Affine Volterra Processes

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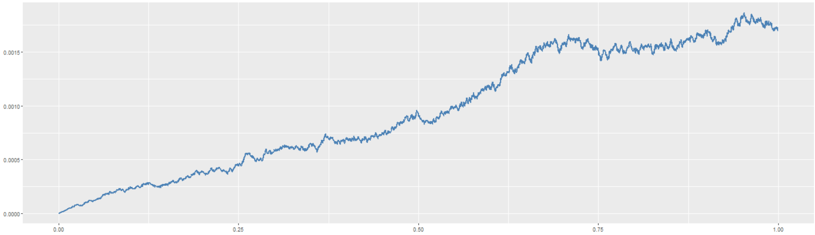
Outline

Rough volatility models



- Empirical studies indicate **volatility is rougher than BM**: **Gatheral et al. (2018); Bennedsen et al. (2016), . . .**
- Subsequent development of **stochastic models with this feature**: **Gatheral et al. (2018); Guennoun et al. (2017); Bayer et al. (2016); Bennedsen et al. (2016); El Euch and Rosenbaum (2016, 2017), . . .**
- **Rough Volatility Literature** (sites.google.com/site/roughvol/home/risks-1)

Features of rough volatility models

- These models are able to
  - **match roughness of time series data**
  - **fit implied volatility smiles remarkably well**
  - **admit in some cases microstructural justification**

- Mathematically, this rests on **fractional Brownian motion** in the tradition of **Kolmogorov (1940), Mandelbrot and Van Ness (1968), . . .**

Rough Heston model

- The **Heston model** is the stock price model

$$\frac{dS_t}{S_t} = \sqrt{V_t} dB_t$$

where the **volatility follows a CIR process**

$$V_t = V_0 + \int_0^t \lambda(\theta - V_s) ds + \int_0^t \sigma \sqrt{V_s} dW_s$$

- **El Euch and Rosenbaum (2016)** study the **rough Heston model** obtained by replacing the CIR process by the **rough CIR process**

$$V_t = V_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\lambda(\theta - V_s) ds + \sigma \sqrt{V_s} dW_s \Big)$$

where $\alpha \in (\frac{1}{2}, 1)$

Roughness of time series data

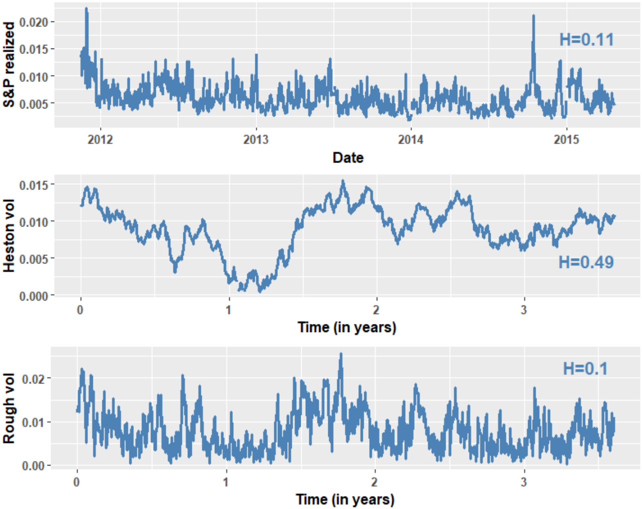


Figure: S&P vol. vs. simulated paths of Heston and rough Heston

The volatility skew

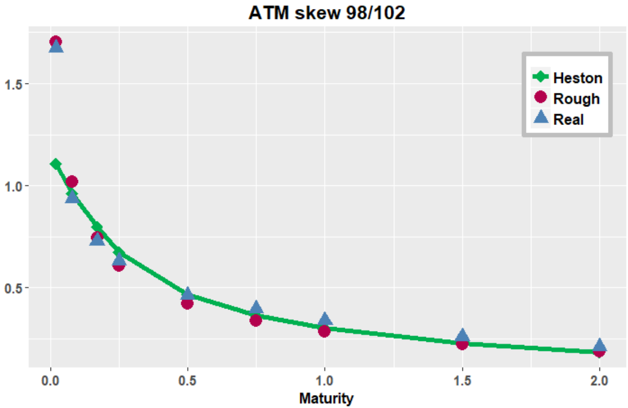


Figure: ATM implied vol. skew $\left|\frac{\partial \sigma_{BS}}{\partial k}(\tau, k)\right|_{k=0}$: Heston vs. rough Heston calibration

Rough Heston model

- Inspired by the **Riemann–Liouville fractional Brownian motion** introduced by **Lévy (1953)**
- Hölder continuous paths of any order less than $H = \alpha - \frac{1}{2}$
- Microstructural foundation as scaling limit of Hawkes processes

But:

- Existence and uniqueness is non-trivial: **El Euch and Rosenbaum (2016) construct the rough CIR using Hawkes processes**
- **NOT A SEMIMARTINGALE, NOT MARKOVIAN !**
- ... not clear how to usefully describe its law

Warmup: The standard Heston model

Characteristic function of the (standard) Heston model I

- The Heston model is tractable because $(\log S_t, V_t)$ **is affine**
- This gives **explicit characteristic function** and option prices via Fourier methods:

$$\mathbb{E}[e^{u \log S_T}] = e^{\phi(T)+\psi(T)V_0}$$

for $u \in \mathrm{i}\mathbb{R}$ and $S_0 = 1$

- (ϕ, ψ) solve the **Riccati equations**

$$\phi' = \lambda\theta\psi \qquad \phi(0) = 0$$

$$\psi' = \frac{1}{2}(u^2 - u) + (u\rho\sigma - \lambda)\psi + \frac{\sigma^2}{2}\psi^2 \qquad \psi(0) = 0$$

Characteristic function of the (standard) Heston model II

Proof:

1. **Define**

$$M_t = e^{\phi(T-t)+\psi(T-t)V_t+u\log S_t}.$$

2. **Apply Itô:**

$$\frac{dM_t}{M_t} = -\left\{(\phi' - \lambda\theta\psi) + \left(\psi' - \left[\frac{1}{2}(u^2 - u) - \cdots\right]\right)V_t\right\}dt + (\text{local mgle.})$$

3. M **is a martingale** because $\operatorname{Re}\psi \leq 0$
4. Since $\psi(0) = \phi(0) = 0$, $M_T = e^{u\log S_T}$ and

$$\mathbb{E}[e^{u\log S_T}] = \mathbb{E}[M_T] = M_0 = e^{\phi(T)+\psi(T)V_0}$$

What about the rough Heston model?

- Remarkably, **El Euch and Rosenbaum (2016)** obtain an analogous result for the rough Heston model
- **Notation:** $D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds$

Theorem (El Euch and Rosenbaum (2016))

Assume we are given a solution ψ of the **"fractional Riccati equation"**

$$D^\alpha \psi = \frac{1}{2}(u^2 - u) + (u\rho\sigma - \lambda)\psi + \frac{\sigma^2}{2}\psi^2$$

and define ϕ and χ by

$$\phi' = \lambda\theta\chi, \quad \phi(0) = 0; \qquad \chi' = D^\alpha \psi, \quad \chi(0) = 0$$

Then

$$\mathbb{E}[e^{u \log S_T}] = e^{\phi(T) + \chi(T) V_0}$$

Proof: Rather involved. Uses the Hawkes approximation

Questions

1. Can the proof of this result be simplified?
2. Can the Hawkes approximation be avoided?
3. What about the joint characteristic function of $(\log S_T, V_T)$?
4. What about conditional characteristic function?
5. What about more general specifications: higher dimensions, other kernels?

Stochastic Volterra Equations (SVEs)

Equations of interest (in $\mathbb{R}$):

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s \qquad (1)$$

- **Initial condition:** $X_0 \in \mathbb{R}$
- **Kernel:** $K \in L^2_{loc}(\mathbb{R}_+)$
- **Martingale driver:** $W = (W_t)_{t\geq 0}$ a **Brownian motion**
- **Coefficients:** b, σ **real continuous with linear growth**

$$|b(x)| \vee |\sigma(x)| \leq c_{LG}(1+|x|) \qquad (2)$$

Remark: *Solutions to* (1) *will be understood to have* **continuous paths**

Examples of Kernels

1. **Exponential kernel:**

$$K(t) = \exp(-\gamma t), \quad (\gamma = 0 \Rightarrow K \equiv 1)$$

Observation: In this case we can rewrite (1) as

$$X_t = X_0 + \int_0^t (b(X_s) - \gamma(X_s - X_0))ds + \int_0^t \sigma(X_s)dW_s \tag{3}$$

2. **Fractional kernel:**

$$K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha \in \left(\frac{1}{2}, 1\right)$$

3. **Gamma kernel:**

$$K(t) = \exp(-\gamma t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

Remark: *Case 1 - Markovian* structure of (1). *Cases 2-3 non-Markovian*

Examples of SVEs

1. **Volterra Brownian motion:** $b(x) \equiv 0$, $\sigma(x) \equiv \sigma$ constant

$$X_t = X_0 + \sigma \int_0^t K(t-s) dW_s$$

Observation: Fractional kernel $\Rightarrow$ **Riemann–Liouville fractional Brownian motion – Lévy (1953)**

2. **Volterra Ornstein–Uhlenbeck process:** $b(x) = \lambda(\theta - x), \sigma(x) \equiv \sigma$

$$X_t = X_0 + \lambda \int_0^t K(t-s)(\theta - X_s) ds + \sigma \int_0^t K(t-s) dW_s$$

Examples of SVEs (cont.)

3. **Volterra square root/CIR process:** $b(x) = \lambda(\theta - x), \sigma(x) = \sigma\sqrt{x}$

$$X_t = X_0 + \lambda \int_0^t K(t-s)(\theta - X_s)ds + \sigma \int_0^t K(t-s)\sqrt{X_s}dW_s \quad (4)$$

Remark: Delicate questions of **existence of (nonnegative) solutions**

4. **Volterra Heston model:** The **price process** S satisfies

$$dS_t = S_t\sqrt{V_t}dB_t$$

where V, **the square volatility process**, is a Volterra CIR process as in (4), and $d\langle B, W\rangle = \rho dt$

Remark: K a fractional kernel $\Rightarrow$ **Rough Heston Model (El Euch and Rosenbaum (2016))**

Function and measure convolution

- For a (real) kernel K and a measure L (of locally bounded variation) on $\mathbb{R}_+$ we write

$$(K * L)(t) = (L * K)(t) = \int_{[0,t]} K(t-s)L(ds)$$

Example: $L(ds) = \delta_0(ds) \Rightarrow (K * L)(t) = K(t)$

- If F is a function on $\mathbb{R}_+$ we write

$$(K * F)(t) = (K * (F ds))(t) = \int_0^t K(t-s)F(s)ds$$

Example: $K(t) = t^{\alpha-1}$, $F(t) = t^{-\alpha}$, $\alpha \in \left(\frac{1}{2}, 1\right)$, then

$$(K * F)(t) = \Gamma(\alpha)\Gamma(1-\alpha)$$

Convolution with local martingales

Let M be a one-dimensional continuous local martingale then

$$(K * dM)_t = \int_0^t K(t-s)dM_s$$

is well-defined as long as $\int_0^t |K(t-s)|^2 d\langle M \rangle_s < \infty$

(e.g. $K \in L^2_{loc}(\mathbb{R}_+)$ and $\langle M \rangle_t = \int_0^t a_s ds$ for some locally bounded process a)

Observation:

- $(K * dM)_t$ is the **final value of** (the martingale)

$$N_u = \mathbb{E}[(K * dM)_t | \mathcal{F}_u] = \int_0^u K(t-s)dM_s, \quad 0 \le u \le t$$

but in general $N_s \neq (K * dM)_s$ for $s < t$

Associativity of the convolution

Lemma (Associativity of convolution)

Assume

1. $K \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$
2. L *a measure on* $\mathbb{R}_+$ *of locally bounded variation*
3. M *be a continuous local martingale with* $\langle M \rangle_t = \int_0^t a_s ds$ *for some locally bounded process* a

Then

$$(L * (K * dM))_t = ((L * K) * dM)_t, \quad t \geq 0 \tag{5}$$

Stochastic Fubini Theorem

The previous lemma is a consequence of

Theorem (Stochastic Fubini – Veraar (2012))

(X, Σ, μ) a σ-finite measure space, M a continuous local martingale, $\psi(t, x, \omega)$ progressively measurable s.t.

$$\int_X \left(\int_0^T |\psi(x,t,\omega)|^2 d\langle M \rangle_t(\omega) \right)^{\frac{1}{2}} \mu(dx) < \infty, \quad a.s.$$

Then

$$\int_X \left(\int_0^T \psi(x,t,\omega) dM_t \right) \mu(dx) = \int_0^T \left(\int_X \psi(x,t,\omega)\mu(dx) \right) dM_t$$

Conditions on K for existence of solutions of SVEs

1. **Behaviour around zero:**

$$K \in L^2_{\mathrm{loc}}(\mathbb{R}_+) \text{ and } \exists \gamma \in (0,2] \text{ s.t. } \int_0^h K(t)^2 dt = O(h^\gamma)$$
$$\text{and } \int_0^T (K(t+h) - K(t))^2 dt = O(h^\gamma) \text{ for every } T < \infty \tag{6}$$

2. **Existence of a resolvent of the first kind:**

$$\exists L \text{ of locally bounded variation s.t.}$$
$$K * L = L * K \equiv \mathbf{1} \tag{7}$$

Definition (Resolvent of the first kind)

The function L is known as the **resolvent of the first kind**

Comments on the conditions for K

Condition (6)

- Locally Lipschitz kernels K clearly satisfy (6) with $\gamma = 1$
- $K(t) = t^{\alpha-1}$ with $\alpha \in (\frac{1}{2}, 1]$ satisfies (6) with $\gamma = 2\alpha - 1$

Condition (7)

- $K \equiv \mathbf{1}$, then $L(ds) = \delta_0(ds)$
- $K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, $\alpha \in (\frac{1}{2}, 1)$, then $L(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$

Theorem (Theorem 5.5.4, Gripenberg et al. (1990))

*K **completely monotone and not identically zero**, then*

$$L(ds) = c\delta_0(ds) + l(s)\, ds$$

with l a completely monotone function

- **Recall:** K is **completely monotone** if

$$(-1)^n \frac{d^n}{dt^n} K(t) \geq 0, \quad \forall n \in \mathbb{N}$$

Examples: $t^{\alpha-1}$ with $\alpha \in (\frac{1}{2}, 1)$, and $\mathrm{e}^{-\beta t}$ with $\beta > 0$

Strong solutions of SVEs: Lipschitz case

Theorem (Strong existence SVEs, Lipschitz coefficients)

Assume that

1. b *and* σ *are* **Lipschitz continuous**
2. K *satisfy* (6)

Then (1) admits a unique continuous strong solution X for any initial condition $X_0 \in \mathbb{R}$

The proof parallels that of **Proposition 2.1 in Mytnik and Salisbury (2015), using a Picard iteration scheme and the following two lemmas**

Lemma: Existence of Hölder continuous versions

Lemma (Hölder continuous version)

Assume

1. K *satisfies* (6)
2. $X = K * (bdt + dM)$, b *predictable and* M *a continuous local martingale with* $\langle M \rangle_t = \int_0^t a_s ds$ *for some predictable* a
3. $T \geq 0$ *and* $p > 2/\gamma$ *s.t.* $\sup_{t \leq T} \mathbb{E}[|a_t|^{p/2} + |b_t|^p] < \infty$

Then X **admits a Hölder continuous version on** $[0, T]$ **of any order** $\alpha < \gamma/2 - 1/p$ *and for this version*

$$\mathbb{E}\left[\left(\sup_{0 \leq s < t \leq T} \frac{|X_t - X_s|}{|t - s|^\alpha}\right)^p\right] \leq c \sup_{t \leq T} \mathbb{E}[|a_t|^{p/2} + |b_t|^p] \tag{8}$$

for all $\alpha \in [0, \gamma/2 - 1/p)$, *where* c *is a constant that only depends on* p, K, *and* T

Lemma: Finite moments of solutions of SVEs

Lemma (Moment bound)

Assume

1. b *and* σ *are continuous and satisfy the linear growth condition* (2) *for some constant* c_{LG}
2. X *be a continuous solution of* (1) *with initial condition* $X_0 \in \mathbb{R}$

Then for any $p \geq 2$ *and* $T < \infty$ *one has*

$$\sup_{t \leq T} \mathbb{E}[|X_t|^p] \leq c$$

for some constant c *that only depends on* $|X_0|$, $K|_{[0,T]}$, c_{LG}, p *and* T

Main existence theorem for SVEs

Theorem (Weak existence SVEs – linear growth – Abi Jaber et al. (2017))

Consider the SVE (1) compactly written as

$$X = X_0 + K * (b(X)dt + \sigma(X)dW)$$

Assume that:

1. b *and* σ *are continuous and satisfy the linear growth condition (2)*
2. K *satisfies (6)*
3. K *admits a resolvent of the first kind (see (7))*

Then (1) admits a continuous weak solution for any initial condition $X_0 \in \mathbb{R}$

Stability of SVEs

Lemma (Stability of SVEs)

Assume that

1. K *admits a resolvent of the first kind* L *– see* (7)
2. X^n *be a weak solution of* (1) *with coefficients* b^n *and* σ^n *that satisfy* (2) *with a common constant* c_{LG}.
3. $b^n \to b$ *and* $\sigma^n \to \sigma$ *locally uniformly for some coefficients* b *and* σ
4. $X^n \Rightarrow X$ *for some continuous process* X

Then X *is a weak solution of* (1)

Resolvent of the first kind and SVEs

Assume:

1. X a continuous process
2. $dZ = b\,dt + \sigma\,dW$ a continuous semimartingale with b, σ, and $K * dZ$ continuous
3. K admits a resolvent of the first kind L – see (7)

Then

$$X - X_0 = K * dZ \qquad \Longleftrightarrow \qquad L * (X - X_0) = Z. \tag{9}$$

In this case, for any $F \in L^2_{\text{loc}}(\mathbb{R}_+)$ such that $F * L$ **is right-continuous and of locally bounded variation**

$$F * dZ = (F * L)(0)X - (F * L)X_0 + d(F * L) * X \quad dt \otimes \mathbb{P}\text{-a.e.} \tag{10}$$

SVEs - the semimartingale case

Important consequence:

- $K(0) < \infty$
- $K' \in L^2_{loc}(\mathbb{R}_+)$
- $K' * L$ **right-continuous and of locally bounded variation**

Then (1) becomes

$$dX_t = ((K' * L)(0)X - (K' * L)X_0 + d(K' * L) * X)dt + K(0)(b(X)dt + \sigma(X)dW)$$

so X **is a semimartingale**.

Remark: In particular when $K(t) = \exp(-\gamma t)$ this agrees with (3)

Conditions for an invariance result of SVEs on $\mathbb{R}_+$

State space: $E = \mathbb{R}_+$

Extra conditions on K:

1. K satisfies (6)
2. K is nonnegative, not identically zero, non-increasing and continuous on $(0, \infty)$, and its resolvent of the first kind L is nonnegative and non-increasing in that $s \mapsto L([s, s + t])$ is non-increasing for all $t \geq 0$ (11)

Conditions on the coefficients:

1. b and σ are continuous and satisfy the linear growth condition (2)
2. **Inward pointing condition:**

$$x = 0 \text{ implies } b(x) \geq 0 \text{ and } \sigma(x) = 0 \tag{12}$$

An invariance result of SVEs on $\mathbb{R}_+$

Theorem (SVEs on $\mathbb{R}_+$ – Abi Jaber et al. (2017))

Under the conditions above on K, b and σ, the SVE (1) admits an $\mathbb{R}_+$-valued continuous weak solution for any initial condition $X_0 \in \mathbb{R}_+$

Remarks:

- Observe that (12) **is independent of K!**
- For a **Volterra square root process** as in (4) with $b(x) = \lambda(\theta - x)$ and $\sigma(x) = \sigma\sqrt{x}$ the conditions are satisfied if $\lambda\theta \geq 0$

Affine Volterra processes

- **State space** $E \subseteq \mathbb{R}^d$
- **Affine diffusion and drift coefficients**

$$a(x) = A^0 + A^1 x_1 + \cdots + A^d x_d$$
$$b(x) = b^0 + b^1 x_1 + \cdots + b^d x_d$$

with $A^i \in \mathbb{S}^d$, $b^i \in \mathbb{R}^d$, and $a(x) \succeq 0$ on E

- $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ continuous with $\sigma(x)\sigma(x)^\top = a(x)$ on E
- **Matrix-valued kernel** $K \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$

Definition (Affine Volterra process)

A continuous E-valued solution X of the stochastic Volterra equation

$$X_t = X_0 + \int_0^t K(t-s) b(X_s) ds + \int_0^t K(t-s)\sigma(X_s) dW_s$$

is called an **affine Volterra process** (of convolution type)

Affine Volterra processes: Examples

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

- **Example:** For usual affine diffusions, take $K(t) \equiv \mathrm{id}$
- **Example:** The volatility process in the rough Heston model by **El Euch and Rosenbaum (2016)** is obtained with

$$K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$$

- **Example:** More generally, the full rough Heston model uses $d = 2$ and the kernel

$$K(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \end{pmatrix}$$

The one-dimensional case

For simplicity we suppose $d = 1$ so that

$$b(x) = \beta - \lambda x \text{ and } \sigma(x)^2 = \alpha + ax \tag{13}$$

for some real parameters $\beta, \lambda, \alpha, a$

Existence: Weak existence of a solution such that $\alpha + aX_t \geq 0$ for all t if

$$\alpha + aX_0 \geq 0, \quad a\beta + \lambda\alpha \geq 0$$

Important functions:

$$\mathcal{R}_\phi(y) = \beta y + \frac{\alpha}{2} y^2, \qquad \mathcal{R}_\Psi(y) = -\lambda y + \frac{a}{2} y^2 \tag{14}$$

Fourier–Laplace transform

Theorem (Fourier–Laplace transform – Abi Jaber et al. (2017))

X be a solution of (1) with $b(x)$ and $\sigma(x)$ as in (13) and $K \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$. Fix $T > 0$ and $v \in \mathbb{C}$, and assume that the **Riccati–Volterra equation**

$$\psi = vK + K * \mathcal{R}_\Psi(\psi) \tag{15}$$

has a solution $\psi \in L^2(0,T)$. Then the process

$$M_t = \exp\left(v\,\mathbb{E}\left[X_T \mid \mathcal{F}_t\right] + \frac{1}{2}\int_t^T (\alpha + a\,\mathbb{E}[X_s \mid \mathcal{F}_t])\psi(T-s)^2 ds\right) \tag{16}$$

is a local martingale on $[0,T]$, and satisfies

$$\frac{dM_t}{M_t} = \psi(T-t)\sigma(X_t)dW_t \tag{17}$$

If M is a true martingale, the **Fourier–Laplace transform** *of X_T is $\mathbb{E}[\exp(vX_T) \mid \mathcal{F}_t] = M_t$*

Uniqueness in law

We can extend to the previous result to show a formula of the form

$$\mathbb{E}\left[\exp\left(vX_T + (f * X)_T\right) \mid \mathcal{F}_t\right] = M_t$$

where $f \in L^1_{loc}(\mathbb{R}_+)$ and

$$M_t = \exp\left(\mathbb{E}[X_T + (f * X)_T \mid \mathcal{F}_t] + \frac{1}{2}\int_t^T (\alpha + a\,\mathbb{E}[X_s \mid \mathcal{F}_t])\psi(T-s)^2 ds\right)$$

where ψ solves

$$\psi = vK + K * (\mathcal{R}_\Psi(\psi) + f)$$

Remark: Existence Riccati Volterra eqn. $\Rightarrow$ **Uniqueness in law!**

Unconditional Fourier–Laplace transform

Taking $t = 0$ in the previous theorem one can show that

$$E[\exp(vX_T)] = M_0 = \exp(\phi(T) + \chi(T)X_0) \tag{18}$$

where

$$\phi' = \mathcal{R}_\Phi(\psi), \quad \chi' = \mathcal{R}_\Psi(\psi) \tag{19}$$

and

$$\phi(0) = 0, \quad \chi(0) = v$$

Recall: $\mathcal{R}_\Phi, \mathcal{R}_\Psi$ defined in (14)

Characteristic function: Short derivation

Proof.

1. **Ansatz:** Fix T and consider the semimartingale $M_t = e^{Y_t}$, where

$$Y_t = \phi(T-t) + \chi(T)X_0 - \int_0^t \chi'(T-s)X_s ds + \int_0^t \psi(T-s)dZ_s$$

with $Z_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$

2. **Itô** yields

$$\frac{dM_t}{M_t} = \left(-\phi' + \mathcal{R}_\Phi\right) dt + \left(-\chi' + \mathcal{R}_\Psi\right) X_t dt + (dW_t \text{ term})$$

a local martingale **by** (19)

3. By **"martingale condition"**, a martingale

4. **Def. of** Y **and** (15) yield $Y_T = \phi(0) + \chi(0)X_T = vX_T$. Hence

$$\mathbb{E}[e^{vX_T}] = \mathbb{E}[M_T] = M_0 = e^{\phi(T)+\chi(T)X_0}$$

The classical case

- **Classical case:** $K \equiv 1$, $\chi = \psi$ and (18) is the classical exponential affine formula in terms of the **Riccati equations** (19)
- In this case we have the **exponential affine formula for the conditional Fourier–Laplace transform**

$$\mathbb{E}\left[\exp(vX_T) \mid \mathcal{F}_t\right] = \exp(\phi(T-t) + \psi(T-t)X_t) \tag{20}$$

  where $\phi, \psi = \chi$ satisfy (19)
- **Variation of constants:**

$$\mathbb{E}[X_s \mid \mathcal{F}_t] = \exp(-\lambda(s-t))X_t + \beta \int_0^{s-t} \exp(-\lambda r)\, dr, \quad s \geq t$$

$$\psi(t) = v \exp(-\lambda t) + \frac{a}{2} \int_0^t \exp(-\lambda(t-s))\psi^2(s) ds$$

  These formulas can be used to show the equivalence between (16) and (20)

Forward process: Definition

- **General case:**

$$\mathbb{E}[X_s \mid \mathcal{F}_t] = X_0 + \int_0^s K(s-u)(\beta - \lambda \mathbb{E}[X_u \mid \mathcal{F}_t])du + \int_0^t K(s-u)\sigma(X_u)dW_u$$

Observation: The martingale property of $\int_0^{\cdot} K(s-u)\sigma(X_u)dW_u$ follows from the moment bounds for X

- In order to find $\mathbb{E}[X_s \mid \mathcal{F}_t]$ explicitly **we need a variation of constants analogue**
- The same applies for the Riccati Volterra equation (15) in order to simplify the linear term

Definition (Forward process)

We call

$$\xi_t(T) = \mathbb{E}[X_T \mid \mathcal{F}_t]$$

the forward process of X

Resolvent of second kind

Definition (Resolvent of the second kind)

For $K \in L^1_{\text{loc}}(\mathbb{R}_+)$, the *resolvent*, or *resolvent of the second kind*, corresponding to K is the kernel $R \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$K * R = R * K = K - R \tag{21}$$

Remarks:

- Rather than (21), it is sometimes required $K * R = R * K = R - K$ in the definition of resolvent. We use (21) to remain consistent with **Gripenberg et al. (1990)**
- **The resolvent always exists and is unique**
- The resolvent R **allows to derive a variation of constants formula**

Resolvent of the second kind : Examples

- The kernel λK admits a **resolvent of the second kind** $R_\lambda \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$:

$$(\lambda K) * R_\lambda = R_\lambda * (\lambda K) = \lambda K - R_\lambda$$

- **Example:** If $K \equiv 1$ then $R_\lambda(t) = \lambda e^{-\lambda t}$
- **Example:** If $K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$ then

$$R_\lambda = f^{\alpha,\lambda}$$

is the so-called **Mittag-Leffler density function**

Table: Resolvents of the first and second kind

$K(t)$	$R(t)$	$L(dt)$
(Const.) c	$c\mathrm{e}^{-ct}$	$c^{-1}\delta_0(dt)$
(Fract.) $c\,\frac{t^{\alpha-1}}{\Gamma(\alpha)}$	$ct^{\alpha-1}E_{\alpha,\alpha}(-ct^{\alpha})$	$\frac{c^{-1}t^{-\alpha}}{\Gamma(1-\alpha)}dt$
(Exp.) $c\mathrm{e}^{-\lambda t}$	$c\mathrm{e}^{-\lambda t}\mathrm{e}^{-ct}$	$c^{-1}(\delta_0(dt)+\lambda\,dt)$
(Gamma) $c\mathrm{e}^{-\lambda t}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$	$c\mathrm{e}^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-ct^{\alpha})$	$\frac{c^{-1}\mathrm{e}^{-\lambda t}}{\Gamma(1-\alpha)}\frac{d}{dt}(t^{-\alpha}*\mathrm{e}^{\lambda t})(t)dt$

Table: Some kernels K and their resolvents R and L of the second and first kind. Here $E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$ denotes the Mittag–Leffler function, and the constant c may be an invertible matrix.

Variation of constants

Lemma (Variation of constants)

Assume

1. X *a continuous process*
2. F *a continuous function on* $\mathbb{R}_+$
3. $B \in \mathbb{R}$
4. $Z = \int b\,dt + \int \sigma\,dW$ *a continuous semimartingale with* b, σ, *and* $K * dZ$ *continuous*

Then

$$X = F + (KB) * X + K * dZ \qquad \Longleftrightarrow \qquad X = F - R_B * F + E_B * dZ$$

where R_B *is the resolvent of* $-KB$ *and* $E_B = K - R_B * K$

Forward process: Main result

Proposition (Forward process)

Let R_λ be the resolvent of λK. The forward process

$$\xi_t(T) = \mathbb{E}[X_T \mid \mathcal{F}_t]$$

satisfies

$$d\xi_t(T) = \lambda^{-1} R_\lambda(T-t)\sigma(X_t)dW_t$$

with initial condition

$$\xi_0(T) = X_0 \left(1 - \int_0^T R_\lambda(s)ds\right) + \beta \int_0^T \lambda^{-1} R_\lambda(s)ds$$

If $\lambda = 0$, interpret $\lambda^{-1} R_\lambda = K$, and note that $R_\lambda = 0$ in this case

This proposition + Variation of constants in Riccati Volterra Eqn.
$\Rightarrow$ Fourier–Laplace formula

Exponential affine formula w.r.t. the past: Preliminaries

Notation:

1. **Shift:**
$$\Delta_h f(t) = f(t+h).$$
2. R_λ the resolvent of λK and $E_\lambda = \lambda^{-1} R_\lambda = K - R_\lambda$
3. $\Pi_h = (\Delta_h E_\lambda) * L - \Delta_h (E_\lambda * L)$
4. $\pi_h = \Delta_h \psi * L - \Delta_h (\psi * L)$

Assumption on the kernel: K is continuous on $(0, \infty)$, admits a resolvent of the first kind L, and that one has the total variation bound

$$\sup_{h \leq T} \|\Delta_h K * L\|_{\mathrm{TV}(0,T)} < \infty, \quad T \geq 0 \tag{22}$$

Exponential affine formula w.r.t. the past

Theorem (Affine w.r.t past – Abi Jaber et al. (2017))

Under the above conditions, the following hold with $h = T - t$:

1. **Forward process:**

$$\mathbb{E}[X_T \mid \mathcal{F}_t] = (1*E_\lambda)(h)\beta+(\Delta_h E_\lambda*L)(0)X_t-\Pi_h(t)X_0+(d\Pi_h * X)_t \tag{23}$$

2. **Fourier–Laplace transform:** *Suppose that ψ solves the Riccati Volterra equation (15) then*

$$\mathbb{E}[\exp(vX_T) \mid \mathcal{F}_t] = \exp(Y_t) \tag{24}$$

where

$$Y_t = \phi(h) + (\Delta_h\psi * L)(0)X_t - \pi_h(t)X_0 + (d\pi_h * X)_t$$

and $\phi(h) = \int_0^h \mathcal{R}_\Phi(\psi(s))ds$

Back to Volterra Heston

Let $\xi_t(s) = \mathbb{E}[V_s \mid \mathcal{F}_t]$ and $Q(u,z) = \frac{1}{2}(u^2 - u) + \sigma\rho u z + \frac{\sigma^2}{2} z^2$

Theorem (Volterra Heston Fourier–Laplace formula)

Consider the Volterra–Heston model. Fix $T > 0$ and $u \in \mathbb{C}$, and assume that the **Riccati–Volterra equation**

$$\psi = K * (Q(u,\psi) - \lambda\psi) \tag{25}$$

has a solution $\psi \in L^2(0,T)$. Then the auxiliary process

$$M_t = \exp\left(u \log(S_t) + \int_t^T \xi_t(s) Q(u, \psi(T-s)) ds\right) \tag{26}$$

is a local martingale on $[0,T]$. If it is a true martingale, the **Fourier–Laplace transform** *of $\log(S_T)$ is $\mathbb{E}[\exp(u\log(S_T)) \mid \mathcal{F}_t] = M_t$*

Variance and integrated variance processes

Extension to variance and integrated variance:

$$\mathbb{E}\left[\exp\left(u\log(S_T) + vV_T + w\int_0^T V_s\,ds\right)\,\Big|\,\mathcal{F}_t\right] = M_t$$

where

$$M_t = \exp\left(u\log(S_t) + v\xi_t(T) + w\int_0^T \xi_t(s)ds + \int_t^T \xi_t(s)Q(u,\psi(T-s))ds\right)$$

and ψ solves

$$\psi = vK + K * (Q(u,\psi) - \lambda\psi + w)$$

Remark:

- $$\frac{dM_t}{M_t} = u\sqrt{V_t}dB_t + \sigma\psi(T-t)\sqrt{V_t}dW_t$$
- **Gatheral and Keller-Ressel (2018):** If the Laplace-transform formula above holds $\Rightarrow (\log S, V)$ is **Volterra Heston**

Fractional calculus and the rough Heston model

- **Power law kernel** $K(t) = t^{\alpha-1}/\Gamma(\alpha)$ used in the **rough Heston model**
- **Riemann–Liouville fractional integral:** $I^\alpha f = K * f$
- **Riemann–Liouville fractional derivative:** $D^\alpha f = \frac{d}{dt} I^{1-\alpha} f$
- The **Riccati Volterra equation** (25) is

$$D^\alpha \psi = Q(u, \psi) - \lambda \psi$$

  which is precisely the **fractional Riccati equation** derived by **El Euch and Rosenbaum (2016)**
- **Fourier–Laplace formula:** See (18)

$$\mathbb{E}[e^{u \log(S_T)}] = \exp\left(u \log(S_0) + \lambda\theta \int_0^T \psi(s) ds + V_0 \, I^{1-\alpha} \psi(T) \right)$$

Riccati Volterra equations: Existence result

Theorem (Volterra equation of quadratic growth; Abi Jaber et al. (2017))

Assume that $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C})$, $p(\,\cdot\,, 0) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, *and that for all* $T \in \mathbb{R}_+$ *there exist a positive constant* Θ_T *and a function* $\Pi_T \in L^2([0,T], \mathbb{R}_+)$ *such that*

$$|p(t,x) - p(t,y)| \leq \Pi_T(t)|x-y| + \Theta_T|x-y|(|x|+|y|), \qquad \forall x, y, \; t \leq T \tag{27}$$

The **Volterra integral equation**

$$\psi = g + K * p(\,\cdot\,, \psi) \tag{28}$$

has a unique non-continuable solution $\psi \in L^2_{\mathrm{loc}}([0, T_{\max}))$

Remark: In the **Lipschitz case**

$$p(\,\cdot\,, \psi) = G(\,\cdot\,)q(\,\cdot\,, \psi), \quad q \text{ Lipschitz and } q(\,\cdot\,, 0), G \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$$

we have a unique global solution, i.e. $T_{max} = \infty$

Deterministic Volterra equations: Invariance result

Theorem (Invariance of Linear Volterra equation)

Assume $K \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ *satisfies* (6) *and the shifted kernel* $\Delta_h K$ *satisfy* (11) *for all* $h \in [0,1]$. *Let* $u, v \in \mathbb{R}$, $F \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *and* $G \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ *be such that* $u, v, F \geq 0$. *Then the linear Volterra equation*

$$\chi = Ku + v + K * (F + G\chi) \tag{29}$$

has a unique solution $\chi \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ *with* $\chi \geq 0$

Observation: Proof *uses a stability result for Volterra integral equations* to reduce to the case χ continuous – see **Abi Jaber et al. (2017)**

Volterra–Ornstein–Uhlenbeck process

- With $\sigma(x) \equiv \sigma$ constant we obtain

$$X_t = X_0 + \lambda \int_0^t K(t-s)(\theta - X_s)ds + \int_0^t K(t-s)\sigma dW_s$$

- This is a **Gaussian process:** If $E_\lambda = K - R_\lambda * K$ with R_λ the resolvent of λK

$$X_t = \left(1 - \int_0^t R_\lambda(s)ds\right) X_0 + \left(\int_0^t E_\lambda(s)ds\right) \lambda\theta + \int_0^t E_\lambda(t-s)\sigma dW_s$$

- The Riccati–Volterra equation has an **explicit solution**: $\psi = uE_\lambda$
- The quadratic variation of the process $Y = \log(M)$ is deterministic

$$\langle Y \rangle_t = \int_0^t \psi(T-s)\sigma\sigma^\top \psi(T-s)^\top ds$$

Thus the martingale condition holds and we have the Fourier–Laplace formula

Volterra square-root process

- With $\sigma(x) = \sigma\sqrt{x}$ we obtain:

$$X_t = X_0 + \lambda \int_0^t K(t-s)(\theta - X_s)ds + \int_0^t K(t-s)\sigma\sqrt{X_s}dW_s \quad (30)$$

- **Inward-pointing drift condition:**

$$\lambda\theta \geq 0$$

- **Assumption on the kernel:** K satisfies (6) and the shifted kernels $\Delta_h K_i$ satisfy (11) for all $h \in [0, 1]$

Volterra square-root process

Theorem (Volterra square-root process – Abi Jaber et al. (2017))

- **The stochastic Volterra equation** (30) **has a unique in law** $\mathbb{R}_+$**-valued weak solution** *for any initial condition* $X_0 \in \mathbb{R}_+$
- *The paths of* X *are Hölder continuous of any order less than* $H = 1/2 - \gamma$
- *For any* $u \in \mathbb{C}$ *with* $\operatorname{Re} u \leq 0$, *the* **Riccati–Volterra equation**

$$\psi(t) = uK(t) + \int_0^t K(t-s)\mathcal{R}_\Psi(\psi(s))ds$$

  has a unique global solution $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ *with* $\operatorname{Re} \psi \leq 0$
- **The martingale condition holds**, *as does the affine transform formula*

Volterra Heston model

- $X = (\log S, V)$ with state space $\mathbb{R} \times \mathbb{R}_+$, where

$$\frac{dS_t}{S_t} = \sqrt{V_t}\, dB_t$$

$$V_t = V_0 + \int_0^t K(t-s)\left(\lambda(\theta - V_s)ds + \sigma\sqrt{V_s}\, dW_s\right)$$

with $d\langle B, W\rangle_t = \rho\, dt$

- **Riccati–Volterra equation:**

$$\psi = u_2 K + K * \left(\frac{1}{2}\left(u_1^2 - u_1\right) + (\rho\sigma u_1 - \lambda)\psi + \frac{1}{2}\sigma^2\psi^2\right)$$

Volterra Heston model

Theorem (Volterra Heston model – Abi Jaber et al. (2017))

Under the same assumptions of the previous theorem

- **The stochastic Volterra equation has a unique in law** $\mathbb{R} \times \mathbb{R}_+$*-valued continuous weak solution* $(\log S, V)$ *for any initial condition* $(\log S_0, V_0) \in \mathbb{R} \times \mathbb{R}_+$
- *The* **paths of V are Hölder continuous** *of any order less than* $H = 1/2 - \gamma$
- *For any* $u \in (\mathbb{C}^2)^*$ *such that*

$$\operatorname{Re} u_1 \in [0,1] \text{ and } \operatorname{Re} u_2 \leq 0$$

  the Riccati–Volterra equation has a unique global solution $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C})$, *which satisfies* $\operatorname{Re} \psi \leq 0$
- **The martingale condition holds,** *as does the affine transform formula*
- **The process S is a martingale**

Musiela parameterization of the forward curve

- **Musiela parametrization of the forward process:**

$$\begin{aligned}\tilde{u}_t(x) &= \xi_t(t+x) = \mathbb{E}[X_{t+x} \mid \mathcal{F}_t] \\ &= \xi_0(t+x) + \int_0^t \lambda^{-1} R_\lambda(T-s)\sigma(X_s)dW_s, \quad t, x \geq 0\end{aligned}$$

Observation: The variable x is **time to maturity**

- **Modified Musiela parametrization of the forward process**

$$u_t(x) = \mathbb{E}\left[X_{t+x} - \int_t^{t+x} K(t-s+x)b(X_s)\,ds \,\middle|\, \mathcal{F}_t\right]$$

- We have

$$u_t(x) = X_0 + \int_0^t K(t-s+x)b(X_s)\,ds + \int_0^t K(t-s+x)\sigma(X_s)\,dW_s \tag{31}$$

Observation: No need to have an affine drift / No use of R_λ

SPDEs - Infinite dimensional affine process

- The process $u_t(x)$ in (31) is a **mild solution of the SPDE**

$$du_t(x) = (\partial_x u_t(x) + K(x)b(u_t(0)))dt + K(x)\sigma(u_t(0))dW_t \quad (32)$$

- The SPDE (32) suggests that the process $\{u_t(\,\cdot\,)\}_{t\geq 0}$ is an **infinite dimensional Markov process**
- In the affine case (13), we expect an **exponential affine formula:**

$$\mathbb{E}\left[\mathrm{e}^{\int_0^\infty h(x)u_T(x)dx} \,\middle|\, \mathcal{F}_t\right] = \mathrm{e}^{\phi(T-t)+\int_0^\infty \Psi(T-t,x)u_t(x)dx} \quad (33)$$

  where $\phi(\tau)$ and $\Psi(\tau,x)$ are solutions of appropriate **Riccati equations**
- **Abi Jaber and El Euch (2018a)** treats the **rough Heston case**

Riccati PDE

- **Riccati equations:**

$$\partial_t \phi(t) = \mathcal{R}_\phi \left(\int_0^\infty \Psi(t,y) K(y) dy \right) \tag{34}$$
$$\Psi(t,x) = h(x-t) \mathbf{1}_{\{x \geq t\}} + \mathcal{R}_\Psi \left(\int_0^\infty \Psi(t-x,y) K(y) dy \right) \mathbf{1}_{\{x<t\}} \tag{35}$$

with $\phi(0) = 0$ and $\mathcal{R}_\phi$, $\mathcal{R}_\Psi$ as in (14)

- **Heuristic PDE:**

$$\partial_t \Psi(t,x) = -\partial_x \Psi(t,x) + \mathcal{R}_\Psi \left(\int_0^\infty \Psi(t,y) K(y)\, dy \right) \delta_0(x)$$

with initial condition $\Psi(0,x) = h(x)$

- **Relation to Riccati Volterra equation:**

$$\psi(t) = \int_0^\infty \Psi(t,x) K(x) dx \tag{36}$$

Laplace representation of the kernel

- Assume that K is the **Laplace transform of some measure** μ, that is,

$$K(t) = \int_0^\infty e^{-xt}\mu(dx), \quad t > 0. \tag{37}$$

- **Examples:**

$$K(t) = 1 \Rightarrow \mu = \delta_0$$

$$K(t) = t^{\alpha-1}/\Gamma(\alpha) \Rightarrow \mu(dx) = \frac{x^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\,dx, \quad \alpha \in (\frac{1}{2}, 1)$$

Theorem (Bernstein–Widder theorem)

K *is* **completely monotone** *on* $(0, \infty) \Leftrightarrow$ *there exists* μ *positive such that* (37) *holds*

Mixture of mean-reverting processes

- **For simplicity assume $X_0 = 0$**
- **Representation as a mixture of mean reverting processes:** Suppose X satisfies (1) (with $X_0 = 0$), then

$$X_t = \int_0^\infty u_t(x)\mu(dx) \tag{38}$$

where

$$u_t(x) = \int_0^t e^{-x(t-s)} b(X_s)ds + \int_0^t e^{-x(t-s)} \sigma(X_s)dW_s \tag{39}$$

Observation: $\{u_t(x)\}_{t\geq 0}$ is a **semimartingale,** even if X is not!

Infinite dimensional system of SDEs

- We have

$$du_t(x) = (-xu_t(x) + b(X_t))dt + \sigma(X_t)dW_t$$

Plugging (38) into this expression gives

$$du_t(x) = \left(-xu_t(x) + b\left(\int_0^\infty u_t(y)\mu(dy)\right)\right)dt + \sigma\left(\int_0^\infty u_t(y)\mu(dy)\right)dW_t \tag{40}$$

- **Affine case:** Plugging (13) into this expression gives

$$du_t(x) = \left(\beta - xu_t(x) - \lambda\int_0^\infty u_t(y)\mu(dy)\right)dt + \sqrt{\alpha + a\int_0^\infty u_t(y)\mu(dy)}dW_t \tag{41}$$

- Gaussian case $a = 0$ is treated by **Carmona et al. (2000); Harms and Stefanovits (2018)**

Approximation with finite dimensional Markov processes

- **Useful for numerical purposes:** Replace μ by an approximation μ_n that is supported on finitely many points $x_1, \dots, x_n$

$$\mu(dx) \approx \sum_{i=1}^{n} c_i \delta_{x_i}(dx)$$

- The system (40) then becomes an SDE for the n-dimensional Markov process $\{u_t(x_1), \dots, u_t(x_n)\}_{t \geq 0}$. For $i = 1, \dots, n$

$$\begin{aligned} du_t(x_i) = &\left(-x_i u_t(x_i) + b\left(\sum_{i=1}^{n} c_i u_t(x_i) \right) \right) dt \\ &+ \sigma\left(\sum_{i=1}^{n} c_i u_t(x_i) \right) dW_t \end{aligned} \tag{42}$$

- **Approximate Volterra process:** $X \approx \sum_{i=1}^{n} c_i u_t(x_i)$ – see **Abi Jaber and El Euch (2018b), Cuchiero and Teichmann (2018)**

Approximation with f.d. Markov processes (cont.)

- The process

$$X_t^n = \sum_{i=1}^{n} c_i u_t(x_i)$$

solves the SVE

$$X_t^n = K_n * (b(X_t^n)dt + \sigma(X_t^n)dW_t)$$

with

$$K_n(t) = \sum_{i=1}^{n} c_i \mathrm{e}^{-tx_i}$$

- If K satisfies (6) and $\|K^n - K\|_{L^2} \to 0$, then $(X^n)_n$ **is tight and its limit points satisfy** (1) – see **Abi Jaber and El Euch (2018b)**

Fourier–Laplace transform and Riccati equations

- In the **affine case** (42) suggests that the process $\{u_t(\cdot)\}_{t\geq 0}$ **is an affine Markov process, possibly infinite-dimensional**
- **Fourier–Laplace formula:**

$$\mathbb{E}\left[\mathrm{e}^{\int_0^\infty h(x)u_T(x)\mu(dx)} \,\middle|\, \mathcal{F}_t\right] = \mathrm{e}^{\phi(T-t)+\int_0^\infty \Psi(T-t,x)u_t(x)\mu(dx)} \tag{43}$$

- **Riccati equations:** $\phi(\tau)$ and $\Psi(\tau,x)$ solutions to

$$\begin{aligned} \partial_t\phi(t) &= \mathcal{R}_\phi\left(\textstyle\int_0^\infty \Psi(t,y)\mu(dy)\right), & \phi(0) &= 0 \\ \partial_t\Psi(t,x) &= -x\Psi(t,x) + \mathcal{R}_\Psi\left(\textstyle\int_0^\infty \Psi(t,y)\mu(dy)\right), & \Psi(0,x) &= h(x) \end{aligned} \tag{44}$$

- **Relation to Riccati Volterra equation:**

$$\psi(t) = \int_0^\infty \Psi(t,x)\mu(dx)$$

Summary

- Brownian paths are too smooth for volatility modeling – Motivation for **rough volatility models**, e.g. rough Heston model
- **Affine Volterra processes** generalize known rough volatility models
- **Existence, invariance results** despite lack of Markov / semimartingale property - Using the theory of convolution equations
- **Affine transform formulas + Riccatica Volterra equations:** full justification for Volterra OU, CIR and Heston
- **Infinite dimensional lifts:** Forward curve and Laplace representation

Future research

- Modeling with Volterra equations **beyond rough volatility**
- **Numerical methods** for SVEs and the Riccati–Volterra equations
- **Statistics** of stochastic Volterra equations
- Hedging and optimal investment, or in general **control problems,** in these models (or infinite dimensional lifts)
- **Boundary attainment** for Volterra square-root processes
- **Non-convolution kernels** $K(t, s)$
- **Jumps**
- Etc.

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