Mini Course
Arbitrage Opportunities Relative to the Market

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An earlier summer school...

Outline

Some references

- Fernholz, E. R. (2002). Stochastic Portfolio Theory, Springer.
- Fernholz, R. and Karatzas, I. (2009). Stochastic Portfolio Theory: an overview. In Bensoussan, A., editor, Handbook of Numerical Analysis.
- Fernholz, R., Karatzas, I., and Ruf, J. (2016). Volatility and arbitrage. Preprint.
- Karatzas, I. and Ruf, J. (2016). Trading strategies generated by Lyapunov functions. Preprint.
- Banner, A., Fernholz, R., Papathanakos, V., Ruf, J., and Schofield, D. (2016+). Diversification, volatility, and 'surprising' Alpha. In preparation.
- The papers cited in these references.

Stochastic Portfolio Theory (SPT)

A rich and flexible framework introduced by Bob Fernholz for analyzing portfolio behavior and equity market structure.

Two important components of SPT

Research in SPT focuses on two main areas.

1. Abstract markets: building models that reflect properties of real equity markets.
2. Arithmetics of returns: relevance of logarithmic returns, the role of diversification, constructing relative arbitrages.

In this mini course we will see only a very short overview of abstract markets. Instead we focus on some aspects of relative arbitrage.

General setup

- A probability space $(\Omega, \mathscr{F}, \mathrm{P})$ equipped with a right-continuous filtration $\mathfrak{F}$.
- $d \in \mathbb{N}$: number of assets at time zero. E.g., $d = 505$ (S&P 500) or $d = 8000$.

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- A probability space $(\Omega, \mathscr{F}, \mathrm{P})$ equipped with a right-continuous filtration $\mathfrak{F}$.
- $d \in \mathbb{N}$: number of assets at time zero. E.g., $d = 505$ (S&P 500) or $d = 8000$.
- Nonnegative continuous semimartingales $S_1(\cdot), \cdots, S_d(\cdot)$, representing the capitalization (stock-price, multiplied by the number of shares outstanding) of each company.
- For example, $S_i(\cdot)$ might be an Itô process of the form

$$\mathrm{d}S_i(t) = S_i(t)\left[b_i(t)\mathrm{d}t + \sum_{\nu=1}^{N} \sigma_{i,\nu}(t)\mathrm{d}W_\nu(t)\right],$$

  where $W(\cdot)$ denotes an N–dimensional vector of independent Brownian motions.
- For simplicity, there is no traded bond.

Market weights

- Let $\Sigma(t)$ denote the total market capitalization at time t; i.e.:

$$\Sigma(t) = S_1(t) + \cdots + S_d(t).$$

- We shall assume, throughout, that $S_1(t) + \cdots + S_d(t) > 0$.

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- We shall assume, throughout, that $S_1(t) + \cdots + S_d(t) > 0$.
- Then the relative market weights $\mu_1(\cdot), \cdots, \mu_d(\cdot)$ of each asset are given by

$$\mu_i(t) = \frac{S_i(t)}{\Sigma(t)}$$

and take values in

$$\mathbf{\Delta}^d = \left\{ \left(x_1, \cdots, x_d\right)' \in [0,1]^d \; : \; \sum_{i=1}^{d} x_i = 1 \right\}.$$

An important empirical property of equity markets

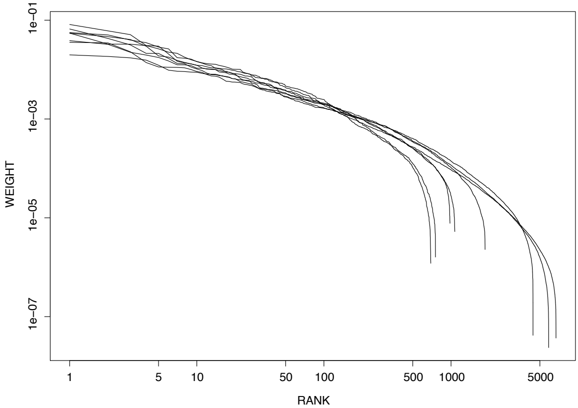

Figure: The capital distribution curve — Market weights $\mu_i(\cdot)$ against ranks on logarithmic scale, 1929 - 1999 — Thanks to Bob Fernholz!

Abstract market models, I

- It is not easy to write down tractable mathematical models for $S_1(\cdot), \cdots, S_d(\cdot)$ whose capital market curves (and especially their dynamics) resemble the empirical ones.
- In financial mathematics very good and helpful models have been developed that yield realistic dynamics for one-dimensional stock dyncamics.
  (Samuelson-Black-Scholes-Merton, stochastic volatility, rough volatility, ...).
- Unfortunately, just combining such models does not yield realistic market models.

Abstract market models, II

- The two most important market models in SPT:
  - volatitility-stabilized market model:

$$\mathrm{d}\log S_i(t) = \frac{\alpha}{2\mu_i(t)}\mathrm{d}t + \frac{1}{\sqrt{\mu_i}}\mathrm{d}W_i(t),$$

    where $\alpha > 0$, with the alternative representation

$$\mathrm{d}S_i(t) = \frac{1+\alpha}{2}(S_1(t) + \cdots S_d(t))\mathrm{d}t + \sqrt{S_i(t)(S_1(t) + \cdots S_d(t))}\mathrm{d}W_i$$

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  - rank-based models: drift and volatility of S_i depend also on the relative rank that the i-th company takes in the market.
- Lots of interesting mathematical properties and questions. E.g, relationships to Bessel processes, the Wright-Fisher diffusion model of population genetics, interacting particle systems, asymptotics for $d \uparrow \infty$, ...
- However, no time left here to discuss this further :-(

Outline

Returns: An MBA overview

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- Suppose we wish to calculate the average annual return of an investment over several years, where the annual returns are given by $r_1, r_2, \ldots, r_n$.
- Several common methods are available.
  1. *Arithmetic return:* $\frac{1}{n}\Big((1 + r_1) + \cdots + (1 + r_n)\Big) - 1.$
  2. *Geometric return:* $\sqrt[n]{(1 + r_1) \times \cdots \times (1 + r_n)} - 1.$
  3. *Logarithmic return:* $\frac{1}{n}\Big(\log(1 + r_1) + \cdots + \log(1 + r_n)\Big).$

Some remarks on computing the average returns

1. *Arithmetic return:* $\frac{1}{n}\Big((1+r_1)+\cdots+(1+r_n)\Big)-1$.
   Used in Modern Portfolio Theory. Compatible with the linear models used to calculate the Sharpe ratio and beta. But leads to absurd estimates in some cases.

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Jensen's inequality yields

$$\text{arithmetic return} \;\geq\; \text{geometric return} \;\geq\; \text{logarithmic return.}$$

The dynamics of return

Let $S(t)$ represent the price of a stock at time t. Assume that

$$\mathrm{d}S(t) = S(t)\Big[b(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\Big].$$

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- Itô's formula implies that

$$\mathrm{d}\log S(t) = g(t)\,\mathrm{d}t + \sigma(t)\mathrm{d}W(t),$$

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  g is called the *rate of log-return*, or *growth rate*, of S.
- The process g determines the long-term behavior of S:

  $$\lim_{T\uparrow\infty} \frac{1}{T}\Big(\log S(T) - \int_0^T g(t)\mathrm{d}t\Big) = 0$$

  (under appropriate assumptions).

Portfolio return and log-return

Suppose we have assets $S_1, \ldots, S_d$ and a portfolio π with weights $\pi_1(t) + \cdots + \pi_d(t) = 1$ and value $V^\pi(t)$ at time t. Then the portfolio return satisfies

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_i \pi_i(t) \frac{\mathrm{d}S_i(t)}{S_i(t)}$$

(Markowitz (1952)).

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(Markowitz (1952)). The analogous equation for log-return is

$$\mathrm{d} \log V^\pi(t) = \sum_i \pi_i(t) \, \mathrm{d} \log S_i(t) + \gamma_\pi^*(t) \, \mathrm{d}t,$$

where γ_π^* is called the *excess growth rate* of the portfolio (Fernholz and Shay (1982)).

The dynamics of portfolio log-return

$$
\begin{aligned}
\mathrm{d}\log V^{\pi}(t) &= \frac{\mathrm{d}V^{\pi}(t)}{V^{\pi}(t)} - \frac{1}{2}\sigma_{\pi}^{2}(t)\,\mathrm{d}t \\
&= \sum_{i} \pi_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} - \frac{1}{2}\sigma_{\pi}^{2}(t)\,\mathrm{d}t \\
&= \sum_{i} \pi_i(t)\Big(\mathrm{d}\log S_i(t) + \frac{1}{2}\sigma_i^{2}(t)\,\mathrm{d}t\Big) - \frac{1}{2}\sigma_{\pi}^{2}(t)\,\mathrm{d}t \\
&= \sum_{i} \pi_i(t)\mathrm{d}\log S_i(t) + \gamma_{\pi}^{*}(t)\,\mathrm{d}t,
\end{aligned}
$$

with $$\gamma_{\pi}^{*}(t) = \frac{1}{2}\Big(\sum_{i} \pi_i(t)\sigma_i^{2}(t) - \sigma_{\pi}^{2}(t)\Big).$$

The excess growth rate

The excess growth rate measures the efficacy of diversification in a portfolio.

$$\begin{aligned}
\gamma_\pi^*(t) &= \frac{1}{2}\Big(\sum_i \pi(t)\sigma_i^2(t) - \sigma_\pi^2(t)\Big) \\
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- The excess growth rate is higher for portfolios of volatile stocks with low correlation. If all else is equal, a higher excess growth rate will increase the long-term performance of a portfolio.
- The formulas assume an implicit rebalancing to the target weights $\pi(t)$ at an infinitesimal time scale reflecting the underlying Brownian motion. Without rebalancing, there's no excess growth.

Decomposition of portfolio log-return

There is a natural decomposition of the log-return of a portfolio into two components. For the interval $[0, T]$,

$$\text{Log-return} = \int_0^T \sum_i \pi_i(t)\,\mathrm{d}\log S_i(t) + \int_0^T \gamma_\pi^*(t)\mathrm{d}t.$$

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Hence the log return of a portfolio is not only the average of the log returns of its constituents but an additional term appears. (In financial marketing, this phenomenon is sometimes called "volatility harvesting," "volatility pumping," "smart beta, " "volatility capture," "rebalancing premium," or "diversification premium")

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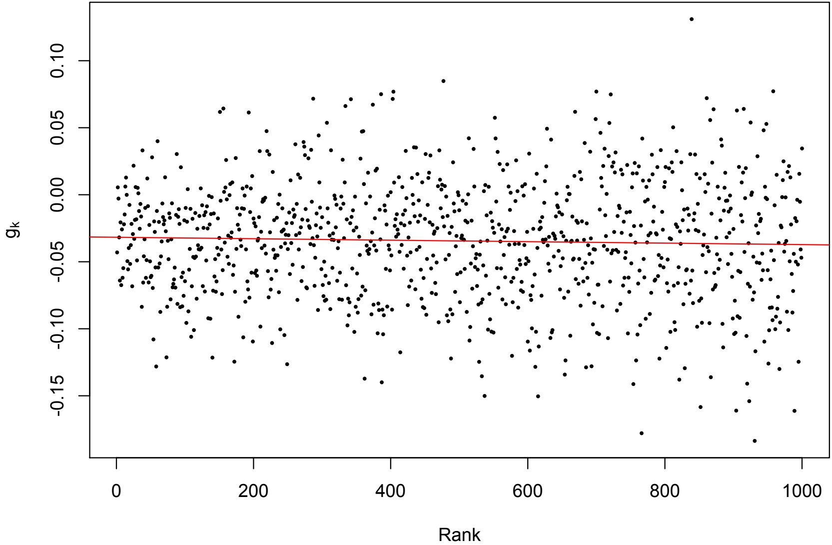
The first question of this mini course is: how can we construct portfolios with a high log-return, that are additionally tractable.

An Excursion: Rank-based analysis of logarithmic returns

Let us briefly look at the "size effect" of the top 1000 stocks. Accordingly, let $r_t(i)$ be the rank of $S_i(t)$, and define the *average rank-based growth rates* $\boldsymbol{g}_k$ over $[0, T]$ by

$$\boldsymbol{g}_k = \frac{1}{T} \int_0^T \sum \mathbf{1}_{\{r_t(i)=k\}} \mathrm{d} \log S_i(t).$$

Estimated g_k, 1964–2012 (relative)



Outline

Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
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- No bond.

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- No frictions; in particular, "small investor" and no trading costs (!)

Trading strategies

For an $\mathbb{R}^d$–valued predictable process $\vartheta(\cdot)$ write

$$V^{\vartheta}(t;S) := \sum_{i=1}^{d} \vartheta_i(t) S_i(t).$$

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Definition

Suppose that $\vartheta(\cdot) \in \mathscr{L}(S)$ ($\vartheta(\cdot)$ is integrable with respect to $S(\cdot)$) and that

$$V^{\vartheta}(T;S) - V^{\vartheta}(0;S) = \int_0^T \langle \vartheta(t), \mathrm{d}S(t) \rangle$$

holds. Then $\vartheta(\cdot)$ is called trading strategy (with respect to $S(\cdot)$) and we write $\vartheta(\cdot) \in \mathscr{T}(S)$.

Change of numéraire

Recall

$$V^\vartheta(t;S) := \sum_{i=1}^d \vartheta_i(t)S_i(t); \qquad V^\vartheta(t;\mu) := \sum_{i=1}^d \vartheta_i(t)\mu_i(t).$$

Proposition

An $\mathbb{R}^d$–valued process ϑ is a trading strategy with respect to S if and only if it is a trading strategy with respect to μ.

In particular, $\mathscr{T}(\mathcal{S}) = \mathscr{T}(\mu)$ and

$$V^\vartheta(\cdot\,;S) = \Sigma(\cdot)V^\vartheta(\cdot\,;\mu)\,.$$

Notational convention below: $V^\vartheta(\cdot) := V^\vartheta(\cdot\,;\mu)$.

Arbitrage relative to the market

Definition

A trading strategy $\varphi(\cdot)$ is a relative arbitrage with respect to the market portfolio over the time horizon $[0, T]$ if

$$V^{\varphi}(0; S) = \Sigma(0); \qquad V^{\varphi}(t; S) \geq 0;$$

and

$$\mathrm{P}\left(V^{\varphi}(T; S) \geq \Sigma(T)\right) = 1; \qquad \mathrm{P}\left(V^{\varphi}(T; S) > \Sigma(T)\right) > 0.$$

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Alternatively:

$$V^{\varphi}(0; \mu) = 1; \qquad V^{\varphi}(t; \mu) \ \geq \ 0$$

and

$$\mathrm{P}\Big(V^{\varphi}(T; \mu) \ \geq 1\Big) = 1; \qquad \mathrm{P}\Big(V^{\varphi}(T; \mu) \ > 1\Big) > 0.$$

Deflator

Some results below require the notion of a deflator for $\mu(\cdot)$.

Definition

A deflator is a continuous, adapted, strictly positive process $Z(\cdot)$ with $Z(0) = 1$ for which

$$\text{all products} \quad Z(\cdot)\,\mu_i(\cdot)\,, \quad i = 1, \cdots, d \quad \text{are local martingales.}$$

Warming up ...

From integrands to trading strategies

- Given: $\vartheta(\cdot) \in \mathcal{L}(\mu)$.
- Recall

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- Consider the quantity

$$Q^{\vartheta}(\cdot) = V^{\vartheta}(\cdot) - V^{\vartheta}(0) - \int_0^{\cdot} \left\langle \vartheta(t), \mathrm{d}\mu(t) \right\rangle,$$

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  which measures the "defect of self-financibility" of $\vartheta(\cdot)$.
- If $Q^{\vartheta}(\cdot) = 0$ fails, the process $\vartheta(\cdot)$ is not a trading strategy.
- However, for any $c \in \mathbb{R}$,

$$\varphi_i(\cdot) = \vartheta_i(\cdot) - Q^{\vartheta}(\cdot) + c$$

  is a trading strategy and satisfies

$$V^{\varphi}(\cdot) \;=\; V^{\vartheta}(0) + c + \int_0^{\cdot} \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle.$$

Regular functions

Definition

A continuous function $\boldsymbol{G} : \operatorname{supp}(\mu) \to \mathbb{R}$ is *regular* if

1. there exists a measurable function

$$D\boldsymbol{G} = \big(D_1\boldsymbol{G}, \cdots, D_d\boldsymbol{G}\big)^{\mathsf{T}} : \operatorname{supp}(\mu) \to \mathbb{R}^d$$

such that the process $\boldsymbol{\vartheta}(\cdot) \in \mathcal{L}(\mu)$ with

$$\boldsymbol{\vartheta}_i(\cdot) = D_i\boldsymbol{G}\big(\mu(\cdot)\big), \qquad i = 1, \cdots, d;$$

2. the continuous, adapted process

$$\Gamma^{\boldsymbol{G}}(\cdot) = \boldsymbol{G}\big(\mu(0)\big) - \boldsymbol{G}\big(\mu(\cdot)\big) + \int_0^{\cdot} \langle \vartheta(t), \mathrm{d}\mu(t)\rangle$$

has finite variation on compact intervals.

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Remark:
Assume there exists a deflator $Z(\cdot)$ for $\mu(\cdot)$ and $\boldsymbol{G}$ is nonnegative a Lyapunov function for $\mu(\cdot)$.

Lyapunov functions

Definition

We call a regular function $\boldsymbol{G}$ a *Lyapunov function* if the process $\Gamma^{\boldsymbol{G}}(\cdot)$ is non-decreasing.

Remark:

Assume there exists a deflator $Z(\cdot)$ for $\mu(\cdot)$ and $\boldsymbol{G}$ is nonnegative a Lyapunov function for $\mu(\cdot)$. Then

$$Z(\cdot)\boldsymbol{G}(\mu(\cdot)) = Z(\cdot)\left(\boldsymbol{G}(\mu(0)) + \int_0^{\cdot} \sum_{i=1}^{d} D_i \boldsymbol{G}(\mu(t))\mathrm{d}\mu_i(t)\right) - \int_0^{\cdot} \Gamma^{\boldsymbol{G}}(t)\mathrm{d}Z(t) - \int_0^{\cdot} Z(t)\mathrm{d}\Gamma^{\boldsymbol{G}}(t)$$

is a P–local supermartingale, thus also a P–supermartingale as it is nonnegative.

An example for regular and Lyapunov functions

Example

For instance, if $\boldsymbol{G}$ is of class $\mathcal{C}^2$, in a neighbourhood of $\boldsymbol{\Delta}^d$, Itô's formula yields

$$\Gamma^{\boldsymbol{G}}(\cdot) = -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^{\cdot} D_{ij}^2\boldsymbol{G}\big(\mu(t)\big)\,\mathrm{d}\langle \mu_i, \mu_j\rangle(t)$$

with $D_{ij}^2\boldsymbol{G} = \frac{\partial^2 \boldsymbol{G}}{\partial x_i\,\partial x_j}$.

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with $D^2_{ij}\boldsymbol{G} = \frac{\partial^2 \boldsymbol{G}}{\partial x_i\,\partial x_j}$. Therefore, $\boldsymbol{G}$ is regular; if it is also concave, then $\boldsymbol{G}$ becomes a Lyapunov function.

Functionally generated strategies (additive case)

For a regular function $\boldsymbol{G}$ consider the trading strategy $\varphi(\cdot)$ with

$$\varphi_i(\cdot) = D_i \boldsymbol{G}(\mu(\cdot)) - Q^{\vartheta}(\cdot) + c$$

with $c = \boldsymbol{G}\big(\mu(0)\big) - \sum_{j=1}^{d} \mu_j(0)\, D_j \boldsymbol{G}\big(\mu(0)\big)$.

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We say that the trading strategy $\varphi(\cdot)$ is *additively generated* by the regular function $\boldsymbol{G}$. $\square$

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Proposition

The value process generated by the strategy $\varphi(\cdot)$ *is given by*

$$V^{\varphi}(\cdot) = \boldsymbol{G}\big(\mu(\cdot)\big) + \Gamma^{\boldsymbol{G}}(\cdot).$$

and

$$\varphi_i(\cdot) \;=\; D_i \boldsymbol{G}\big(\mu(\cdot)\big) + \Gamma^{\boldsymbol{G}}(\cdot) + \boldsymbol{G}\big(\mu(\cdot)\big) - \sum_{j=1}^{d} \mu_j(\cdot) D_j \boldsymbol{G}\big(\mu(\cdot)\big).$$

Functionally generated strategies (multiplicative case)

For a regular function $\boldsymbol{G}$ such that $1/\boldsymbol{G}(\mu(\cdot))$ is locally bounded, consider

$$\widetilde{\boldsymbol{\vartheta}}_i(\cdot) = \boldsymbol{\vartheta}_i(\cdot) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^{\boldsymbol{G}}(t)}{\boldsymbol{G}\left(\mu(t)\right)}\right) = D_i\boldsymbol{G}(\mu(\cdot)) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^{\boldsymbol{G}}(t)}{\boldsymbol{G}\left(\mu(t)\right)}\right)$$

and the trading strategy $\psi(\cdot)$ with

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We say that the trading strategy $\psi(\cdot)$ is *multiplicatively generated* by the regular function $\boldsymbol{G}$. □

Proposition (Master equation of Fernholz, 1999, 2002)

The value process generated by the strategy $\psi(\cdot)$ is given by

$$V^{\psi}(\cdot) = \boldsymbol{G}\left(\mu(\cdot)\right) \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^{\boldsymbol{G}}(t)}{\boldsymbol{G}\left(\mu(t)\right)}\right) > 0.$$

Functionally generated arbitrage (additive case)

Theorem

Fix a Lyapunov function $\boldsymbol{G} : \mathbf{supp}\,(\mu) \to [0, \infty)$ *satisfying* $\boldsymbol{G}(\mu(0)) = 1$, *and suppose that for* $T_* > 0$ *we have*

$$\mathsf{P}\big(\Gamma^{\boldsymbol{G}}(T_*) > 1\big) = 1.$$

Then the additively generated strategy $\varphi(\cdot)$ *strongly outperforms the market over every time-horizon* $[0, T]$ *with* $T \geq T_*$.

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Idea of proof:

$$V^{\varphi}(T) = \boldsymbol{G}\big(\mu(T)\big) + \Gamma^{\boldsymbol{G}}(T) \geq \Gamma^{\boldsymbol{G}}(T_*) > 1.$$

An important remark

As long as the market model $\mu(\cdot)$ satisfies

$$\mathrm{P}\big(\Gamma^{\boldsymbol{G}}(T_*) > 1\big) = 1$$

the arbitrage strategy

$$\varphi_i(\cdot) \;=\; D_i\boldsymbol{G}\big(\mu(\cdot)\big) + \Gamma^{\boldsymbol{G}}(\cdot) + \boldsymbol{G}\big(\mu(\cdot)\big) - \sum_{j=1}^{d} \mu_j(\cdot) D_j\boldsymbol{G}\big(\mu(\cdot)\big).$$

does *n*ot depend on the model parameters or the time horizon.

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Then there exists a constant $d > 0$ *such that the trading strategy* $\psi^{(d)}(\cdot)$, *multiplicatively generated by the regular function*

$$\boldsymbol{G}^{(d)} = \frac{\boldsymbol{G} + d}{1 + d}$$

strongly outperforms the market over every time-horizon $[0, T]$ *with* $T \geq T_*$.

Example: entropy function

- Consider the (nonnegative) Gibbs *entropy function*

$$\boldsymbol{H}(x) = \sum_{j=1}^{d} x_j \log\left(\frac{1}{x_j}\right).$$

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- Assuming that either $\mu(\cdot) \in \boldsymbol{\Delta}_+^d$ or the existence of an SDF $Z(\cdot)$, $\boldsymbol{H}$ is a Lyapunov function with nondecreasing

$$\Gamma^{\boldsymbol{H}}(\cdot) = \frac{1}{2}\sum_{j=1}^{d}\int_0^{\cdot} \mathbf{1}_{\{\mu_j(t)>0\}} \frac{\mathrm{d}\langle \mu_j \rangle(t)}{\mu_j(t)}.$$

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- So-called *cumulative excess growth of the market.*
- If

$$\mathrm{P}\left(\Gamma^{\boldsymbol{H}}(t) \geq \eta\, t\,,\ \ \forall\ t \geq 0\right) = 1$$

for some real constant $\eta > 0$, then arbitrage exists over any time-horizon $[0, T]$ with $T > \frac{\boldsymbol{H}(\mu(0))}{\eta}$.

Cumulative excess growth of the market

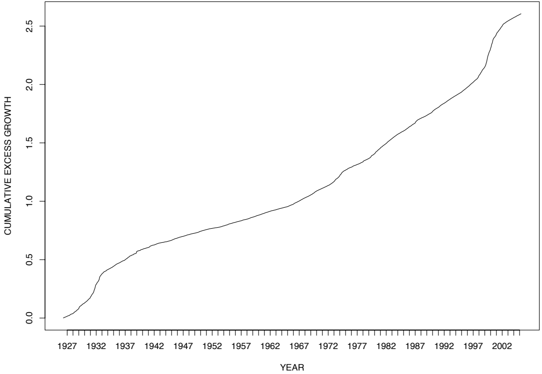

Figure: Cumulative Excess Growth $\Gamma^{\boldsymbol{H}}(\cdot)$ for the U.S. Equity Market, during the period 1926 –1999. — Thanks to Bob Fernholz!

Discussion: entropy function

Recall:

$$\Gamma^{\boldsymbol{H}}(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_0^{\cdot} \frac{\mathrm{d}\langle \mu_j \rangle(t)}{\mu_j(t)};$$

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- Under this condition there exists one (*horizon-independent*) trading strategy, which is an arbitrage over any time-horizon $[0, T]$ with
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- Fernholz & Karatzas (2005) asked whether then there is also arbitrage possible over any time horizon.

Concave functions are Lyapunov

Theorem

A continuous function $\boldsymbol{G} : \operatorname{supp}(\mu) \to \mathbb{R}$ *is Lyapunov if it can be extended to a continuous, concave function on*

1. $\boldsymbol{\Delta}_+^d = \boldsymbol{\Delta}^d \cap (0,1)^d$ *and*

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3. $\boldsymbol{\Delta}^d$, *and there exists a deflator* $Z(\cdot)$.

Functions based on rank

- "Rank operator" $\mathfrak{R} : \boldsymbol{\Delta}^d \mapsto \mathbb{W}^d$, where

$$\mathbb{W}^d = \Big\{ \big(x_1, \cdots , x_d \big) \in \boldsymbol{\Delta}^d : 1 \geq x_1 \geq x_2 \geq \cdots \geq x_{d-1} \geq x_d \geq 0 \Big\}.$$

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- Then $\overline{\boldsymbol{\mu}}(\cdot)$ can be interpreted again as a market model; however, without a deflator.

Theorem

Consider a function $\overline{\boldsymbol{G}} : \mathbf{supp}\,(\overline{\boldsymbol{\mu}}) \to \mathbb{R}$, which is regular for $\overline{\boldsymbol{\mu}}(\cdot)$. Then $\boldsymbol{G} = \overline{\boldsymbol{G}} \circ \mathfrak{R}$ is a regular function for $\mu(\cdot)$.

Remarks on the proof that a concave function is Lyapunov

Dellacherie & Meyer:

REMARKS. (a) The same argument would show that, if $X^1, X^2, \ldots, X^n$ are semimartingales and f is a convex function on $\mathbb{R}^n$, the process $f(X_t^1, \ldots, X_t^n)$ is a semimartingale; it is only necessary to know that f is locally Lipschitz, which is true, but rather more delicate[1] than on $\mathbb{R}$.

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Also, Rockefeller has a proof.

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- Bouleau (1981, 1984): If $\boldsymbol{G}$ is twice continuously differentiable in some "open" $A \in \boldsymbol{\Delta}^d$,

$$\Gamma^{\boldsymbol{G}}(\cdot) = \frac{1}{2} \sum_{i,j=1}^{d} \int_0^{\cdot} \mathbf{1}_A(\mu(t)) D_{i,j} \boldsymbol{G}(\mu(t)) \,\mathrm{d}\langle \mu_i, \mu_j \rangle(t) + \int_0^{\cdot} \mathbf{1}_{\boldsymbol{\Delta}^d \setminus A}(\mu(t)) \,\mathrm{d}\Gamma^{\boldsymbol{G}}(t).$$

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- Our conjecture: quadratic covariation:

$$\Gamma^{\boldsymbol{G}}(\cdot) = \frac{1}{2} \sum_{i,j=1}^{d} [D_i \boldsymbol{G}(\mu(\cdot)), \mu_j(\cdot)].$$

Outline

Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
- No bond.
- Relative market weights modeled by nonnegative continuous semimartingales $\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_d(\cdot))$ taking values in

$$\boldsymbol{\Delta}^d = \left\{ \left(x_1, \cdots, x_d\right)' \in [0,1]^d \; : \; \sum_{i=1}^{d} x_i = 1 \right\}.$$

- No frictions; in particular, "small investor" and no trading costs (!)

Recalling regular functions

- *Regular function:* a continuous mapping $\boldsymbol{G} : \boldsymbol{\Delta}^d \to \mathbb{R}$ that satisfies a generalized Itô rule:

$$\boldsymbol{G}(\mu(\cdot)) = \boldsymbol{G}(\mu(0)) + \int_0^{\cdot} \sum_{i=1}^{d} D_i \boldsymbol{G}(\mu(t))\, \mathrm{d}\mu_i(t) - \Gamma^{\boldsymbol{G}}(\cdot),$$

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  where $\Gamma^{\boldsymbol{G}}(\cdot)$ which has finite variation on compact time-intervals.

- If $\boldsymbol{G}$ is smooth (we will assume this from now on) then

$$\Gamma^{\boldsymbol{G}}(\cdot) = -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^{\cdot} D^2_{i,j} \boldsymbol{G}\left(\mu(t)\right) \mathrm{d}\langle \mu_i, \mu_j \rangle(t).$$

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- If $\Gamma^{\boldsymbol{G}}(\cdot)$ is nondecreasing then $\boldsymbol{G}$ is called Lyapunov function.

Functionally generated trading strategies

- *Additive generation:* The process $\varphi^{\boldsymbol{G}}(\cdot)$ with components

$$\varphi_i^{\boldsymbol{G}}(\cdot) := D_i \boldsymbol{G}\big(\mu(\cdot)\big) + \Gamma^{\boldsymbol{G}}(\cdot) + \boldsymbol{G}\big(\mu(\cdot)\big) - \sum_{j=1}^{d} \mu_j(\cdot) D_j \boldsymbol{G}\big(\mu(\cdot)\big)$$

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  Then the process $\psi^{\boldsymbol{G}}(\cdot)$ with components

$$\psi_i^{\boldsymbol{G}}(\cdot) := Z^{\boldsymbol{G}}(\cdot)\left(1 + \frac{1}{\boldsymbol{G}\left(\mu(\cdot)\right)}\left(D_i \boldsymbol{G}\left(\mu(\cdot)\right) - \sum_{j=1}^{d} D_j \boldsymbol{G}\left(\mu(\cdot)\right) \mu_j(\cdot)\right)\right),$$

  is a trading strategy with $V^{\psi^{\boldsymbol{G}}}(\cdot) = Z^{\boldsymbol{G}}(\cdot)$.

Example: quadratic function

Consider

$$\boldsymbol{Q}(x) := 1 - \sum_{i=1}^{d} x_i^2, \qquad x \in \boldsymbol{\Delta}^d.$$

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$$\Gamma^{\boldsymbol{Q}}(\cdot) = \sum_i \langle \mu_i \rangle(\cdot).$$

- The additively generated strategy equals

$$\begin{aligned}\varphi_i^{\boldsymbol{Q}}(\cdot) &= D_i \boldsymbol{Q}\big(\mu(\cdot)\big) + \Gamma^{\boldsymbol{Q}}(\cdot) + \boldsymbol{Q}\big(\mu(\cdot)\big) - \sum_j \mu_j(\cdot) D_j \boldsymbol{Q}\big(\mu(\cdot)\big) \\ &= -2\mu_i(\cdot) + \sum_j \langle \mu_j \rangle(\cdot) + 1 + \sum_j \mu_j^2(\cdot).\end{aligned}$$

Relative arbitrage

Definition

Given a real constant $T > 0$, we say that a trading strategy $\vartheta(\cdot)$ is a *relative arbitrage with respect to the market* over the time horizon $[0, T]$ if $V^{\vartheta}(0) = 1$, $V^{\vartheta}(\cdot) \geq 0$, and

$$\mathsf{P}\big(V^{\vartheta}(T) \geq 1\big) = 1, \qquad \mathsf{P}\big(V^{\vartheta}(T) > 1\big) > 0.$$

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If in fact $\mathsf{P}\big(V^{\vartheta}(T) > 1\big) = 1$ holds, this relative arbitrage is called *strong*.

Strong relative arbitrage over sufficiently long time horizons

Theorem

Suppose that $\boldsymbol{G} : \boldsymbol{\Delta}^d \to [0, \infty)$ is a regular function with $\boldsymbol{G}(\mu(0)) > 0$ such that

$$\mathrm{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \ \text{ is nondecreasing}\right) = 1$$

for some $\eta > 0$. Then strong relative arbitrage with respect to the market exists, over any time horizon $[0, T]$ with

$$T > \frac{\boldsymbol{G}(\mu(0))}{\eta}.$$

Arbitrage over arbitrary time horizons??

Consider the condition

$$\mathrm{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \;\text{ is nondecreasing}\right) = 1$$

for some $\eta > 0$.

Does there exist arbitrage with respect to the market portfolio over time horizon $[0, T]$, for any $T > 0$?

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Does there exist arbitrage with respect to the market portfolio over time horizon $[0, T]$, for any $T > 0$?

Answer: Under additional assumptions, yes. In general, no.

Outline of the rest of this lecture

$$\mathrm{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \text{ is nondecreasing}\right) = 1$$

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1. Existence of short-term relative arbitrage
   1.1 Existence of strong relative arbitrage
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2. Lack of short-term relative arbitrage

One asset with sufficient variation

Theorem

Suppose there exists a constant $\eta > 0$ *such that* $\langle \mu_1 \rangle(t) \geq \eta t$ *holds on the stochastic interval* $[\![0, \mathscr{D}^*[\![$ *with*

$$\mathscr{D}^* := \inf \left\{ t \geq 0 : \ \mu_1(t) \leq \frac{\mu_1(0)}{2} \right\}.$$

Then, given any real number $T > 0$ *there exists a long-only trading strategy* $\varphi(\cdot)$ *which is strong relative arbitrage with respect to the market over the time horizon* $[0, T]$.

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Then, given any real number $T > 0$ there exists a long-only trading strategy $\varphi(\cdot)$ which is strong relative arbitrage with respect to the market over the time horizon $[0, T]$.

Some intuition why the theorem could be true:

- $\mu_1(\cdot)$ is bounded by above from one.
- There needs to be a very large drift that forces $\mu(\cdot)$ to not become larger than one.
- Hence, an arbitrage strategy should under-invest into asset 1.

Proof

- It suffices to argue that the market $\nu(\cdot) = \mu(\cdot \wedge \mathscr{D}^*)$ allows for arbitrage.

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$$\boldsymbol{F}(x) := x_1^q, \qquad x \in \boldsymbol{\Delta}^d.$$

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- For $q \geq 1$, consider the regular function

$$\boldsymbol{F}(x) := x_1^q, \qquad x \in \boldsymbol{\Delta}^d.$$

- $\boldsymbol{F}$ generates multiplicatively the strategy

$$\psi_1^{\mathbf{F}}(\cdot) = \left(\frac{q}{\nu_1(\cdot)} + 1 - q\right) Z^{\mathbf{F}}(\cdot); \qquad \psi_i^{\mathbf{F}}(\cdot) = (1-q) Z^{\mathbf{F}}(\cdot), \quad i \geq 2,$$

where

$$V^{\psi^{\boldsymbol{F}}}(\cdot) = Z^{\mathbf{F}}(\cdot) = \big(\nu_1(\cdot)\big)^q \exp\left(-\frac{1}{2} q(q-1) \int_0^{\cdot} \big(\nu_1(t)\big)^{-2} \mathrm{d}\langle \nu_1 \rangle(t)\right)$$

Proof (cont'd)

- Introduce now the trading strategy

$$\varphi_i(\cdot) = 1 + \big(\nu_1(0)\big)^q - \psi_i^{\boldsymbol{F}}(\cdot), \qquad i = 1, \cdots, d$$

with associated wealth process

$$V^{\varphi}(\cdot) = 1 + \big(\nu_1(0)\big)^q - Z^{\boldsymbol{F}}(\cdot).$$

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- On the event $\{\mathscr{D}^* \leq T\}$ we have

$$V^\varphi(T) \geq 1 + \big(\nu_1(0)\big)^q - \big(\nu_1(T)\big)^q = 1 + \big(\nu_1(0)\big)^q - \left(\frac{\nu_1(0)}{2}\right)^q > 1$$

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- On the event $\{\mathscr{D}^* > T\}$, for sufficiently large q, we have

$$\begin{aligned} V^{\varphi}(\cdot) &\geq 1 + \big(\nu_1(0)\big)^q - \exp\left(-\frac{1}{2}q(q-1)\langle\nu_1\rangle(T)\right) \\ &\geq 1 + \big(\nu_1(0)\big)^q - \left(\exp\left(-\frac{\eta}{2}(q-1)T\right)\right)^q > 1. \quad \square \end{aligned}$$

Outline of this section

$$\mathrm{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \text{ is nondecreasing}\right) = 1$$

for some $\eta > 0$.

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Time-homogeneous support

Theorem

Suppose that for a given generating function $\boldsymbol{G}$ *and appropriate real constants* $\eta > 0$ *and* $h \geq 0$,

$$\mathrm{P}\left(\textit{the mapping } [0,\infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \ \textit{ is nondecreasing}\right) = 1$$

is satisfied, along with the lower bound

$$\mathrm{P}\big(\boldsymbol{G}(\mu(t)) \geq h, \quad t \geq 0\big) = 1$$

and the "time homogeneous support" property

$$\mathrm{P}\Big(\boldsymbol{G}\big(\mu(t)\big) \in \big[h, h{+}\varepsilon\big), \ \textit{ for some } t \in [0,T]\Big) > 0, \quad \textit{for all } T > 0,\, \varepsilon > 0$$

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Then arbitrage relative to the market exists over the time horizon $[0, T]$, *for every real number* $T > 0$.

Proof

- Fix $T > 0$ and introduce the regular function

$$\boldsymbol{G}^{\star} := (\boldsymbol{G} - h)\frac{3}{\eta T},$$

and denote

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- Introduce the stopping time

$$\tau := \inf\left\{t \in \left[0, \frac{T}{2}\right] : \boldsymbol{G}(\mu(t)) < h + \frac{\eta T}{3}\right\}.$$

- Then

$$\mathrm{P}\left(\tau \leq \frac{T}{2}\right) > 0.$$

Proof (cont'd)

- Let $\varphi^\star(\cdot) := \varphi^{\boldsymbol{G}^\star}(\cdot)$ denote the additively generated trading strategy and consider

$$\varphi_i(\cdot) := 1 + \big(\varphi_i^\star(\cdot) - \boldsymbol{G}^\star(\mu(\tau)) - \Gamma^\star(\tau)\big)\mathbf{1}_{[\![\tau,\infty[\![}.$$

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- Hence, $V^\varphi(\cdot) \geq 0$ and $V^\varphi(T) \geq 3/2$ on the event $\{\tau \leq T/2\}$; moreover, $V^\varphi(T) = 1$ holds on $\{\tau > T/2\}$.

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- Hence, $V^\varphi(\cdot) \geq 0$ and $V^\varphi(T) \geq 3/2$ on the event $\{\tau \leq T/2\}$; moreover, $V^\varphi(T) = 1$ holds on $\{\tau > T/2\}$.
- Since $\mathrm{P}(\tau \leq T/2) > 0$the trading strategy $\varphi(\cdot)$ is relative arbitrage.

Outline of this section

$$\mathrm{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \text{ is nondecreasing}\right) = 1$$

for some $\eta > 0$.

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Some necessary notation

Recall

$$\Gamma^{\boldsymbol{Q}}(\cdot) = \sum_{i=1}^{d} \langle \mu_i \rangle (\cdot).$$

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Consider the sequence of stopping times

$$\mathscr{D}^n := \inf\left\{ t \geq 0 : \min_{1 \leq i \leq d} \mu_i(t) < \frac{1}{n} \right\}.$$

A strict nondegeneracy condition

Theorem

Suppose that for a given generating function $\boldsymbol{G}$ and $\eta > 0$,

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Moreover, suppose that there exists a deflator for $\mu(\cdot)$ and that the $d-1$ largest eigenvalues of the matrix-valued process $\alpha(\cdot)$ are bounded away from zero on $[\![0, \mathscr{D}^n[\![$ uniformly in (t,ω), for each $n \in \mathbb{N}$.

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Then relative arbitrage with respect to the market exists over $[0, T]$, for every real number $T > 0$.

Attention: If we replace "strict nondegeneracy" by nondegeneracy, then the theorem is not correct.

Outline of the proof

- By contradiction. Assume that $\mu(\cdot)$ is a martingale under an equivalent measure.
- Then prove that $\boldsymbol{G}(\mu(\cdot))$ reaches the minimum of $\boldsymbol{G}$ with positive probability arbitrarily close, arbitrarily fast. (repeated changes of measures)
- This then contradicts the time-homogeneous support property of the previous result.

Outline of this section

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The case of two assets

Proposition

Assume that $d = 2$ and that that for a given generating function $\boldsymbol{G}$ and $\eta > 0$,

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Then strong arbitrage relative to the market can be realized by a long-only trading strategy over the time horizon $[0, T]$, for any given real number $T > 0$.

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Idea of proof (does not necessarily yield strong arbitrage):

- Recall
$$\langle \mu_i, \mu_j \rangle(\cdot) = \int_0^{\cdot} \alpha_{i,j}(t)\, \mathrm{d}\Gamma^{\boldsymbol{Q}}(t).$$
- Note $\mu_2(\cdot) = 1 - \mu_1(\cdot)$, hence $\langle \mu_1, \mu_2 \rangle = -\langle \mu_1 \rangle(\cdot)$.

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- Note $\mu_2(\cdot) = 1 - \mu_1(\cdot)$, hence $\langle \mu_1, \mu_2 \rangle = -\langle \mu_1 \rangle(\cdot)$.
- Hence, $\alpha_{1,1}(\cdot) = \alpha_{2,2}(\cdot) = 1/2$ and $\alpha_{1,2}(\cdot) = \alpha_{2,1}(\cdot) = -1/2$, so the eigenvalues of the matrix $\alpha(\cdot)$ are then indeed 0 and 1.

Outline of this section

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Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
- Relative market weights modeled by nonnegative continuous semimartingales $\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_d(\cdot))$ taking values in

$$\mathbf{\Delta}^d = \left\{ \left(x_1, \cdots, x_d\right)' \in [0,1]^d \ : \ \sum_{i=1}^{d} x_i = 1 \right\}.$$

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for some $\eta > 0$.
- E.g., if $\boldsymbol{G}(x) = \boldsymbol{Q}(x) = 1 - \sum_{j=1}^{d} x_j^2$, then

$$\Gamma^{\boldsymbol{Q}}(t) = \sum_{j=1}^{d} \langle \mu_j \rangle (t).$$

The goal

- Goal: Construct process $\mu(\cdot)$ with each component a martingale such that $\Gamma^{\boldsymbol{Q}}(t) = t$, $t \in [0, T^*]$ for some $T^* > 0$.
- This then yields a counterexample since then no arbitrage is possible with respect to the market, but

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$$\mathrm{P}\left(\text{the mapping } [0, T^*) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - t \text{ is nondecreasing}\right) = 1.$$

- The process $\mu(\cdot)$ is not allowed to have full support (otherwise, we know by previous results that short-term arbitrage is possible).
- For $d = 2$, such a construction is impossible.

An Itô diffusion

- Consider $d = 3$ (three assets).
- Consider SDEs:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
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- Define $r(v) = \sqrt{\sum_{i=1}^{3}\left(v_i - \frac{1}{3}\right)^2}$.

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- Define $r(v) = \sqrt{\sum_{i=1}^{3}\left(v_i - \frac{1}{3}\right)^2}$.
- If $v_1(0) + v_2(0) + v_3(0) = 1$, then Itô's formula yields

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An Itô diffusion

- Consider $d = 3$ (three assets).
- Consider SDEs:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
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- A solution:

$$v_i(t) = \frac{1}{3} + \delta \mathrm{e}^{t/2} \cos\left(\Theta(t) + 2\pi\left(u + \frac{i-1}{3}\right)\right).$$

An Itô diffusion (cont'd)

- A slight modification:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}r(t)}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$

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$$(v_1(0), v_2(0), v_3(0)) \neq \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

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- Market model $\mu(\cdot)$: stopped version of $v(\cdot)$.

Possible extensions

- It is possible to modify the model for $\mu(\cdot)$ such that the covariance matrix has two positive eigenvalues (instead of one) – however, we know that it is not possible that both positive eigenvalues are bounded away from zero uniformly.

Possible extensions

- It is possible to modify the model for $\mu(\cdot)$ such that the covariance matrix has two positive eigenvalues (instead of one) – however, we know that it is not possible that both positive eigenvalues are bounded away from zero uniformly.
- For a general Lyapunov function $\boldsymbol{G}$, construct a market model $\mu(\cdot)$ with each component a martingale such that $\Gamma^{\boldsymbol{G}}(t) = t$, $t \in [0, T^*]$ for some $T^* > 0$.

Спасибо!