Introduction to affine processes.

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Structure of the lecture

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Introduction (blackboard)

Affine diffusions on $\mathbb{R}$ (blackboard)

Affine diffusions on $\mathbb{R}^d$ I

$$t \geq 0,\ X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s.$$

Assumptions : domain $\mathbb{D} \subset \mathbb{R}^d$,
$\forall x \in \mathbb{D}$, there exists a unique solution s.t.$\mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1$;
$b_i(x), (\sigma(x)\sigma^\top(x))_{i,j} \in \mathcal{C}^0$.

Definition 1

X is affine if there exists functions $\phi(t,u)$ and $\psi(t,u)$ taking respectively their values in $\mathbb{C}$ and $\mathbb{C}^d$ with $\mathcal{C}^0$ time derivatives such that

$$\forall u \in i\mathbb{R}^d, 0 \leq t \leq T, x \in \mathbb{D},\ \mathbb{E}[e^{u^\top X_T^x}|X_t^x] = e^{\phi(T-t,u)+\psi(T-t,u)^\top X_t^x}$$

Affine diffusions on $\mathbb{R}^d$ II

Theorem 2

If X is affine, there exists matrices $a, \alpha_1, \dots, \alpha_d \in \mathcal{M}_d(\mathbb{R})$ and vectors $b, \beta_1, \dots, \beta_d \in \mathbb{R}^d$ such that

$$b(x) = b + \sum_{i=1}^d x_i\beta_i,\ \sigma(x)\sigma^\top(x) = a + \sum_{i=1}^d x_i\alpha_i.$$

(Part of Thm 10.1 in Filipović (2009)). To have a more precise statement, we need to specify the domain $\mathbb{D}$.
Question : On which domain $\mathbb{D}$ can we define affine processes ?
$\mathbb{R}$, $\mathbb{R}_+$, $\mathcal{S}_d^+(\mathbb{R})$, symmetric cones on Euclidean Jordan Algebras, and any product of these sets [Faraut and Koranyi (1994), Grasselli and Tebaldi (2008), Cuchiero, Keller-Ressel, Mayerhofer, Teichmann (2016).]

Affine diffusions on $\mathbb{R}^d$ III

$\mathbb{D} = \mathbb{R}^m_+ \times \mathbb{R}^n$ with $m + n = d$.

Theorem 3

If X is an affine diffusion on $\mathbb{D}$, iff there exists $a, \alpha_1, \dots, \alpha_d \in \mathcal{S}_d^+(\mathbb{R})$ and vectors $b, \beta_1, \dots, \beta_d \in \mathbb{R}^d$ satisfying

- $a_{kl} = 0$ *if* $k \le m$ *or* $l \le m$,
- $\alpha_i = 0$, $i > m$ *and for* $i \le m$, $(\alpha_i)_{kl} = 0$ *if* $k \le m, k \neq i$ *or* $l \le m, l \neq i$,
- $b \in \mathbb{R}^m_+ \times \mathbb{R}^n$, *for* $i \le m$, $(\beta_i)_l \ge 0$ *if* $l \le m, l \neq i$, *for* $i > m$, $(\beta_i)_l = 0$ *if* $l \le m$.

such that

$$b(x) = b + \sum_{i=1}^d x_i \beta_i, \ \sigma(x)\sigma^\top(x) = a + \sum_{i=1}^d x_i \alpha_i.$$

Affine processes

- On $\mathbb{D} = \mathbb{R}^m_+ \times \mathbb{R}^n$: Duffie, Filipović, Schachermayer (2003). Time-inhomogeneous, Filipović (2005)
- On $\mathbb{D} = \mathcal{S}_d^+(\mathbb{R})$: Cuchiero, Filipović, Mayerhofer, Teichmann (2011).
- On general symmetric cones : Cuchiero, Keller-Ressel, Mayerhofer, Teichmann (2016).
- Regularity of affine processes : Keller-Ressel, Teichmann, Schachermayer (2013).

Ornstein-Uhlenbeck processes

$\mathbb{D} = \mathbb{R}^d$, $x, a \in \mathbb{R}^d$, $K \in \mathcal{M}_d(\mathbb{R})$, $\Sigma \in \mathcal{S}_d^+(\mathbb{R})$, W d-dimensional Brownian motion.

$$t \geq 0, X_t^x = x + \int_0^t (a - KX_s^x)ds + \Sigma W_t.$$

This is a Gaussian process, and in particular $X_t^x \sim \mathcal{N}_d(xe^{-Kt} + \int_0^t e^{K(s-t)}dsa, \int_0^t e^{K(s-t)}\Sigma^2 e^{K^\top(s-t)}ds)$. The law of $(X_t^x, \int_0^t X_s^x ds)$ is also a Gaussian vector. Explicit Laplace transform and moments.

The CIR process, first properties

$\mathbb{D} = \mathbb{R}_+$, $x, a \in \mathbb{R}_+$, $k \in \mathbb{R}$, $\sigma \geq 0$.

$$t \geq 0, X_t^x = x + \int_0^t (a - kX_s^x)ds + \sigma \int_0^t \sqrt{X_s^x} dW_s.$$

Theorem 4

There exists a unique nonnegative C^0 process X^x that solves this SDE.

Proposition 5

When $a > 0$, the density of X_t^x is given by

$$p(t,x,z) = \sum_{i=0}^{\infty} \frac{e^{-d_t x/2}(d_t x/2)^i}{i!} \frac{c_t/2}{\Gamma(i + \frac{2a}{\sigma^2})} \left(\frac{c_t z}{2}\right)^{i-1+\frac{2a}{\sigma^2}} e^{-c_t z/2}, \ z > 0.$$

where $c_t = \frac{4}{\sigma^2 \zeta_k(t)}$ and $d_t = c_t e^{-kt}$, i.e. $c_t X_t$ chi-square distr. with degree $\frac{4a}{\sigma^2}$ and noncentrality $d_t x$.

The CIR process : joint characteristic function

Set of convergence :

$\mathcal{D}_t = \left\{(u,v) \in \mathbb{R}, \ \mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] < \infty\right\}.$

Proposition 6

Let $\gamma_v = \sqrt{k^2 - 2\sigma^2 v}$ and $\psi_0 = \frac{k+\gamma_v}{\sigma^2}$. The set of convergence is given by

$$\mathcal{D}_t = \left\{(u,v) \in \mathbb{R}, \ s.t. \ v \leq \frac{k^2}{2\sigma^2}, \frac{2}{\zeta_{-\gamma_v}(t)} > \sigma^2 u - (k + \gamma_v)\right\}$$
$$\bigcup \left\{(u,v) \in \mathbb{R}, \ s.t. \ v > \frac{k^2}{2\sigma^2}, \forall s \in [0,t], \cos\left(|\gamma_v|\frac{s}{2}\right) - \frac{\sigma^2 u - k}{|\gamma_v|} \sin\left(|\gamma_v|\frac{s}{2}\right) > 0\right\}.$$

For $(u,v) \in \mathcal{D}_t$, we have $\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] =$

$$= \left(\frac{e^{\frac{\gamma_v + k}{2}t}}{1 - \frac{\sigma^2}{2}(u - \psi_0)\zeta_{-\gamma_v}(t)}\right)^{\frac{2a}{\sigma^2}} \exp\left(x\left[\psi_0 + \frac{(u-\psi_0)e^{\gamma_v t}}{1 - \frac{\sigma^2}{2}(u-\psi_0)\zeta_{-\gamma_v}(t)}\right]\right).$$

The CIR process : further properties I

- Explicit characteristic function of $(X_t^x, \int_0^t X_s^x ds, 1/X_t^x, \int_0^t (1/X_s^x)ds)$: Hurd and Kuznetsov (2006), Craddock and Lennox (2009).
- Feller condition. Let $x > 0$ and $\tau_0 = \inf\{t \geq 0, X_t^x = 0\}$ with $\inf \emptyset = +\infty$. Then, $\tau_0 = +\infty$ a.s. if, and only if

$$2a \geq \sigma^2 \quad (1)$$

  When $\sigma^2 > 2a$, we have $\tau_0 < \infty$ a.s. if, and only if $k \geq 0$.
- Let $x \geq 0$, $k \in \mathbb{R}$, $\sigma > 0$, and $p \in \mathbb{N}^*$ independent Ornstein-Uhlenbeck processes $dY_t^i = -\frac{k}{2}Y_t^i dt + \frac{\sigma}{2}dW_t^i$, $Y_0^i = \sqrt{x/p}$, for $1 \leq i \leq p$. Then $X_t = \sum_{i=1}^p (Y_t^i)^2$ solves

$$dX_t = \left(p\frac{\sigma^2}{4} - kX_t\right)dt + \sigma\sqrt{X_t}dW_t.$$

The CIR process : further properties II

- "Remove the linear drift by a time change" :

$$\begin{aligned} X_t^x &= x + \int_0^t (a - kX_s^x)ds + \int_0^t \sigma\sqrt{X_s^x}dW_s, \\ \tilde{X}_t^x &= x + at + \int_0^t \sigma\sqrt{\tilde{X}_s^x}dW_s,\ x \in \mathbb{R}_+, t \geq 0. \end{aligned}$$

The processes $(e^{-kt}\tilde{X}^x_{\zeta_{-k}(t)})_{t\geq 0}$ and $(X_t^x)_{t\geq 0}$ have the same law. ($\zeta_k(t) = (1-e^{-kt})/k$)

CIR and Wright-Fisher/Jacobi processes I

- The Wright-Fisher process is valued in $\mathbb{D} = [0,1]$ and defined by

$$X_t^x = x + \int_0^t (a - kX_t^x)dt + \int_0^t \sigma\sqrt{X_t^x(1-X_t^x)}dW_t,\ t \geq 0,$$

with $x \in [0,1]$, $0 \leq a \leq k$ and $\sigma \in \mathbb{R}$.
- The Jacobi process is valued in $\mathbb{D} = [-1,1]$ and defined by

$$X_t^x = x + \int_0^t (a - kX_t^x)dt + \int_0^t \sigma\sqrt{1-(X_t^x)^2}dW_t,\ t \geq 0, \quad (2)$$

with $x \in [0,1]$,$-k \leq a \leq k$ and $\sigma \in \mathbb{R}$.
- X^x is a Jacobi process iff $(1+X^x)/2$ is a Wright-Fisher process.
- If X^x is a Jacobi process with $a=0$, $(X^x)^2$ is a Wright-Fisher process.

CIR and Wright-Fisher/Jacobi processes II

- Explicit moments. For Jacobi processes,

$$\frac{d}{dt}\mathbb{E}[(X_t^x)^m] = m\mathbb{E}[a(X_t^x)^{m-1} - k(X_t^x)^m] + \frac{m(m-1)\sigma^2}{2}\mathbb{E}[(X_t^x)^{m-2} - (X_t^x)^m].$$

- By using the family of Jacobi orthogonal polynomials

$$P_n^{\gamma,\delta}(x) = \frac{1}{2^n}\sum_{k=0}^n \binom{n+\gamma-1}{k}\binom{n+\delta-1}{n-k}(x-1)^{n-k}(x+1)^k,$$

with $\gamma = (k-a)/\sigma^2$, $\delta = (k+a)/\sigma^2$: $\mathbb{E}[P_n^{\gamma,\delta}(X_t)] = \mathbb{E}[P_n^{\gamma,\delta}(X_0)]e^{-n(n+\gamma+\delta-1)\sigma^2 t/2}$.
- Belong to the class of polynomial process (Cuchiero, Keller-Ressel and Teichmann (2012)), which includes affine processes.

CIR and Wright-Fisher/Jacobi processes III

Proposition 7

Let B^1 and B^2 two independent real Brownian motions. Let $b_1, b_2, z_1, z_2 \geq 0$ and $\sigma > 0$ such that $\sigma^2 \leq 2(b_1+b_2)$ and $z_1 + z_2 > 0$. We consider the following CIR processes

$$Z_t^i = z_i + b_i t + \int_0^t \sigma\sqrt{Z_s^i}dB_s^i,\ i = 1, 2.$$

Then, $Y_t = Z_t^1 + Z_t^2$ is a CIR process that never reaches 0, and we define

$$t \geq 0,\ X_t = \frac{Z_t^1}{Y_t},\ \phi(t) = \int_0^t \frac{1}{Y_s}ds.$$

Then, ϕ is bijective on $\mathbb{R}_+$ and the process $(X_{\phi^{-1}(t)}, t \geq 0)$ is a Wright-Fisher diffusion with parameters $a = b_1$, $k = b_1 + b_2$ and σ that is independent of $(Y_t, t \geq 0)$.

Proposition 8

Let B^1 and B^2 two independent real Brownian motions. Let $b_2, z_2 \geq 0$, $\tilde{z}_1 \in \mathbb{R}$ and $\sigma > 0$ such that $\sigma^2 \leq 4b_2$ and $z_2 + (\tilde{z}_1)^2 > 0$. Let $\tilde{Z}^1_t = \tilde{z}_1 + \frac{\sigma}{2}B^1_t$, $Z^1_t = (\tilde{Z}^1_t)^2$ and Z^2 be the following CIR process

$$Z^2_t = z_2 + b_2 t + \int_0^t \sigma \sqrt{Z^2_s} dB^2_s.$$

Then, $Y_t = Z^1_t + Z^2_t$ is a CIR process that never reaches 0, and we define

$$t \geq 0, \ \tilde{X}_t = \frac{\tilde{Z}^1_t}{\sqrt{Y_t}}, \ \phi(t) = \int_0^t \frac{1}{Y_s} ds.$$

Then, ϕ is bijective on $\mathbb{R}_+$ and the process $(\tilde{X}_{\phi^{-1}(t)}, t \geq 0)$ is a Jacobi diffusion with parameters $a = 0$, $k = b_2/2$ and $\sigma/2$ that is independent of $(Y_t, t \geq 0)$.

Hawkes processes I

We start with the simplest form of Hawkes processes : N_t is a unit jump process, with intensity λ_t s.t.

$$d\lambda_t = (a - k\lambda_t)dt + dN_t,$$

with $a \geq 0$, $k \in \mathbb{R}$. Self-exciting process with exponential kernel :

$$\lambda_t = \lambda_0 e^{-kt} + a\zeta_k(t) + \int_0^t e^{-k(t-s)} dN_s.$$

- **Characteristic function :** $u_1, u_2 \in i\mathbb{R}$, $t \leq T$
$$\mathbb{E}[e^{u_1 N_T + u_2 \lambda_T} | \mathcal{F}_t] = e^{\phi(u,T-t) + u_1 N_t + \psi_2(u,T-t)\lambda_t},$$
with $-\partial_t \phi + a\psi_2 = 0, \ -\partial_t \psi_2 - k\psi_2 + e^{u_1 + \psi_2} = 0$. (Blackboard)
- Example of mutually exciting Hawkes processes → Blackboard.
- Initially used to model Earthquakes. Application in Credit Risk and Microstructure.

Hawkes processes II

Multi-exponential kernels : N_t is unit jump process with intensity $\lambda_t = \sum_{i=1}^p \alpha_i \lambda^i_t$, with

- $k_i, \alpha_i > 0$, $i = 1, \dots, p$, $\theta > 0$
- $d\lambda^i_t = k_i \left(\frac{\theta}{p\alpha_i} - \lambda^i_t \right) dt + dN_t$, so that
$$\lambda^i_t = (\lambda^i_0 - \frac{\theta}{p\alpha_i})e^{-k_i t} + \frac{\theta}{p\alpha_i} + \int_0^t e^{-k_i(t-s)} dN_s.$$
- $\lambda_t = \sum_{i=1}^p (\alpha_i \lambda^i_0 - \theta_i/p)e^{-k_i t} + \theta + \int_0^t \varphi(t-s) dN_s$.
- Hawkes process with kernel $\varphi(u) = \sum_{i=1}^p \alpha_i e^{-k_i u}$. The process $(N, \lambda^1, \dots, \lambda^p)$ is affine.
- Extension to completely monotone kernels. Application : Market microstructure, survey by Bacry, Mastromatteo and Muzy (2015).

Wishart processes

Wishart processes have initially been introduced and studied by Bru in her phD thesis on Escherichia Coli (1987), and have recently been extended by Cuchiero, Filipovic, Mayerhofer and Teichmann (2009). A Wishart process $(X_t)_{t \geq 0}$ of dimension d is defined on nonnegative symmetric matrices $\mathcal{S}^+_d(\mathbb{R})$ and solves the following SDE :

$$\begin{aligned} dX_t &= (\alpha a^T a + bX_t + X_t b^T)dt + \sqrt{X_t} dW_t a + a^T dW_t^T \sqrt{X_t}, \ t \geq 0, \\ X_0 &= x \in \mathcal{S}^+_d(\mathbb{R}). \end{aligned}$$

Here, $\alpha \in \mathbb{R}$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $\sqrt{X_t}$ is the square root of the nonnegative matrix X_t : if $X_t = O_t diag(\Lambda^1_t, \dots, \Lambda^d_t) O_t^{-1}$, $\sqrt{X_t} := O_t diag(\sqrt{\Lambda^1_t}, \dots, \sqrt{\Lambda^d_t}) O_t^{-1}$. W_t denotes a $d \times d$ matrix whose components are independent standard Brownian motions.
$\underline{d = 1 \textbf{ CIR diffusion :}}$ $dX_t = (\alpha a^2 + 2bX_t)dt + 2a\sqrt{X_t} dW_t, t \geq 0$.

Existence, uniqueness and properties

We have the following results (Bru (1991), Cuchiero and al. (2009), Mayerhofer and al. (2011)) :

- When $\alpha \geq d+1$, the SDE has a unique strong solution on the positive symmetric matrices $\mathcal{S}_d^{+,*}(\mathbb{R})$.
- When $d-1 < \alpha < d+1$, the SDE has a unique weak solution on $\mathcal{S}_d^{+}(\mathbb{R})$.

Explicit characteristic function :

$$\forall v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(i\mathrm{Tr}(vX_t))] = \frac{\exp(\mathrm{Tr}[iv(I_d - 2iq_tv)^{-1}m_txm_t^T])}{\det(I_d - 2iq_tv)^{\alpha/2}},$$

where $q_t = \int_0^t \exp(sb)a^Ta\exp(sb^T)ds$, $m_t = \exp(tb)$.

Wishart processes and Matrix Ornstein-Uhlenbeck processes

Let Z_t be a standard Brownian process in $\mathcal{M}_{d,d'}(\mathbb{R})$. We define an Ornstein-Uhlenbeck process Y on $\mathcal{M}_{d,d'}(\mathbb{R})$ as follows :

$$dY_t = bY_tdt + adZ_t, \ Y_0 = y_0,$$

where $a, b \in \mathcal{M}_d(\mathbb{R})$.

Proposition 9

When $d' \geq d-1$, $X_t = Y_tY_t^\top$ is distributed as a Wishart process and we have

$$X \sim WIS_d(y_0y_0^\top, d', b, a^\top). \tag{3}$$

Rk : When $d' \in \{1, \dots, d-2\}$, we remark that the process X is still defined. This gives a way to define Wishart processes for $\alpha \in \{1, \dots, d-2\}$. In this case, X_t is a matrix of maximal rank d' and is never invertible.

Affine diffusions on $\mathcal{S}_d^+(\mathbb{R})$ (Cuchiero et al. (2009))

$$dX_t = (\bar{\alpha} + B(X_t))dt + \sqrt{X_t}dW_ta + a^TdW_t^T\sqrt{X_t}, X_0 = x \in \mathcal{S}_d(\mathbb{R}).$$

$\bar{\alpha} \in \mathcal{S}_d(\mathbb{R})$, $a \in \mathcal{M}_d(\mathbb{R})$ and $B : \mathcal{S}_d(\mathbb{R}) \to \mathcal{S}_d(\mathbb{R})$ is a linear mapping such that $\mathrm{Tr}(B(x)z) \geq 0$ if $\mathrm{Tr}(xz) = 0$ for $x, z \in \mathcal{S}_d^+(\mathbb{R})$.
Wishart SDE if $\bar{\alpha} = \alpha a^Ta$, $B(x) = bx + xb^T$.

- If $\bar{\alpha} - (d+1)a^Ta \in \mathcal{S}_d^+(\mathbb{R})$, unique strong solution.
- If $\bar{\alpha} - (d-1)a^Ta \in \mathcal{S}_d^+(\mathbb{R})$, unique weak solution.

Characteristic function : $\mathbb{E}[\exp(\mathrm{Tr}(uX_T^x))] = \exp(\phi_u(T) + \mathrm{Tr}(\psi_u(T)x))$ with

$$\begin{array}{rcll} \psi_u'(t) & = & B^*(\psi_u(t)) + 2\psi_u(t)a^\top a\psi_u(t) & ; \quad \psi_u(0) = u \\ \phi_u'(t) & = & \mathrm{Tr}(\overline{\alpha}\psi_u(t)) & ; \quad \phi_u(0) = 0. \end{array}$$

In general, no explicit solution $\to$ numerical integration.

Vasicek (1977) I

Short rate model : $dr_t = k(\theta - r_t)dt + \sigma dW_t$. Bank-account $dB(t) = r_t B(t)dt$.
Zero-coupon price :

$$P(t,T) = \mathbb{E}[\exp(-\int_t^T r_s ds)|\mathcal{F}_t] = P^{\text{Vas}}(r_t, T-t),$$

with

$$P^{\text{Vas}}(r,t) = A^{\text{Vas}}(t)\exp\left(-rB^{\text{Vas}}(t)\right), \quad r \geq 0,\ t \geq 0, \tag{4}$$
$$A^{\text{Vas}}(t) = \exp\left[\left(\frac{\sigma^2}{2k^2} - \theta\right)(t - \zeta_k(t)) - \frac{\sigma^2}{4k}\zeta_k(t)^2\right] \text{ and } B^{\text{Vas}}(t) = \zeta_k(t).$$

Vasicek (1977) II

- T-forward measure : $\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{e^{-\int_0^T r_s ds}}{P^{\text{Vas}}(r_0,T)}$.
- For $t \leq T$, $r_t \sim \mathcal{N}(m_{t,T}, \Sigma_t^2)$ under the T-forward probability measure with

$$m_{t,T} = r_0 e^{-kt} + \left(k\theta - \frac{\sigma^2}{k}\right)\zeta_k(t) + \frac{\sigma^2}{k}e^{-k(T-t)}\zeta_{2k}(t) \text{ and } \Sigma_t^2 = \sigma^2\zeta_{2k}(t).$$

- Explicit prices for call/put options on Zero-Coupon bonds (and thus for floorlets/caplets) :

$$\begin{aligned}&\mathbb{E}[e^{-\int_0^{T_0} r_s ds}(P^{\text{Vas}}(r_{T_0}, T_1 - T_0) - K)^+]\\ &= P^{\text{Vas}}(r_0, T_1)\Phi\left(\frac{r^\star(T_1 - T_0) - m_{T_0,T_1}}{\Sigma_{T_0}}\right)\\ &- KP^{\text{Vas}}(r_0, T_0)\Phi\left(\frac{r^\star(T_1 - T_0) - m_{T_0,T_0}}{\Sigma_{T_0}}\right).\end{aligned}$$

Cox-Ingersoll-Ross model (1985)

Short rate model : $dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$. Bank-account $dB(t) = r_t B(t)dt$.
Zero-coupon price :

$$P(t,T) = \mathbb{E}[\exp(-\int_t^T r_s ds)|\mathcal{F}_t] = P^{\text{CIR}}(r_t, T-t),$$

with

$$P^{\text{CIR}}(r,t) = A^{\text{CIR}}(t)\exp\left(-rB^{\text{CIR}}(t)\right), \quad r \geq 0,\ t \geq 0,$$
$$A^{\text{CIR}}(t) = \left(\frac{2\gamma e^{\frac{\gamma+k}{2}t}}{\gamma - k + (\gamma+k)e^{\gamma t}}\right)^{\frac{2k\theta}{\sigma^2}}, \quad B^{\text{CIR}}(t) = \frac{2(e^{\gamma t} - 1)}{\gamma - k + (\gamma+k)e^{\gamma t}},$$

where $\gamma = \sqrt{k^2 + 2\sigma^2}$.

Again, explicit T-forward measure and law of r_t under this measure :
$\Longrightarrow$ explicit prices for floorlets/caplets.

The Linear Gaussian Model

El Karoui et al. (1991) and El Karoui and Lacoste (1992) : Short rate model : $r_t = \varphi + \sum_{i=1}^d Y_t^i$ with

$$Y_t = y + \int_0^t \kappa(\theta - Y_s)ds + \int_0^t \sqrt{V}dZ_s,$$

where $\kappa = \text{diag}(\kappa_1, \dots, \kappa_d)$ with $0 < \kappa_1 < \cdots < \kappa_d$, V is a semidefinite positive matrix of order d and $\theta \in \mathbb{R}^d$.

- Zero Coupon prices :

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right] = \exp(E(T-t) + B(T-t)^\top Y_t),$$

  where $B(\tau) = -(\kappa^\top)^{-1}(I_d - e^{-\kappa^\top\tau})\mathbf{1}_d$ and $E(\tau) = -\varphi\tau + \int_0^\tau B(s)^\top\kappa\theta + \frac{B(s)^\top VB(s)}{2}ds$ for $\tau \geq 0$.
- Explicit prices for caplets/floorlets.

Affine Term-Structure Models

- $r_t = \varphi + \sum_{i=1}^{p} Y_t^i$ with Y affine diffusion on $(\mathbb{R}_+)^m \times \mathbb{R}^n$, $m+n=d$: Duffie and Kan (1996), Duffie and Singleton (1999), Dai and Singleton (2000), ...
- Typical results : Y is an affine process iff $\mathbb{E}[\exp\left(-\int_0^T r_0 ds\right)] = \exp(\phi(T) + \psi(T)^\top Y_0)$.
- Recent extensions based on Wishart processes : Gnoatto (2012) $r_t = a + \mathrm{Tr}[BX_t]$ where X is a Wishart proc., Ahdida, A. Palidda (2017) : replace V in the LGM by a Wishart process to generate smile.

Black-Scholes-Merton model (1973)

Affine structure on the log-price $X_t = \log(S_t)$

- $X_T = X_0 + rT + \sigma W_T$.
- Explicit call/put option prices.

The Heston model (1993) I

We use again the log-price $X_t = \log(S_t)$.

- Parameters $a \geq 0$, $\rho \in [-1,1]$, $k \in \mathbb{R}$.

$$\begin{cases} X_t^x = X_0^x + \int_0^t (r - \frac{1}{2}V_u)du + \int_0^t \sqrt{V_u}(\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2) \\ V_t = V_0 + \int_0^t (a - kV_u)du + \sigma \int_0^t \sqrt{V_u} dW_u^1 \end{cases}, \ t \geq 0.$$

- The infinitesimal generator is given by

$$\begin{aligned} Lf(x,v) = {} & (r - \frac{v}{2})\partial_x f(x,v) + (a - kv)\partial_v f(x,v) + \frac{\sigma^2}{2} v \partial_v^2 f(x,v) \\ & + \frac{v}{2}\partial_x^2 f(x,v) + \rho\sigma v \partial_x \partial_v f(x,v) \end{aligned}$$

and is affine with respect to (x,v).

The Heston model (1993) II

- We work with the following SDE : $t \geq 0$,

$$\begin{cases} (X_t^x)_1 = x_1 + \int_0^t (r - \lambda (X_s^x)_2) ds + \int_0^t \sqrt{(X_s^x)_2} (\rho dW_s^1 + \sqrt{1-\rho^2} dW_s^2) \\ (X_t^x)_2 = x_2 + \int_0^t (a - k(X_s^x)_2) ds + \sigma \int_0^t \sqrt{(X_s^x)_2} dW_s^1 \end{cases}.$$

- $\lambda = 1/2$: Heston model. We can show that under $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \frac{e^{-rT} S_T}{S_0}$, $(X_t^x, V_t)_{t \in [0,T]}$ follows the same SDE with $\lambda = -1/2$ and $k - \rho\sigma$ instead of k.

The Heston model (1993) III

Proposition 10

The characteristic function is given by

$$u \in i\mathbb{R}^2, \ \mathbb{E}[\exp(u^\top X_t^x)] = \exp(\phi_u(t) + \psi_u(t)^\top x),$$

$$\phi_u(t) = \left(r u_1 + a(\Psi - \frac{2\sqrt{\Delta}}{\sigma^2}) \right) t - \frac{2a}{\sigma^2} \log\left(\frac{\exp(-\sqrt{\Delta} t) - g}{1-g} \right),$$

$$(\psi_u(t))_1 = u_1, \ (\psi_u(t))_2 = u_2 + (\Psi - u_2) \frac{1 - \exp(\sqrt{\Delta} t)}{1 - g \exp(\sqrt{\Delta} t)},$$

with $\Delta = (\rho\sigma u_1 - k)^2 - \sigma^2(u_1^2 - 2\lambda u_1)$, $\Psi = \frac{k - \rho\sigma u_1 + \sqrt{\Delta}}{\sigma^2}$ *and* $g = \frac{k - \rho\sigma u_1 + \sqrt{\Delta} - \sigma^2 u_2}{k - \rho\sigma u_1 - \sqrt{\Delta} - \sigma^2 u_2}$. *These formulas are valid when* $\Delta \neq 0$, *considering that* $\frac{\exp(-\sqrt{\Delta}t) - g}{1-g} = 1$ *and* $\frac{1-\exp(\sqrt{\Delta}t)}{1 - g\exp(\sqrt{\Delta}t)} = 0$ *when the denominator of* g *is zero.*

The Heston model (1993) IV

Remarks :

- When $\Delta = 0$, one has the following formulas

$$\phi_u(t) = (r u_1 + a\Psi) t - \frac{2a}{\sigma^2} \log\left(1 + \frac{\sigma^2}{2} t (\Psi - u_2) \right),$$

$$(\psi_u(t))_1 = u_1, \ (\psi_u(t))_2 = u_2 + (\Psi - u_2)^2 \frac{\sigma^2 t}{2 + \sigma^2 t(\Psi - u_2)}.$$

- Correct formula, especially for the complex logarithm (Lord ans Kahl (2010)).

The Heston model (1993) V

Corollary 11

The price of a European call option in the Heston model is given by

$$\begin{aligned} C(T,K) = & S_0 \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-iv\log(K)} \tilde{\Phi}(v)}{iv} \right) dv \right) \\ & - K e^{-rT} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-iv\log(K)} \Phi(v)}{iv} \right) dv \right), \end{aligned}$$

where $\Phi(v)$ *(resp.* $\tilde{\Phi}(v)$*) is given by the previous formulas with* $u_1 = iv$, $u_2 = 0$, k *and* $\lambda = 1/2$ *(resp.* $k - \rho\sigma$ *and* $\lambda = -1/2$*).*

The Heston model (1993) VI

Explosion of moments : Andersen and Piterbarg (2007).

Corollary 12

For $p \in \mathbb{R}$, we set $v(p) = (k\frac{\rho}{\sigma} - \frac{1}{2})p + (1-\rho^2)\frac{p^2}{2}$. We also define $\bar{\gamma}_v = \sqrt{|k^2 - 2\sigma^2 v|}$ for $v \in \mathbb{R}$. In the Heston model, the moment of order p is finite at time $t > 0$, i.e. $\mathbb{E}[S_t^p] < \infty$, if, and only if one of these three condition holds.

1. $v(p) \leq \frac{k^2}{2\sigma^2}$ *and* $\rho\sigma p \leq k + \bar{\gamma}_{v(p)}$,
2. $v(p) \leq \frac{k^2}{2\sigma^2}$, $\rho\sigma p > k + \bar{\gamma}_{v(p)}$ *and* $t < \frac{1}{\bar{\gamma}_{v(p)}} \log\left(1 + \frac{2\bar{\gamma}_{v(p)}}{\rho\sigma p - (k + \bar{\gamma}_{v(p)})}\right)$,
3. $v(p) > \frac{k^2}{2\sigma^2}$, *and* $t < \frac{2}{\bar{\gamma}_{v(p)}} \arctan\left(\frac{\bar{\gamma}_{v(p)}}{\rho\sigma p - k}\right) + \pi \mathbf{1}_{\{\rho\sigma p - k < 0\}}$.

In particular, $v(1) \leq \frac{k^2}{2\sigma^2}$ *and* $\rho\sigma \leq k + \bar{\gamma}_{v(1)}$ *which gives* $\mathbb{E}[S_t] = e^{rt} S_0$.

Large deviations and asymptotics on the smile : Jacquier with Forde, Lee and Roome (2009,2011,2012,2013).

Pricing with the Fourier Transform : general principle

- Goal : calculate $\mathbb{E}[f(X_T)]$, $X_T : \Omega \to \mathbb{R}^d$.
- Assumption $\mathbb{E}[e^{iu^\top X_T}]$ is known explicitly or can be computed quickly.
- Under suitable conditions, $f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(u) e^{-iu^\top x} du$, with $\hat{f}(u) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(x) e^{iu^\top x} dx$, and thus

$$\mathbb{E}[f(X_T)] = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(u)\mathbb{E}[e^{-iu^\top X_T}] du.$$

- In practice very efficient for $d = 1$ (Pricing with FFT, Carr and Madan (1999)), can be competitive for $d = 2, 3$ but suffers then from the curse of dimensionality.

Multi-asset Wishart stochastic volatility model I

Gourieroux and Sufana (2004) have proposed the following model for d assets :

$$t \geq 0, 1 \leq l \leq d, \ S_t^l = S_0^l + r\int_0^t S_u^l du + \int_0^t S_u^l (\sqrt{X_u} dB_u)_l,$$

where

$$X_t = X_0 + \int_0^t \left(\alpha a^\top a + bX_u + X_u b^\top\right) du + \int_0^t \left(\sqrt{X_u} dW_u a + a^\top dW_u^\top \sqrt{X_u}\right)$$

is a Wishart process. B Brownian motion on $\mathbb{R}^d$ indep of W.

- Laplace transform obtained by a system of ODEs,
- Instantaneous quadratic covariation matrix between the log-prices :

$$\langle d\log(S_t^k), d\log(S_t^l)\rangle = (X_t)_{k,l} dt.$$

- $d = 1$: Heston model with zero correlation.

Multi-asset Wishart stochastic volatility model II

Da Fonseca, Grasselli and Tebaldi (2007) extended this model, assuming :

$$S_t^l = S_0^l + r\int_0^t S_u^l du + \int_0^t S_u^l (\sqrt{X_u}[\sqrt{1 - \|\rho\|_2^2} dB_u + dW_u \rho])_l,$$

with $\rho \in \mathbb{R}^d$ such that $\|\rho\|_2^2 = \rho^\top \rho = \sum_{i=1}^d \rho_i^2 \leq 1$.
The model is still affine :

$$\langle d(X_t)_{i,j}, d(Y_t)_m\rangle = [(a^\top \rho)_j (X_t)_{i,m} + (a^\top \rho)_i (X_t)_{j,m}] dt.$$

Introduction

The simulation of affine diffusions is :

- obvious for Ornstein-Uhlenbeck processes !
- more involved for CIR, affine diffusions on $\mathbb{R}_+^m \times \mathbb{R}^n$, and Wishart processes. This is our goal here.

Exact simulation

- We recall the probability density function :

$$p(t,x,z) = \sum_{i=0}^{\infty} \frac{e^{-d_t x/2}(d_t x/2)^i}{i!} \frac{c_t/2}{\Gamma(i+\frac{2a}{\sigma^2})} \left(\frac{c_t z}{2}\right)^{i-1+\frac{2a}{\sigma^2}} e^{-c_t z/2}, \; z>0,$$

with $c_t = \frac{4}{\sigma^2 \zeta_k(t)}$, $d_t = c_t e^{-kt}$,

- the $\Gamma(\alpha,\beta)$ with shape $\alpha>0$ and rate $\beta>0$:

$$\frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}, \; z>0.$$

Proposition 13

Let N be a Poisson random variable with parameter $d_t x/2$ and Z be such that the conditional law of Z given N is $\Gamma\left(N+\frac{2a}{\sigma^2}, \frac{c_t}{2}\right)$. Then, Z and X_t^x have the same law.

Exact simulation without the Poisson r.v.

Proposition 14

Suppose that $\frac{4a}{\sigma^2} \geq 1$. Let $G \sim \mathcal{N}(0,1)$. We consider an independent random variable $Z \sim \Gamma\left(\frac{2a}{\sigma^2} - \frac{1}{2}, \frac{c_t}{2}\right)$ when $\frac{4a}{\sigma^2} > 1$ and set $Z = 0$ if $\sigma^2 = 4a$. Then, $(e^{-kt/2}\sqrt{x} + (\sigma/2)\sqrt{\zeta_k(t)}G)^2 + Z$ and X_t^x have the same law.

Proposition 15

(Shao (2012)) Let $U \sim \mathcal{U}([0,1])$ and $Z \sim \Gamma\left(\frac{2a}{\sigma^2}, \frac{c_t}{2}\right)$ be independent random variables that are independent from $\tilde{X}$, CIR process with $\tilde{a} = a + \sigma^2/2$ instead of a. Then, $\mathbf{1}_{U \leq e^{-d_t x/2}} Z + \mathbf{1}_{U > e^{-d_t x/2}} \tilde{X}_t^{x + \frac{2}{d_t}\log(U)}$ and X_t^x have the same law.

But the simulation of Gamma r.v. is still longer than one Gaussian r.v.

The Euler scheme for the CIR process

We use the following parametrization of the CIR :

$$X_t^x = x + \int_0^t (a - kX_s^x)ds + \sigma \int_0^t \sqrt{X_s^x} dW_s, t \in [0,T]$$
$$x \geq 0, a > 0, k \in \mathbb{R}, \sigma > 0.$$

Time discretization : $t_i^n = \frac{iT}{n}, i = 0, \dots, n.$

Euler scheme : $\hat{X}^n_{t^n_{i+1}} = \hat{X}^n_{t^n_i} + (a - k\hat{X}^n_{t^n_i})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{t^n_i}}\underbrace{(W_{t^n_{i+1}} - W_{t^n_i})}_{\text{may be large and } <0}$

It is thus not well defined $\implies$ need to introduce bespoke schemes for the CIR.

"Corrections" of the Euler scheme.

- Deelstra and Delbaen (1998) :

$$\hat{X}^n_{t_0} = x,\ \hat{X}^n_{t_{i+1}} = \hat{X}^n_{t_i} + (a - k\hat{X}^n_{t_i})\frac{T}{n} + \sigma\sqrt{\left(\hat{X}^n_{t_i}\right)^+}(W_{t_{i+1}} - W_{t_i}),\ 1 \leq i \leq n-1.$$

- Higham and Mao (2005) :

$$\hat{X}^n_{t_0} = x,\ \hat{X}^n_{t_{i+1}} = \hat{X}^n_{t_i} + (a - k\hat{X}^n_{t_i})\frac{T}{n} + \sigma\sqrt{|\hat{X}^n_{t_i}|}(W_{t_{i+1}} - W_{t_i}).$$

- Berkaoui, Bossy and Diop (2008) :

$$\hat{X}^n_{t_0} = x,\ \hat{X}^n_{t_{i+1}} = \left|\hat{X}^n_{t_i} + (a - k\hat{X}^n_{t_i})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{t_i}}(W_{t_{i+1}} - W_{t_i})\right|.$$

- Lord, Koekkoek and Van Dijk (2010) :

$$\hat{X}^n_{t_0} = x,\ \hat{X}^n_{t_{i+1}} = \hat{X}^n_{t_i} + (a - k(\hat{X}^n_{t_i})^+)\frac{T}{n} + \sigma\sqrt{\left(\hat{X}^n_{t_i}\right)^+}(W_{t_{i+1}} - W_{t_i}).$$

Correction of the Milstein scheme

- Milstein scheme for the CIR :

$$\begin{aligned}\hat{X}^n_{t_{i+1}} &= \hat{X}^n_{t_i} + (a - k\hat{X}^n_{t_i})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{t_i}}(W_{t_{i+1}} - W_{t_i}) \\ &\quad + \frac{\sigma^2}{4}((W_{t_{i+1}} - W_{t_i})^2 - \frac{T}{n}) \\ &= \hat{X}^n_{t_i} + (a - \frac{\sigma^2}{4} - k\hat{X}^n_{t_i})\frac{T}{n} + \left(\sqrt{\hat{X}^n_{t_{i+1}}} + \frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i})\right)^2.\end{aligned}$$

Well defined for $\sigma^2 \leq 4a$ and $k \leq 0$.

- A. (2005) uses the following correction for $\sigma^2 \leq 4a$ and $n \geq kT/2$

$$\hat{X}^n_{t_{i+1}} = \left(\left(1 - \frac{kT}{2n}\right)\sqrt{\hat{X}^n_{t_i}} + \frac{\sigma(W_{t_{i+1}} - W_{t_i})}{2(1 - \frac{kT}{2n})}\right)^2 + (a - \sigma^2/4)\frac{T}{n}.$$

Correction of the Milstein scheme

- Milstein scheme for the CIR :

$$\begin{aligned}\hat{X}^n_{t_{i+1}} &= \hat{X}^n_{t_i} + (a - k\hat{X}^n_{t_i})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{t_i}}(W_{t_{i+1}} - W_{t_i}) \\ &\quad + \frac{\sigma^2}{4}((W_{t_{i+1}} - W_{t_i})^2 - \frac{T}{n}) \\ &= (a - \frac{\sigma^2}{4} - k\hat{X}^n_{t_i})\frac{T}{n} + \left(\sqrt{\hat{X}^n_{t_{i+1}}} + \frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i})\right)^2.\end{aligned}$$

  Well defined for $\sigma^2 \le 4a$ and $k \le 0$.
- A. (2005) uses the following correction for $\sigma^2 \le 4a$ and $n \ge kT/2$

$$\hat{X}^n_{t_{i+1}} = \left(\left(1 - \frac{kT}{2n}\right)\sqrt{\hat{X}^n_{t_i}} + \frac{\sigma(W_{t_{i+1}} - W_{t_i})}{2(1 - \frac{kT}{2n})}\right)^2 + (a - \sigma^2/4)\frac{T}{n}.$$

Implicit Euler schemes

- On the CIR diffusion (Brigo and A., 2005) :

$$\hat{X}^n_{t_{i+1}} = \left(\frac{\sigma(W_{t_{i+1}} - W_{t_i}) + \sqrt{\sigma^2(W_{t_{i+1}} - W_{t_i})^2 + 4(\hat{X}^n_{t_i} + (a - \frac{\sigma^2}{2})\frac{T}{n})(1 + k\frac{T}{n})}}{2(1 + k\frac{T}{n})}\right)^2.$$

  Well-defined for $\sigma^2 \le 2a$.
- On the square root $d\sqrt{X^x_t} = \left(\frac{a - \sigma^2/4}{2\sqrt{X^x_t}} - k\sqrt{X^x_t}\right)dt + \frac{\sigma}{2}dW_t$ (A. 2005) :

$$\hat{X}^n_{t_{i+1}} = \left(\frac{\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{\hat{X}^n_{t_i}} + \sqrt{(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{\hat{X}^n_{t_i}})^2 + 4(1 + \frac{kT}{2n})\frac{a - \sigma^2/4}{2}\frac{T}{n}}}{2(1 + \frac{kT}{2n})}\right)^2.$$

  Well-defined for $\sigma^2 \le 4a$.

Strong convergence results

It has been shown that all these schemes converges strongly to the CIR, i.e.

$$\mathbb{E}\left[\max_{0 \le i \le n} |\hat{X}^n_{t_i} - X^x_{t_i}|\right] \to 0,$$

some of them with a cv rate.
Best known strong cv rates for the implicit scheme on the square root (A. (2013) and Neuenkirch and Szpruch (2014)) :

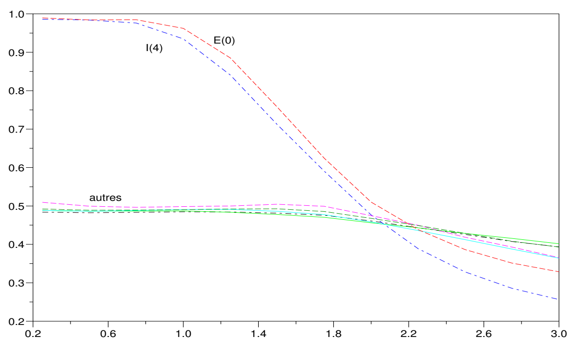
Theorem 16

Let $x > 0$, $2a > \sigma^2$ and $T > 0$. Then,

$$2a > \sigma^2 : \forall p \in [1, \frac{2a}{\sigma^2}), \exists K_p > 0, \left(\mathbb{E}\left[\max_{0 \le i \le n} |\hat{X}^n_{t_i} - X^x_{t_i}|^p\right]\right)^{1/p} \le K_p\sqrt{\frac{T}{n}},$$

$$a > \sigma^2 : \forall p \in [1, \frac{4a}{3\sigma^2}), \exists K_p > 0, \left(\mathbb{E}\left[\max_{0 \le i \le n} |\hat{X}^n_{t_i} - X^x_{t_i}|^p\right]\right)^{1/p} \le K_p\frac{T}{n}.$$

Empirical strong cv rate in fct of $\sigma^2/2a$



Common shortcoming of these schemes

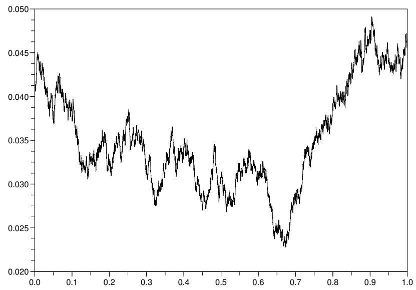
- All of these schemes can be written for some φ :

$$\hat{X}^n_{t^n_{i+1}} = \varphi(\hat{X}^n_{t^n_i}, T/n, W_{t^n_{i+1}} - W_{t^n_i}).$$

- These schemes have not the same convergence behaviour, but most of them bring satisfactory convergence properties only for $\sigma^2 \leq 4a$.
- **For $\sigma^2 \gg 4a$, none of them has a fast convergence.**
- Heuristic reason : the larger is sigma, the more the CIR spend time around zero where the square-root derivative blows up.

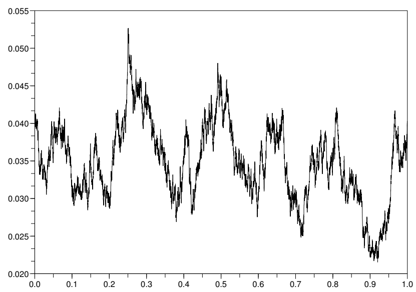
Example of CIR paths I

$x = 0.04$, $k = 0.5$, $a = 0.02$ and $\sigma = 0.1$, i.e. $\sigma^2 = a/2$.



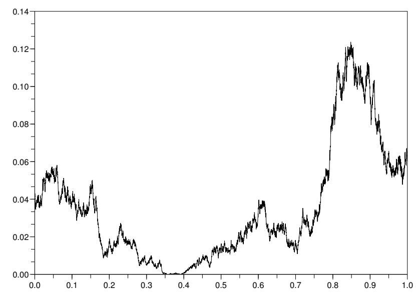
Example of CIR paths II

$\sigma = 0.2$ i.e. $\sigma^2 = 2a$



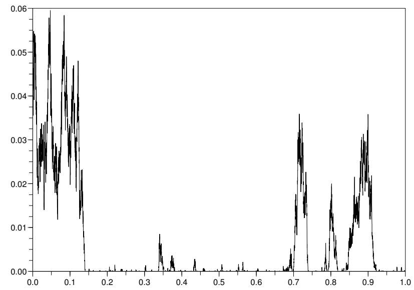
Example of CIR paths III

$\sigma = 0.4$ i.e. $\sigma^2 = 8a$



Example of CIR paths IV

$\sigma = 1$ i.e. $\sigma^2 = 25a$



A recent result by Hefter and Jentzen (2017)

Theorem 17

Let $a, \sigma > 0$ such that $\sigma^2 > 2a$. There exists $c > 0$ such that for all $n \geq 1$,

$$\inf_{\varphi:\mathbb{R}^n \to \mathbb{R} \text{ mesurable}} \mathbb{E}[|X_T^x - \varphi(W_{T/n}, W_{2T/n} \dots, W_T)| \geq cN^{-2a/\sigma^2}.$$

$\Longrightarrow$ To get efficient schemes for large values of σ, it is wise to consider schemes that approximates the law rather than the path for a given W.

- Andersen (2008) has proposed an ad-hoc scheme that works well without restriction on σ, but no convergence result is given.
- We present here a framework to analyze the weak error, and high order schemes for the weak error.

Introduction

- To discretize an SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, the most common choice is to use the Euler-Maruyama scheme :

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + b(\hat{X}_{t_i})(t_{i+1} - t_i) + \sigma(\hat{X}_{t_i})(W_{t_{i+1}} - W_{t_i}).$$

  It is basically used to compute expectations with a Monte-Carlo algorithm.
- To speed up the computation, one may desire to consider sharper schemes. To do so, one way is to find a better pathwise approximation by using iterated stochastic Taylor expansions (Milstein scheme and further expansions). However, a concrete implementation of these schemes is not easy in general.
- An alternative is to approximate the law of the SDE increments. We will use this point of view here.

Assumptions

$$t \geq 0,\ X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s.$$

Assumptions : domain $\mathbb{D} \subset \mathbb{R}^d$, $\forall x \in \mathbb{D}, \mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1$;
$b_i(x), (\sigma(x)\sigma^*(x))_{i,j} \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$, sublinear growth :
$\|b(x)\| + \|\sigma(x)\| \leq K(1 + \|x\|)$

$$\begin{aligned} \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}) \quad = \quad & \{f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}), \forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, e_\alpha \in \mathbb{N}^*, \forall x \in \mathbb{D}, \\ & |\partial_\alpha f(x)| \leq C_\alpha(1 + \|x\|^{e_\alpha})\} \end{aligned}$$

Associated operator :
$Lf(x) = \sum_{i=1}^d b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^{d_W} \sigma_{i,k}(x)\sigma_{j,k}(x)\partial_i\partial_j f(x)$.
Remark : $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}) \implies Lf \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$.
This framework embeds affine diffusions (i.e. when $b(x)$ and $\sigma\sigma^*(x)$ affine functions of x).

Notations for discretization schemes

Definition 18

A family of transition probabilities $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ on $\mathbb{D}$ is s.t. $\hat{p}_x(t)$ is a probability law on $\mathbb{D}$ for $t > 0$ and $x \in \mathbb{D}$. We note $\hat{X}_t^x$ a r.v. with law $\hat{p}_x(t)(dz)$.

<u>Associated discretization scheme</u> on the regular time grid $t_i^n = iT/n$:
$(\hat{X}^n_{t_i^n}, 0 \leq i \leq n)$ sequence of $\mathbb{D}$-valued r.v. s.t. : $\hat{X}^n_{t_{i+1}^n}$ is sampled according to $\hat{p}_{\hat{X}^n_{t_i^n}}(T/n)(dz)$.

Example (Euler) : $\hat{X}_t^x = x + b(x)t + \sigma(x)W_t$, $\hat{p}_x(t)$: law of $\hat{X}_t^x$.

Talay-Tubaro Theorem (1990) I

If

1. $f : \mathbb{D} \to \mathbb{R}$ s. t. $u(t,x) = \mathbb{E}[f(X_{T-t}^x)]$ is defined on $[0,T] \times \mathbb{D}$, solves for $t \in [0,T], x \in \mathbb{D}, \partial_t u(t,x) = -Lu(t,x)$, and has "good bounds" on all its derivatives $\partial_t^l \partial_\alpha u$, i.e.

$$\forall l \in \mathbb{N}, \alpha \in \mathbb{N}^d, \exists C_{l,\alpha}, e_{l,\alpha} > 0, \forall x \in \mathbb{D}, t \in [0,T],\ |\partial_t^l \partial_\alpha u(t,x)| \leq C_{l,\alpha}(1 + \|x\|^{e_{l,\alpha}}).$$

2. the scheme is a **potential weak νth-order discr. scheme** for L :

$$\mathbb{E}[f(\hat{X}_t^x)] = f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x) + Remainder\ "O(t^{\nu+1})"$$

and $(\hat{X}^n_{t_i^n}, i = 0, \dots, n)$ has uniformly bounded moments.

then, $|\mathbb{E}[f(\hat{X}^n_{t_n^n})] - \mathbb{E}[f(X_T^x)]| \leq K/n^\nu$.

Talay-Tubaro Theorem (1990) II

Proof : $\mathbb{E}[f(\hat{X}^n_{t_n^n})] - \mathbb{E}[f(X_T^x)] = \mathbb{E}[u(t_n^n, \hat{X}^n_{t_n^n})] - u(0,x) =$
$\sum_{i=0}^{n-1} \mathbb{E}\left[u(t_{i+1}^n, \hat{X}^n_{t_{i+1}^n}) - u(t_i^n, \hat{X}^n_{t_i^n})\right]$

Remark : Talay-Tubaro theorem is originally stated

- for the Euler-Maruyama scheme $\hat{X}_t^x = x + b(x)t + \sigma(x)W_t$ which is a first-order scheme :

$$\mathbb{E}[f(\hat{X}_t^x)] = f(x) + tLf(x) + "O(t^2)".$$

  This can be easily checked by a Taylor expansion of f around x. It also gives an expansion of the weak error.
- for $b, \sigma \in \mathcal{C}^\infty$ with bounded derivatives, which ensures (1).

Composition of discretization schemes I

$\hat{X}_t^{1,x}$, $\hat{X}_t^{2,x}$: potential νth-order schemes for L_1, L_2.
$\hat{X}_{\lambda_2 t}^{2,\hat{X}_{\lambda_1 t}^{1,x}}$: scheme that amounts to first use the scheme 1 with a time step $\lambda_1 t$ and then the scheme 2 with a time step $\lambda_2 t$. Probability law :
$\hat{p}^2(\lambda_2 t) \circ \hat{p}_x^1(\lambda_1 t)(dz) = \int_{\mathbb{D}} \hat{p}_y^2(\lambda_2 t)(dz)\hat{p}_x^1(\lambda_1 t)(dy)$

Proposition 19

$$\mathbb{E}[f(\hat{X}_{\lambda_2 t}^{2,\hat{X}_{\lambda_1 t}^{1,x}})] = \sum_{l_1+l_2\leq\nu} \frac{\lambda_1^{l_1}\lambda_2^{l_2}}{l_1!l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x) + \text{"}O(t^{\nu+1})\text{"}$$

$(= [I + \lambda_1 t L_1 f + \ldots + \frac{(\lambda_1 t)^\nu}{\nu!} L_1^\nu f][I + \lambda_2 t L_2 f + \ldots + \frac{(\lambda_2 t)^\nu}{\nu!} L_2^\nu f] + \text{"}O(t^{\nu+1})\text{"})$
Csq : a scheme acts on f “as” the operator $I + tLf + \ldots + \frac{t^\nu}{\nu!} L^\nu f + Rem.$
$\boxed{\textbf{Comp. of schemes} = \textbf{Comp. of operators}}$ (in the reverse order).

Composition of discretization schemes II

Proof : Tower property of the conditional expectation :

$$\mathbb{E}[f(\hat{X}_{\lambda_2 t}^{2,\hat{X}_{\lambda_1 t}^{1,x}})] = \mathbb{E}[\mathbb{E}[f(\hat{X}_{\lambda_2 t}^{2,\hat{X}_{\lambda_1 t}^{1,x}})|\hat{X}_{\lambda_1 t}^{1,x}]]$$

We have $\mathbb{E}[f(\hat{X}_{\lambda_2 t}^{2,\hat{X}_{\lambda_1 t}^{1,x}})|\hat{X}_{\lambda_1 t}^{1,x}] = \sum_{l=0}^{\nu} \frac{(\lambda_2 t)^l}{l!} L_2^l f(\hat{X}_{\lambda_1 t}^{1,x}) + \text{"}O(t^{\nu+1})\text{"}$
and $\mathbb{E}[L_2^l f(\hat{X}_{\lambda_1 t}^{1,x})] = \sum_{k=0}^{\nu-l} \frac{(\lambda_1 t)^k}{k!} L_1^k L_2^l f(x) + \text{"}O(t^{\nu+1-l})\text{"}$ $\square$.
When $L_1 L_2 = L_2 L_1$, $\lambda_1 = \lambda_2 = 1$,

$$\mathbb{E}[f(\hat{X}_t^{2,\hat{X}_t^{1,x}})] = \sum_{l=0}^{\nu} \frac{t^l}{l!}(L_1 + L_2)^l f(x) + \text{"}O(t^{\nu+1})\text{"},$$

i.e. $\hat{X}_t^{2,\hat{X}_t^{1,x}}$ is a potential νth-order schemes for $L_1 + L_2$.

Recursive construction of 2^{nd} order schemes

Theorem 20

$\hat{p}_x^1, \hat{p}_x^2$: potential 2^{nd} order schemes for L_1, L_2. Then,

$$\hat{p}^2(t/2) \circ \hat{p}^1(t) \circ \hat{p}_x^2(t/2) \quad (\textit{Strang 1968}) \tag{5}$$
$$\frac{1}{2}\left(\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}^1(t) \circ \hat{p}_x^2(t)\right) \tag{6}$$

are potential second order schemes for $L_1 + L_2$.

Proof for (6) : $(I + tL_1 + t^2/2L_1^2 + \ldots)(I + tL_2 + t^2/2L_2^2 + \ldots) =$
$I + t(L_1 + L_2) + t^2/2(L_1^2 + L_2^2 + 2L_1L_2) + \ldots$
Remark : By a recursive application of these rules, we can get from m potential second order schemes for $L_1, \ldots, L_m$, a potential second order schemes for $L = L_1 + L_2 + \cdots + L_m$, provided that the composition is well-defined.

A general method to split an operator L

Let $I \subset \{1, \dots, d_W\}$. Suppose that $b(x) = b^I(x) + b^{I^c}(x)$. Let

- $(W_t^I)_i = (W_t)_i$ if $i \in I$ and $(W_t^I)_i = 0$ otherwise,
- $(W_t^{I^c})_i = (W_t)_i$ if $i \notin I$ and $(W_t^{I^c})_i = 0$ if $i \in I$ ($W_t = W_t^I + W_t^{I^c}$).

Then,

$$\boxed{L = L^I + L^{I^c}}$$

where L^I (resp. L^{I^c}) is the operator associated to

$$dX_t^I = b^I(X_t^I)dt + \sigma(X_t^I)dW_t^I \text{ (resp. } dX_t^{I^c} = b^{I^c}(X_t^{I^c})dt + \sigma(X_t^{I^c})dW_t^{I^c}\text{)}$$

Remark : When the original diffusion associated to L is defined on a domain $\mathbb{D} \subsetneq \mathbb{R}^d$, one has to take care that the diffusions associated to L^I and L^{I^c} are not necessarily well defined on $\mathbb{D}$.

The scheme of Ninomiya and Victoir (2008) I

It is a splitting that gives a 2^{nd}-order scheme by solving only ODEs.
Assume that $\sigma(x)$ is s.t. the operators $V_k f(x) = \sum_{i=1}^d \sigma_{i,k}(x)\partial_i f$ for $k = 1, \dots, d_W$, $V_0 f(x) = \sum_{i=1}^d b_i(x)\partial_i f(x) - \frac{1}{2}\sum_{i,j=1}^d \sum_{k=1}^{d_W} \partial_j \sigma_{i,k}\sigma_{j,k}\partial_i f(x)$ are well defined on $\mathbb{D}$ and satisfy the same assumptions as L on $\mathbb{D}$. Then, we have

$$L = V_0 + \frac{1}{2}\sum_{k=1}^{d_W} V_k^2.$$

Let v_k s.t. $V_k f(x) =: v_k(x).\nabla f$. We assume that $\exists K > 0, \|v_k(x)\| \leq K(1 + \|x\|)$ and that $X_0(t,x)$ (resp. $X_k(t,x)$, $k = 1, \dots, d_W$) is a $\mathbb{D}$-valued solution to the ODE

$$\frac{dX_0(t,x)}{dt} = v_0(X_0(t,x)),\ t \geq 0 \quad \text{(resp. } \frac{dX_k(t,x)}{dt} = v_k(X_k(t,x)),\ t \in \mathbb{R})$$

that starts from $x \in \mathbb{D}$ at $t = 0$.

The scheme of Ninomiya and Victoir (2008) II

Theorem 21

$X_0(t,x)$ (resp. $X_k(\sqrt{t}N, x)$ where $N \sim \mathcal{N}(0,1)$, for $k = 1, \dots, d_W$) is an exact scheme for the ODE (resp. SDE) associated to the operator V_0 (resp. $\frac{1}{2}V_k^2$) and is in particular a potential νth-order scheme. Moreover,

$$\frac{1}{2}\left(\hat{p}^0(t/2) \circ \hat{p}^m(t) \circ \dots \circ \hat{p}^1(t) \circ \hat{p}_x^0(t/2) + \hat{p}^0(t/2) \circ \hat{p}^1(t) \circ \dots \circ \hat{p}^m(t) \circ \hat{p}_x^0(t/2)\right)$$

is a potential second order scheme on $\mathbb{D}$ for L.

Remarks : • To get a 2nd order scheme for L, it is not necessary to solve exactly the ODEs X_0 and X_k. A third (resp. sixth) order scheme enough to approximate X_0 (resp. X_k). [Refined in Al Gerbi, Clément, Jourdain (2016)].
• If Y is s.t. $\mathbb{E}[Y^q] = \mathbb{E}[N^q]$ for $q = 1, \dots, 5$, $X_k(\sqrt{t}Y, x)$ is a potential 2nd-order scheme for $\frac{1}{2}V_k^2$: the same result holds replacing N by Y.

NV's scheme for the CIR

$$L_{\text{CIR}}f(x) = \underbrace{(a - \frac{\sigma^2}{4} - kx)\partial_x f(x)}_{V_0} + \underbrace{\frac{\sigma^2}{4}\partial_x f(x) + \frac{1}{2}\sigma^2 x \partial_x^2 f(x)}_{V_1^2/2,\ V_1 = \sigma\sqrt{x}\partial_x}$$

• $X_0'(t,x) = a - \frac{\sigma^2}{4} - kX_0(t,x),\ X(0,x) = x$
$\rightsquigarrow\ X_0(t,x) = xe^{-kt} + (a - \sigma^2/4)\psi_k(t)$ with $\psi_k(t) = \frac{1-e^{-kt}}{k}$ is an exact scheme for V_0.
• $(\sqrt{x} + \frac{\sigma}{2}\sqrt{t}N)^2 := X_1(\sqrt{t}N, x)$ with $N \sim \mathcal{N}(0,1)$ is an exact scheme for $V_1^2/2$.
(5) $\implies \hat{X}_t^x = \varphi(x, t, \sqrt{t}N)$ with
$\varphi(x,t,w) = X_0(t/2, X_1(w, X_0(t/2,x))) =$
$e^{-\frac{kt}{2}}\left(\sqrt{(a-\sigma^2/4)\psi_k(t/2) + e^{-\frac{kt}{2}}x} + \frac{\sigma}{2}w\right)^2 + (a - \sigma^2/4)\psi_k(t/2)$ is a potential 2$^{\text{nd}}$ order scheme for the CIR.
Problem : when $a < \sigma^2/4$, $\mathbb{D} = \mathbb{R}_+$ is not stable by the scheme for V_0 and the composition is not well-defined.

A second-order scheme without restriction on σ I

Our guideline is to preserve nonnegativity. (A. 2010)

- We replace the standard Gaussian N by a bounded r.v. that matches the five first moments : $\mathbb{E}[Y^q] = \mathbb{E}[N^q]$ for $q = 1, \dots, 5$. **When x is large enough,** $\varphi(x,t,\sqrt{t}Y) \geq 0$ and
$$\mathbb{E}[f(\varphi(x,t,\sqrt{t}Y))] = f(x) + tL_{\text{CIR}}f(x) + \frac{t^2}{2}L_{\text{CIR}}^2 f(x) + \text{"}O(t^3)\text{"}.$$
- **For small values of** x, we consider an ad-hoc positive scheme that matches the two first moments of the CIR.

A second-order scheme without restriction on σ II

Scheme for the "large values" :
We take Y s.t. $\mathbb{P}(Y = \pm\sqrt{3}) = \frac{1}{6}$, and $\mathbb{P}(Y=0) = 2/3$ and have $\varphi(x,t,\sqrt{t}Y) \geq 0$ iff $x \geq \mathbf{K}_2(t)$, where

$$\mathbf{K}_2(t) = \mathbf{1}_{\{\sigma^2 > 4a\}} e^{\frac{kt}{2}} \left((\frac{\sigma^2}{4} - a)\psi_k(t/2) + \left[\sqrt{e^{\frac{kt}{2}}[(\frac{\sigma^2}{4} - a)\psi_k(t/2)]} + \frac{\sigma}{2}\sqrt{3t} \right]^2 \right) = O(t).$$

The scheme composition is then well defined and we get :

$$\forall x \geq \mathbf{K}_2(t), \mathbb{E}[f(\varphi(x,t,\sqrt{t}Y))] = f(x) + tL_{\text{CIR}}f(x) + \frac{t^2}{2}L_{\text{CIR}}^2 f(x) + \text{"}O(t^3)\text{"}.$$

A second-order scheme without restriction on σ III

Scheme near 0 :
For $0 \leq x < \mathbf{K}_2(t)$, we take a scheme that takes two values $0 \leq x_-(t,x) < x_+(t,x)$ with respective probabilities $1 - \pi(t,x)$ and $\pi(t,x)$ s.t.

$$\pi(t,x)x_+(t,x)^i + (1-\pi(t,x))x_-(t,x)^i = \mathbb{E}((X_t^x)^i),\ i = 1,2.$$

$\pi(t,x) = \frac{1-\sqrt{1-\mathbb{E}(X_t^x)^2/\mathbb{E}((X_t^x)^2)}}{2}$, $x_-(t,x) = \frac{\mathbb{E}(X_t^x)}{2(1-\pi(t,x))}$, $x_+(t,x) = \frac{\mathbb{E}(X_t^x)}{2\pi(t,x)}$.
Why is it sufficient to match the two first moments there ?
From a Taylor expansion of f in 0 :
$|\mathbb{E}[f(X_t^x)] - \mathbb{E}[f(\hat{X}_t^x)]| = \mathbb{E}|\int_0^{X_t^x} \frac{(x-y)^2}{2} f^{(3)}(y)dy - \int_0^{\hat{X}_t^x} \frac{(x-y)^2}{2} f^{(3)}(y)dy| \leq$
$C\mathbb{E}[(\hat{X}_t^x)^3 + (\hat{X}_t^x)^{q+3} + (X_t^x)^3 + (X_t^x)^{q+3}] \leq C't^3$ for $x \leq \mathbf{K}_2(t) = O(t)$.

A second-order scheme without restriction on σ IV

We thus get a potential 2^{nd} order scheme, changing scheme whether $x \leq \mathbf{K}_2(t)$ or not. We can check that we have bounded moments and good bounds on the derivatives of $u(t,x) = \mathbb{E}[f(X^x_{T-t})]$ when $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ (A. 2005). From slide 5, we get :

Theorem 22

The scheme $(\hat{X}^n_{t^n_i}, 0 \leq i \leq n)$ starting from $\hat{X}^n_{t^n_0} = x \in \mathbb{R}_+$ is well defined and nonnegative. One has,

$$\forall f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R}_+), \exists C > 0, \forall n \in \mathbb{N}, |\mathbb{E}[f(\hat{X}^n_{t^n_n})] - \mathbb{E}[f(X^x_T)]| \leq C/n^2.$$

A second-order scheme for a time-dependent CIR I

- Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ and $k, \sigma : \mathbb{R}_+ \to \mathbb{R}$. We consider :

$$X^x_t = x + \int_0^t (a(s) - k(s)X^x_s)ds + \int_0^t \sigma(s)\sqrt{X^x_s}dW_s, \; x, t \geq 0, \qquad (7)$$

  Used in Maghsoodi (1996), Benhamou Gobet and Miri (2010).
- This can be seen as a two-dimensional time-homogeneous SDE :

$$\begin{cases} X^x_t &= x + \int_0^t (a(Y_s) - k(Y_s)X^x_s)ds + \int_0^t \sigma(Y_s)\sqrt{X^x_s}dW_s, \; x, t \geq 0. \\ Y_t &= t. \end{cases}$$

- Its infinitesimal generator is given by $L = L_1 + L_2$, where

$$L_1 = (a(y) - k(y)x)\partial_x + \frac{\sigma(y)^2}{2}\partial^2_x, \; L_2 = \partial_y.$$

A second-order scheme for a time-dependent CIR II

- Strang's splitting : the second order scheme for the CIR with frozen parameters at time $\frac{t_i + t_{i+1}}{2}$, i.e. with constant parameters $a(\frac{t_i+t_{i+1}}{2})$, $k(\frac{t_i+t_{i+1}}{2})$ and $\sigma(\frac{t_i+t_{i+1}}{2})$ is a weak second-order scheme.
- Works more generally with (affine) diffusions with time-dependent parameters.

A third-order scheme without restriction on σ I

Let assume that $L_1L_2 = L_2L_1 + L_3^2$ and denote $S_i(t) = I + tL_i + \frac{t^2}{2}L_i^2 + \frac{t^3}{6}L_i^3 + \dots$ Then, we have

$$\frac{1}{6}\sum_{\varepsilon \in \{-1,1\}} [S_2(t)S_1(t)S_3(\varepsilon t) + S_2(t)S_3(\varepsilon t)S_1(t) + S_3(\varepsilon t)S_2(t)S_1(t)]$$
$$= I + t(L_1 + L_2) + \frac{t^2}{2}(L_1+L_2)^2 + \frac{t^3}{6}(L_1+L_2)^3 + \dots. \qquad (8)$$

Csq : if one has a third-order scheme for L_1, L_2 and L_3, we can get a third-order scheme for $L_1 + L_2$ with the scheme :

$$\frac{1}{6}\left(\sum_{\varepsilon \in \{-1,1\}} \hat{p}^3(\varepsilon t) \circ \hat{p}^1(t) \circ \hat{p}^2_x(t) + \hat{p}^1(t) \circ \hat{p}^3(\varepsilon t) \circ \hat{p}^2_x(t) + \hat{p}^1(t) \circ \hat{p}^2(t) \circ \hat{p}^3_x(\varepsilon t) \right),$$

if the compositions are well-defined.

A third-order scheme without restriction on σ II

CIR with $k=0$: $\frac{1}{2}(V_0V_1^2 - V_1^2V_0) = \frac{\sigma^2}{2}\left(a - \frac{\sigma^2}{4}\right)\partial_x^2$.
We are in the previous situation with $L_1 = V_0$ (resp. $L_1 = V_1^2/2$), $L_2 = V_1^2/2$ (resp. $L_2 = V_0$) and $L_3 = \frac{\sigma}{\sqrt{2}}\sqrt{\left|a - \frac{\sigma^2}{4}\right|}\partial_x$ if $\sigma^2 \leq 4a$ (resp. $\sigma^2 > 4a$).

- $x + (a - \sigma^2/4)t$ exact scheme for V_0
- $\tilde{X}(t,x) = x + t\frac{\sigma}{\sqrt{2}}\sqrt{\left|a - \frac{\sigma^2}{4}\right|}$ exact scheme for L_3
- $(\sqrt{x} + \frac{\sigma}{2}\sqrt{t}Y)^2$ with Y s.t. $\mathbb{E}[Y^q] = \mathbb{E}[N^q]$ for $q = 1, \dots, 7$ is a third order scheme for $V_1^2/2$.

A third-order scheme without restriction on σ III

- This remark allows to define a potential third-order scheme that preserves nonnegativity for $x \geq \mathbf{K}_3(t)$, using a r.v. that matches the 7 first moments of $\mathcal{N}(0,1)$: Y s.t
$\mathbb{P}(Y = \sqrt{3+\sqrt{6}}) = \mathbb{P}(Y = -\sqrt{3+\sqrt{6}}) = \frac{\sqrt{6}-2}{4\sqrt{6}}$, and
$\mathbb{P}(Y = \sqrt{3-\sqrt{6}}) = \mathbb{P}(Y = -\sqrt{3-\sqrt{6}}) = \frac{1}{2} - \frac{\sqrt{6}-2}{4\sqrt{6}}$.
We can show that the scheme composition is well-defined and preserve nonnegativity iff $x \geq \mathbf{K}_3(t)$ where

$$\mathbf{K}_3(t) = t \times \left[\mathbf{1}_{\{4a/3<\sigma^2<4a\}}\left(\sqrt{\frac{\sigma^2}{4} - a + \frac{\sigma}{\sqrt{2}}\sqrt{a - \frac{\sigma^2}{4}}} + \frac{\sigma}{2}\sqrt{3+\sqrt{6}}\right)^2\right.$$
$$\left. + \mathbf{1}_{\{\sigma^2 \leq 4a/3\}}\frac{\sigma}{\sqrt{2}}\sqrt{a - \sigma^2/4} + \mathbf{1}_{\{4a<\sigma^2\}}\left[\frac{\sigma^2}{4} - a + \left(\sqrt{\frac{\sigma}{\sqrt{2}}\sqrt{\frac{\sigma^2}{4} - a}} + \frac{\sigma}{2}\sqrt{3+\sqrt{6}}\right)^2\right]\right] = O(t).$$

A third-order scheme without restriction on σ IV

- For $0 < x \leq \mathbf{K}_3(t)$, we take a nonnegative r.v. that matches the 3 first moments of the CIR. This can be explicitly done taking a r.v. that takes two values $x_-(t,x) < x_+(t,x)$:

$$\pi(t,x)x_+(t,x)^i + (1 - \pi(t,x))x_-(t,x)^i = \mathbb{E}((X_t^x)^i),\ i = 1,2,3.$$

Lemma 23

Let X s.t. for $i \in \{1,2,3\}$, $\mathbb{E}[|X|^i] < \infty$, and set $m_i = \mathbb{E}[X^i]$. Let $s = \frac{m_3 - m_1m_2}{m_2 - m_1^2}$ and $p = \frac{m_1m_3 - m_2^2}{m_2 - m_1^2}$. Then, $\Delta = s^2 - 4p > 0$ and defining $x_\pm = \frac{s \pm \sqrt{\Delta}}{2}$ and $\pi = \frac{m_1 - x_-}{x_+ - x_-}$, the r.v. defined by :

$$x_+\mathbf{1}_{\{U \leq \pi\}} + x_-\mathbf{1}_{\{U > \pi\}} \text{ with } U \sim \mathcal{U}([0,1])$$

matches the three first moments of X. Moreover, it is nonnegative if $X \geq 0$.

A third-order scheme without restriction on σ V

To extend to $k \neq 0$, we use the fact that
$(X_t^x, t \geq 0) \overset{law}{=} (e^{-kt}X^{x,k=0}_{\psi_{-k}(t)}, t \geq 0)$, and define the scheme as

$$\hat{X}_t^x := e^{-kt}\hat{X}^{x,k=0}_{\psi_{-k}(t)}.$$

Theorem 24

The scheme $(\hat{X}^n_{t_i^n}, 0 \leq i \leq n)$ given by this construction and starting from $\hat{X}^n_{t_0^n} = x \in \mathbb{R}_+$ is a third-order scheme :

$$\forall f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R}_+), \exists K > 0, \forall n \in \mathbb{N}^*,\ |\mathbb{E}[f(\hat{X}^n_{t_n^n})] - \mathbb{E}[f(X_T^x)]| \leq K/n^3.$$

A third-order scheme without restriction on σ VI

$\texttt{function X0}(x) : x \leftarrow x + (a - \sigma^2/4)\psi_{-k}(t)$
$\texttt{function X1}(x) : x \leftarrow ((\sqrt{x} + \sigma\sqrt{\psi_{-k}(t)}Y/2)^+)^2$
$\texttt{function Xt}(x) : x \leftarrow x + \frac{\sigma}{\sqrt{2}}\sqrt{|a - \sigma^2/4|}\varepsilon\psi_{-k}(t)$
$\texttt{function CIR\_O3}(x) :$
$\texttt{if}\ (x \geq \mathbf{K}_3(t))\ \{$
$\quad \texttt{if}\ (\zeta = 1)\ \{\ \texttt{if}\ (\sigma^2 \leq 4a)\ \{\ \texttt{X1}(x)\,\texttt{X0}(x)\,\texttt{Xt}(x)\ \}\ \texttt{else}\ \{\ \texttt{X0}(x)\,\texttt{X1}(x)\,\texttt{Xt}(x)\ \}\ \}$
$\quad \texttt{if}\ (\zeta = 2)\ \{\ \texttt{if}\ (\sigma^2 \leq 4a)\ \{\ \texttt{X1}(x)\,\texttt{Xt}(x)\,\texttt{X0}(x)\ \}\ \texttt{else}\ \{\ \texttt{X0}(x)\,\texttt{Xt}(x)\,\texttt{X1}(x)\ \}\ \}$
$\quad \texttt{if}\ (\zeta = 3)\ \{\ \texttt{if}\ (\sigma^2 \leq 4a)\ \{\ \texttt{Xt}(x)\,\texttt{X1}(x)\,\texttt{X0}(x)\ \}\ \texttt{else}\ \{\ \texttt{Xt}(x)\,\texttt{X0}(x)\,\texttt{X1}(x)\ \}\ \}$
$x \leftarrow xe^{-kt}\ \}$
$\texttt{else}\ \{\ s \leftarrow \frac{\tilde{u}_3(t,x) - \tilde{u}_1(t,x)\tilde{u}_2(t,x)}{\tilde{u}_2(t,x) - \tilde{u}_1(t,x)^2}, p \leftarrow \frac{\tilde{u}_1(t,x)\tilde{u}_3(t,x) - \tilde{u}_2(t,x)^2}{\tilde{u}_2(t,x) - \tilde{u}_1(t,x)^2}, \delta = \sqrt{s^2 - 4p}, \pi \leftarrow \frac{\tilde{u}_1 - (s-\delta)/2}{\delta}$
$\texttt{if}\ (U < \pi)\ x \leftarrow (s+\delta)/2\ \texttt{else}\ x \leftarrow (s-\delta)/2\ \}$

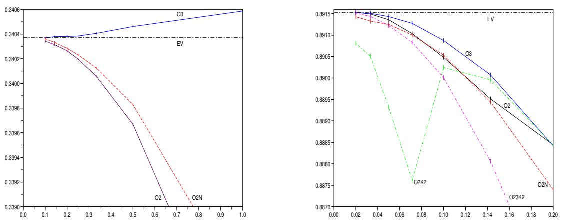


FIGURE: $\mathbb{E}(\exp(-\hat{X}^n_{t^n_n}))$ in function of $1/n$ with $x_0 = 3/2$, $k = 1/2$, $a = 1/2$ and $\sigma = 0.8$ (left) and $x_0 = 0.3$, $k = 0.1$, $a = 0.04$ and $\sigma = 2$ (right).
Scheme 1 : 2$^{\text{nd}}$ order, scheme 2 : 3$^{\text{rd}}$ order.

Heston model : find a second order scheme for

$$\begin{cases} X^1_t = X^1_0 + \int_0^t (a - kX^1_s)ds + \sigma\int_0^t \sqrt{X^1_s}dW_s \\ X^2_t = \int_0^t X^1_s ds \\ X^3_t = X^3_0 + \int_0^t rX^3_s ds + \int_0^t \sqrt{X^1_s}X^3_s(\rho dW_s + \sqrt{1-\rho^2}dZ_s) \\ X^4_t = \int_0^t X^3_s ds \end{cases}$$

$X^1_0 \geq 0$, $X^3_0 > 0$, $r \in \mathbb{R}$, $\rho \in [-1, 1]$ and $(a, k, \sigma) \in \mathbb{R}^*_+ \times \mathbb{R} \times \mathbb{R}^*_+$.
Remark : We will construct in fact a potential second order scheme for $(X^1, \log(X^3))$.

The algorithm

We split the operator without splitting the CIR nested part in order to use our CIR discretization. We write it as the sum $L^W + L^Z$ of the operators associated to :

$$\begin{cases} dX^1_t = (a - kX^1_t)dt + \sigma\sqrt{X^1_t}dW_t \\ dX^2_t = X^1_t dt \\ dX^3_t = (r - \frac{1}{2}(1-\rho^2)X^1_t)X^3_t dt + \rho\sqrt{X^1_t}X^3_t dW_t \\ dX^4_t = X^3_t dt \end{cases} \qquad \begin{cases} dX^1_t = 0 \\ dX^2_t = 0 \\ dX^3_t = \frac{1}{2}(1-\rho^2)X^1_t X^3_t dt + X^3_t\sqrt{(1-\rho^2)X^1_t}dZ_t \\ dX^4_t = 0. \end{cases}$$

`function HW` (x_1, x_2, x_3, x_4) :

$\Delta x_1 \leftarrow -x_1$, `CIR_O2` (x_1), $\Delta x_1 \leftarrow \Delta x_1 + x_1$ `//` `CIR_O3` can be used instead of `CIR_O2`

$x_2 \leftarrow x_2 + (x_1 + 0.5\Delta x_1)t$

$x_4 \leftarrow x_4 + 0.5x_3 t$

$x_3 \leftarrow x_3 \exp\left[(r - \rho a/\sigma)t + \rho\Delta x_1/\sigma + (\rho k/\sigma - 0.5)(x_1 + 0.5\Delta x_1)t\right]$

$x_4 \leftarrow x_4 + 0.5x_3 t$

$x_1 \leftarrow x_1 + \Delta x_1$

`function HZ` $(x_1, x_2, x_3, x_4) : x_3 \leftarrow x_3 \exp(\sqrt{(1-\rho^2)x_1 t}N)$

`function` (x_1, x_2, x_3, x_4) :

`if` $(B = 1)$ `HZ` (x_1, x_2, x_3, x_4) `HW` (x_1, x_2, x_3, x_4) `else HW` (x_1, x_2, x_3, x_4) `HZ` (x_1, x_2, x_3, x_4)

Scheme 1 (resp. 2) with `CIR_O2` (resp. `CIR_O3`)

Affine Term Structure Models I

We consider the following canonical form (Dai Singleton (2000))

$$dX_t = (A - KX_t)dt + \sqrt{D_t}dW_t, \tag{9}$$

Domain : $\mathbb{D} = \mathbb{R}^{d'}_+ \times \mathbb{R}^{d-d'}$. In the canonical form, the operator can be written $Lf = L_A f + L_B f + L_C f$, where

$$L_A f = \sum_{i=1}^{d'} \left((A_i - K_{ii}x_i)\partial_i + \frac{\gamma_{ii}}{2} x_i \partial_i^2 \right), \text{(Sum of indep CIR processes)}$$

$$L_B f = -\sum_{i=1}^{d}\sum_{j=1}^{d} \tilde{K}_{ij} x_j \partial_i f, \text{(Linear ODE)}$$

$$L_C f = \sum_{i=d'+1}^{d} \left(A_i \partial_i f + \frac{1}{2}(\gamma_{i0} + \sum_{j=1}^{d'} \gamma_{ij} x_j) \partial_i^2 f \right) \text{(sum of 1D indep Gaussian proc).}$$

Proposition 25

The scheme $\frac{1}{2}p^B(t/2) \circ p^A(t) \circ p^C(t) \circ p^B_x(t/2) + \frac{1}{2}p^B(t/2) \circ p^C(t) \circ p^A(t) \circ p^B_x(t/2)$ is a potential second-order scheme for the $Lf = L_A f + L_B f + L_C f$ operator defined in (10) on $\mathbb{D}$.

`function Affine` $(x_1, \dots, x_d)$:

$x \leftarrow \exp(-\tilde{K}t/2)x$

`if` $(B = 1)$ { `for` $i = 1$ `to` d', `CIR_O2`(x_i) `//` or `CIR_O3`

`for` $i = d'+1$ `to` $d, x_i \leftarrow x_i + A_i t + \sqrt{\gamma_{i0} + \sum_{j=1}^{d'} \gamma_{ij} x_j}\sqrt{t}N_i$ }

`else` { `for` $i = d'+1$ `to` $d, x_i \leftarrow x_i + A_i t + \sqrt{\gamma_{i0} + \sum_{j=1}^{d'} \gamma_{ij} x_j}\sqrt{t}N_i$

`for` $i = 1$ `to` d', `CIR_O2`(x_i) } `//` or `CIR_O3`

$x \leftarrow \exp(-\tilde{K}t/2)x$

A useful identity

We recall the explicit characteristic function :

$$\forall v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(iTr(vX_t))] = \frac{\exp(Tr[iv(I_d - 2iq_t v)^{-1} m_t x m_t^T])}{\det(I_d - 2iq_t v)^{\alpha/2}},$$

where $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds$, $m_t = \exp(tb)$.
Let $n = \mathrm{Rk}(q_t)$. Then, there is $\theta_t \in \mathcal{M}_d(\mathbb{R})$ invertible such that $q_t = t\theta_t I_d^n \theta_t^\top$, and we have :

$$\implies WIS_d(x, \alpha, b, a; t) \underset{Law}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)\theta_t^T. \quad (10)$$

It is therefore sufficient to simulate exactly a Wishart process with $a = I_d^n$ and $b = 0$, which we call a canonical Wishart process.

A remarkable splitting for canonical Wishart processes (Ahdida and A., 2011)

The infinitesimal generator of a canonical Wishart process is :

$$L = Tr(\alpha I_d^n D) + 2Tr(xDI_d^n D), \text{ with } D_{i,j} = \partial_{i,j}$$

for $f : \mathcal{M}_d(\mathbb{R}) \to \mathbb{R}$ s.t. $\partial_{i,j} f = \partial_{j,i} f$ for $1 \le i, j \le d$. We have

$$L = L_1 + \cdots + L_n, \text{ with } L_i L_j = L_j L_i, \text{ and where}$$

- L_i is the same operator as L_1 by permuting ith and first coordinates.
- L_1 is the operator of an SDE that is well defined on $\mathcal{S}_d(\mathbb{R})$ and that can be solved explicitly.

$\implies$ By composition, we get an exact scheme for the canonical Wishart process.

The case $d = 2$ ($\alpha > d - 1 = 1$)

The operator L_1 is associated to the following SDE when $(X_0)_{2,2} > 0$

$$\begin{cases} d(X_t)_{1,1} = \alpha dt + 2\sqrt{(X_t)_{1,1} - \frac{(X_t)_{1,2}^2}{(X_t)_{2,2}}} dB_t^1 + 2\frac{(X_t)_{1,2}}{\sqrt{(X_t)_{2,2}}} dB_t^2 \\ d(X_t)_{1,2} = \sqrt{(X_t)_{2,2}} dB_t^2, \ (X_t)_{2,1} = (X_t)_{1,2}, \\ d(X_t)_{2,2} = 0 \end{cases}$$

and if $(X_0)_{2,2} = 0$:

$$d(X_t)_{1,1} = \alpha dt + 2\sqrt{(X_t)_{1,1}} dB_t^1, \ d(X_t)_{1,2} = d(X_t)_{2,2} = 0.$$

In the second case : CIR that can be simulated exactly.
In the first case, we set $U_t = (X_t)_{1,1} - ((X_t)_{1,2})^2/(X_t)_{2,2}$:

$$dU_t = (\alpha - 1)dt + \sqrt{U_t} dB_t^1 : \text{CIR indep. of } (X_t)_{1,2} \sim \mathcal{N}((X_t)_{1,2}, (X_0)_{2,2} t).$$

When $d > 2$ ($\alpha > d - 1$)

The SDE associated to L_1 can be solved explicitly as for $d = 2$, and requires the sampling of 1 CIR distribution and $d - 1$ standard Gaussian variables that are independent.
It requires however some additional techniques (Cholesky decomposition, and outer product Cholesky decomposition when the initial condition is not invertible).

Exact scheme for L_1 when $d \geq 3$ ($\alpha \geq d-1$) I

Up to a permutation, $(x)_{2\leq i,j\leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}\begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix} =: cc^T$.
We can show that L_1 is the generator of the SDE :

$$\begin{array}{rcl} d(X_t^x)_{1,1} & = & \alpha dt + 2\sqrt{(X_t^x)_{1,1} - \sum_{k=1}^r \left(\sum_{l=1}^r (c_r^{-1})_{k,l}(X_t^x)_{1,l+1}\right)^2} dZ_t^1 \\ & & +2\sum_{k=1}^r \sum_{l=1}^r (c_r^{-1})_{k,l}(X_t^x)_{1,l+1} dZ_t^{k+1} \\ d(X_t^x)_{1,i} & = & \sum_{k=1}^r c_{i-1,k} dZ_t^{k+1} = d(X_t^x)_{i,1},\ i=2,\dots,d \\ d(X_t^x)_{l,k} & = & 0, \ \text{for } 2 \leq k,l \leq d. \end{array} \tag{11}$$

The SDE associated to L_1 can be solved explicitly as for $d=2$, and requires the sampling of 1 CIR distribution and $r-1$ standard Gaussian variables that are independent :

Exact scheme for L_1 when $d \geq 3$ ($\alpha \geq d-1$) II

$$X_t^x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \times$$
$$\begin{pmatrix} (U_t^u)_{1,1} + \sum_{k=1}^r ((U_t^u)_{1,k+1})^2 & ((U_t^u)_{1,l+1})^T_{1\leq l\leq r} & 0 \\ ((U_t^u)_{1,l+1})_{1\leq l\leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^T & k_r^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix},$$

where

$$\begin{array}{rcl} d(U_t^u)_{1,1} & = & (\alpha - r)dt + 2\sqrt{(U_t^u)_{1,1}} dZ_t^1, \\ u_{1,1} & = & x_{1,1} - \sum_{k=1}^r (u_{1,k+1})^2 \geq 0, \\ d((U_t^u)_{1,l+1})_{1\leq l\leq r} & = & (dZ_t^{l+1})_{1\leq l\leq r}, \\ (u_{1,l+1})_{1\leq l\leq r} & = & c_r^{-1}(x_{1,l+1})_{1\leq l\leq r}. \end{array} \tag{12}$$

Second and third order schemes for Wishart processes

In the sampling of the exact scheme for L_1, we replace the sampling of the CIR by a second (resp. third) order scheme, and the sampling of the Normal variables by a moment-matching r.v. s.t. $\mathbb{E}[Y^k] = \mathbb{E}[G^k]$ for $k = 1,\dots,5$ (resp. for $k=1,\dots,7$), we get :

$$\mathbb{E}(f(\hat{X}_t^x)) = \sum_{k=0}^{2} \frac{t^k}{k!} L_1^k f(x) + O(t^3) \ \ (\text{resp. } \mathbb{E}(f(\hat{X}_t^x)) = \sum_{k=0}^{3} \frac{t^k}{k!} L_1^k f(x) + O(t^4))$$

By composition rule, we get a second (resp. third) order scheme for canonical Wishart processes and then for General Wishart processes.

A faster second order scheme when $\alpha \geq d$

All the previous schemes rely on the splitting given by the remarkable splitting and require thus $O(d^4)$ operations.
Remark : We can check that if $c^T c = x$, $(c + W_t I_d^n)^T (c + W_t I_d^n)$ is a Wishart process with $\alpha = d$, $a = I_d^n$, $b = 0$ starting from x. Also, $(c + \sqrt{t}\hat{G} I_d^n)^T (c + \sqrt{t}\hat{G} I_d^n)$ is a potential second order scheme for $WIS_d(x, d, 0, I_d^n)$ where $\hat{G}$ is a matrix with independent elements matching the five first moments of the Normal r. v.
Consequence : By using the splitting :

$$L = \underbrace{\mathrm{Tr}((\alpha - d) I_d^n D^S)}_{\tilde{L}_{ODE}} + \underbrace{d\mathrm{Tr}(D^S) + 2\mathrm{Tr}(x D^S I_d^n D^S)}_{L_{WIS_d(x,d,0,I_d^n)}},$$

we get a by Corollary 20 a second order scheme for $WIS_d(x, \alpha, 0, I_d^n)$ (and then for $WIS_d(x, \alpha, b, a)$) in $O(d^3)$ operations.

A modified Euler scheme (As a comparison)

The Euler scheme for the Wishart diffusion (3) is :

$$\begin{aligned}\hat{X}_{t_{i+1}} &= \hat{X}_{t_i} + (\alpha a^T a + b\hat{X}_{t_i} + \hat{X}_{t_i} b^T)(t_{i+1} - t_i) \\ &\quad + \sqrt{\hat{X}_{t_i}}(W_{t_{i+1}} - W_{t_i})a + a^T(W_{t_{i+1}} - W_{t_i})^T \sqrt{\hat{X}_{t_i}}.\end{aligned}$$

It is not well-defined since $\hat{X}_{t_{i+1}}$ may not be nonnegative.
Corrected Euler scheme :

$$\begin{aligned}\hat{X}_{t_{i+1}} &= \hat{X}_{t_i} + (\alpha a^T a + b\hat{X}_{t_i} + \hat{X}_{t_i} b^T)(t_{i+1} - t_i) \\ &\quad + \sqrt{(\hat{X}_{t_i})^+}(W_{t_{i+1}} - W_{t_i})a + a^T(W_{t_{i+1}} - W_{t_i})^T \sqrt{(\hat{X}_{t_i})^+},\end{aligned}$$

where $\sqrt{x^+} := odiag(\sqrt{\lambda_1^+}, \dots, \sqrt{\lambda_d^+})o^{-1}$ for $x \in \mathcal{S}_d(\mathbb{R})$ and $x = odiag(\lambda_1, \dots, \lambda_d^+)o^{-1}$.

A time comparison (10^6 samples, N time-steps)

	$N = 10$			$N = 30$		
Schemes	R. value	Im. value	Time	R. value	Im. value	Time
Exact (1 step)	-0.526852	-0.227962	12			
2^{nd} order bis	-0.526229	-0.228663	41	-0.526486	-0.229078	125
2^{nd} order	-0.526577	-0.228923	76	-0.526574	-0.228133	229
3^{rd} order	-0.527021	-0.227286	82	-0.527613	-0.228376	244
Exact (N steps)	-0.526963	-0.228303	123	-0.526891	-0.227729	369
Corrected Euler	-0.525627^*	-0.233863^*	225	-0.525638^*	-0.231449^*	687
$\alpha = 3.5, d = 3, \Delta_R = 10^{-3}, \Delta_{Im} = 10^{-3}$, exact value R. = -0.527090 and Im.= -0.228251						
Exact (1 step)	-0.591579	-0.037651	12			
2^{nd} order	-0.590444	-0.037024	77	-0.590808	-0.036487	229
3^{rd} order	-0.591234	-0.034847	82	-0.590818	-0.036210	246
Exact (N steps)	-0.591169	-0.036618	174	-0.592145	-0.037411	920
Corrected Euler	-0.589735^*	-0.042002^*	223	-0.590079^*	-0.039937^*	680
$\alpha =2.2, d = 3, \Delta_R = 0.9 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}$, exact value R. = -0.591411 and Im.= -0.036346						
Exact (1 step)	0.062712	-0.063757	181			
2^{nd} order bis	0.064237	-0.063825	921	0.064573	-0.062747	2762
2^{nd} order	0.064922	-0.064103	1431	0.063534	-0.063280	4283
3^{rd} order	0.064620	-0.064543	1446	0.064120	-0.063122	4343
Exact (N steps)	0.063418	-0.064636	1806	0.063469	-0.064380	5408
Corrected Euler	0.068298^*	-0.058491^*	2312	0.061732^*	-0.056882^*	7113
$\alpha = 10.5, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}$, exact value R. = 0.063960 and Im.= -0.063544						
Exact (1 step)	-0.036869	-0.094156	177			
2^{nd} order	-0.036246	-0.094196	1430	-0.035944	-0.092770	4285
3^{rd} order	-0.035408	-0.093479	1441	-0.036277	-0.093178	4327
Exact (N steps)	-0.036478	-0.092860	1866	-0.036145	-0.093003	6385
Corrected Euler	-0.028685^*	-0.094281^*	2321	-0.030118^*	-0.088988^*	7144
$\alpha = 9.2, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.4 \times 10^{-3}$, exact value R. = -0.036064 and Im.= -0.093275						

Weak convergence

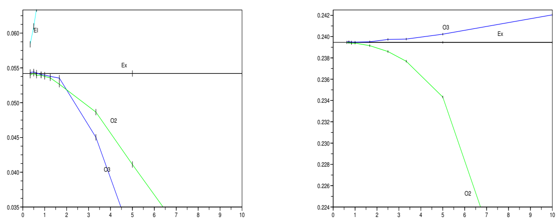


FIGURE: $d = 3$, 10^7 MC samples, $T = 10$. $\Re(\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t_N}))])$ in fct of T/N. Left : $v = 0.05I_d$, $x = 0.4I_d$, $\alpha = 4.5$, $a = I_d$ and $b = 0$. Exact value : 0.054277. Right : $v = 0.2I_d + 0.04q$ $x = 0.4I_d + 0.2q$, $\alpha = 2.22$, $a = I_d$ and $b = -0.5I_d$. Exact value : 0.239836. $q_{i,j} = \mathbf{1}_{i \neq j}$.

A scheme for the Gourieroux-Sufana model

The joint operator of the Gourieroux-Sufana model (S_t, X_t) is

$$L = L^S + L^X, \text{ where } L^S = \sum_{i=1}^d r s_i \partial_{s_i} + \frac{1}{2}\sum_{i,j=1}^d s_i s_j x_{i,j} \partial_{s_i}\partial_{s_j},$$

and L^X is the generator of a Wishart process. We can solve explicitly the SDE associated to $L^S : S^l_t = S^l_0 \exp[(r - x_{l,l}/2)t + (\sqrt{x}Z_t)_l]$.
By using a second order scheme for L^X, we get a second order scheme for L by Corollary 20.

Put option in the Gourieroux-Sufana model

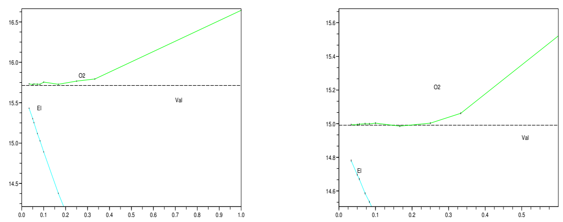


FIGURE: $\mathbb{E}[e^{-rT}(K - \max(\hat{S}^{1,N}_{t_N^N}, \hat{S}^{2,N}_{t_N^N}))^+]$ in fct of T/N. $d = 2$, $T = 1$, $K = 120$, $S_0^1 = S_0^2 = 100$, and $r = 0.02$. $x = 0.04I_d + 0.02q$ with $q_{i,j} = \mathbf{1}_{i \neq j}$, $a = 0.2I_d$, $b = 0.5I_d$ and $\alpha = 4.5$ (left), $\alpha = 1.05$ (right). 10^6 Monte-Carlo samples.