Mini Course
Arbitrage Opportunities Relative to the Market

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An earlier summer school...
Outline

1. Stochastic Portfolio Theory: an overview
   1.1 Abstract markets
   1.2 The arithmetics of returns

2. Functional generation of trading strategies

3. The question of arbitrage over arbitrary time horizons
Some references

- The papers cited in these references.
Stochastic Portfolio Theory (SPT)

A rich and flexible framework introduced by Bob Fernholz for analyzing portfolio behavior and equity market structure.
Two important components of SPT

Research in SPT focuses on two main areas.

1. Abstract markets: building models that reflect properties of real equity markets.

2. Arithmetics of returns: relevance of logarithmic returns, the role of diversification, constructing relative arbitrages.

In this mini course we will see only a very short overview of abstract markets. Instead we focus on some aspects of relative arbitrage.
General setup

• A probability space \((\Omega, \mathcal{F}, P)\) equipped with a right-continuous filtration \(\mathcal{F}\).

• \(d \in \mathbb{N}\) : number of assets at time zero. E.g., \(d = 505\) (S&P 500) or \(d = 8000\).

• Nonnegative continuous semimartingales \(S_1(\cdot), \cdots, S_d(\cdot)\), representing the capitalization (stock-price, multiplied by the number of shares outstanding) of each company.

• For example, \(S_i(\cdot)\) might be an Itô process of the form

\[
dS_i(t) = S_i(t) \left[ b_i(t) \, dt + \sum_{\nu=1}^{N} \sigma_{i,\nu}(t) \, dW_{\nu}(t) \right],
\]

where \(W(\cdot)\) denotes an \(N\)-dimensional vector of independent Brownian motions.

• For simplicity, there is no traded bond.
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Market weights

• Let $\Sigma(t)$ denote the total market capitalization at time $t$; i.e.:

$$\Sigma(t) = S_1(t) + \cdots + S_d(t).$$

• We shall assume, throughout, that $S_1(t) + \cdots + S_d(t) > 0$. 
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• We shall assume, throughout, that $S_1(t) + \cdots + S_d(t) > 0$.

• Then the relative market weights $\mu_1(\cdot), \cdots, \mu_d(\cdot)$ of each asset are given by

$$\mu_i(t) = \frac{S_i(t)}{\Sigma(t)}$$

and take values in

$$\Delta^d = \left\{(x_1, \cdots, x_d)' \in [0, 1]^d : \sum_{i=1}^{d} x_i = 1 \right\}.$$
An important empirical property of equity markets

Figure: The capital distribution curve — Market weights $\mu_i(\cdot)$ against ranks on logarithmic scale, 1929 - 1999 — Thanks to Bob Fernholz!
• It is not easy to write down tractable mathematical models for $S_1(\cdot), \ldots, S_d(\cdot)$ whose capital market curves (and especially their dynamics) resemble the empirical ones.

• In financial mathematics very good and helpful models have been developed that yield realistic dynamics for one-dimensional stock dynamics. (Samuelson-Black-Scholes-Merton, stochastic volatility, rough volatility, ...).

• Unfortunately, just combining such models does not yield realistic market models.
Abstract market models, II

- The two most important market models in SPT:
  - Volatility-stabilized market model:
    
    \[
    d \log S_i(t) = \frac{\alpha}{2\mu_i(t)} dt + \frac{1}{\sqrt{\mu_i}} dW_i(t),
    \]

    where \( \alpha > 0 \), with the alternative representation

    \[
    dS_i(t) = \frac{1 + \alpha}{2} (S_1(t) + \cdots S_d(t)) dt + \sqrt{S_i(t)(S_1(t) + \cdots S_d(t))} dW_i(t)
    \]

  - Rank-based models: drift and volatility of \( S_i \) depend also on the relative rank that the \( i \)-th company takes in the market.

- Lots of interesting mathematical properties and questions. E.g., relationships to Bessel processes, the Wright-Fisher diffusion model of population genetics, interacting particle systems, asymptotics for \( d \uparrow \infty \), ...

- However, no time left here to discuss this further :-(...
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Returns: An MBA overview

- The classical definition of return on an investment is

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\text{Return} = \frac{\text{final value} - \text{initial value}}{\text{initial value}}.
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- Suppose we wish to calculate the average annual return of an investment over several years, where the annual returns are given by \( r_1, r_2, \ldots, r_n \).

- Several common methods are available.
  1. Arithmetic return: \( \frac{1}{n} \left( (1 + r_1) + \cdots + (1 + r_n) \right) - 1. \)
  2. Geometric return: \( \sqrt[n]{(1 + r_1) \times \cdots \times (1 + r_n)} - 1. \)
  3. Logarithmic return: \( \frac{1}{n} \left( \log(1 + r_1) + \cdots + \log(1 + r_n) \right). \)
Some remarks on computing the average returns

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   Used in Modern Portfolio Theory. Compatible with the linear models used to calculate the Sharpe ratio and beta. But leads to absurd estimates in some cases.
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2. *Geometric return:* \( \sqrt[n]{(1 + r_1) \times \cdots \times (1 + r_n)} - 1. \)
   Sometimes very difficult to compute.

3. *Logarithmic return:* \[ \frac{1}{n} \left( \log(1 + r_1) + \cdots + \log(1 + r_n) \right). \]
   Used in stochastic portfolio theory. Jensen's inequality yields arithmetic return \( \geq \) geometric return \( \geq \) logarithmic return.
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Some remarks on computing the average returns

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Jensen’s inequality yields

\[
\text{arithmetic return} \geq \text{geometric return} \geq \text{logarithmic return}.
\]
The dynamics of return

Let $S(t)$ represent the price of a stock at time $t$. Assume that

$$dS(t) = S(t) \left[ b(t) dt + \sigma(t) dW(t) \right].$$

Then $b$ is called the rate of return of $S$. 
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- Itô’s formula implies that

$$d \log S(t) = g(t) dt + \sigma(t) dW(t),$$

where

$$g(t) = b(t) - \frac{1}{2} \sigma^2(t).$$

$g$ is called the rate of log-return, or growth rate, of $S$. 
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- The process \( g \) determines the long-term behavior of \( S \):

\[
\lim_{T \to \infty} \frac{1}{T} \left( \log S(T) - \int_0^T g(t)dt \right) = 0
\]

(under appropriate assumptions).
Portfolio return and log-return

Suppose we have assets $S_1, \ldots, S_d$ and a portfolio $\pi$ with weights $\pi_1(t) + \cdots + \pi_d(t) = 1$ and value $V^\pi(t)$ at time $t$. Then the portfolio return satisfies

$$\frac{dV^\pi(t)}{V^\pi(t)} = \sum_i \pi_i(t) \frac{dS_i(t)}{S_i(t)}$$

(Markowitz (1952)).
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(Markowitz (1952)). The analogous equation for log-return is

$$d \log V^\pi(t) = \sum_i \pi_i(t) d \log S_i(t) + \gamma^*_\pi(t) dt,$$

where $\gamma^*_\pi$ is called the excess growth rate of the portfolio (Fernholz and Shay (1982)).
The dynamics of portfolio log-return

\[ \text{d } \log V^\pi(t) = \frac{\text{d}V^\pi(t)}{V^\pi(t)} - \frac{1}{2}\sigma_\pi^2(t)\,\text{d}t \]

\[ = \sum_i \pi_i(t) \frac{\text{d}S_i(t)}{S_i(t)} - \frac{1}{2}\sigma_\pi^2(t)\,\text{d}t \]

\[ = \sum_i \pi_i(t) \left( \text{d} \log S_i(t) + \frac{1}{2}\sigma_i^2(t)\,\text{d}t \right) - \frac{1}{2}\sigma_\pi^2(t)\,\text{d}t \]

\[ = \sum_i \pi_i(t) \text{d} \log S_i(t) + \gamma^*_\pi(t)\,\text{d}t, \]

with \( \gamma^*_\pi(t) = \frac{1}{2} \left( \sum_i \pi_i(t)\sigma_i^2(t) - \sigma_\pi^2(t) \right). \)
The excess growth rate

The excess growth rate measures the efficacy of diversification in a portfolio.

\[ \gamma^*_\pi(t) = \frac{1}{2} \left( \sum_i \pi(t) \sigma^2_i(t) - \sigma^2_\pi(t) \right) \]

\[ = \frac{1}{2} \left( \text{weighted average variance} - \text{portfolio variance} \right) \]

\[ \geq 0 \quad \text{for a long-only portfolio.} \]
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- The excess growth rate is higher for portfolios of volatile stocks with low correlation. If all else is equal, a higher excess growth rate will increase the long-term performance of a portfolio.
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- The excess growth rate is higher for portfolios of volatile stocks with low correlation. If all else is equal, a higher excess growth rate will increase the long-term performance of a portfolio.
- The formulas assume an implicit rebalancing to the target weights \( \pi(t) \) at an infinitesimal time scale reflecting the underlying Brownian motion. Without rebalancing, there’s no excess growth.
Decomposition of portfolio log-return

There is a natural decomposition of the log-return of a portfolio into two components. For the interval \([0, T]\),

\[
\text{Log-return} = \int_0^T \sum_i \pi_i(t) \, d\log S_i(t) + \int_0^T \gamma^*_\pi(t) \, dt.
\]

Hence the log return of a portfolio is not only the average of the log returns of its constituents but an additional term appears. (In financial marketing, this phenomenon is sometimes called "volatility harvesting," "volatility pumping," "smart beta," "volatility capture," "rebalancing premium," or "diversification premium").

There are hundreds (probably thousands) of empirical papers that somehow discuss this effect. Also, the popular press (e.g., FT Alphaville) likes to report about these "surprising" observations.

The first question of this mini course is: how can we construct portfolios with a high log-return, that are additionally tractable.
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Let us briefly look at the “size effect” of the top 1000 stocks. Accordingly, let $r_t(i)$ be the rank of $S_i(t)$, and define the average rank-based growth rates $g_k$ over $[0, T]$ by

$$g_k = \frac{1}{T} \int_0^T \sum 1\{r_t(i)=k\} d \log S_i(t).$$
Estimated $g_k$, 1964–2012 (relative)
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Recalling the setup

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- No bond.
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  \text{ where } \Sigma(t) = S_1(t) + \cdots + S_d(t).
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  and taking values in
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  and taking values in
  \[ \Delta^d = \left\{ (x_1, \cdots, x_d)' \in [0, 1]^d : \sum_{i=1}^d x_i = 1 \right\}. \]
- No frictions; in particular, “small investor” and no trading costs (!)
Trading strategies

For an $\mathbb{R}^d$–valued predictable process $\vartheta(\cdot)$ write

$$V^{\vartheta}(t; S) := \sum_{i=1}^{d} \vartheta_i(t)S_i(t).$$
Trading strategies

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Definition
Suppose that $\vartheta(\cdot) \in \mathcal{L}(S)$ ($\vartheta(\cdot)$ is integrable with respect to $S(\cdot)$) and that

$$V^{\vartheta}(T; S) - V^{\vartheta}(0; S) = \int_0^T \langle \vartheta(t), dS(t) \rangle$$

holds. Then $\vartheta(\cdot)$ is called trading strategy (with respect to $S(\cdot)$) and we write $\vartheta(\cdot) \in \mathcal{T}(S)$. 
Change of numéraire

Recall

\[ V^\vartheta(t; S) := \sum_{i=1}^{d} \vartheta_i(t)S_i(t); \quad V^\vartheta(t; \mu) := \sum_{i=1}^{d} \vartheta_i(t)\mu_i(t). \]

Proposition

An \( \mathbb{R}^d \)–valued process \( \vartheta \) is a trading strategy with respect to \( S \) if and only if it is a trading strategy with respect to \( \mu \).

In particular, \( \mathcal{T}(S) = \mathcal{T}(\mu) \) and

\[ V^\vartheta(\cdot; S) = \Sigma(\cdot)V^\vartheta(\cdot; \mu). \]

Notational convention below: \( V^\vartheta(\cdot) := V^\vartheta(\cdot; \mu). \)
Arbitrage relative to the market

Definition
A trading strategy \( \varphi(\cdot) \) is a relative arbitrage with respect to the market portfolio over the time horizon \([0, T]\) if

\[
V^\varphi(0; S) = \Sigma(0); \quad V^\varphi(t; S) \geq 0;
\]

and

\[
P(V^\varphi(T; S) \geq \Sigma(T)) = 1; \quad P(V^\varphi(T; S) > \Sigma(T)) > 0.
\]
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\]

Alternatively:

\[
V^\varphi(0; \mu) = 1; \quad V^\varphi(t; \mu) \geq 0
\]

and

\[
P \left( V^\varphi(T; \mu) \geq 1 \right) = 1; \quad P \left( V^\varphi(T; \mu) > 1 \right) > 0.
\]
Some results below require the notion of a deflator for $\mu(\cdot)$.

**Definition**
A deflator is a continuous, adapted, strictly positive process $Z(\cdot)$ with $Z(0) = 1$ for which

all products $Z(\cdot)\mu_i(\cdot)$, $i = 1, \cdots, d$ are local martingales.
Warming up ...
From integrands to trading strategies

- Given: $\vartheta(\cdot) \in \mathcal{L}(\mu)$.
- Recall

$$V^{\vartheta}(\cdot) = \sum_{i=1}^{d} \vartheta_i(\cdot) \mu_i(\cdot).$$

- Consider the quantity

$$Q^{\vartheta}(\cdot) = V^{\vartheta}(\cdot) - V^{\vartheta}(0) - \int_{0}^{\cdot} \langle \vartheta(t), d\mu(t) \rangle,$$

which measures the "defect of self-financibility" of $\vartheta(\cdot)$.

- If $Q^{\vartheta}(\cdot) = 0$ fails, the process $\vartheta(\cdot)$ is not a trading strategy.

- However, for any $c \in \mathbb{R}$, $\varphi(\cdot) = \vartheta(\cdot) + Q^{\vartheta}(\cdot) + c$ is a trading strategy and satisfies

$$V^{\varphi}(\cdot) = V^{\vartheta}(0) + c + \int_{0}^{\cdot} \langle \vartheta(t), d\mu(t) \rangle.$$
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  \[ V^{\varphi}(\cdot) = V^{\vartheta}(0) + c + \int_{0}^{\cdot} \langle \vartheta(t), d\mu(t) \rangle. \]
Regular functions

Definition

A continuous function $G : \text{supp}(\mu) \to \mathbb{R}$ is regular if

1. there exists a measurable function $D G = (D_1 G, \cdots, D_d G)^T : \text{supp}(\mu) \to \mathbb{R}^d$

such that the process $\vartheta(\cdot) \in \mathcal{L}(\mu)$ with

$$\vartheta_i(\cdot) = D_i G(\mu(\cdot)), \quad i = 1, \cdots, d;$$

2. the continuous, adapted process

$$\Gamma^G(\cdot) = G(\mu(0)) - G(\mu(\cdot)) + \int_0^\cdot \langle \vartheta(t), \, d\mu(t) \rangle$$

has finite variation on compact intervals.
Regular functions

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Remark:
Assume there exists a deflator $Z(\cdot)$ for $\mu(\cdot)$ and $G$ is nonnegative a Lyapunov function for $\mu(\cdot)$. Then

$$Z(\cdot)G(\mu(\cdot)) = Z(\cdot)(G(\mu(0)) + \int_0^\cdot \sum_{i=1} dD_i G(\mu(t)) \, d\mu_i(t)) - \int_0^\cdot \Gamma G(t) \, dZ(t) - \int_0^\cdot Z(t) \, d\Gamma G(t)$$

is a P–local supermartingale, thus also a P–supermartingale as it is nonnegative.
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$$- \int_0^\cdot \Gamma^G(t) dZ(t) - \int_0^\cdot Z(t) d\Gamma^G(t)$$

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Example

For instance, if $G$ is of class $C^2$, in a neighbourhood of $\Delta^d$, Itô’s formula yields

$$\Gamma^G(\cdot) = -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^\cdot D^2_{ij} G(\mu(t)) \, d\langle \mu_i, \mu_j \rangle(t)$$

with $D^2_{ij} G = \frac{\partial^2 G}{\partial x_i \partial x_j}$.
An example for regular and Lyapunov functions

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For instance, if \( G \) is of class \( C^2 \), in a neighbourhood of \( \Delta^d \), Itô’s formula yields

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\Gamma^G(\cdot) = -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^\cdot D_{ij}^2 G(\mu(t)) \, d\langle \mu_i, \mu_j \rangle(t)
\]

with \( D_{ij}^2 G = \frac{\partial^2 G}{\partial x_i \partial x_j} \). Therefore, \( G \) is regular; if it is also concave, then \( G \) becomes a Lyapunov function.
Functionally generated strategies (additive case)

For a regular function $G$ consider the trading strategy $\varphi(\cdot)$ with

$$\varphi_i(\cdot) = D_i G(\mu(\cdot)) - Q^{\varphi}(\cdot) + c$$

with $c = G(\mu(0)) - \sum_{j=1}^d \mu_j(0) D_j G(\mu(0))$.
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**Definition**

We say that the trading strategy $\varphi(\cdot)$ is *additively generated* by the regular function $G$. 

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Definition
We say that the trading strategy $\varphi(\cdot)$ is *additively generated* by the regular function $G$.

Proposition
The value process generated by the strategy $\varphi(\cdot)$ is given by

$$V^{\varphi}(\cdot) = G(\mu(\cdot)) + \Gamma^G(\cdot).$$

and

$$\varphi_i(\cdot) = D_i G(\mu(\cdot)) + \Gamma^G(\cdot) + G(\mu(\cdot)) - \sum_{j=1}^d \mu_j(\cdot) D_j G(\mu(\cdot)).$$
Functionally generated strategies (multiplicative case)

For a regular function $G$ such that $1/G(\mu(\cdot))$ is locally bounded, consider

$$\tilde{\vartheta}_i(\cdot) = \vartheta_i(\cdot) \times \exp \left( \int_0 \frac{d\Gamma^G(t)}{G(\mu(t))]} \right) = D_i G(\mu(\cdot)) \times \exp \left( \int_0 \frac{d\Gamma^G(t)}{G(\mu(t))]} \right)$$

and the trading strategy $\psi(\cdot)$ with

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We say that the trading strategy $\psi(\cdot)$ is multiplicatively generated by the regular function $G$. 

\[\square\]
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**Definition**

We say that the trading strategy $\psi(\cdot)$ is *multiplicatively generated* by the regular function $G$.

**Proposition (Master equation of Fernholz, 1999, 2002)**

*The value process generated by the strategy $\psi(\cdot)$ is given by*

$$V^\psi(\cdot) = G(\mu(\cdot)) \exp\left( \int_0^\cdot \frac{d\Gamma^G(t)}{G(\mu(t))} \right) > 0.$$
Functionally generated arbitrage (additive case)

Theorem

Fix a Lyapunov function $G : \text{supp}(\mu) \to [0, \infty)$ satisfying $G(\mu(0)) = 1$, and suppose that for $T_* > 0$ we have

$$P(\Gamma^G(T_*) > 1) = 1.$$

Then the additively generated strategy $\varphi(\cdot)$ strongly outperforms the market over every time-horizon $[0, T]$ with $T \geq T_*$. 
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Idea of proof:

$$V^\varphi(T) = G(\mu(T)) + \Gamma^G(T) \geq \Gamma^G(T_*) > 1.$$
An important remark

As long as the market model $\mu(\cdot)$ satisfies

$$P(\Gamma^G(T_*) > 1) = 1$$

the arbitrage strategy

$$\varphi_i(\cdot) = D_i G(\mu(\cdot)) + \Gamma^G(\cdot) + G(\mu(\cdot)) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G(\mu(\cdot)).$$

does not depend on the model parameters or the time horizon.
Functionally generated arbitrage (multiplicative case)

Theorem

Fix a Lyapunov function $G : \text{supp} (\mu) \to [0, \infty)$ satisfying $G(\mu(0)) = 1$, and suppose that for $T^* > 0$ and $\varepsilon > 0$ we have

$$P(\Gamma^G(T^*) > 1 + \varepsilon) = 1.$$
Functionally generated arbitrage (multiplicative case)

Theorem
Fix a Lyapunov function \( G : \text{supp} (\mu) \rightarrow [0, \infty) \) satisfying \( G(\mu(0)) = 1 \), and suppose that for \( T_\ast > 0 \) and \( \varepsilon > 0 \) we have
\[
P\left( \Gamma^G (T_\ast) > 1 + \varepsilon \right) = 1.
\]

Then there exists a constant \( d > 0 \) such that the trading strategy \( \psi^{(d)} (\cdot) \), multiplicatively generated by the regular function
\[
G^{(d)} = \frac{G + d}{1 + d}
\]
strongly outperforms the market over every time-horizon \([0, T]\) with \( T \geq T_\ast \).
Example: entropy function

- Consider the (nonnegative) Gibbs entropy function

\[ H(x) = \sum_{j=1}^{d} x_j \log \left( \frac{1}{x_j} \right). \]
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- Assuming that either \( \mu(\cdot) \in \Delta^d_+ \) or the existence of an SDF \( Z(\cdot) \), \( H \) is a Lyapunov function with nondecreasing

\[ \Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_0^\cdot \mathbf{1}_{\{\mu_j(t) > 0\}} \frac{d \langle \mu_j \rangle(t)}{\mu_j(t)}. \]
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- So-called cumulative excess growth of the market.
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- So-called cumulative excess growth of the market.

- If

\[ P \left( \Gamma^H(t) \geq \eta t, \ \forall \ t \geq 0 \right) = 1 \]

for some real constant \( \eta > 0 \), then arbitrage exists over any time-horizon \( [0, T] \) with \( T > \frac{H(\mu(0))}{\eta} \).
Cumulative excess growth of the market

Figure: Cumulative Excess Growth $\Gamma^H(\cdot)$ for the U.S. Equity Market, during the period 1926 –1999. — Thanks to Bob Fernholz!
Recall:

\[ \Gamma^H(\cdot) = \frac{1}{2} \sum_{j=1}^{d} \int_{0}^{\cdot} \frac{d\langle \mu_j \rangle(t)}{\mu_j(t)} ; \]

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Discussion: entropy function

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- Under this condition there exists one (\textit{horizon-independent}) trading strategy, which is an arbitrage over any time-horizon \([0, T]\) with

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• Fernholz & Karatzas (2005) asked whether then there is also arbitrage possible over any time horizon.
Concave functions are Lyapunov

Theorem

A continuous function $G : \text{supp} (\mu) \to \mathbb{R}$ is Lyapunov if it can be extended to a continuous, concave function on

1. $\Delta^d_+ = \Delta^d \cap (0,1)^d$ and

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3. $\Delta^d$, and there exists a deflator $Z(\cdot)$. 


Functions based on rank

- “Rank operator” $\mathcal{R} : \Delta^d \mapsto \mathbb{W}^d$, where

$$\mathbb{W}^d = \left\{ (x_1, \cdots, x_d) \in \Delta^d : 1 \geq x_1 \geq x_2 \geq \cdots \geq x_{d-1} \geq x_d \geq 0 \right\}.$$
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• Process of market weights ranked in descending order, namely

$$\overline{\mu}(\cdot) = \mathcal{R}(\mu(\cdot)).$$
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\]

• Then \( \bar{\mu}(\cdot) \) can be interpreted again as a market model; however, without a deflator.

Theorem

Consider a function \( \overline{G} : \text{supp}(\bar{\mu}) \rightarrow \mathbb{R} \), which is regular for \( \bar{\mu}(\cdot) \). Then \( G = \overline{G} \circ \mathcal{R} \) is a regular function for \( \mu(\cdot) \).
Remarks on the proof that a concave function is Lyapunov

Dellacherie & Meyer:

REMARKS. (a) The same argument would show that, if $X^1, X^2, \ldots, X^n$ are semimartingales and $f$ is a convex function on $\mathbb{R}^n$, the process $f(X^1_t, \ldots, X^n_t)$ is a semimartingale; it is only necessary to know that $f$ is locally Lipschitz, which is true, but rather more delicate than on $\mathbb{R}$. 
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A paper from 1972:

EVERY CONVEX FUNCTION IS LOCALLY LIPSCHITZ

Wayne State University, Mathematics Department Coffee Room
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Also, Rockefeller has a proof.
On the process $\Gamma^G(\cdot)$

- If there exists a stochastic discount factor, then the process $\Gamma^G(\cdot)$ is independent of the choice of the supergradient.

Bouleau (1981, 1984): If $G$ is twice continuously differentiable in some "open" $A \in \Delta^d$, then

$$\Gamma^G(\cdot) = \frac{1}{2} \sum_{i,j=1}^d \int_0^1 A(\mu(t)) D_{ij} G(\mu(t)) d\langle \mu_i, \mu_j \rangle(t) + \int_0^1 \Delta dA(\mu(t)) d\Gamma^G(t).$$

Our conjecture: quadratic covariation: $\Gamma^G(\cdot) = \frac{1}{2} \sum_{i,j=1}^d [D_{ij} G(\mu(\cdot)), \mu(\cdot)].$
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+ \int_0^\cdot 1_{\Delta^d \setminus A}(\mu(t)) \, d\Gamma^G(t).
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&\quad + \int_{0}^{\cdot} 1_{\Delta^d \setminus A}(\mu(t)) \, d\Gamma^G(t).
\end{align*}
$$

- Our conjecture: quadratic covariation:

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\Gamma^G(\cdot) = \frac{1}{2} \sum_{i,j=1}^{d} [D_i G(\mu(\cdot)), \mu_j(\cdot)].
$$
Outline

1. Stochastic Portfolio Theory: an overview
   1.1 Abstract markets
   1.2 The arithmetics of returns
2. Functional generation of trading strategies
3. The question of arbitrage over arbitrary time horizons
Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
- No bond.
- Relative market weights modeled by nonnegative continuous semimartingales $\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_d(\cdot))$ taking values in $\Delta^d = \left\{(x_1, \cdots, x_d) \in [0, 1]^d : \sum_{i=1}^{d} x_i = 1 \right\}$. 
- No frictions; in particular, “small investor” and no trading costs (!)
Recalling regular functions

- **Regular function**: a continuous mapping $G : \Delta^d \rightarrow \mathbb{R}$ that satisfies a generalized Itô rule:

$$G(\mu(\cdot)) = G(\mu(0)) + \int_0^\cdot \sum_{i=1}^d D_i G(\mu(t)) \ d\mu_i(t) - \Gamma^G(\cdot),$$

where $\Gamma^G(\cdot)$ which has finite variation on compact time-intervals.
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where $\Gamma^G(\cdot)$ which has finite variation on compact time-intervals.

- If $G$ is smooth (we will assume this from now on) then

$$\Gamma^G(\cdot) = -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^\cdot D_{i,j}^2 G(\mu(t)) \, d\langle \mu_i, \mu_j \rangle(t).$$
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• If $\Gamma^G(\cdot)$ is nondecreasing then $G$ is called Lyapunov function.
Functionally generated trading strategies

- *Additive generation:* The process $\varphi^G(\cdot)$ with components

$$\varphi_i^G(\cdot) := D_i G(\mu(\cdot)) + \Gamma^G(\cdot) + G(\mu(\cdot)) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G(\mu(\cdot))$$

is a trading strategy with $V\varphi^G(\cdot) = G(\mu(\cdot)) + \Gamma^G(\cdot)$. 
Functionally generated trading strategies

- **Additive generation:** The process $\varphi^G(\cdot)$ with components

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\varphi_i^G(\cdot) := D_i G(\mu(\cdot)) + \Gamma^G(\cdot) + G(\mu(\cdot)) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G(\mu(\cdot))
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is a trading strategy with $V\varphi^G(\cdot) = G(\mu(\cdot)) + \Gamma^G(\cdot)$.

- **Multiplicative generation:** Assume that $1/G(\mu(\cdot))$ is locally bounded and define the process

$$
Z^G(\cdot) := G(\mu(\cdot)) \exp \left( \int_{0}^{\cdot} \frac{d\Gamma^G(t)}{G(\mu(t))} \right) > 0.
$$

Then the process $\psi^G(\cdot)$ with components

$$
\psi_i^G(\cdot) := Z^G(\cdot) \left( 1 + \frac{1}{G(\mu(\cdot))} \left( D_i G(\mu(\cdot)) - \sum_{j=1}^{d} D_j G(\mu(\cdot)) \mu_j(\cdot) \right) \right),
$$

is a trading strategy with $V\psi^G(\cdot) = Z^G(\cdot)$. 
Example: quadratic function

Consider

\[ Q(x) := 1 - \sum_{i=1}^{d} x_i^2, \quad x \in \Delta^d. \]

- \( Q \) takes values in \([0, 1 - 1/d]\).
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- \( Q \) takes values in \([0, 1 - 1/d]\).
- The corresponding aggregated measure of cumulative volatility is given by
  \[ \Gamma^Q(\cdot) = \sum_i \langle \mu_i \rangle(\cdot). \]
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  \[ \Gamma^Q(\cdot) = \sum_{i} \langle \mu_i \rangle(\cdot). \]
- The additively generated strategy equals
  \[ \varphi^Q_i(\cdot) = D_i Q(\mu(\cdot)) + \Gamma^Q(\cdot) + Q(\mu(\cdot)) - \sum_{j} \mu_j(\cdot) D_j Q(\mu(\cdot)) \]
  \[ = -2\mu_i(\cdot) + \sum_{j} \langle \mu_j \rangle(\cdot) + 1 + \sum_{j} \mu_j^2(\cdot). \]
Relative arbitrage

Definition
Given a real constant $T > 0$, we say that a trading strategy $\vartheta(\cdot)$ is a relative arbitrage with respect to the market over the time horizon $[0, T]$ if $V^\vartheta(0) = 1$, $V^\vartheta(\cdot) \geq 0$, and

$$P(V^\vartheta(T) \geq 1) = 1, \quad P(V^\vartheta(T) > 1) > 0.$$
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$$P(V^\vartheta(T) \geq 1) = 1, \quad P(V^\vartheta(T) > 1) > 0.$$ 

If in fact $P(V^\vartheta(T) > 1) = 1$ holds, this relative arbitrage is called strong.
Strong relative arbitrage over sufficiently long time horizons

Theorem

Suppose that $G : \Delta^d \to [0, \infty)$ is a regular function with $G(\mu(0)) > 0$ such that

$$P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1$$

for some $\eta > 0$. Then strong relative arbitrage with respect to the market exists, over any time horizon $[0, T]$ with

$$T > \frac{G(\mu(0))}{\eta}.$$
Arbitrage over arbitrary time horizons??

Consider the condition
\[ P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1 \]
for some \( \eta > 0 \).

**Does there exist arbitrage with respect to the market portfolio over time horizon \([0, T]\), for any \( T > 0\)??**
Arbitrage over arbitrary time horizons??

Consider the condition

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for some \( \eta > 0 \).

**Does there exist arbitrage with respect to the market portfolio over time horizon \([0, T]\), for any \( T > 0\)?**

**Answer:** Under additional assumptions, yes. In general, no.
Outline of the rest of this lecture

\[ P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1 \]

for some \( \eta > 0 \).

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2. Lack of short-term relative arbitrage
One asset with sufficient variation

**Theorem**

*Suppose there exists a constant $\eta > 0$ such that $\langle \mu_1 \rangle(t) \geq \eta t$ holds on the stochastic interval $[0, D^*]$ with*

$$D^* := \inf \left\{ t \geq 0 : \mu_1(t) \leq \frac{\mu_1(0)}{2} \right\}.$$

*Then, given any real number $T > 0$ there exists a long-only trading strategy $\varphi(\cdot)$ which is strong relative arbitrage with respect to the market over the time horizon $[0, T]$.***
One asset with sufficient variation

Theorem

Suppose there exists a constant \( \eta > 0 \) such that \( \langle \mu_1 \rangle(t) \geq \eta t \) holds on the stochastic interval \([0, D^*] \) with

\[
D^* := \inf \left\{ t \geq 0 : \mu_1(t) \leq \frac{\mu_1(0)}{2} \right\}.
\]

Then, given any real number \( T > 0 \) there exists a long-only trading strategy \( \varphi(\cdot) \) which is strong relative arbitrage with respect to the market over the time horizon \([0, T]\).

Some intuition why the theorem could be true:

- \( \mu_1(\cdot) \) is bounded by above from one.
- There needs to be a very large drift that forces \( \mu(\cdot) \) to not become larger than one.
- Hence, an arbitrage strategy should under-invest into asset 1.
Proof

- It suffices to argue that the market \( \nu(\cdot) = \mu(\cdot \wedge \mathcal{D}^*) \) allows for arbitrage.
Proof

- It suffices to argue that the market $\nu(\cdot) = \mu(\cdot \wedge D^*)$ allows for arbitrage.
- For $q \geq 1$, consider the regular function

$$F(x) := x_1^q, \quad x \in \Delta^d.$$
Proof

• It suffices to argue that the market \( \nu(\cdot) = \mu(\cdot \wedge \mathbb{D}^*) \) allows for arbitrage.

• For \( q \geq 1 \), consider the regular function

\[
F(x) := x_1^q, \quad x \in \Delta^d.
\]

• \( F \) generates multiplicatively the strategy

\[
\psi_1^F(\cdot) = \left( \frac{q}{\nu_1(\cdot)} + 1 - q \right) Z^F(\cdot); \quad \psi_i^F(\cdot) = (1 - q) Z^F(\cdot), \quad i \geq 2,
\]

where

\[
V^{\psi^F} (\cdot) = Z^F (\cdot) = (\nu_1(\cdot))^q \exp \left( -\frac{1}{2} q (q - 1) \int_0^\cdot (\nu_1(t))^{-2} d\langle \nu_1 \rangle(t) \right).
\]
Proof (cont’d)

• Introduce now the trading strategy

\[ \varphi_i(\cdot) = 1 + (\nu_1(0))^q - \psi_i^F(\cdot), \quad i = 1, \ldots, d \]

with associated wealth process

\[ V^\varphi(\cdot) = 1 + (\nu_1(0))^q - Z^F(\cdot). \]
Proof (cont’d)

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• In particular, note \( V^{\varphi}(0) = 1 \) and \( V^{\varphi}(\cdot) \geq 0. \)
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• In particular, note \( V^{\varphi}(0) = 1 \) and \( V^{\varphi}(\cdot) \geq 0 \).
• On the event \( \{ D^* \leq T \} \) we have

\[ V^{\varphi}(T) \geq 1 + (\nu_1(0))^q - (\nu_1(T))^q = 1 + (\nu_1(0))^q - \left( \frac{\nu_1(0)}{2} \right)^q > 1. \]
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- On the event \( \{ \mathcal{D}^* \leq T \} \) we have
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- On the event \( \{ \mathcal{D}^* > T \} \), for sufficiently large \( q \), we have
  \[ V^\varphi(\cdot) \geq 1 + (\nu_1(0))^q - \exp \left( -\frac{1}{2} q (q - 1) \langle \nu_1 \rangle(T) \right) \geq 1 + (\nu_1(0))^q - \left( \exp \left( -\frac{\eta}{2} (q - 1) T \right) \right)^q > 1. \]
P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1

for some \eta > 0.

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2. Lack of short-term relative arbitrage
Theorem

Suppose that for a given generating function \( G \) and appropriate real constants \( \eta > 0 \) and \( h \geq 0 \),

\[
P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1
\]

is satisfied, along with the lower bound

\[
P \left( G(\mu(t)) \geq h, \quad t \geq 0 \right) = 1
\]

and the “time homogeneous support” property

\[
P \left( G(\mu(t)) \in [h, h+\varepsilon), \text{ for some } t \in [0, T] \right) > 0, \quad \text{for all } T > 0, \varepsilon > 0
\]
Time-homogeneous support

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Then arbitrage relative to the market exists over the time horizon $[0, T]$, for every real number $T > 0$. 
Proof

- Fix $T > 0$ and introduce the regular function

$$G^* := (G - h) \frac{3}{\eta T},$$

and denote

$$\Gamma^*(\cdot) := \Gamma^{G^*}(\cdot) = \frac{3}{\eta T} \Gamma^{G}(\cdot).$$
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  and denote
  \[ \Gamma^*(\cdot) := \Gamma^{G^*}(\cdot) = \frac{3}{\eta T} \Gamma^G(\cdot). \]

- Introduce the stopping time
  \[ \tau := \inf \left\{ t \in \left[ 0, \frac{T}{2} \right] : G(\mu(t)) < h + \frac{\eta T}{3} \right\}. \]

- Then
  \[ P \left( \tau \leq \frac{T}{2} \right) > 0. \]
Proof (cont’d)

• Let $\varphi^*(\cdot) := \varphi^{G^*}(\cdot)$ denote the additively generated trading strategy and consider

$$\varphi_i(\cdot) := 1 + (\varphi^*_i(\cdot) - G^*(\mu(\tau)) - \Gamma^*(\tau))1_{[\tau,\infty[}.$$
Proof (cont’d)

• Let $\varphi^*(\cdot) := \varphi^{G^*}(\cdot)$ denote the additively generated trading strategy and consider

$$\varphi_i(\cdot) := 1 + (\varphi^*_i(\cdot) - G^*(\mu(\tau)) - \Gamma^*(\tau)) 1_{[\tau, \infty[.}$$

• Then

$$V^\varphi(t) = 1 + (G^*(\mu(t)) + \Gamma^*(t) - G^*(\mu(\tau)) - \Gamma^*(\tau)) 1_{[\tau, \infty[}(t)$$

$$\geq 1_{[0, \tau[}(t) + \frac{3}{\eta T} \left( \Gamma^G(t) - \Gamma^G(\tau) \right) 1_{[\tau, \infty[}(t)$$

$$\geq 1_{[0, \tau[}(t) + \frac{3(t - \tau)}{T} 1_{[\tau, \infty[}(t).$$
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• Let $\varphi^*(\cdot) := \varphi^G(\cdot)$ denote the additively generated trading strategy and consider

$$\begin{align*}
\varphi_i(\cdot) &:= 1 + \left( \varphi_i^*(\cdot) - G^*(\mu(\tau)) - \Gamma^*(\tau) \right) 1_{[\tau, \infty]}.
\end{align*}$$

• Then

$$\begin{align*}
V^\varphi(t) &= 1 + \left( G^*(\mu(t)) + \Gamma^*(t) - G^*(\mu(\tau)) - \Gamma^*(\tau) \right) 1_{[\tau, \infty]}(t) \\
&\geq 1_{[0, \tau]}(t) + \frac{3}{\eta T} \left( \Gamma^G(t) - \Gamma^G(\tau) \right) 1_{[\tau, \infty]}(t) \\
&\geq 1_{[0, \tau]}(t) + \frac{3(t - \tau)}{T} 1_{[\tau, \infty]}(t).
\end{align*}$$

• Hence, $V^\varphi(\cdot) \geq 0$ and $V^\varphi(T) \geq 3/2$ on the event $\{\tau \leq T/2\}$; moreover, $V^\varphi(T) = 1$ holds on $\{\tau > T/2\}$. 
Proof (cont’d)

• Let \( \varphi^*(\cdot) := \varphi^G(\mu(\tau)) - \Gamma(\tau) \) denote the additively generated trading strategy and consider

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\varphi_i(\cdot) := 1 + (\varphi_i^*(\cdot) - \varphi^G(\mu(\tau)) - \Gamma(\tau)) 1_{[\tau, \infty[}.
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• Then

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V^{\varphi}(t) = 1 + (\varphi(\mu(t)) + \Gamma(t) - \varphi^G(\mu(\tau)) - \Gamma(\tau)) 1_{[\tau, \infty[}(t)
\geq 1_{[0, \tau[}(t) + \frac{3}{\eta T} \left( \Gamma^G(t) - \Gamma(t) \right) 1_{[\tau, \infty[}(t)
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• Hence, \( V^{\varphi}(\cdot) \geq 0 \) and \( V^{\varphi}(T) \geq 3/2 \) on the event \( \{\tau \leq T/2\} \); moreover, \( V^{\varphi}(T) = 1 \) holds on \( \{\tau > T/2\} \).

• Since \( P(\tau \leq T/2) > 0 \) the trading strategy \( \varphi(\cdot) \) is relative arbitrage.
The mapping \([0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t\) is nondecreasing for some \(\eta > 0\).

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2. Lack of short-term relative arbitrage
Some necessary notation

Recall

\( \Gamma^Q(\cdot) = \sum_{i=1}^{d} \langle \mu_i \rangle(\cdot). \)
Some necessary notation

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Then we have

\[ \langle \mu_i, \mu_j \rangle(\cdot) = \int_0^\cdot \alpha_{i,j}(t) \, d\Gamma^Q(t). \]
Some necessary notation

Recall

$$\Gamma^Q(\cdot) = \sum_{i=1}^{d} \langle \mu_i \rangle(\cdot).$$

Then we have

$$\langle \mu_i, \mu_j \rangle(\cdot) = \int_{0}^{\cdot} \alpha_{i,j}(t) \, d\Gamma^Q(t).$$

Consider the sequence of stopping times

$$D^n := \inf \left\{ t \geq 0 : \min_{1 \leq i \leq d} \mu_i(t) < \frac{1}{n} \right\}.$$
A strict nondegeneracy condition

Theorem

Suppose that for a given generating function $G$ and $\eta > 0$,

$$P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1.$$

Moreover, suppose that there exists a deflator for $\mu(\cdot)$ and that the $d-1$ largest eigenvalues of the matrix-valued process $\alpha(\cdot)$ are bounded away from zero on $[0, D^n]$ uniformly in $(t, \omega)$, for each $n \in \mathbb{N}$. 

Attention: If we replace "strict nondegeneracy" by nondegeneracy, then the theorem is not correct.
Theorem

Suppose that for a given generating function $G$ and $\eta > 0$,

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Then relative arbitrage with respect to the market exists over $[0, T]$, for every real number $T > 0$. 

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Suppose that for a given generating function $G$ and $\eta > 0$,

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Attention: If we replace “strict nondegeneracy” by nondegeneracy, then the theorem is not correct.
Outline of the proof

• By contradiction. Assume that $\mu(\cdot)$ is a martingale under an equivalent measure.

• Then prove that $G(\mu(\cdot))$ reaches the minimum of $G$ with positive probability arbitrarily close, arbitrarily fast. (repeated changes of measures)

• This then contradicts the time-homogeneous support property of the previous result.
Outline of this section

\[ P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1 \]

for some \( \eta > 0 \).

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2. Lack of short-term relative arbitrage
The case of two assets

Proposition

Assume that $d = 2$ and that for a given generating function $G$ and $\eta > 0$,

$$P \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1.$$

Then strong arbitrage relative to the market can be realized by a long-only trading strategy over the time horizon $[0, T]$, for any given real number $T > 0$. 
The case of two assets

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Then strong arbitrage relative to the market can be realized by a long-only trading strategy over the time horizon $[0, T]$, for any given real number $T > 0$.

Idea of proof (does not necessarily yield strong arbitrage):

- Recall
  $$\langle \mu_i, \mu_j \rangle(\cdot) = \int_0^\cdot \alpha_{i,j}(t) \, d\Gamma^Q(t).$$
- Note $\mu_2(\cdot) = 1 - \mu_1(\cdot)$, hence $\langle \mu_1, \mu_2 \rangle = -\langle \mu_1 \rangle(\cdot)$. 
The case of two assets

Proposition

Assume that \( d = 2 \) and that for a given generating function \( G \) and \( \eta > 0 \),

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Then strong arbitrage relative to the market can be realized by a long-only trading strategy over the time horizon \([0, T]\), for any given real number \( T > 0 \).

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• Recall

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\langle \mu_i, \mu_j \rangle(\cdot) = \int_0^\cdot \alpha_{i,j}(t) d\Gamma^Q(t).
\]

• Note \( \mu_2(\cdot) = 1 - \mu_1(\cdot) \), hence \( \langle \mu_1, \mu_2 \rangle = -\langle \mu_1 \rangle(\cdot) \).

• Hence, \( \alpha_{1,1}(\cdot) = \alpha_{2,2}(\cdot) = 1/2 \) and \( \alpha_{1,2}(\cdot) = \alpha_{2,1}(\cdot) = -1/2 \), so the eigenvalues of the matrix \( \alpha(\cdot) \) are then indeed 0 and 1.
Outline of this section

\[
\mathbb{P}\left(\text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1
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2. Lack of short-term relative arbitrage
Recalling the setup

- \( d \in \mathbb{N} \): number of assets at time zero.
- Relative market weights modeled by nonnegative continuous semimartingales \( \mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_d(\cdot)) \) taking values in

\[
\Delta^d = \left\{ (x_1, \cdots, x_d)' \in [0, 1]^d : \sum_{i=1}^d x_i = 1 \right\}.
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$$\Delta^d = \left\{ (x_1, \cdots, x_d)' \in [0, 1]^d : \sum_{i=1}^{d} x_i = 1 \right\}.$$

- The condition we study:

$$\mathbb{P} \left( \text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondecreasing} \right) = 1$$

for some $\eta > 0$. 

- E.g., if $G(x) = Q(x) = 1 - \sum_{j=1}^{d} x_j^2$, then

$$\Gamma^Q(t) = d \sum_{j=1}^{d} \langle \mu_j \rangle(t).$$
Recalling the setup

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- E.g., if \( G(x) = Q(x) = 1 - \sum_{j=1}^d x_j^2 \), then

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\Gamma^Q(t) = \sum_{j=1}^d \langle \mu_j \rangle(t).
\]
The goal

- Goal: Construct process $\mu(\cdot)$ with each component a martingale such that $\Gamma^Q(t) = t$, $t \in [0, T^*]$ for some $T^* > 0$.
- This then yields a counterexample since then no arbitrage is possible with respect to the market, but
The goal

- Goal: Construct process $\mu(\cdot)$ with each component a martingale such that $\Gamma^Q(t) = t, \ t \in [0, T^*]$ for some $T^* > 0$.
- This then yields a counterexample since then no arbitrage is possible with respect to the market, but

$$P\left(\text{the mapping } [0, T^*) \ni t \mapsto \Gamma^G(t) - t \text{ is nondecreasing}\right) = 1.$$

- The process $\mu(\cdot)$ is not allowed to have full support (otherwise, we know by previous results that short-term arbitrage is possible).
- For $d = 2$, such a construction is impossible.
An Itô diffusion

- Consider $d = 3$ (three assets).
- Consider SDEs:

  \[
  d\nu_1(t) = \frac{1}{\sqrt{3}}(\nu_2(t) - \nu_3(t))d\Theta(t);
  \]

  \[
  d\nu_2(t) = \frac{1}{\sqrt{3}}(\nu_3(t) - \nu_1(t))d\Theta(t);
  \]

  \[
  d\nu_3(t) = \frac{1}{\sqrt{3}}(\nu_1(t) - \nu_2(t))d\Theta(t).
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  \[ \text{d}v_2(t) = \frac{1}{\sqrt{3}}(v_3(t) - v_1(t)) \text{d}\Theta(t); \]
  
  \[ \text{d}v_3(t) = \frac{1}{\sqrt{3}}(v_1(t) - v_2(t)) \text{d}\Theta(t). \]

• Define $r(v) = \sqrt{\sum_{i=1}^{3} (v_i - \frac{1}{3})^2}$. 

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  \text{d}v_3(t) &= \frac{1}{\sqrt{3}}(v_1(t) - v_2(t))\text{d}\Theta(t).
  \end{align*}
  \]

• Define $r(v) = \sqrt{\sum_{i=1}^{3} (v_i - \frac{1}{3})^2}$.

• If $v_1(0) + v_2(0) + v_3(0) = 1$, then Itô’s formula yields

  \[\langle v_1 \rangle(t) + \langle v_2 \rangle(t) + \langle v_3 \rangle(t) = r^2(v(t)) = r^2(v(0))e^t.\]
An Itô diffusion

- Consider \( d = 3 \) (three assets).
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  &d\nu_2(t) = \frac{1}{\sqrt{3}} (\nu_3(t) - \nu_1(t))d\Theta(t); \\
  &d\nu_3(t) = \frac{1}{\sqrt{3}} (\nu_1(t) - \nu_2(t))d\Theta(t).
  \end{align*}
  \]
- Define \( r(\nu) = \sqrt{\sum_{i=1}^{3} (\nu_i - \frac{1}{3})^2} \).
- If \( \nu_1(0) + \nu_2(0) + \nu_3(0) = 1 \), then Itô’s formula yields
  \[
  \langle \nu_1 \rangle(t) + \langle \nu_2 \rangle(t) + \langle \nu_3 \rangle(t) = r^2(\nu(t)) = r^2(\nu(0))e^t.
  \]
- A solution:
  \[
  \nu_i(t) = \frac{1}{3} + \delta e^{t/2} \cos \left( \Theta(t) + 2\pi \left( u + \frac{i-1}{3} \right) \right).
  \]
An Itô diffusion (cont’d)

• A slight modification:

\[
\begin{align*}
\quad dv_1(t) &= \frac{1}{\sqrt{3}r(t)}(v_2(t) - v_3(t))d\Theta(t); \\
\quad dv_2(t) &= \frac{1}{\sqrt{3}r(t)}(v_3(t) - v_1(t))d\Theta(t); \\
\quad dv_3(t) &= \frac{1}{\sqrt{3}r(t)}(v_1(t) - v_2(t))d\Theta(t).
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- A slight modification:

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\end{align*}
\]

- Assume that

\[
(v_1(0), v_2(0), v_3(0)) \neq \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).
\]
An Itô diffusion (cont’d)

• A slight modification:

\[
\begin{align*}
\frac{dv_1(t)}{dt} &= \frac{1}{\sqrt{3}r(t)}(v_2(t) - v_3(t))d\Theta(t); \\
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\frac{dv_3(t)}{dt} &= \frac{1}{\sqrt{3}r(t)}(v_1(t) - v_2(t))d\Theta(t).
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• Assume that

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• Now,

\[\langle v_1 \rangle(t) + \langle v_2 \rangle(t) + \langle v_3 \rangle(t) = r^2(v(t)) = t.\]
An Itô diffusion (cont’d)

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• Now,

\[
\langle v_1 \rangle(t) + \langle v_2 \rangle(t) + \langle v_3 \rangle(t) = r^2(v(t)) = t.
\]

• Market model \(\mu(\cdot):\) stopped version of \(v(\cdot).\)
Possible extensions

- It is possible to modify the model for $\mu(\cdot)$ such that the covariance matrix has two positive eigenvalues (instead of one) – however, we know that it is not possible that both positive eigenvalues are bounded away from zero uniformly.
Possible extensions

- It is possible to modify the model for $\mu(\cdot)$ such that the covariance matrix has two positive eigenvalues (instead of one) – however, we know that it is not possible that both positive eigenvalues are bounded away from zero uniformly.

- For a general Lyapunov function $G$, construct a market model $\mu(\cdot)$ with each component a martingale such that $\Gamma^G(t) = t$, $t \in [0, T^*]$ for some $T^* > 0$. 
Спасибо!