Introduction to Continuous-time Corporate Finance

11th European Summer School of Financial Mathematics

Lecture 1: Stéphane Villeneuve

INTRODUCTION

These notes are an introduction to continuous-time corporate finance which is mainly concerned with the decisions taken by firm managers in terms of investment, hedging and dividend distribution. Unlike standard option-pricing models, which rule out frictions or market imperfections, Corporate finance is relevant only when the markets exhibit frictions such as tax subsidies, liquidation costs, costs of issuing new shares and agency costs. This is the famous Modigliani-Miller theorem that claims: in the absence of frictions, the value of a firm is independent of its capital structure that is the proportion of debt and equity in the liability side of its balance sheet. As a consequence, in a world without frictions, all managerial decisions that would modify the structure of the balance sheet, have no impact on the total value of the firm. In these notes, we will focus on two type of frictions: issuance costs and agency costs keeping in mind that agency costs are a way to explain why issuance costs are high in practice.
In the first chapter, we present the base liquidity-management problem where the issuance costs are so high that firm are liquidated when they run out of cash reserves. We apply the base model to banks to study the impact of regulatory capital requirements and to investment in a more productive technology. From a mathematical viewpoint, the liquidity-management problem necessitates to solve stochastic singular control problems whose difficulty increases as and when the model becomes more realistic. While the first chapter assumes exogenous costly refinancing, a deeper approach consists in focusing on endogenous costs due to a conflict of interest between shareholders and managers such as moral hazard issues when the firm's owner cannot observe the effort made by his/her manager. As a consequence, we must study the contracting problem, called principal-agent problem that defines the relationship between firm's owners and firm's managers. The practical implementation of the optimal contract when it exists gives a role to cash reserves which establishes a connection between the two chapters.

CHAPTER 1

LIQUIDITY-MANAGEMENT PROBLEMS

1.1 A workhorse model

A firm has a single investment project that generates random cash-flows over time. The cumulative cash-flow process $X = (X_t)_{t\geq 0}$ is an arithmetic Brownian motion with strictly positive drift μ,

$$X_t = \mu\, t + \sigma\, W_t \tag{1.1}$$

In absence of any frictions (in particular, new equity issuances are costless, the firm's value is defined by the sum of initial cash holdings of the firm x plus the present value of future cash-flows:

$$V(x) = x + \mathbb{E}_x\left[\int_0^{\infty} e^{-rt} dX_t\right] = x + \frac{\mu}{r}. \tag{1.2}$$

Assume now that the firm is cash constrained in the sense that it cannot afford to issue new equity or debt. To meet the operating costs and avoid bankruptcy, the cash reserves must always remain non negative. The cash reserves of the firm $M = (M_t)_{t\geq 0}$ evolves according to

$$dM_t \;=\; dX_t - dZ_t \quad M_0 = x \tag{1.3}$$

where $Z = (Z_t)_{t\geq 0}$ is the cumulative payment process to equity-holders (here we assume that the manager acts on behalf of equity-holders) to be endogenously determined. The process Z is assumed to be non decreasing meaning that the losses for equity-holders are limited to their initial investment, which is called limited liability. For a given payment policy Z the value of the firm is defined by the present value of future payments up to the liquidation time τ_L

$$V(x;Z) \;=\; \mathbb{E}_x\left[\int_0^{\tau_L} e^{-rt} dZ_t\right] \tag{1.4}$$

where

$$\tau_L \;=\; \{t \geq 0 \;\; M_t < 0\} \tag{1.5}$$

Thus the Stochastic Control Problem that defines the firm's value

$$\sup_Z V(x;Z) \;=\; \sup_Z \mathbb{E}_x\left[\int_0^{\tau_L} e^{-rt} dZ_t\right] \equiv V^*(x) \tag{1.6}$$

1.1.1 A stochastic singular control problem

Bounded dividend rates

In that case, the dividend payment process Z_t is given by $dZ_t = u(X_t)\,dt$ where $0 \le u(.) \le M$. Then, the value function is now given by

$$V^*(x) = \sup_u \mathbb{E}\left(\int_0^{\tau_L} e^{-rs} u(X_s)\,ds\right).$$

Let $V(x,u) = \mathbb{E}\left(\int_0^{\tau_L} e^{-rs} u(X_s)\,ds\right)$ be the expected total discounted value corresponding to the manager's strategy u. To prove that a strategy u^* is optimal, we will proceed in two steps:

1. Find a function ϕ such that $V(x,u) \le \phi$ for all $x \ge 0$ and for all u.
2. Prove that $\phi(x) = V(x,u^*)$.

Verification Theorem Denote by $\mathcal{A}$ the differential operator

$$\mathcal{A}f(x) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \mu\frac{\partial f}{\partial x} - rf.$$

Assume that there is a smooth function ϕ such that the following variational inequalities hold:

- For all bounded dividend rate u,

$$\mathcal{A}\phi(x) + u(x)(1 - \phi'(x)) \le 0. \tag{1.7}$$

Then, the function ϕ satisfies $\phi \ge V(x,u)$ for all bounded dividend rate u.

Proof: Apply Itô formula to $e^{-rt}\phi(X_t)$.

Free Boundary Problem It is clear from variational inequalities (1.7) that the optimal control u^* depends on the the sign of $1 - V'$ in the following way

$$u^*(x) = \begin{cases} 0 & \text{if} \quad V' > 1 \\ M & \text{if} \quad V' < 1. \end{cases}$$

When $V' = 1$ the optimal control switches. Let (V_M, x_M^*) be a solution to the following free boundary problem

$$\begin{aligned} \mathcal{A}V_M &= 0, \quad x \le x_M^* \\ \mathcal{A}V_M + M(1 - V_M') &= 0, \quad x \ge x_M^* \\ V_M(0) &= 0 \quad \text{limited liability condition} \\ V(x_M^*) &= 1 \quad \text{switching condition.} \end{aligned}$$

Then, the shareholder value function is V_M and the optimal strategy is $u^*(x) = M\mathbb{1}_{\{x \ge x_M^*\}}$.

Unbounded dividend rates

Heuristic derivation of Bellman Equation.

- The policy $Z_0 = z \in (0,x)$ yields to

$$V^*(x) \ge V^*(x - z) + z.$$

  Letting z go to 0 yields that $V'(x) \ge 1$ for all $x > 0$.

- Dynamic programming principle yields to $LV \le 0$.

Let

$$V(x,Z) = \mathbb{E}\left(\int_0^{\tau_L} e^{-rs}\, dZ_s\right).$$

and ϕ such that $\phi' \geq 1$ and $L\phi \leq 0$ then $\phi \geq V(x,Z)$ for all admissible strategy Z.

Solution of the Bellman Equation. Let (V_0, x_0) the solution to the following boundary problem:

$$\begin{array}{rcl} \mathcal{A}V_0(x) &=& 0, \quad x \leq x_0 \\ V_0'(x) &=& 1 \quad x \geq x_0 \\ V_0(0) &=& 0\, V_0'(x_0) = 1 \text{ and } V_0''(x_0) = 0 \end{array}$$

Then, the shareholder value function is given by V_0 and the optimal strategy is given by

$$Z_t^{x_0} = \max\left[0, \max_{0\leq s\leq t}\left(\mu s + \sigma W_s - x_0\right)\right].$$

Computations are explicit and give

$$V_0(x) = \mathbb{E}_x\left[\int_0^{\tau_L} e^{-rs} dZ_s^{x_0}\right] = \frac{f_0(x)}{f_0'(x_0)} \quad 0 \leq x \leq x_0, \tag{1.8}$$

with

$$f_0(x) = e^{\alpha_0^+ x} - e^{\alpha_0^- x} \quad \text{and} \quad x_0 = \frac{1}{\alpha_0^+ - \alpha_0^-} \ln \frac{\left(\alpha_0^-\right)^2}{\left(\alpha_0^+\right)^2}, \tag{1.9}$$

where $\alpha_0^- < 0 < \alpha_0^+$ are the roots of the characteristic equation

$$\mu x + \frac{1}{2}\sigma^2 x^2 - r = 0.$$

If the firm starts with cash reserves x above x_0, the optimal dividend policy distributes immediately the amount $(x - x_0)$ as exceptional dividend and then follows the dividend policy defined by the process Z^{x_0}. Thus, for $x \geq x_0$,

$$V_0(x) = x - x_0 + \frac{\mu}{r}.$$

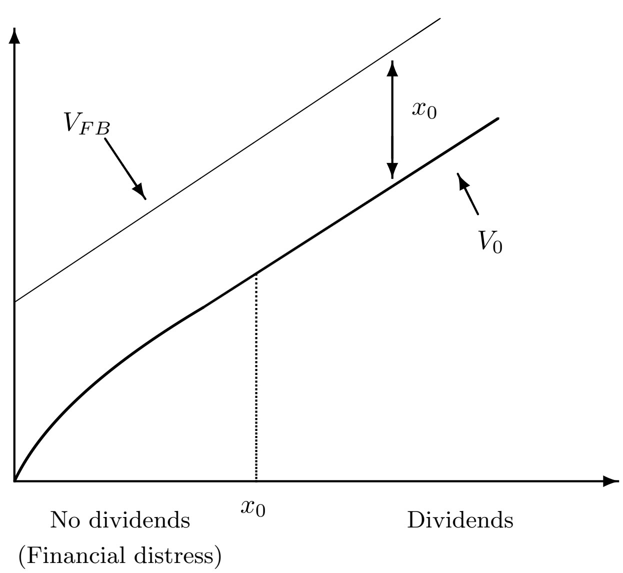

Figure 1: Shareholder value $V_0(x)$

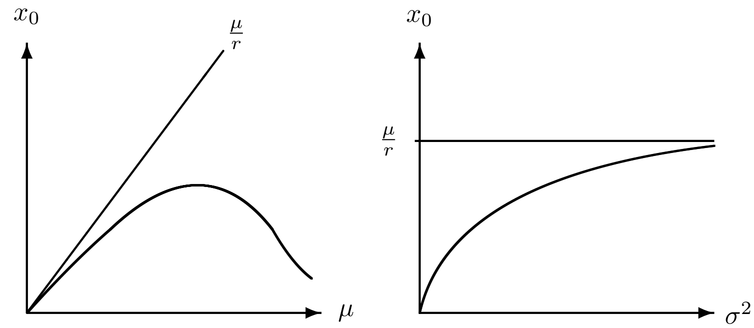

Figure 3: The cost of financial frictions
as a function of μ and σ^2.

1.1.2 Application to Banking

This part is based on the book by Rochet and Santiago-Moreno, Chapter 4. Let us consider a bank with a fixed volume of deposit and a fixed volume of risky assets. It is assume that the assets generate cashflows

$$\mu dt + \sigma dW_t,$$

where $\mu, \sigma > 0$. The structure of the balance sheet implies that the book value of equity E_t satisfies

$$E_t = M_t + A - D.$$

If we assume for simplicity, that neither the cash nor the deposit are remunerated, we obtain the dynamics of cash reserves

$$dM_t = \mu dt + \sigma dW_t - dZ_t,$$

where $Z = \{Z_t, t \geq 0\}$ represents the cumulative dividends paid to shareholders.
There is a deposit insurance system that protects depositors in case the bank is liquidated. Here, we make the strong assumption that the bank cannot issue new equity and is forced to liquidate its assets at a lower price than the market value of these assets when $M_t < 0$. The losses are modeled by an exogenous parameter α and $/alphaA$ represents the proceeds of liquidation. Assuming $\alpha A < D$, the deposit insurance system has to provide the shortfall $D - \alpha A$ and to avoid this situation, it imposes a minimum capital requirement. More precisely, we assume that the bank is allowed to operate as long as $E_t \geq \underline{e}$, where $\underline{e} = A(1 - \alpha)$ to avoid costly deposit insurance. This is equivalent to impose a minimum level of cash reserves $\underline{m} = D - \alpha A$.
The shareholder value function under regulation is

$$V_R(m) = \sup_Z \mathbb{E}_m \left[\int_0^{\tau_L} e^{-rt} dZ_t \right]$$

where now

$$\tau_L = \inf\{t \geq 0, M_t \leq \underline{m}\}.$$

Because the parameters of the model are constant, it is not difficult to see that for $m \geq \underline{m}$

$$V_R(m) = V^*(m - \underline{m}),$$

and that the distribution threshold is $m_R = x_0 + \underline{m}$. The target cash level is shifted to the right by the minimum-liquidity ratio. In particular, if the parameter α is very low (high cost of fire sale) or if the level of deposit is high, the shareholders have to wait a longer time to distribute dividends.

1.2 Optimal Investment for cash-constrained Firm

So far, we have considered firms with a fixed technology. A natural question arises: how liquidity management interacts with investment decision. This introduces a real-option component in the liquidity management problem. We will study a model based on Décamps and Villeneuve (2007) where a cash-constrained firm can adopt a new technology. The optimal time to invest is a classical real option problem but it is complicated here by the assumption that the firm has no access to external financing.

1.2.1 The model

A firm has an investment option to improve its profitability at a sunk cost I. We assume that the firm must finance the investment cost with the funds generated by its own activity. This is a crucial difference with the base model because now retaining earnings is not only precautionary but can be used to make the firm more profitable.
The manager who again acts on behalf of shareholders chooses a control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$ that models a dividend/investment policy. Z_t^π therefore corresponds to the total amount of dividends paid out by the firm up to time t and the control component τ^π represents the investment time in the growth opportunity. Therefore, cash reserves of the firm at time t evolves as

$$dX_t^\pi = (\mu_0 1\!\!1_{t \leq \tau^\pi} + \mu_1 1\!\!1_{t > \tau^\pi})dt + \sigma dW_t - dZ_t^\pi - dI_t^\pi,$$

A given control policy $(Z_t^\pi, \tau^\pi; t \geq 0)$ fully characterizes the associated investment process $(I_t^\pi)_{t\geq 0}$ by relation $I_t = I 1\!\!1_{t \geq \tau^\pi}$. As before, we define the bankruptcy time as the first time cash reserves are depleted $\tau_0^\pi = \inf\{t \geq 0 \ : \ X_t^\pi \leq 0\}$. The shareholder value along a fixed control π is given by

$$V_\pi(x) = \mathbb{E}_x\left[\int_0^{\tau_0^\pi} e^{-rs} dZ_s^\pi\right].$$

and thus the shareholder value function is

$$V^*(x) = \sup_\pi V_\pi(x),$$

and the optimal policy $\pi^\star = (\tau^{\pi^\star}, Z_t^{\pi^\star}; t \geq 0)$ is such that

$$V_{\pi^\star}(x) = V^*(x).$$

There are two simple scenarios that play a key role in the analysis. The first one is

- *Never invest in the growth option (and follow the associated optimal dividend policy).*
  This gives the following value function

$$\begin{cases} V_0(x) = \dfrac{f_0(x)}{f_0'(x_0)}, \quad 0 \leq x \leq x_0, \\ V_0(x) = x - x_0 + V_0(x_0), \quad x \geq x_0. \end{cases}$$

  The second is

- *Invest immediately in the growth option (and follow the optimal dividend policy associated to the new profit rate).*
  This gives the following value function

$$\begin{cases} V_1(x-I) = \max\left(0, \dfrac{f_1(x-I)}{f_1'(x_1)}\right), \quad 0 \leq x \leq x_1 + I, \\ V_1(x-I) = x - I - x_1 + V_1(x_1), \quad x \geq x_1 + I. \end{cases}$$

Clearly, we have $V^*(x) \geq V_0(x)$ and $V^*(x) \geq V_1(x-I)$. Note that the second strategy exposes the firm to a higher risk of liquidation as its cash reserves are reduced by I.

Now let consider the following optimal stopping problem

$$\phi(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} \max(V_0(R_{\tau \wedge \tau_0}), V_1(R_{\tau \wedge \tau_0} - I)) \right]$$

where $(R_t)_t$ represents the cash reserves process generated by the activity in place in absence of dividend distribution:

$$dR_t = \mu_0 dt + \sigma dW_t$$

and $\tau_0 = \inf\{t \geq 0 \ : \ R_t \leq 0\}$. We have,

Theorem 1.2.1. *For all $x \in [0, \infty)$, $V^*(x) = \phi(x)$.*

A road map towards the proof

A first observation is that Optimal stopping theory implies $V^*(x) \geq \phi(x)$. The difficulty is to prove the reverse inequality. We introduce the HJB equation

$$\max(1 - v', \, \mathcal{L}_0 v - rv, \, V_1(. - I) - v) \; = \; 0.$$

First, we have a standard verification result

Lemma 1.2.2. *Suppose we can find a positive function $\tilde{V}$ piecewise C^2 on $(0, +\infty)$ with bounded first derivatives and such that for all $x > 0$,*

(i) $\mathcal{L}_0 \tilde{V} - r\tilde{V} \leq 0$ in the sense of distributions,

(ii) $\tilde{V}(x) \geq V_1(x - I)$,

(iii) $\tilde{V}'(x) \geq 1$,

with the initial condition $\tilde{V}(0) = 0$ then, $\tilde{V}(x) \geq V^*(x)$ for all $x \in [0, \infty)$.

Lemma 1.2.3. *ϕ is a supersolution and thus $V^* = \phi$.*

Some observations. First, the growth option is worthless if and only if

$$\left(\frac{\mu_1 - \mu_0}{r} \right) < (x_1 + I) - x_0.$$

We focus on the case where the investment is valuable by assuming

$$\textbf{(H1)} \qquad \frac{\mu_1 - \mu_0}{r} \geq (x_1 + I) - x_0.$$

If the manager decides to postpone dividend distribution until investment, we can define the associated value function

$$\theta(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} V_1(R_{\tau \wedge \tau_0} - I) \right],$$

that can be explicitly computed

$$\begin{cases} \theta(x) = \dfrac{f_0(x)}{f_0(b)} V_1(b - I) & x \leq b, \\ \theta(x) = V_1(x - I), \quad x \geq b, \end{cases}$$

where $b > I$ is defined by the smooth-fit principle

$$\frac{V_1'(b - I)}{f_0'(b)} = \frac{V_1(b - I)}{f_0(b)}. \tag{1.10}$$

This allows us to determine the optimal strategies.

Proposition 1.2.4. *If $\theta(x_0) > V_0(x_0)$ then, the policy $\pi^\star = (Z_t^{\pi^\star}, \tau^{\pi^\star})$ defined by the increasing right-continuous process*

$$Z_t^{\pi^\star} \;=\; ((R_{\tau_b} - I) - x_1)_+ \mathbb{1}_{t=\tau_b} + L_t^{x_1}(\mu_1)\mathbb{1}_{t>\tau_b},$$

and by the stopping time $\tau^{\pi^\star} \;=\; \tau_b$ satisfies for all positive x the relation $\phi(x) = V_{\pi^\star}(x)$.

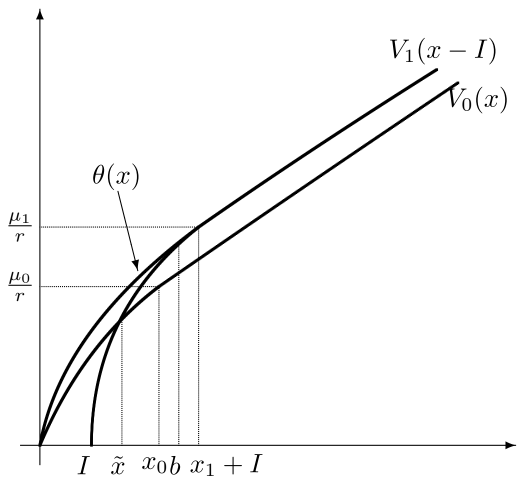

Figure 1: $\theta(x_0) > V_0(x_0)$

Proposition 1.2.5. *If $\theta(x_0) \leq V_0(x_0)$ then, the policy $\pi^\star = (Z_t^{\pi^\star}, \tau^{\pi^\star})$ defined by the increasing right-continuous process*

$$Z_t^{\pi^\star} = \left[(R_{\tau_a} - x_0)_+ \mathbb{1}_{t=\tau_a} + (L_t^{x_0}(\mu_0) - L_{\tau_a}^{x_0}(\mu_0))\mathbb{1}_{t>\tau_a}\right] \mathbb{1}_{\tau_a<\tau_c}$$
$$+ \left[((R_{\tau_c} - I) - x_1)_+ \mathbb{1}_{t=\tau_c} + L_t^{x_1}(\mu_1, W)\mathbb{1}_{t>\tau_c}\right] \mathbb{1}_{\tau_c<\tau_a},$$

and by the stopping time

$$\tau^{\pi^\star} = \begin{cases} \tau_c \text{ if } \tau_c < \tau_a \\ \infty \text{ if } \tau_c > \tau_a \end{cases}$$

satisfies for all positive x the relation $\phi(x) = V_{\pi^\star}(x)$.

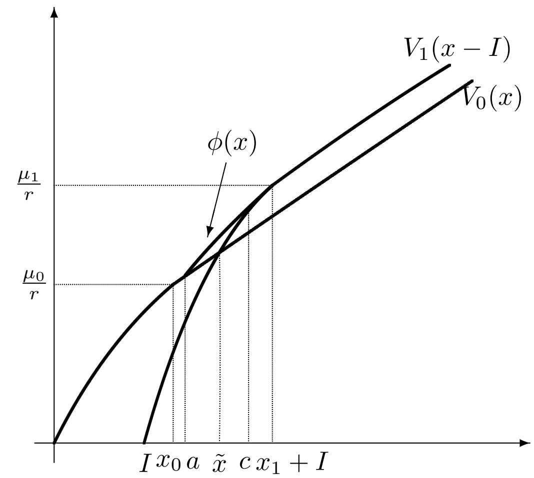

Figure 2: $\theta(x_0) < V_0(x_0)$

CHAPTER 2

AGENCY FRICTIONS IN A CONTINUOUS-TIME FRAMEWORK

The models in Chapter 1 assume an exogenous friction, that is a costly refinancing. A way to understand why refinancing is costly is to focus on endogenous frictions due for instance to conflicts of interest between managers and shareholders. These frictions are called agency frictions and we will focus here more particularly on moral hazard issue, when the firm's owner is unable to observe if the manager is exerting effort to improve the earnings of a project. As a consequence, we have to regulate this friction by entering into an agreement or a contract specifying the remuneration of the manager in order to give him the right incentives to make the desirable effort.

2.1 Principal-Agent model

We will consider a standard framework where an entrepreneur, refereed as the *Agent* has the expertise but not the funds to start a project. An investor, referred as the *Principal* has the required funds but lack of expertise to steer the project. We assume that both actors are risk-neutral but they differ by their preference with respect to time. We assume that the agent is less patient than the principal and thus has a discount rate ρ that is higher than the principal's discount rate r. We also assume that the agent has limited liability which imposes that all the losses are the responsibility of the principal who is assumed to be able to finance any shortfall. Once the contract has been signed, the project starts and generates cash flows that evolve as

$$dX_t = (\mu - (1 - e_t)\delta)dt + \sigma dZ_t^e. \tag{2.1}$$

where $\mu, \delta > 0$ which makes the effort profitable, Z_t^e is a Brownian motion and $e = \{e_t, t \geq 0\}$ is an effort process adapted to the filtration generated by Z^e that takes values in $[0, 1]$. It is assumed that the agent has a private benefit $B\,dt$ where B is a positive constant, whenever he shirks. Finally, once the contract is terminated, the principal receives a liquidation payoff $L \leq \frac{\mu}{r}$.

Probabilistic model. Formally, we consider the probability space $\Omega = \mathcal{C}([0, \infty), \mathbb{R})$, the set of continuous real functions on $[0, +\infty)$ endowed with the Wiener measure denoted by $\mathbb{P}$. Let $Z = (Z_t)_{t\geq 0}$ be a Brownian motion under $(\mathbb{P}, \mathcal{F}_t)$ where $\mathcal{F}_t$ is the completion of the natural filtration generated by Z. Under $\mathbb{P}$, we assume that the project's cash flows evolve as

$$dX_t = \mu\, dt + \sigma dZ_t.$$

Thus, $\mathbb{P}$ corresponds to the probability distribution of the profitability when the agent chooses to make effort at any time. For any effort process $e = \{e_t, t \geq 0\}$ which is assumed to be a $\mathcal{F}_t$

adapted process with values in $[0,1]$, we define

$$\gamma_t^e = \exp\left[\int_0^t -\left(\frac{\delta(1-e_s)}{\sigma}\right) dZ_s - \frac{1}{2}\int_0^t \left(\frac{\delta(1-e_s)}{\sigma}\right)^2 ds\right].$$

Because the effort process is bounded, the process $(\gamma_t^e)_{t\geq 0}$ is an $\mathcal{F}_t$- martingale. We then define a probability $\mathbb{P}^e$ on Ω such that

$$\frac{d\mathbb{P}^e}{d\mathbb{P}}|\mathcal{F}_t = \gamma_t^e.$$

The process $(Z_t^e)_{t\geq 0}$ with

$$Z_t^e = Z_t + \int_0^t \left(\frac{\delta(1-e_s)}{\sigma}\right) ds$$

is a Brownian motion under $\mathbb{P}^e$. Therefore, any action process e induces a probability measure $\mathbb{P}^e$ on Ω for which the dynamics of cash flows is given by Equation (2.16).

Problem formulation. A contract is a triplet (C, τ_L, e) that specifies nonnegative transfers $C = (C_t)_{t\geq 0}$ (remuneration) from the principal to the agent, a stopping time τ_L at which the project is liquidated and an effort process e that the principal recommends to the agent. The process C is $\mathcal{F}^X$-adapted, nondecreasing (reflecting agent's limited liability), τ_L is an $\mathcal{F}^X$-stopping time, and, for any effort process e, we assume

$$\mathbb{E}^e\left(\int_0^{\tau_L} e^{-rs} dC_s\right) < +\infty. \tag{2.2}$$

Throughout the paper $\mathcal{F}^X$ denotes the $\mathbb{P}^e$-augmentation of the filtration generated by $(X_t)_{t\geq 0}$ and $\mathcal{T}^X$ the set of $\mathcal{F}^X$-stopping times.

For a fixed contract $\Gamma = (C, \tau_L, e)$. The agent's expected profit and the principal's expected profit associated to Γ are respectively,

$$V_A(\Gamma) = \mathbb{E}^e\left(\int_0^{\tau_L} e^{-\rho t}(B(1-e_t)\,dt + dC_t)\right),$$

and

$$V_P(\Gamma) = \mathbb{E}^e\left(\int_0^{\tau_L} e^{-rt}(dX_t - dC_t) + e^{-r\tau_L} L\right).$$

An *incentive-compatible* effort process $e^*(C,\tau_L) = (e_t^*(C,\tau_L))_{t\geq 0}$ is an agent best reply in term of effort to a given remuneration and liquidation policy (C, τ_L). That is, for any effort process a, the effort process $e^*(C,\tau_L)$ satisfies

$$\mathbb{E}^e\left(\int_0^{\tau_L} e^{-\rho t}(B(1-e_t)\,dt + dC_t)\right) \leq \mathbb{E}^{e^*(C,\tau_L)}\left(\int_0^{\tau_L} e^{-\rho t}(B(1-e_t^*(C,\tau_L))\,dt + dC_t)\right).$$

We say that a contract (C, τ_L, e) is incentive compatible or (C, τ_L) induces an effort strategy $e^*(C,\tau_L)$ if $e = e^*(C,\tau_L)$. An optimal contract is an incentive compatible contract that maximizes the expected principal's profit at date 0 subject to delivering to the agent a payoff larger than her reservation utility $w_0 > 0$. The principal problem is then to find, if any, an optimal contract. Formally, the principal studies the problem

$$\sup_{C,\tau_L} \mathbb{E}^{e^*(C,\tau_L)}\left(\int_0^{\tau_L} e^{-rt}(dX_t - dC_t) + e^{-r\tau_L} L\right) \tag{2.3}$$

$$\text{s.t } \mathbb{E}^{e^*(C,\tau_L)}\left(\int_0^{\tau_L} e^{-\rho t}(B(1-e_t^*(C,\tau_L))\,dt + dC_t)\right) \geq w_0. \tag{2.4}$$

We refer inequality (2.4) as the agent's participation constraint.

2.2 Incentive compatibility and Markov formulation

This section develops in our setting a standard result due to Sannikov, generalized by Cvitanic, Possamaï and Touzi: the continuation value of the agent (defined below) characterizes the incentive compatible effort and allows for a Markov formulation of the principal's problem (2.3)-(2.4).

Fix a contract $\Gamma = (C, \tau_L, e)$ and assume for a while that effort process e is incentive compatible which yields that both players have the same set of information. Let us define the process $W^\Gamma = (W_t^\Gamma)_{t\geq 0}$ as

$$W_t^\Gamma = \mathbb{E}^e \left(\int_t^{\tau_L} e^{-\rho(s-t)} (B(1-e_s)ds + dC_s) \,|\, \mathcal{F}_t^X \right).$$

The process W^Γ corresponds to the agent's continuation value process associated to contract Γ. Because C is an increasing process and effort process e takes values bounded by 1, $W_t^\Gamma \geq 0$ for all $t \leq \tau_L$ while $W_{\tau_L}^\Gamma = 0$ by construction. Moreover, because we assumed the manager has limited liability, we deduce that a contract Γ is terminated the first time W^Γ hits zero. The following holds.

Lemma 2.2.1. *The continuation value process W^Γ associated to the incentive compatible contract Γ satisfies under $\mathbb{P}^e$ the dynamics*

$$dW_t^\Gamma = (\rho W_t^\Gamma - B(1-e_t))\,dt + \beta_t^\Gamma\,dZ_t^e - dC_t \quad \textit{for} \quad t \leq \tau_L, \tag{2.5}$$

where the process $\beta^\Gamma = (\beta_t^\Gamma)_{t\geq 0}$ is $\mathcal{F}^X$ predictable and uniquely defined. It is called hereafter the sensitivity process.

Proof of Lemma 2.2.1. By assumption (2.2) and because $\rho > r$, the process

$$U_t = e^{-\rho t} W_t^\Gamma + \int_0^t e^{-\rho s}(B(1-e_s)ds + dC_s) = \mathbb{E}^e \left(\int_0^{\tau_L} e^{-\rho s}(B(1-e_s)ds + dC_s) | \mathcal{F}_t^X \right)$$

is a uniformly integrable martingale under $\mathbb{P}^e$. By the martingale Representation theorem, there exists a unique $\mathcal{F}_t^X$ predictable process β^Γ such that

$$U_t = Y_0 + \int_0^t e^{-\rho s} \beta_s^\Gamma\,dZ_s^e,$$

with

$$\mathbb{E}^e \left(\int_0^{\tau_L} e^{-2\rho s} (\beta_s^\Gamma)^2\,ds \right) < +\infty.$$

Then, Itô's formula, yields (2.5). □

Thus, any contract $\Gamma = (C, \tau_L, e)$ defines a unique sensitivity process $\beta^\Gamma = (\beta_t^\Gamma)_{t\geq 0}$ by the representation theorem for Brownian martingale that yields (2.5). We could interpret Lemma 2.2.1 in the framework of BSDE as follows: for any given incentive compatible contract $\Gamma = (C, \tau_L, e)$, there exists an unique pair of $\mathcal{F}_t^X$ adapted process $(W_t^\Gamma, \beta_t^\Gamma)$ such that

$$\begin{cases} W_{\tau_L}^\Gamma &= 0, \\ dW_t^\Gamma &= (\rho W_t^\Gamma - B(1-e_t))\,dt + \beta_t^\Gamma\,dZ_t^e - dC_t. \end{cases}$$

However, the question of characterizing incentive-compatible contracts that satisfy the agent's participation constraint (2.4) remains unanswered: we have to characterize the set $\Gamma(w_0)$ of contracts Γ for which $e = e^*(\tau_L, (C_t)_t)$ and, W_0^Γ is greater than the participation constraint w_0.

To solve this problem, the idea of Sannikov (2008) has been to see the sensitivity process β^{Γ} as a control. To this end, let us consider the class of $\mathcal{F}^X$ measurable processes $\beta = (\beta_t)_{t\geq 0}$ such that

$$\mathbb{E}^a\left(\int_0^{\infty} e^{-2rs}\beta_s^2\, ds\right) < +\infty, \tag{2.6}$$

and, for any fixed increasing process C, let us consider the process $W^{\beta} = (W_t^{\beta})_{t\geq 0}$ that satisfies the controlled stochastic differential equation under $\mathbb{P}^0$,

$$dW_t^{\beta} = (\rho W_t^{\beta} + h(\beta_t))\, dt + \beta_t\, dZ_t - dC_t \ \text{ and } \ W_0^{\beta} \geq w_0,$$

with $h(\beta) = \inf_{0\leq e\leq 1}(\frac{\delta}{\sigma}\beta - B)(1-e)$. We would like the process $(W_t^{\beta})_{t\geq 0}$ to play the role of the agent continuation value associated to some incentive compatible contract $\Gamma \in \Gamma(w_0)$. By limited liability, this requires $W_t^{\beta} \geq 0$ up to the termination date of the contract Γ. Therefore, we introduce

$$\tau_0^{\beta}(C) = \inf\{t \geq 0\,,\, W_t^{\beta} = 0\}.$$

Let us recall that we have assumed so far that action process a is incentive compatible. The next lemma characterizes incentive compatible contracts as a deterministic function of the control process β.

Lemma 2.2.2. *For any compensation process $(C_t)_{t\geq 0}$ satisfying (2.2) and any process $(\beta_t)_{t\geq 0}$ satisfying (2.6), the contract $\Gamma = (C, \tau_0^{\beta}(C), \mathbb{1}_{\beta_t \geq \sigma\lambda})$ is incentive compatible and belongs to $\Gamma(w_0)$.*

Proof of Lemma 2.2.2. The proof follows from a standard application of the martingale optimality principle. For any compensation process $(C_t)_{t\geq 0}$ satisfying (2.2) and any process $(\beta_t)_{t\geq 0}$ satisfying (2.6), the process

$$R_t^e = e^{-\rho t}W_t^{\beta} + \int_0^t e^{-\rho s}(B(1-e_s)ds + dC_s)$$

is a uniformly integrable $\mathbb{P}^e$-supermartingale for every effort process $(e_t)_{t\geq 0}$ and a uniformly integrable $\mathbb{P}^{e^*}$-martingale where, for any $t \geq 0$, $e_t^* = \mathbb{1}_{\beta_t \geq \sigma\lambda}$ with $\lambda = \frac{B}{\delta}$. Therefore

$$\begin{aligned}
\mathbb{E}^e\left(\int_0^{\tau_0^{\beta}(C)} e^{-\rho t}(B(1-e_t)dt + dC_t)\right) &\leq R_0^e \\
&= W_0^{\beta} \\
&= R_0^{e^*} \\
&= \mathbb{E}^{e^*}\left(\int_0^{\tau_0^{\beta}(C)} e^{-\rho t}(B(1-e_t^*)dt + dC_t)\right)
\end{aligned} \tag{2.7}$$

□

Therefore, the principal's problem is to find a contract $\Gamma = (C, \tau_0^{\beta}(C), \mathbb{1}_{\beta \geq \lambda\sigma})$ that maximizes her expected profit at date 0. This leads to the following Markov formulation of problem (2.3)-(2.4).

$$V_P(w_0) = \max\nolimits_{w \geq w_0} V_P(w) \tag{2.8}$$

where

$$V_P(w) = \sup_{C,\beta} \mathbb{E}^{e^*}\left(\int_0^{\tau_0^{\beta}(C)} e^{-rs}(dX_s \ - \ dC_s) + e^{-r\tau_0^{\beta}(C)}L\right)$$

$$\text{with} \ \ e_t^* = \mathbb{1}_{\beta_t \geq \sigma\lambda},$$

such that

$$dW_t = (\rho W_t - B\mathbb{1}_{\beta_t < \sigma\lambda})\, dt + \beta_t dZ_t - dC_t \text{ with } W_0 = w. \tag{2.9}$$

2.2.1 Full-effort contracts

From now, we will focus on contracts that induce full effort $e_t = 1$ for all t. From the above Markov formulation, the best full-effort contract is obtained by solving the singular stochastic control problem

$$V_P(w) = \sup_{C, \beta \geq \lambda\sigma} \mathbb{E}^{e^*} \left(\int_0^{\tau_0^\beta(C)} e^{-rs}(dX_s \;-\; dC_s) + e^{-r\tau_0^\beta(C)} L \right) \tag{2.10}$$

where the dynamics of the agent's continuation utility is given by

$$dW_t = \rho W_t \, dt + \beta_t dZ_t - dC_t \text{ with } W_0 = w.$$

There are similarities between the Principal problem and those of the shareholders value function in the liquidity management models. In particular, the value function V_P can be characterized in terms of the HJB equation:

$$\max \left\{ \max_{\beta \geq \lambda\sigma} \frac{\beta^2}{2} v''(w) + \rho w v'(w) - r v(w) + \mu, -v' - 1 \right\} = 0 \tag{2.11}$$

together with the boundary condition $v(0) = L$.
The main result, first obtained in a continuous-time framework by De Marzo and Sannikov is that the Principal value V_P is a smooth and concave solution of the above HJB equation. Note that the concavity of V_P implies that the principal chooses the sensitivity parameter $\beta = \lambda\sigma$. More precisely, we state the main result as follows (see De Marzo and Sannikov, Proposition 8)

Proposition 2.2.3. *There exists a unique function v_0 defined on $(0, +\infty)$ and a unique threshold $w^* > 0$ such that*

- $v_0(0) = L$, $\frac{\lambda^2\sigma^2}{2} v_0''(w) + \rho w v_0'(w) - r v_0(w) + \mu = 0$, $v_0' \geq -1$ *on* $(0, w^*)$ *and* $v_0'(w^*) = -1$,
- v_0 *is concave and twice continuously differentiable.*

Some words about concavity. Assume there is a point $\tilde{w}$ such that $v_0''(\tilde{w}) \geq 0$. Because $v_0' \geq -1$, this implies

$$v_0(\tilde{w}) \geq \frac{\mu - \rho\tilde{w}}{r}.$$

But, in absence of agency frictions, the right-hand side corresponds to the first best value, i.e. choose the admissible remuneration policy $C_t = \rho\tilde{w}t$. Thanks to the concavity, it is clear that v_0 is a solution of (2.11) and thus a standard verification result implies that $v = V_P$. Termination occurs when the continuation utility hits zero where the manager is fired. Unlike liquidity management models, it is here a matter of incentives and not a question of liquidity because the shareholders have deep-pockets. Observe that V_P is not always increasing because w belongs to the agent and at some point, increases in cash flows no longer offset the promised payments which triggers the remuneration at a point where the marginal cost of an immediate payment equals that of increased continuation utility, $v'(w^*) = -1$.
Therefore, it could happen that V_P falls below the liquidation value L before w^*. Without agency frictions, the shareholders would prefer to liquidate the firm. However, because of agency frictions, they have committed to a long-term contract which prevents them from stopping the project outside the rules of the contract.
If it happens that $V_P \leq I$ then the shareholders have no incentives to start the contract at time zero. This could happen for low profitability firms, when μ is small relative to I. To sum up, the project is undertaken if the set $\{w \geq 0, V_P(w) > I\}$ is not empty. If we assume that the labor market for skilled managers is competitive which is equivalent to assume that the shareholders have the bargaining power, they will offer a contract that will promise ex-ante

$$w_0 = \max\{w \geq 0, V_P(w) > I\}.$$

2.2.2 Implementation

The theoretical characterization of the optimal contract using the agent's continuation utility needs a more concrete implementation which uses a combination of limited-liability securities, such as stocks and bonds that can be sold to the financial market. We present the implementation proposed by Bias et al. where all managerial decisions are contingent on the level of cash reserves. As a nice consequence, the capital structure of the firm looks like that of liquidity management models.
The idea of Biais et al. is to use cash reserves to model agent's continuation utility. We assume that these reserves M_t are kept in an escrow account whose balance is observable by posing $W_t = \lambda\sigma M_t$. Setting $\alpha = \lambda\sigma$, we get

$$dM_t = \rho M_t dt + \sigma dZ_t - \frac{dL_t}{\alpha}.$$

But cash reserves grow with the interest rate earned ($\rho M_t dt$) and with the earnings ($\mu dt + \sigma dZ_t$) and decreases with the payout to stakeholders. In order to stick with this accounting evolution of cash reserves, we observe that the evolution of M_t can be written as follows:

$$dM_t = \rho M_t dt + \mu dt + \sigma dZ_t - \frac{dL_t}{\alpha} - \mu dt.$$

The payouts are thus $\frac{dL_t}{\alpha} + \mu dt$. The flow μdt corresponds to a perpetual bond with coupon μ paid out to bondholders.
To sum up, the optimal full-effort contract is implemented with a combination of debt, equity and liquid reserves. The firm issue debt with a coupon equals to the expected profitability to discipline the manager. The manager receive a fraction α of the firm's equity. Observe that the greater the moral hazard problem, the greater the stake held by the manager. Dividend are paid when cash reserves hit the target level $\frac{w^*}{\alpha}$. The greater the moral hazard problem, the higher the target cash level. Finally, the firm is liquidated the first time the cash reserves hit zero. Although the last feature is similar to liquidity management models, it is not for liquidity reasons that the firm is liquidated but rather to give the incentives to do effort.

2.3 Moral Hazard with a Risk-Averse Agent

Consider the following dynamic principal-agent model in continuous time based on Sannikov (RES 2008) who considers a setting where a risk- averse agent works for a risk-neutral principal for an infinite time horizon. The principal and the agent discount future utility at a common rate r.

The total output X_t produced up to time t evolves according to

$$dX_t = A_t\,dt + dB_t^A,$$

where $(B_t^A)_{t\geq 0}$ is a Brownian motion under $\mathbb{P}^A$. The process $(A_t)_t$ is measurable with respect to the filtration generated by X and represents the agent's choice of effort level. We assume that A_t takes values in a bounded interval $[0, \bar{A}]$. The agent has a cost of effort given by $h(A_t)$ where h is a continuous, increasing and convex function. Furthermore, it is assumed that $h(0) = 0$ and there exists some γ such that $h(a) \geq \gamma a$.
Using the same probabilistic formulation as above, we observe that X is a Brownian motion under $\mathbb{P}^0$.
The principal compensates the agent by a stream of consumption $(C_t)_{t\geq 0}$, the consumption needs to be nonnegative which reflects the limited liability of the agent. The agent cannot save and his utility function u is assumed to be smooth, increasing and concave with $u(0) = 0$.

The output process X is publicly observable by both the principal and the agent. The principal does not observe the agent's effort, and uses the observations of X to give the agent incentives to make costly effort. Because the cost function and utility function are common knowledge, the principal knows the set of all effort processes that are optimal for the agent given this compensation contract. In cases where the optimal effort process is not uniquely determined, we assume that we can choose arbitrarily one such A. This is due to the common assumption in the literature that the agent will choose what the principal prefers if there is more than one optimal choice for the agent, since he will have no incentives to deviate from that choice.Before the agent starts working for the principal, the principal offers him a full-commitment contract that specifies a nonnegative flow of consumption $(C_t)_t$ and a termination date τ.

A contract $\Gamma = (\tau, (C_t)_t)$ is said to be incentive-compatible if there exists at least an effort process $A^*(\Gamma)$ that maximizes the agent's total expected utility

$$\mathbb{E}^A \left(\int_0^\tau e^{-rt}(u(C_t) - h(A_t))\, dt \right).$$

The principal's problem then becomes

$$\sup_{\Gamma I.C.} \mathbb{E}^{A^*(\Gamma)} \left(\int_0^\tau e^{-rt}(A_t^*(\Gamma) - C_t)\, dt \right),$$

Moreover, an incentive-compatible contract is admissible if

$$\mathbb{E}^{A^*(\Gamma)} \left(\int_0^\tau e^{-rt}(u(C_t) - h(A_t^*(\Gamma)))\, dt \right) \geq w_0,$$

where w_0 is the agent's reservation utility that could be interpreted as the utility the agent benefits from not working.

2.3.1 Characterization of incentive-compatible contracts

Given the principal-agent setting, the intuitive formulation is that the principal will offer a compensation at time t depends solely on the path of the output process up to time t. Knowing this contract, the agent chooses his action which maximizes his expected payoff. The process $(A_t)_t$ is restricted to be progressively measurable with respect to the filtration generated by X.

Proceeding analogously as in the risk-neutral case, we introduce the control $\pi = ((\beta_t)_t, (C_t)_t)$ and define the controlled SDE W^π as follows:

$$\begin{cases} W_0^\pi &= w_0 \\ dW_t^\pi &= (rW_t^\pi + f(\beta_t) - u(C_t))\, dt + \beta_t dB_t, \end{cases}$$

where $f(z) = \inf_{a\in[0,\bar{A}]}(h(a) - za)$. We also define $\tau_0^\pi = \inf\{t \geq 0,\, W_t^\pi = 0\}$ such that the control π induces a contract $(\tau_0^\pi, (C_t)_t)$.

The martingale optimality principle gives that the family

$$R_t^A = e^{-rt}W_t^\pi + \int_0^t e^{-rt}(u(C_t) - h(A_t))\, dt$$

is a uniformly integrable $\mathbb{P}^A$ supermartingale and a uniformly integrable $\mathbb{P}^{A^*}$ martingale for

$$A_t^*(\pi) = (h')^{-1}(\beta_t)\mathbb{1}_{\beta_t \geq \gamma} = a^*(\beta_t).$$

Optional sampling Theorem gives

$$w_0 = \mathbb{E}^{A^*} \left(\int_0^{\tau_0^\pi} e^{-rt}(u(C_t) - h(A_t^*))\, dt \right)$$

and thus the contract $(\tau_0^\pi, (C_t)_t)$ is incentive-compatible with the associated best reply $a^*(\beta_t)$. It is moreover admissible because $W_0^\pi = w_0$ by definition.

The principal's problem becomes

$$F(w_0) = \sup_{\beta, C} \mathbb{E}^{a^*(\beta)} \left(\int_0^{\tau_0^\pi} e^{-rt}(a_t^*(\beta) - C_t)\, dt \right).$$

This is a Markov control problem, while not a singular one unlike the risk-neutral case.
The principal value dominates the value associated to the shirking action $A_t = 0$ for all t. Denote by F_0 this value. Because the agent must at least receive w, a constant flow of consumption $c_0 = u^{-1}(rw_0)$ is an admissible contract which gives $F_0(w_0) = -\frac{u^{-1}(rw_0)}{r}$. Observe that F_0 is a decreasing and concave function with $F_0(0) = 0$. The associated HJB equation is

$$\begin{cases} 0 & = \sup_{c>0,\beta\in\mathbb{R}} \left(\frac{\beta^2}{2} F''(w) + (rw + f(\beta) + \beta a^*(\beta) - u(c))F'(w) - rF(w) + a^*(\beta) - c \right) \\ 0 & = F(0) \end{cases}$$

Observe also that $f(\beta) + \beta a^*(\beta) = h(a^*(\beta))$ and when $\beta \geq \gamma$, there is a one-to-one correspondence between β and a.
Sannikov has made two important contributions. First, he has conjectured that it is optimal to choose always $\beta \geq \gamma$ and he has shown that there is a solution F_1 and a threshold w_1 such that

- F_1 is strictly concave and twice continuously differentiable.
- F_1 satisfies $F_1(0) = 0$, $F_1(w_1) = F_0(w_1)$ and $F_1'(w_1) = F_0'(w_1)$
- F_1 is a solution of the HJB equation

$$\sup_{a,c>0} \left(\frac{(h'(a))^2}{2} F_1''(w) + (rw + h(a) - u(c))F_1'(w) - rF_1(w) + a - c \right) = 0 \text{ on } (0, w_1).$$

Second, he has shown that $F = F_0$ for large w and has thus linked the Principal value to the value function of the mixed optimal stopping/control problem

$$\hat{F}(w) = \sup_{\tau,a,c} \mathbb{E}^a \left(\int_0^{\tau_0^\pi \wedge \tau} e^{-rt}(a_t - C_t)\, dt + e^{-r(\tau_0^\pi \wedge \tau)} F_0(W_{\tau_0^\pi \wedge \tau}) \right)$$

Applying a verification result, we have $F_1 = \hat{F}$ on $(0, w_1)$. But Sannikov claimed that $F_1 = \hat{F}$ everywhere. From a mathematical viewpoint, a rigorous proof of this claim is still an open question as the question of the optimality to choose always $\beta \geq \gamma$.

To close that section, we sketch the proof of $F = F_0$ for large w.

Lemma 2.3.1. *Assume* $\lim_{c\to\infty} u'(c) = 0$. *Then, for every* $w \geq \hat{w}$, *where* $\hat{w}$ *is defined by* $u'(u^{-1}(r\hat{w}) = \gamma$, *we have* $F(w) = F_0(w)$.

Proof. For $w \geq \hat{w}$, let us define c by the relation $u(c) = rw$. By concavity of u, $u'(c) \leq \gamma$. Then, we have for any incentive pair (A, C),

$$\begin{aligned} w &= \mathbb{E}\left(\int_0^\infty e^{-rt}(u(C_t) - h(A_t))\, dt \right) && (2.12) \\ &\leq \mathbb{E}\left(\int_0^\infty e^{-rt}(c + u'(c)(C_t - c) - \gamma A_t)\, dt \right) && (2.13) \\ &\leq w + u'(c)\left(\mathbb{E}\left(\int_0^\infty e^{-rt}(C_t - A_t)\, dt \right) - \frac{c}{r} \right) && (2.14) \\ &\leq w + u'(c)\left(F_0(w) - \mathbb{E}\left(\int_0^\infty e^{-rt}(A_t - C_t)\, dt \right) \right). && (2.15) \end{aligned}$$

Therefore, for any incentive pair (A, C), we have

$$F_0(w) \geq \mathbb{E}\left(\int_0^\infty e^{-rt}(A_t - C_t)\, dt \right),$$

which concludes the proof.

2.4 Moral Hazard and Real Option

2.4.1 The model

We will study now a dynamic corporate finance contracting model in which firm's profitability fluctuates and is impacted by unobservable managerial effort. Thereby, we introduce in an agency framework the issue of strategic liquidation, a real option to abandon a project. We shall link the principal's problem to a two-dimensional fully degenerate optimal stopping problem.
Principal and agent. We consider a firm that hires a manager to operate a project. The firm's owner, or the principal, has access to unlimited funds and the manager, or agent, is protected by limited liability. The agent and the principal both agree on the same discount rate r. We assume that, at any time t, the project produces observable cash flows if and only if the manager is in charge. In particular, the project is abandoned when the manager is fired and we assume without loss of generality that its scrap value is zero. The cumulative cash flows process $(Y_t)_{t\geq 0}$ and the profitability process $(X_t)_{t\geq 0}$ evolve as

$$dY_t = X_t\,dt \text{ and } dX_t = -\delta a_t\,dt + \sigma dZ_t^a, \;\; X_0 = x \tag{2.16}$$

where δ and σ are positive constants, Z_t^a is a Brownian motion, and $a_t \in [0,1]$ is the agent's unobservable action. The unobservable action $a_t = 0$ is called the effort action, the unobservable action $a_t > 0$ is called the shirking action. Thus, shirking has a negative effect $-\delta a_t$ on profitability. Whenever the agent shirks, he receives a private benefit $Ba_t\,dt$ where B is a positive constant.
Proceeding analogously as before, we obtain the following Markov formulation of our problem: find a contract $\Gamma = (C, \tau_0^\beta(C), \mathbb{1}_{\beta<\lambda\sigma})$ that maximizes her expected profit at date 0.

$$V_P(x, w_0) = \max\nolimits_{w\geq w_0} V_P(x,w) \tag{2.17}$$

where

$$V_P(x,w) = \sup_{C,\beta} \mathbb{E}^{a^*}\left(\int_0^{\tau_0^\beta(C)} e^{-rs}(X_s\,ds - \,dC_s)\right)$$

$$\text{with } \; a^* = (a_t^*)_{t\geq 0} \text{ and } \; a_t^* = \mathbb{1}_{\beta_t<\sigma\lambda},$$

such that

$$dX_t = -\delta\mathbb{1}_{\beta_t<\sigma\lambda}\,dt \; + \sigma dZ_t \text{ with } X_0 = x, \tag{2.18}$$

$$dW_t = (rW_t - B\mathbb{1}_{\beta_t<\sigma\lambda})\,dt + \beta_t dZ_t - dC_t \text{ with } W_0 = w. \tag{2.19}$$

Our first result concerns the possibility to postpone the payment to a terminal lump sum transfer.

Lemma 2.4.1. *It is always optimal for the principal to postpone payments and to pay the agent only at the liquidation time with a lump-sum payment.*

Proof of Lemma 2.4.1. First, observe that, from (2.7), the Principal's value function (2.17) can be re-written as $V_P(x,w_0) = \max_{w\geq w_0}(v(x,w) - w)$ where

$$v(x,w) = \sup_{C,\beta} \mathbb{E}^{a^*}\left(\int_0^{\tau_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\,ds\right) \quad \text{s.t (2.18) and (2.19).} \tag{2.20}$$

The amount $v(x,w)$ corresponds to the total surplus generated by the project in our moral hazard framework.

Second, note that $\tau_0^\beta(C) = \sigma_0^\beta \wedge \tilde\tau_0^\beta(C)$ where for any fixed increasing process $(C_t)_{t\geq 0}$, we have

$$\tilde\tau_0^\beta(C) = \inf\{t \geq 0,\, W_{t-}^\beta = 0\},$$

and

$$\sigma_0^\beta = \inf\{t \geq 0,\, (\Delta C)_t = W_{t-}^\beta \text{ and } (\Delta C)_t > 0\}.$$

Third, with no loss of generality, a remuneration process can be written under the form $(C_t)_{t<\tau_0^\beta(C)} + W_{(\tau_0^\beta(C))^-}\mathbb{1}_{t=\tau_0^\beta(C)}$. Therefore, a control policy can be viewed as a pair (C,β) and a stopping time τ at which the Principal pays $W_{\tau-}^\beta$ and liquidate. Thus, we have

$$v(x,w) = \sup_{C,\beta,\tau} \mathbb{E}^{a^*}\left(\int_0^{\tau\wedge\tilde\tau_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\,ds\right).$$

Observe that $\tilde\tau_0^\beta(0)$ corresponds to the liquidation time when the principal postpones payments up to liquidation. We have

$$v(x,w) \geq \sup_{\beta,\tau}\mathbb{E}^{a^*}\left(\int_0^{\tau\wedge\tilde\tau_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\,ds\right), \text{ choosing } C=0 \tag{2.21}$$

$$\geq \mathbb{E}^{a^*}\left(\int_0^{(\sigma_0^\beta\wedge\tilde\tau_0^\beta(C))\wedge\tilde\tau_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\,ds\right), \text{ choosing } \tau=\sigma_0^\beta\wedge\tilde\tau_0^\beta(C)$$

$$= \mathbb{E}^{a^*}\left(\int_0^{\sigma_0^\beta\wedge\tilde\tau_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\,ds\right), \text{ observing } \tilde\tau_0^\beta(C)\leq\tilde\tau_0^\beta(0). \tag{2.22}$$

Taking the supremum over the controls C, β in (2.22) yields $v(x,w)$. It follows from (2.21) that

$$v(x,w) = \sup_{\beta,\tau}\mathbb{E}^{a^*}\left(\int_0^{\tau\wedge\tilde\tau_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\,ds\right), \tag{2.23}$$

which proves that it is optimal to postpone payments. □

2.4.2 Full-effort contracts

We focus on *full-effort* contracts, that is the class of contracts that induces the agent to exert effort at any time. It follows from Lemma 2.2.2 that the full-effort action process $a=0$ is incentive compatible if and only if $\beta_t \geq \lambda\sigma$. Restricting the analysis to contracts that incentivize the full-effort action leads to re-write the Principal problem as follows:

Find a contract $\Gamma = (W_{\tau-}\mathbb{1}_{t=\tau}, \tau\wedge\tilde\tau_0^\beta, 0)$ where the pair (τ,β) is solution to

$$v(x,w) = \sup_{\beta\geq\lambda\sigma,\tau} \mathbb{E}^0\left(\int_0^{\tau\wedge\tilde\tau_0^\beta} e^{-rs}X_s\,ds\right) \tag{2.24}$$

such that

$$dX_t = \sigma dZ_t \text{ with } X_0 = x, \tag{2.25}$$

$$dW_t = rW_t + \beta_t dZ_t \text{ with } W_0 = w, \tag{2.26}$$

where

$$\tilde\tau_0^\beta = \inf\{t\geq 0,\, W_{t^-}=0\}.$$

The above problem is an Optimal Stopping problem with a random maturity $\tilde\tau_0^\beta$ and the rest of the lecture will consists in solving this problem.
First remember that the unconstrained stopping problem

$$v_0(x) = \sup_\tau \mathbb{E}^0\left(\int_0^\tau e^{-rs}X_s\,ds\right) \tag{2.27}$$

which corresponds to the firm value in a frictionless world in which there are no asymmetry of information and private benefits is a standard real option problem that has an explicit solution (see for instance Dixit and Pindyck). We have

$$v_0(x) = \frac{x}{r} - \frac{x^*}{r} e^{\theta(x-x^*)}, \text{ with } \theta = \frac{-\sqrt{2r}}{\sigma} \text{ and } x^* = \frac{1}{\theta}.$$

The threshold x^* is the profitability threshold below which it is optimal to abandon the project when the profitability is observable. The stopping time

$$\tau^* = \inf\{t \geq 0,\ X_t \leq x^*\} \tag{2.28}$$

is optimal for (2.27).

Second, we consider the sub-solution to problem (2.24)-(2.26) where β_t equals $\lambda\sigma$. This yields the two-dimensional constrained optimal stopping problem

$$u(x, w) = \sup_{\tau} \mathbb{E}^0 \left(\int_0^{\tau \wedge \tilde{\tau}_0^{\lambda\sigma}} e^{-rs} X_s \, ds \right) \tag{2.29}$$

such that

$$dX_t = \sigma dZ_t \text{ with } X_0 = x,$$

$$dW_t = rW_t \, dt + \lambda\sigma dZ_t \text{ with } W_0 = w,$$

and

$$\tilde{\tau}_0^{\lambda\sigma} = \inf\{t \geq 0,\ W_{t^-} = 0\}. \tag{2.30}$$

The main result states

Theorem 2.4.2. *The following holds*

(i) For all $(x, w) \in \mathbb{R} \times \mathbb{R}^+$, $v(x, w) = u(x, w)$. Furthermore, $u(x, w) = v_0(x)$ for all $(x, w) \in \mathbb{R} \times \mathbb{R}^+$ such that $w \geq \lambda(x - x^*)$.

(ii) The contract $((W_{\tau^*_-} \mathbb{1}_{t=\tau^*})_{t \geq 0}, \tau^* \wedge \tilde{\tau}_0^{\lambda\sigma}, 0)$ is the optimal contract that induces full effort.

The proof of Theorem 2.4.2 consist in three steps that we briefly describe below. For a rigorous proof, we refer to Décamps and Villeneuve (2018).

Step 1 *We explicitly solve the optimal stopping problem (2.29).*
The exit time $\tau_R = \tau^* \wedge \tau_0$ of the open rectangle $R = (x^*, +\infty) \times (0, +\infty)$ is optimal for (2.29). That is,

$$u(x, w) = \mathbb{E}^0 \left(\int_0^{\tau_R} e^{-rs} X_s \, ds \right).$$

Moreover, if $w \geq \lambda(x - x^*)$ then, $u(x, w) = v_0(x)$.

Step 2 *We prove regularity results for u.* We have u is jointly continuous over $[x^*, +\infty) \times [0, \infty)$ and C^∞ over $R = (x^*, +\infty) \times (0, +\infty)$. Furthermore, it satisfies

$$\max(\mathcal{L}(\lambda\sigma)u, -u) = 0 \tag{2.31}$$

almost everywhere on $\mathbb{R} \times \mathbb{R}_+$ where $\mathcal{L}(\beta)$ is the fully degenerate differential operator

$$\mathcal{L}(\beta)V \equiv -rV(x, w) + x + rw\frac{\partial V}{\partial w}(x, w) + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}(x, w) + \frac{1}{2}\beta^2\frac{\partial^2 V}{\partial w^2}(x, w) + \sigma\beta\frac{\partial^2 V}{\partial x \partial w}(x, w).$$

Step 3 *We show concavity results for u.* For any $x > x^*$, $\dfrac{\partial u}{\partial w}(x, 0)$ exists and is finite. Moreover, for any $(x, w) \in R$, the value function u satisfies

(i) $\dfrac{\partial u}{\partial w}(x,w) = \mathbb{E}^0\left(\mathbb{1}_{\tau_0 \le \tau^*}\dfrac{\partial u}{\partial w}(X_{\tau_0},0)\right) \ge 0,$

(ii) $\dfrac{\partial^2 u}{\partial w^2}(x,w) < 0,$

(iii) $(\dfrac{\partial^2 u}{\partial x \partial w} + \lambda\dfrac{\partial^2 u}{\partial w^2})(x,w) < 0.$

We thus conclude that u is a smooth solution of

$$\max(\max_{\beta \ge \lambda\sigma}\mathcal{L}(\beta)v, -v) = 0 \text{ on } \mathbb{R} \times \mathbb{R}_+, \tag{2.32}$$

with boundary conditions $v(x,0) = 0$ Then, a standard verification argument based on Itô's formula yields $u = v$.

Optimal payment policies. It follows from the proof of Theorem 2.4.2 that, once the agent's continuation payoff W_t reaches $\lambda(X_t - x^*)$, payment policy $W_{\tau^*_-}\mathbb{1}_{t=\tau^*}$ guarantees optimal liquidation of the firm. Another policy that leads to optimal liquidation once W_t reaches $\lambda(X_t - x^*)$ is to pay the agent at continuous rate $rW_t\,dt$ up to the optimal liquidation time τ^*. To see this, assume that couple (x,w) satisfies relation $w = \lambda(x - x^*)$ and consider payment policy $dC_t = rW_t\,dt$. The dynamics

$$dX_t = \sigma dZ_t \;\; X_0 = x,$$

$$dW_t = rW_t + \lambda\sigma dZ_t - dC_t \;\; W_0 = w,$$

imply that $dW_t = \lambda dX_t$, from which we deduce that $W_t = \lambda(X_t - x^*)$. In turn, $\tau_0 = \tau^*$ a.s., that is, the continuation value process W reaches zero at the optimal liquidation time τ^*. Consistent with (2.7), a direct computation yields that

$$w = \mathbb{E}^0\left(\int_0^{\tau^*} e^{-rt} rW_s\,ds\right), \tag{2.33}$$

and, accordingly, the value of the principal satisfies

$$V_P(x,w) = \mathbb{E}^0\left(\int_0^{\tau^*} e^{-rs}(X_s ds - dC_s)\right) = v_0(x) - w.$$

The above remark shows that paying cash earlier is not costly for the principal provided that principal can ensure that profitability process reaches optimal liquidation threshold x^* before continuation value of the agent falls to zero.

To summarize, when initial values (x,w) satisfy $w < \lambda(x - x^*)$, deferring payments give incentives to the agent, the continuation value rises and the risk of inefficient liquidation is reduced. Once the agent has established a high performance record, that is when continuation value W_t reaches $\lambda(X_t - x^*)$, any payment policy that leads to liquidation at τ^* is optimal. The same idea is present in He and in DeMarzo and Sannikov (2017) where payments to the agent are postponed until inefficient liquidation can be avoided. Note that, as it is usual in dynamic contracting models, the agent may be fired without any compensation at all after a series of bad outcomes leading to inefficient liquidation.