=1}^{d} \big(\mu_j(t)\big)^2} \right).$$

Since $H^{(1)} \geq 0$, we obtain as in Example 5.3 that the condition

$$\mathsf{P}\left(\sum_{i=1}^{d} \langle \mu_i \rangle (T) > H^{(1)}(\mu(0))\right) = 1 \tag{5.4}$$

yields a strategy which strongly outperforms the market on $[0, T]$, and is additively generated by the function $H^{(1)}/H^{(1)}(\mu(0))$. Moreover, the requirement

$$\mathsf{P}\left(\sum_{i=1}^{d} \langle \mu_i \rangle (T) > H^{(1)}(\mu(0)) + \varepsilon\right) = 1 \tag{5.5}$$

for some $\varepsilon > 0$, yields a strategy which strongly outperforms the market on $[0, T]$, and is multiplicatively generated by the function $H^{(c)}/H^{(c)}(\mu(0))$ for some sufficiently large $c > 1$.

For example, if $\mathsf{P}\big(\sum_{i=1}^{d} \langle \mu_i \rangle (t) \geq \eta t, \ \forall\ t \geq 0 \big) = 1$ holds, then there exist both additively and multiplicatively generated strong outperformance of the market over *any* time-horizon $[0, T]$ with

$$T > \frac{1}{\eta}\left(1 - \sum_{i=1}^{d} \big(\mu_i(0)\big)^2\right). \tag{5.6}$$

Let us assume now that the market is diverse, namely,

$$\max_{i=1,\cdots,d} \mu_i(t) \ < \ 1 - \delta\,, \qquad t \geq 0$$

holds for some real constant $\delta \in (0, 1)$. Then we have the bound $H^{(c)} \geq c - 1 + 2\delta(1 - \delta)$. Thus, in particular, $H^{(1-2\delta(1-\delta))} \geq 0$ and we may replace $H^{(1)}$ in (5.4) and (5.5) by $H^{(1-2\delta(1-\delta))}$. This in turn allows us to replace the bound in (5.6) by the improved bound

$$T > \frac{1}{\eta}\left(1 - 2\delta(1-\delta) - \sum_{i=1}^{d} \big(\mu_i(0)\big)^2\right).$$

Finally, for future reference, we remark that the modification

$$H^\flat(x) := 1 - \frac{1}{2}\sum_{i=1}^{d}\left(x_i - \frac{1}{d}\right)^2, \qquad x \in \boldsymbol{\Delta}^d \tag{5.7}$$

of the above quadratic function, satisfies $H^\flat = H^{(2+1/d)}/2$. □

6 Further examples

In this section, we collect several examples that illustrate a variety of Lyapunov functions and their corresponding trading strategies.

Example 6.1 (Gini function and sum of local times). Let us revisit Example 4.2.2 of Fernholz (2002) in our context. We consider the *Gini function*

$$G^\flat(x) := 1 - \frac{1}{2}\sum_{i=1}^{d}\left|x_i - \frac{1}{d}\right|, \qquad x \in \boldsymbol{\Delta}^d,$$

which is concave on $\boldsymbol{\Delta}^d$. Thanks to Theorem 3.6(ii) $G^\flat$ is a Lyapunov function.

This function is used widely as a measure of inequality; the quadratic function of (5.7) is its "smooth sibling." For this Gini function, and with the help of the Itô-Tanaka formula, the processes of (3.1) and (3.2) take the form

$$\boldsymbol{\vartheta}_i^{G^\flat}(\cdot) = -\frac{1}{2}\,\mathrm{sgn}\Big(\mu_i(\cdot) - \frac{1}{d}\Big), \quad i = 1, \cdots, d \qquad \text{and} \qquad \Gamma^{G^\flat}(\cdot) = \sum_{i=1}^{d} \Lambda_i(\cdot),$$

respectively. Here $\Lambda_i(\cdot)$ stands for the local time accumulated by the process $\mu_i(\cdot)$ at the point $1/d$, and "sgn" for the left-continuous version of the signum function. It is now fairly easy to write down the strategies of (4.4) and (4.10) generated by this function. It is harder, though, to posit a condition of the type (5.1), as the sum of local times $\sum_{i=1}^{d} \Lambda_i(\cdot)$ does not typically admit a strictly positive lower bound. □

In the following we present examples of functional generation of trading strategies based on ranks.

Example 6.2 (Capitalization-weighted portfolio of large stocks). Let us recall the notation of (3.6), fix an integer $m \in \{1, \cdots, d-1\}$ and consider, in the manner of Example 4.3.2 in Fernholz (2002), the function $\mathbf{G}^L : \mathbb{W}^d \to (0, \infty)$ given by

$$\boldsymbol{G}^L\big(x_1, \cdots, x_d\big) \;:=\; x_1 + \cdots + x_m\,.$$

If $m = 1$ then we are exactly in the setup of Example 3.9. The function $G^L := \mathbf{G}^L \circ \mathfrak{R}$ in the notation of (3.8) is regular, thanks to Theorem 3.7 or, alternatively, Example 3.8. In the notation of that example, the corresponding function DG^L can by computed by (3.10) as

$$D_i G^L(x) \;=\; \sum_{\ell=1}^{m} \frac{1}{N_\ell(x)} \mathbf{1}_{x_{(\ell)} = x_i} \;=\; \mathbf{1}_{x_{(m+1)} < x_i} + \frac{\sum_{\ell=1}^{m} \mathbf{1}_{x_{(\ell)} = x_i}}{\sum_{\ell=1}^{d} \mathbf{1}_{x_{(\ell)} = x_i}} \mathbf{1}_{x_{(m+1)} = x_i}$$

for all $x \in \boldsymbol{\Delta}^d, \;\; i = 1, \cdots, d$. Thanks to (3.11), the process $\Gamma^{G^L}(\cdot)$ is given by

$$\begin{aligned}
\Gamma^{G^L}(\cdot) &= \sum_{\ell=2}^{m} \sum_{k=1}^{\ell-1} \int_0^{\cdot} \frac{1}{N_\ell(\mu(t))} \mathrm{d}\Lambda^{(k,\ell)}(t) - \sum_{\ell=1}^{m} \sum_{k=\ell+1}^{d} \int_0^{\cdot} \frac{1}{N_\ell(\mu(t))} \mathrm{d}\Lambda^{(\ell,k)}(t) \\
&= \sum_{\ell=1}^{m-1} \sum_{k=\ell+1}^{m} \int_0^{\cdot} \frac{1}{N_\ell(\mu(t))} \mathrm{d}\Lambda^{(\ell,k)}(t) - \sum_{\ell=1}^{m} \sum_{k=\ell+1}^{d} \int_0^{\cdot} \frac{1}{N_\ell(\mu(t))} \mathrm{d}\Lambda^{(\ell,k)}(t) \\
&= -\sum_{\ell=1}^{m} \sum_{k=m+1}^{d} \int_0^{\cdot} \frac{1}{N_m(\mu(t))} \mathrm{d}\Lambda^{(\ell,k)}(t).
\end{aligned}$$

Here the second equality swaps the summation in the first term, relabels the indices, and uses the fact that $N_\ell(\mu(\cdot)) = N_k(\mu(\cdot))$ holds on the support of the collision local time $\Lambda^{(\ell,k)}(\cdot)$, for each $1 \le \ell < k \le d$. The last equality used the fact that $N_\ell(\mu(\cdot)) = N_m(\mu(\cdot))$ holds on the support of $\Lambda^{(\ell,k)}(\cdot)$, for each $\ell = 1, \cdots, m$ and $k = m+1, \cdots, d$.

If there are no triple points at all, that is, if $\mu_{(\ell)}(\cdot) - \mu_{(\ell+2)}(\cdot) > 0$ holds for all $\ell = 1, \cdots, d-2$, then $N_m(\mu(\cdot)) \in \{1, 2\}$ and we get

$$\begin{aligned}
D_i G^L(x) &= \mathbf{1}_{x_{(m+1)} < x_i} + \frac{1}{2} \mathbf{1}_{x_{(m)} = x_{(m+1)} = x_i}\,, \qquad x \in \boldsymbol{\Delta}^d, \;\; i = 1, \cdots, d\,; \\
\Gamma^{G^L}(\cdot) &= -\frac{1}{2} \Lambda^{(m,m+1)}(\cdot).
\end{aligned}$$

For the additively-generated strategy $\boldsymbol{\varphi}(\cdot)$ in (4.4) we get

$$\boldsymbol{\varphi}_i(\cdot) = D_i G^L(\mu(\cdot)) + \Gamma^{G^L}(\cdot), \qquad i = 1, \cdots, d\,;$$

and for the multiplicatively-generated strategy $\boldsymbol{\psi}(\cdot)$ in (4.10) we have

$$\boldsymbol{\psi}_i(\cdot) = D_i G^L(\mu(\cdot)) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^{G^L}(t)}{G^L(\mu(t))}\right), \qquad i = 1, \cdots, d.$$

Hence, the additively-generated strategy invests in all assets (possibly by selling them), provided that $\Gamma^{G^L}(\cdot)$ is not identically equal to zero; while the multiplicatively-generated strategy only invests in the m largest stocks. Whereas the additively-generated strategy might lead to negative wealth, the multiplicatively-generated strategy yields always strictly positive wealth; cf. (4.9). Thus, we may express the multiplicatively-generated strategy $\boldsymbol{\psi}(\cdot)$ in terms of proportions, as in (4.7), by

$$\boldsymbol{\Pi}_i^{G^L}(\cdot) = \frac{D_i G^L(\mu(\cdot))}{\mu_{(1)}(\cdot) + \cdots + \mu_{(m)}(\cdot)}, \qquad i = 1, \cdots, d.$$

We note that this trading strategy only invests in the m largest stocks, and in proportion to each of these stocks' capitalization, apart from the times when several stocks share the m-th position, in which case the corresponding capital is uniformly distributed over these stocks.

In the context of the present example we might think of $d = 7,500$ as the entire US market; and of $m = 500$, as in S&P 500. Alternatively, we might consider $m = 1$, when we are adamant about investing only in the market's biggest company. The non-increasing process $\Gamma^{G^L}(\cdot)$ captures the "leakage" that such a trading strategy suffers every time it has to sell – at a loss – a stock that has dropped out of the higher-capitalization index and been relegated to the "minor (capitalization) leagues." $\square$

Example 6.3 (Capitalization-weighted portfolio of small stocks). Instead of large stocks, as in Example 6.2, we now consider a portfolio consisting of stocks with small capitalization. With the notation recalled in the previous example, we fix again an integer $m \in \{1, \cdots, d-1\}$ and consider the function $\mathbf{G}^S : \mathbb{W}^d \to (0, \infty)$ given by

$$\boldsymbol{G}^S\big(x_1, \cdots, x_d\big) \;:=\; x_{m+1} + \cdots + x_d\,.$$

The function $G^S := \mathbf{G}^S \circ \mathfrak{R}$ is again regular. Exactly as above, we compute,

$$D_i G^S(x) = \mathbf{1}_{\,x_{(m)} > x_i} + \frac{\sum_{\ell=m+1}^d \mathbf{1}_{\,x_{(\ell)} = x_i}}{\sum_{\ell=1}^d \mathbf{1}_{\,x_{(\ell)} = x_i}}\; \mathbf{1}_{\,x_{(m)} = x_i}\;, \qquad x \in \boldsymbol{\Delta}^d, \quad i = 1, \cdots, d$$

$$\Gamma^{G^S}(\cdot) = \sum_{\ell=m+1}^{d} \sum_{k=1}^{m} \int_0^{\cdot} \frac{1}{N_m(\mu(t))} \mathrm{d}\Lambda^{(k,\ell)}(t).$$

Thus, G^S is not only regular, but also a Lyapunov function. The non-decreasing process $\Gamma^{G^S}(\cdot)$ expresses the cumulative gains that the additively-generated strategy generates; whenever it sells a stock, this strategy sells it at a profit — the stock has been promoted to the "major (capitalization) league."

It is again simple to see that the additively-generated strategy invests in all assets, provided that $\Gamma^{G^S}(\cdot)$ is not identically equal to zero; while the multiplicatively-generated strategy only invests in the $d - m$ smallest stocks. $\square$

Example 6.4 (Small stocks, again). Under the setup of Example 6.3, and a bit more generally, consider a function $\boldsymbol{H} : [0,1]^{d-m} \to \mathbb{R}$, which is regular for the truncated vector process of ranked market weights $(\mu_{(m+1)}(\cdot), \cdots, \mu_{(d)}(\cdot))$; for example, if it is twice continuously differentiable. Then it is clear that the function $\mathbf{G} : \mathbb{W}^d \to (0,\infty)$ given by

$$\boldsymbol{G}\big(x_1, \cdots, x_d\big) \;:=\; \boldsymbol{H}\big(x_{m+1}, \cdots, x_d\big)$$

is regular for the full vector process of ranked market weights $\boldsymbol{\mu}(\cdot) = (\mu_{(1)}(\cdot), \cdots, \mu_{(d)}(\cdot))'$ with $\Gamma^{\boldsymbol{G}}(\cdot) = \Gamma^{\boldsymbol{H}}(\cdot)$, $D_\ell \boldsymbol{G} = 0$ for all $\ell = 1, \cdots, m$, and $D_\ell \boldsymbol{G} = D_\ell \boldsymbol{H}$ for all $\ell = m+1, \cdots, d$. Here we write $D\boldsymbol{H} = (D\boldsymbol{H}_{m+1}, \cdots, D\boldsymbol{H}_d)'$ in Definition 3.1.

Hence, by Theorem 3.7 the function $G := \boldsymbol{G} \circ \mathfrak{R} : \Delta^d \to \mathbb{R}$, that is,

$$G(x) \;=\; \boldsymbol{H}\big(x_{(m+1)}, \cdots, x_{(d)}\big)$$

is also regular for the vector process $\mu(\cdot)$ of market weights. As in (3.11) of Example 3.8, we obtain

$$\begin{aligned}
\Gamma^G(\cdot) &= \Gamma^{\boldsymbol{G}}(\cdot) - \sum_{\ell=1}^{d-1} \sum_{k=\ell+1}^{d} \int_0^{\cdot} \frac{1}{N_\ell(\mu(t))} D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\Lambda^{(\ell,k)}(t) + \sum_{\ell=1}^{d} \sum_{k=1}^{\ell-1} \int_0^{\cdot} \frac{1}{N_\ell(\mu(t))} D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\Lambda^{(k,\ell)}(t) \\
&= \Gamma^{\boldsymbol{H}}(\cdot) + \sum_{\ell=1}^{d-1} \sum_{k=\ell+1}^{d} \int_0^{\cdot} \frac{1}{N_\ell(\mu(t))} \Big(D_k \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) - D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big)\Big) \mathrm{d}\Lambda^{(\ell,k)}(t).
\end{aligned}$$

Permutation Invariance: Let us now assume that $\boldsymbol{H}$ is concave, differentiable, and invariant under permutations of its variables; that is, G is a symmetric function of the $d-m$ smallest components of its argument. Then we may assume that for every $x \in [0,1]^{d-m}$ with $x_k = x_\ell$ we have $D_k \boldsymbol{H}(x) = D_\ell \boldsymbol{H}(x)$, for all $m+1 \le \ell, k \le d$. Then, in particular, we get $D_k \boldsymbol{G}\big(\boldsymbol{\mu}(\cdot)\big) = D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(\cdot)\big)$ on the support of $\Lambda^{(\ell,k)}(\cdot)$ for each $k = \ell+1, \cdots, d$ and $\ell = m+1, \cdots, d$. Since also $D_\ell \boldsymbol{G} = 0$ for each $\ell = 1, \cdots, m$, we now have

$$\Gamma^G(\cdot) = \Gamma^{\boldsymbol{H}}(\cdot) + \sum_{\ell=1}^{m} \sum_{k=m+1}^{d} \int_0^{\cdot} \frac{1}{N_m(\mu(t))} D_k \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\Lambda^{(\ell,k)}(t).$$

Since $\boldsymbol{H}$ is concave this finite-variation process is non-decreasing, thus G is a Lyapunov function. If $\boldsymbol{H}$ is nonnegative and $G(\mu(0)) > 0$, then Theorem 5.1 shows now that for some given real number $T > 0$ strong outperformance exists with respect to the market over the horizon $[0,T]$ if $\mathsf{P}(\Gamma^{\boldsymbol{H}}(T) > G(\mu(0))) = 1$. For example, if $\boldsymbol{H}$ is twice differentiable we have

$$\Gamma^{\boldsymbol{H}}(\cdot) = -\frac{1}{2} \sum_{\ell=m+1}^{d} \sum_{k=m+1}^{d} \int_0^{\cdot} D^2_{\ell k} \boldsymbol{H}\big(\mu_{(m+1)}(t), \cdots, \mu_{(d)}(t)\big) \mathrm{d}\big\langle \mu_{(\ell)}, \mu_{(k)} \big\rangle(t).$$

Section 4 in Vervuurt and Karatzas (2015) develops in detail a special case of such a construction for multiplicatively generating a trading strategy. □

7 Concave transformations of semimartingales

Consider a function $G : \boldsymbol{\Delta}^d \to \mathbb{R}$. The *superdifferential* of G at some point $x \in \boldsymbol{\Delta}^d$, denoted by $\partial G(x)$, is the set of all "supergradients" at that point; namely, the set of all vectors $\xi \in \mathbb{R}^d$ such that

$$\sum_{i=1}^{d} \xi_i (y_i - x_i) \ge G(y) - G(x) \quad \text{holds for all } y \in \boldsymbol{\Delta}^d. \tag{7.1}$$

If G is concave we have $\partial G(x) \neq \emptyset$ for all $x \in \boldsymbol{\Delta}^d_+$.

7.1 The proof of Theorems 3.6 and 3.7 and Proposition 4.4

Proof of Theorem 3.6. We proceed in three steps.
Step 1: We shall find it it useful to identify the set $\mathbf{\Delta}_+^d \subset \mathbb{R}_+^n$ of (2.3) with the set

$$\mathbf{\Delta}_{\flat+}^d := \left\{ \big(x_1, \cdots, x_{d-1}\big)' \in (0,1)^{d-1} : \sum_{i=1}^{d-1} x_i < 1 \right\} \subset \mathbb{R}^{d-1}. \tag{7.2}$$

The identification is based on the one-to-one "projection operator" $\mathfrak{P}$ namely, the mapping $\mathbf{\Delta}_e^d \ni (x_1, \cdots, x_d) \mapsto (x_1, \cdots, x_{d-1}) \in \mathbb{R}^{d-1}$. In this manner, a real-valued function G on $\mathbf{\Delta}_+^d$ or on $\mathbf{\Delta}_e^d$ is identified with the function $G_\flat = G \circ \mathfrak{P}^{-1}$ on $\mathbf{\Delta}_{\flat+}^d$ or on $\mathbb{R}^{d-1}$, respectively; and vice-versa. Note that G is concave on $\mathbf{\Delta}_+^d$ or on $\mathbf{\Delta}_e^d$, if and only if $G_\flat$ is concave on $\mathbf{\Delta}_{\flat+}^d$ or on $\mathbb{R}^{d-1}$, respectively.

Step 2: Let us start by imposing either condition (i) *or* (ii). We recall from Theorem 10.4 in Rockafellar (1970) (see also Wayne State University Mathematics Department Coffee Room (1972) and Roberts and Varberg (1974)) that the concave function $G_\flat = G \circ \mathfrak{P}^{-1}$ is locally Lipschitz on the open set $\mathbf{\Delta}_{\flat+}^d$ of (7.2) or on $\mathbb{R}^{d-1}$, respectively. Theorem VI.8 in Meyer (1976), along with the remark on page 222 of Dellacherie and Meyer (1982), yields that $G(\mu(\cdot))$ is a semimartingale.

We now let $DG = \big(D_1 G, \cdots, D_d G\big)' : \mathbf{\Delta}^d \to \mathbb{R}^d$ denote any measurable "supergradient" of G; that is, DG is measurable and satisfies $DG(x) \in \partial G(x)$ for all $x \in \mathbf{\Delta}_+^d$ in Theorem 3.6(i), and for all $x \in \mathbf{\Delta}_e^d$ in Theorem 3.6(ii). The Itô-type formula implicit in (3.2), namely

$$G\big(\mu(0)\big) = G\big(\mu(\cdot)\big) + \int_0^\cdot \big\langle DG\big(\mu(t)\big), \mathrm{d}\mu(t)\big\rangle - \Gamma^G(\cdot) \tag{7.3}$$

with a continuous, non-decreasing $\Gamma^G(\cdot)$, is established as in Bouleau (1981, 1984); see also Grinberg (2013) for an alternative treatment, and Aboulaïch and Stricker (1983) for the special case where G is once continuously differentiable. With the obvious notation $DG_\flat$, we use here the identity

$$\int_0^\cdot \big\langle DG_\flat(\mu_\flat(t)), \mathrm{d}\mu_\flat(t)\big\rangle = \int_0^\cdot \big\langle DG(\mu(t)), \mathrm{d}\mu(t)\big\rangle$$

of stochastic integration for the process $\mu_\flat(\cdot) = (\mu_1(\cdot), \cdots, \mu_{d-1}(\cdot))'$.

Step 3: We place ourselves now under the assumptions of (iii). For sake of notational simplicity we shall assume here $\mathbf{supp}\,(\mu) = \Delta^d$; the general case follows in exactly the same manner. We recall the stopping time $\mathscr{D}$ in (2.16), and note that any component $\mu_i(\cdot)$ with $\mu_i(\mathscr{D}) = 0$ is absorbed at the origin: $\mu(\mathscr{D} + t) = 0$ holds for all $t \geq 0$ on the event $\{\mathscr{D} < \infty\}$ (see Subsection 2.4). We use the notation $\mathfrak{m} : \Omega \to \{1, \cdots, d\}$ for the $\mathscr{F}(\mathscr{D})$–measurable random variable that records the number of assets which have not been absorbed by time $\mathscr{D}$; namely, the number of all indices $i \in \{1, \cdots, d\}$ such that $\mu_i(\mathscr{D}) > 0$.

Assume we have shown that

$$G\big(\mu(\cdot \wedge \mathscr{D})\big) \quad \text{is a semimartingale.} \tag{7.4}$$

Then, after time $\mathscr{D}$, the process $G\big(\mu(\cdot)\big)$ can be identified with a process $\widetilde{G}\big(\widetilde{\mu}(\cdot)\big)$, where $\widetilde{\mu}(\cdot)$ takes values in $\mathbf{\Delta}^{\mathfrak{m}}$, the domain of a concave function $\widetilde{G}$. An iteration of the argument then yields the statement, since the Itô-type formula in (7.3) follows again, exactly as in Step 2. Indeed, as above, DG may denote any measurable supergradient of G on $\mathbf{\Delta}_+^d$. On $\mathbf{\Delta}^d \setminus \mathbf{\Delta}_+^d$, the concave function G can be identified with a concave function $\widehat{G}$ on $\mathbf{\Delta}^n$ for some $n < d$. Thus, for each $x \in \mathbf{\Delta}^d$ and $i = 1, \cdots, d$, if $x_i \in \{0, 1\}$, we can set the i-th component of $DG(x)$ to zero (any arbitrary number would work); and if $x_i \in (0,1)$, to the corresponding component of the supergradient of $\widehat{G}$.

We still need to justify the claim in (7.4). Since G is continuous and thus bounded on the the compact set $\mathbf{\Delta}^d$, we may assume, without loss of generality, that G is nonnegative. Let $Z(\cdot)$ denote a deflator for the vector process $\mu(\cdot)$. Next, we introduce the increasing sequence of stopping times

$$\mathscr{S}_n = \inf\left\{t \geq 0 : \min_{i=1,\cdots,d} \mu_i(t) < \frac{1}{n}\right\}, \qquad n \in \mathbb{N},$$

satisfying $\lim_{n\uparrow\infty} \mathscr{S}_n = \mathscr{D}$. Thanks to (2.15) and (i), in particular (7.3), the process $Z(\cdot \wedge \mathscr{S}_n))G(\mu(\cdot \wedge \mathscr{S}_n))$ is a local supermartingale for each $n \in \mathbb{N}$, and thus $(Z(t)G(\mu(t)))_{0\leq t<\mathscr{D}}$ is a local supermartingale, bounded from below. The supermartingale convergence theorem, see Lemma 4.14 in Larsson and Ruf (2014), yields that $Z(\cdot \wedge \mathscr{D})G\big(\mu(\cdot \wedge \mathscr{D})\big)$ is also a local supermartingale. From this, and from the fact that the reciprocal $1/Z(\cdot \wedge \mathscr{D})$ is a semimartingale, the claim in (7.4) follows. $\square$

The proof of Theorem 3.6 shows that every continuous, concave function G is regular, and the DG in the corresponding Itô formula of (3.2) may be chosen (at least in the set $\mathbf{\Delta}^d_+$) to be a measurable supergradient of G. This observation also motivates the following conjecture.

Remark 7.1 (An open question concerning DG). Assume that a function G is regular and weakly differentiable with gradient $\widetilde{DG}$. Is it then possible to choose $DG = \widetilde{DG}$ in (3.1) and (3.2)? $\square$

Concerning a representation of the finite-variation process $\Gamma^G(\cdot)$, the proof of Theorem 3.6 does not yield any deep insights. The arguments in Bouleau (1981, 1984) yield a representation of $\Gamma^G(\cdot)$ as a limit of mollified second-order terms.

Remark 7.2 (An open question concerning $\Gamma^G(\cdot)$). In the context of Theorem 3.6 we conjecture that the process $-2\Gamma^G(\cdot)$ of (3.2) can be written as the sum of the covariations of the processes $\boldsymbol{\vartheta}_i(\cdot)$ as in (3.1) and $\mu_j(\cdot)$ as in (2.2); namely,

$$\Gamma^G(\cdot) = -\frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \big[\boldsymbol{\vartheta}_i, \mu_j\big](\cdot),$$

whenever the limits below exist in probability, for all $T \geq 0$ and $1 \leq i,j \leq d$:

$$\big[\boldsymbol{\vartheta}_i, \mu_j\big](T) = (\mathsf{P}) \lim_{N\uparrow\infty} \sum_{n:\, t_n^{(N)} \in \mathbb{D}^{(N)},\, t_n^{(N)} < T} \Big(\boldsymbol{\vartheta}_i\big(t_{n+1}^{(N)}\big) - \boldsymbol{\vartheta}_i\big(t_n^{(N)}\big)\Big)\Big(\mu_j\big(t_{n+1}^{(N)}\big) - \mu_j\big(t_n^{(N)}\big)\Big).$$

Here $\big(\mathbb{D}^{(N)}\big)_{N\in\mathbb{N}}$ is a sequence of partitions of $[0,\infty)$ of the form $0 = t_0^{(N)} < t_1^{(N)} < t_2^{(N)} < \cdots$, with each $\mathbb{D}^{(N+1)}$ a refinement of $\mathbb{D}^{(N)}$, and with mesh $\big\|\mathbb{D}^{(N)}\big\| = \max_{n\in\mathbb{N}_0}\{|t_{n+1}^{(N)} - t_n^{(N)}|\}$ decreasing all the way to zero as $N \uparrow \infty$. $\square$

Proof of Theorem 3.7. First, note that (2.4) is equivalent to the condition

$$\mathsf{P}\big(\mathfrak{R}(\mu(t)) \in \mathbf{\Delta}^d_+,\ \forall\, t \geq 0\big) = 1.$$

Hence, the sufficiency of conditions (i), (ii) here, is a simple corollary of the sufficiency of conditions (i), (ii) in Theorem 3.6 with G replaced by $\boldsymbol{G}$, and applied to the $\mathbf{\Delta}^d$–valued process $\boldsymbol{\mu}(\cdot) = \mathfrak{R}(\mu(\cdot))$.

It remains to be argued that if the function $\boldsymbol{G}$ is regular for the vector process $\boldsymbol{\mu}(\cdot)$, then the function $G = \boldsymbol{G} \circ \mathfrak{R}$ is regular for the vector process $\mu(\cdot)$. Towards this end, we generalize the arguments in Example 3.8. First, in a manner similar to (3.9), we recall from Theorem 2.3 in Banner and Ghomrasni

(2008) the existence of measurable functions $\boldsymbol{h}_\ell : \boldsymbol{\Delta}^d \to [0,1]$ and of finite variation processes $\boldsymbol{B}_\ell(\cdot)$ with $\boldsymbol{B}_\ell(0) = 0$, such that we have

$$\boldsymbol{\mu}_\ell(\cdot) = \boldsymbol{\mu}_\ell(0) + \int_0^\cdot \sum_{i=1}^d \boldsymbol{h}_\ell(\mu(t)) \mathbf{1}_{\{\mu_{(\ell)}(t)=\mu_i(t)\}} \mathrm{d}\mu_i(t) + \boldsymbol{B}_\ell(\cdot)$$

for all $\ell = 1, \cdots, d$. Therefore, we obtain

$$G(\mu(\cdot)) = \boldsymbol{G}(\mathfrak{R}(\mu(\cdot))) = \boldsymbol{G}(\boldsymbol{\mu}(\cdot)) = \boldsymbol{G}(\boldsymbol{\mu}(0)) + \int_0^\cdot \sum_{\ell=1}^d D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\boldsymbol{\mu}_\ell(t) - \Gamma^{\boldsymbol{G}}(\cdot),$$

where $D\boldsymbol{G}$ and $\Gamma^{\boldsymbol{G}}(\cdot)$ are as in Definition 3.1; in particular, $\Gamma^{\boldsymbol{G}}(\cdot)$ is a finite-variation process. By analogy with (3.10) and (3.11), we define now

$$D_i G(x) := \sum_{\ell=1}^d \boldsymbol{h}_\ell(x)\, D_\ell \boldsymbol{G}\big(\mathfrak{R}(x)\big)\, \mathbf{1}_{x_{(\ell)}=x_i}\,, \qquad x \in \mathbf{supp}\,(\mu)\,, \quad i = 1, \cdots, d\,,$$

$$\Gamma^G(\cdot) := \Gamma^{\boldsymbol{G}}(\cdot) - \sum_{\ell=1}^d \int_0^\cdot D_\ell \boldsymbol{G}\big(\boldsymbol{\mu}(t)\big) \mathrm{d}\boldsymbol{B}_\ell(t)\,,$$

and note $\boldsymbol{G}(\boldsymbol{\mu}(0)) = G(\mu(0))$. This yields (3.3), thus also the regularity of G for $\mu(\cdot)$. □

Proof of Proposition 4.4. Theorem 3.6 shows that G is a Lyapunov function; its proof also reveals that DG can be chosen to be a supergradient of G, if (i) or (ii) hold. If neither (i) nor (ii) holds, but (iii) does, we may choose DG to be a supergradient of G in $\boldsymbol{\Delta}^d_+$. In that case, for $x \in \boldsymbol{\Delta}^d \setminus \boldsymbol{\Delta}^d_+$ and $i = 1, \cdots, d$, we declare $D_i G(x)$ to be the corresponding component of a concave function $\widetilde{G}$ with domain $\boldsymbol{\Delta}^m$ for some $m < d$ if $x_i \in (0,1)$, and otherwise to be the term $\sum_{j: x_j \in (0,1)} x_j D_j G(x)$.

Fixing this choice of DG we next note from (4.4) that the non-decrease of $\Gamma^G(\cdot)$ gives

$$\boldsymbol{\varphi}_i(\cdot) \geq G\big(\mu(\cdot)\big) + D_i G\big(\mu(\cdot)\big) - \sum_{j=1}^d \mu_j(\cdot)\, D_j G\big(\mu(\cdot)\big)$$

so it suffices to show, for every fixed $i = 1, \cdots, d$ and $x \in \boldsymbol{\Delta}^d$, the inequality

$$G(x) + D_i G(x) - \sum_{j=1}^d x_j D_j G(x) \geq 0\,. \tag{7.5}$$

If $x_i \in \{0,1\}$ then (7.5) follows directly from the nonnegativity of G. Thus, we now consider the case $x_i \in (0,1)$, and let $\mathfrak{e}^{(i)} \in \boldsymbol{\Delta}^d$ denote the i-th unit vector of $\mathbb{R}^d$. Observe that if $x_j = 0$ for some $j = 1, \cdots, d$, then the j-th component of any linear combination of x and $\mathfrak{e}^{(i)}$ is also zero. This fact, the nonnegativity of G, and the property of supergradients given in (7.1), lead to

$$\begin{aligned} 0 \leq G\big(ux + (1-u)\mathfrak{e}^{(i)}\big) &\leq G(x) + \sum_{j=1}^d \big((u-1)x_j + (1-u)\mathfrak{e}^{(i)}_j\big)\, D_j G(x) \\ &= G(x) + (1-u) D_i G(x) - (1-u) \sum_{j=1}^d x_j D_j G(x) \end{aligned}$$

for all $u \in (0,1]$. Letting $u \downarrow 0$ yields (7.5), and thus the statement. □

7.2 Two counterexamples

Example 7.3 (Lack of deflator in Theorem 3.6(iii)). A condition, such as the existence of a deflator in Theorem 3.6(iii), is needed for the result to hold. Even for a one-dimensional semimartingale $X(\cdot)$ taking values in the unit interval $[0,1]$ and absorbed when it hits one of its endpoints, and with a concave function $G:[0,1]\to[0,1]$, the process $G(X(\cdot))$ need not be a semimartingale.

For example, let X be a deterministic continuous semimartingale with $X(0)=1$ and $X(t)=\lim_{s\uparrow 1}X(s)=0$ for all $t\geq 1$, constructed as follows. Let a_n be the smallest odd integer in the interval $[\sqrt{n},3\sqrt{n})$, for all $n\in\mathbb{N}$. On $[1-1/n,1-1/(n+1)]$ let $X(\cdot)$ have exactly a_n oscillations between $1/n$ and $1/(n+1)$, for each $n\in\mathbb{N}$. In particular, $X(1-1/n)=1/n$ and $X(t)\in[1/(n+1),1/n]$ for all $t\in[1-1/n,1-1/(n+1)]$, for each $n\in\mathbb{N}$. Then X is clearly continuous and takes values in the compact interval $[0,1]$. Since the first variation of $X(\cdot)$ is exactly

$$\sum_{n\in\mathbb{N}}a_n\left(\frac{1}{n}-\frac{1}{n+1}\right)\leq\sum_{n\in\mathbb{N}}\frac{3\sqrt{n}}{n^2+n}<\infty,$$

the process $X(\cdot)$ is indeed a continuous, deterministic finite-variation semimartingale.

Now consider the concave and bounded function $\widehat{G}:[0,1]\to[0,1]$ with $\widehat{G}(x):=\sqrt{x}$. Then the first variation of $\widehat{G}(X(\cdot))$ is exactly

$$\sum_{n\in\mathbb{N}}a_n\left(\sqrt{\frac{1}{n}}-\sqrt{\frac{1}{n+1}}\right)\geq\sum_{n\in\mathbb{N}}\left(1-\sqrt{\frac{n}{n+1}}\right)\geq\sum_{n\in\mathbb{N}}\frac{\boldsymbol{\kappa}}{n}=\infty$$

for some $\boldsymbol{\kappa}>0$, where the last inequality follows from l'Hôpital's rule. Thus $\widehat{G}(X(\cdot))$ is deterministic, but of infinite variation and thus not a semimartingale. It follows that, without further assumptions, a concave and continuous transformation defined on a convex set $[0,1]$, of a continuous semimartingale taking values in $[0,1]$, is not necessarily a semimartingale.

To put this example in the context of Theorem 3.6, just set $d=2$, $\mu_1(\cdot):=X(\cdot)$, and $\mu_2(\cdot):=1-\mu_1(\cdot)$. Then, there exists no deflator for $\mu(\cdot)$ and the concave and continuous function $G(x_1,x_2):=\sqrt{x_1}$, for all $(x_1,x_2)\in\boldsymbol{\Delta}^2$ is indeed not a regular function for the process $(\mu_1(\cdot),\mu_2(\cdot))$. $\square$

Example 7.4 (Existence of deflator in Theorem 5.1, but lack of regularity). We now modify Example 7.3 to obtain a setup in which a deflator for the vector process $\mu(\cdot)$ exists, the function $\boldsymbol{G}:\mathbb{W}^d\to[0,1]$ is continuous and concave, but $\boldsymbol{G}$ is not regular for $\boldsymbol{\mu}(\cdot)=\mathfrak{R}(\mu(\cdot))$ in the notation of (3.6) and (3.8).

To this end, set $d=2$ and let $B(\cdot)$ denote a Brownian motion starting at $B(0)=1$, and stopped when hitting 0 or 2. We set $\mu_1(\cdot):=B(\cdot)/2$ and $\mu_2(\cdot):=1-B(\cdot)/2=1-\mu_1(\cdot)$. Since $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are martingales, there exists a deflator for the vector process $\mu(\cdot)$; indeed, $Z(\cdot)\equiv 1$ will serve as one. Next, consider the function $\boldsymbol{G}(x_1,x_2):=\sqrt{x_1-x_2}$ for all $(x_1,x_2)\in\mathbb{W}^2=\mathbf{supp}\,\boldsymbol{\mu}$. Clearly, $\boldsymbol{G}$ is concave and continuous on $\mathbb{W}^2$. However, by Lemma 7.5 below the process $\boldsymbol{G}(\boldsymbol{\mu}(\cdot))=\sqrt{|1-B(\cdot)|}$ is not a semimartingale, thus $\boldsymbol{G}$ is not regular for $\boldsymbol{\mu}(\cdot)$. $\square$

Lemma 7.5 (Square root of Brownian motion)**.** *Let $W(\cdot)$ denote a Brownian motion starting in zero and τ a strictly positive stopping time. Then the process $\sqrt{|W(\cdot\wedge\tau)|}$ is not a semimartingale.*

Proof. Of course, the function $f:\mathbb{R}\ni x\mapsto\sqrt{|x|}$ is not the difference of two convex functions, which would let us conclude from the results in Çinlar et al. (1980), at least formally. For sake of completeness we provide here a direct proof. Note that the quadratic variation of $2f(W(\cdot\wedge\tau))$ can be bounded from below by the quadratic variation of the semimartingales

$$2\sqrt{\varepsilon\vee|W(\cdot\wedge\tau)|},\qquad\varepsilon>0.$$

Thanks to Itô's formula, their quadratic variation is $\int_0^{\cdot\wedge\tau} \mathbf{1}_{\{|W(t)|>\varepsilon\}} 1/|W(t)|\mathrm{d}t$, for each $\varepsilon > 0$. Thus, the quadratic variation of $2f(W(\cdot\wedge\tau))$ is at least $\int_0^{\cdot\wedge\tau} \mathbf{1}_{\{W(t)\neq 0\}} 1/|W(t)|\mathrm{d}t$. An application of the occupation time formula, in conjunction with the continuity of the local time of W then shows that $f(W(\cdot\wedge\tau))$ has infinite quadratic variation and thus, cannot be a semimartingale. □

For results in a similar vein, see Çinlar et al. (1980), especially Theorems 5.8 and 5.9, and Mijatović and Urusov (2015).

8 Conclusion

Fernholz (1999, 2001, 2002) provides a systematic approach to generate trading strategies that can be implemented without the need of heavy statistical estimates and whose performance in a frictionless market can be guaranteed by suitable and weak assumption on the market's volatility structure. The present paper takes a systematic approach to functional generation and makes the following three contributions.

1. Introduces an alternative, "additive" way to generate trading strategies functionally, and compares it to E.R. Fernholz' "multiplicative" functional generation of trading strategies. Given a sufficiently large time horizon $T_* > 0$ and suitable conditions on the volatility structure of the market, the multiplicative version yields, for each $T > T_*$, a portfolio that strongly outperforms the market on $[0, T]$; this portfolio, however, depends on T. By contrast, the additive version yields a *single* trading strategy that outperforms the market over all horizons $[0, T]$ for $T \geq T_*$.

2. Extends the class of functions that generate a trading strategy. This paper introduces the notion of a regular function. Such a function can generate a trading strategy. Modulo necessary technical conditions on the boundary behavior, concave functions are shown to be regular. This weakens the assumption of twice continuous differentiability, normally used in the literature of the subject, and provides a unified framework for standard and rank-based generation, a long-standing open issue.

3. Weakens the assumptions on the market model. Functional generation is shown to work in markets where asset prices are continuous semimartingales that also might completely devaluate. Moreover, major technical assumptions in rank-based generation are removed; for example, it is not necessary anymore to exclude models where the times when two asset prices are identical have strictly positive Lebesgue measure.

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