Convergence results for the indifference value based on the stability of BSDEs

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Abstract

We study the exponential utility indifference value \( h \) for a contingent claim \( H \) in an incomplete market driven by two Brownian motions. The claim \( H \) depends on a nontradable asset variably correlated with the traded asset available for hedging. We provide an explicit sequence that converges to \( h \), complementing the structural results for \( h \) known from the literature. Our study is based on a convergence result for quadratic backward stochastic differential equations. This convergence result, which we prove in a general continuous filtration under weak conditions, also yields that the indifference value in a setting with trading constraints enjoys a continuity property in the constraints.

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1 Introduction

An important task of mathematical finance is the valuation of contingent claims. In incomplete markets, this is often done via utility indifference. The indifference value \( h_t \) for a contingent claim \( H \) at time \( t \) makes an investor indifferent, in terms of maximal expected utility, between not buying \( H \) and buying \( H \) for the amount \( h_t \). We explain this definition in more detail
in Sections 3 and 4.1, where we use an exponential utility function. An overview of various aspects of indifference valuation with a long literature list is provided by the recently published book [3] edited by Carmona.

In a basic model, the financial market consists of a risk-free bank account and a stock $S$ driven by a Brownian motion $W$. The contingent claim $H$ to be valued via exponential utility indifference depends on another Brownian motion $Y$, which has instantaneous correlation $\rho$ with $W$. If $\rho$ is deterministic and constant in time, an explicit formula for the indifference value $h_t$ is available from Tehranchi [12] or Chapter 4 of Carmona [3]. However, the situation is different for general $\rho$. With stochastic and/or time-dependent $\rho$, one knows for $h_t$ only bounds and a structural formula where one parameter is not explicitly determined; see Frei and Schweizer [4]. In view of this lack of explicit results for $h_t$, the goal of this paper is to give an explicit sequence that converges to $h_t$.

Our starting point to study this problem with general $\rho$ is the known characterisation of $(h_t)_{0 \leq t \leq T}$ via a backward stochastic differential equation (BSDE), which we present in Lemma 4.1. We deduce that if $\rho$ is piecewise constant in time, we can obtain an explicit formula for $h_t$ in the same way as for constant $\rho$, just by considering iteratively the BSDE on intervals where $\rho$ is constant. For general $\rho$, the idea is to approximate $\rho$ pointwise by a sequence $(\rho^n)_{n \in \mathbb{N}}$ of piecewise constant processes and to replace $\rho$ by $\rho^n$ in the BSDE so that the solutions have an explicit form. These solutions then converge to (a transform of) $h_t$ by a convergence result for quadratic BSDEs, which we prove in a general continuous filtration to guarantee a broad framework for applications. We thus have an explicitly known sequence which converges to $h_t$. The only point left is whether we can approximate $\rho$ pointwise by a sequence $(\rho^n)_{n \in \mathbb{N}}$ of piecewise constant processes. We show that the above approximation of $\rho$ and thus that of $h_t$ work in a general way for every deterministic $\rho$ except for “pathological examples”. For stochastic $\rho$, we prove that the approximation of $h_t$ is possible if $\rho$ has left-continuous paths.

The paper is structured as follows. Motivated by the above approximation problem, we state in Section 2 convergence results for quadratic BSDEs in a general continuous filtration. Section 3 gives a first application of these results. We show that the indifference value in a general continuous filtration with trading constraints enjoys a continuity property in the constraints. The results on the indifference valuation in a Brownian setting are contained in Section 4. We lay out the model and prove some preliminary results in Section 4.1. We then study in Section 4.2 the approximation of the indifference value $h_t$ by applying the convergence results of Section 2. Section 4.3 shows a continuity property of $h_t$ in the correlation $\rho$. Finally, the Appendix contains the proofs of the convergence results of Section 2.
2 Convergence results

The financial applications in the subsequent sections are based on convergence results for quadratic BSDEs in the setting of Morlais [11]. We first recall this framework and then state the main convergence theorem.

We work on a finite time interval $[0,T]$ for a fixed $T > 0$, and we fix $t \in [0,T]$ throughout this section. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{0 \leq s \leq T}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions with $\mathcal{F} = \mathcal{F}_T$. We assume that $\mathcal{F}$ is continuous, i.e., all local martingales are continuous.

We fix an $\mathbb{R}^d$-valued local martingale $M = (M_s)_{0 \leq s \leq T}$ and take a nondecreasing and bounded process $D$ (e.g., $D = \arctan(\sum_{j=1}^{d} (M_j)$) such that $d(M) = m \, m' \, dD$ for an $\mathbb{R}^{d \times d}$-valued predictable process $m$.

Let us consider, for $0 \leq s \leq T$, the BSDE

$$\Gamma_s = H + \int_s^T f(r, Z_r) \, dD_r + \frac{\beta}{2} (\langle N \rangle_T - \langle N \rangle_s) - \int_s^T Z_r \, dM_r - (N_T - N_s),$$

(2.1)

where $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable ($\mathcal{P}$ denotes the $\sigma$-field of all predictable sets on $\Omega \times [0, T]$ and $\mathcal{B}(\mathbb{R}^d)$ is the Borel $\sigma$-field on $\mathbb{R}^d$), $\beta \in \mathbb{R}$ is a constant and $H$ is a bounded random variable. A solution of (2.1) is a triple $(\Gamma, Z, N)$ satisfying (2.1), where $\Gamma$ is a real-valued bounded continuous semimartingale, $Z$ is an $\mathbb{R}^d$-valued predictable process with $E[\int_0^T |m_s Z_s|^2 \, dD_s] < \infty$ and $N$ is a real-valued square-integrable martingale null at 0 and strongly orthogonal to $M$.

**Theorem 2.1.** Let $(f^n, \beta^n, H^n)_{n=1,2,\ldots,\infty}$ be a sequence of $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable real-valued mappings, constants, and random variables bounded uniformly in $L^\infty$, such that

(i) there exists a nonnegative predictable $\kappa^1$ with $\| \int_0^T \kappa^1_s \, dD_s \|_{L^\infty} < \infty$ and a constant $c^1$ such that

$$|f^n(s, z)| \leq \kappa^1_s + c^1 |m_s z|^2 \quad \text{for all } s \in [0, T], \quad z \in \mathbb{R}^d$$

(2.2)

and $n = 1, \ldots, \infty$;

(ii) there exists a nonnegative predictable $\kappa^2$ with $\| \int_0^T \kappa^2_s^2 \, dD_s \|_{L^\infty} < \infty$ and a constant $c^2$ such that

$$|f^n(s, z^1) - f^n(s, z^2)| \leq c^2(\kappa^2_s + |m_s z^1| + |m_s z^2|)|m_s(z^1 - z^2)|$$

for all $s \in [0, T], \ z^1, z^2 \in \mathbb{R}^d$ and $n = 1, \ldots, \infty$;

(iii) $\lim_{n \to \infty} \beta^n = \beta^\infty$, $\lim_{n \to \infty} H^n = H^\infty$ a.s. and for $(\mathcal{P} \otimes D)$-almost all $(\omega, s) \in [t, T]$, $\lim_{n \to \infty} f^n(s, z)(\omega) = f^\infty(s, z)(\omega)$ for all $z \in \mathbb{R}^d$. 
Then there are unique solutions \((\Gamma^n, Z^n, N^n)\) to the BSDE (2.1) with parameters \((f^n, \beta^n, H^n)\) for \(n = 1, \ldots, \infty\), and \(\Gamma^n_t\) converges to \(\Gamma^\infty_t\) a.s. as \(n \to \infty\),

\[
\lim_{n \to \infty} E\left[ \int_t^T |m_s(Z^n_s - Z^\infty_s)|^2 dD_s \right] = 0, \quad \lim_{n \to \infty} E\left[ (N^n_t - N^\infty_t) - \langle N^n - N^\infty \rangle_t \right] = 0.
\]

Moreover, \(\sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \to 0\) as \(n \to \infty\) in probability and in \(L^p\), \(p \in [1, \infty)\).

In the above theorem, we assumed only weak conditions for the convergence of the input data \((f^n, \beta^n, H^n)\) for \(n = 1, \ldots, \infty\). In particular, we imposed only pointwise convergence of \((f^n)_{n=1,\ldots,\infty}\), which will be significant for the later applications. The price to be paid is that the solutions of the BSDE converge only in some weak sense. However, in the main application in Section 4, the generators \((f^n)_{n=1,\ldots,\infty}\) of the BSDE have a specific form, and for this case, the following result states stronger convergence properties. The proofs of Theorem 2.1 and of the next corollary are presented in the Appendix.

Corollary 2.2. Suppose in addition to the assumptions of Theorem 2.1 that

(iv) \(H^n\) converges to \(H^\infty\) in \(L^\infty\) as \(n \to \infty\);

(v) there exist sequences \((a^n)_{n \in \mathbb{N}}\) and \((\bar{a}^n)_{n \in \mathbb{N}}\) of deterministic functions which converge to 1 uniformly on \([t, T]\) (up to a \((P \otimes D)\)-nullset) such that \(f^n = a^n f + \bar{a}^n \bar{f}\) for every \(n = 1, \ldots, \infty\), where \(\bar{f}, \bar{f}\) satisfy (2.2) with \(f^n\) replaced by \(f, \bar{f}\).

Then we have \(\sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \to 0\) in \(L^\infty\) as \(n \to \infty\) and there even exists a constant \(K > 0\) such that for all \(n \in \mathbb{N},\)

\[
\left\| \sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \right\|_{L^\infty} \leq K \left( \|a^n - 1\|_{L^\infty(P \otimes D)} + \|\bar{a}^n - 1\|_{L^\infty(P \otimes D)} + |\beta^n - \beta^\infty| + \|H^n - H^\infty\|_{L^\infty} \right). \tag{2.3}
\]

Further, \(\int Z^n\,dM \to \int Z^\infty\,dM\) and \(N^n \to N^\infty\) on \([t, T]\) in \(BMO\) as \(n \to \infty\).

In the literature on BSDEs, convergence results are also called stability results. The main differences between Theorem 2.8 of Kobylanski [8] and our Theorem 2.1 are the following: Kobylanski [8] works in a Brownian setting and imposes locally uniform convergence on the generators, whereas our Theorem 2.1 is stated in a general continuous filtration and for generators \((f^n)_{n=1,\ldots,\infty}\) that converge only pointwise. Moreover, the generators in Kobylanski’s Theorem 2.8 can unlike ours also depend on \(\Gamma^n\), and the a.s. convergence of \(\sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^n_s|\) is proved. Another convergence result in a
Brownian setting is Proposition 7 of Briand and Hu [2], which gives convergence of the moments of \( \exp(\sup_{s \in [t,T]} |\Gamma^n_s - \Gamma^\infty_s|) \) and \( \left( \int_t^T |Z^n_s - Z^\infty_s|^2 \, ds \right)^{1/2} \) for unbounded terminal conditions if the generators are convex.

For a general continuous filtration, a convergence result for an exponential transformation of the BSDE (2.1) is available from Lemma 3.3 and Remark 3.4 of Morlais [11]. Lemma 3.3 serves in [11] as an auxiliary result to show existence of a solution to (2.1) with a more general generator \( f \) which can also depend on \( \Gamma \). The proof of the existence result first establishes a one-to-one correspondence between solutions to (2.1) and those to a simpler BSDE which results from an exponential transformation of the original BSDE. Lemma 3.3 is then used in proving existence of a solution to the simpler BSDE. Due to the one-to-one correspondence between solutions to the original and to the simpler BSDEs, Lemma 3.3 gives also a convergence result for the original BSDE, as Morlais remarks. In particular, its application to (2.1) needs that \( \exp(\beta^n H^n) \) and a certain transform of \( f^n \) are nondecreasing in \( n \), and it yields \( E\left[ \sup_{s \in [t,T]} |e^{\beta^n \Gamma^n_s} - e^{\beta^\infty \Gamma^\infty_s}| \right] \to 0 \), which is equivalent to \( \sup_{s \in [t,T]} |\Gamma^n_s - \Gamma^\infty_s| \to 0 \) in \( L^1 \) for \( \beta^\infty \neq 0 \); the equivalence can be shown using

\[
\min\{e^x, e^y\}|x - y| \leq |e^x - e^y| \leq \max\{e^x, e^y\}|x - y|, \quad x, y \in \mathbb{R},
\]

and that \( \Gamma^n \) is uniformly bounded in \( n = 1, \ldots, \infty \) by Lemma 3.1 of Morlais [11]. In contrast to Morlais [11], who proves existence and uniqueness of solutions to (2.1), we focus on convergence questions and work in the proof of Theorem 2.1 directly with the BSDE (2.1) instead of doing first an exponential transformation. Standard BSDE comparison techniques and the application of \( BMO \)-theory enable us to prove in the Appendix the a.s. convergence of \( \Gamma^n \) under weak assumptions.

### 3 Indifference valuation with convergent constraints

We apply the convergence results in two different financial contexts. While both applications deal with indifference valuation, the underlying settings are different. In this section, we consider an investor facing trading constraints in the same fairly general framework as in the previous section. For the main application of Theorem 2.1, we restrict in Section 4 our study to a Brownian model and derive there results independently from this section.

We now work within the framework of Section 2 with a continuous filtration \( \mathcal{F} \). Recall that \( M \) is a local martingale and \( d\langle M \rangle = mm' \, dD \). We suppose that almost surely, the matrix \( m_s \) is invertible for every \( s \in [0,T] \).
The financial market consists of a risk-free bank account yielding zero interest and \(d\) risky assets whose price process \(S = (S_s)_{0 \leq s \leq T}\) is given by
\[
\frac{dS^j_s}{S^j_s} = dM^j + \sum_{i=1}^{d} \lambda^i_s d\langle M^j, M^i \rangle_s, \quad 0 \leq s \leq T, \quad S^j_0 > 0 \quad \text{for } j = 1, \ldots, d,
\]
where \(\lambda\) is a predictable process which satisfies \(\| \int_0^T |m_s \lambda_s|^2 dD_s \|_{L^\infty} < \infty\), i.e., the mean-variance tradeoff process is bounded. Let \(H\) be a bounded random variable, interpreted as a contingent claim or payoff due at time \(T\), and let \(C \subseteq \mathbb{R}^d\) be a closed set with \(0 \in C\). We assume that our investor has an exponential utility function \(U(x) = -\exp(-\gamma x), x \in \mathbb{R}\), for a fixed \(\gamma > 0\).

Starting at time \(t\) with bounded \(\mathcal{F}_t\)-measurable capital \(x_t\), she runs a self-financing strategy \(\pi = (\pi_s)_{t \leq s \leq T}\) valued in \(C\) such that her wealth at time \(s \in [t, T]\) is \(X^x_{s t,\pi} = x_t + \int_t^s \sum_{j=1}^{d} \pi_s^j \frac{dS^j}{S^j}\), where \(\pi^j\) represents the amount invested in \(S^j, j = 1, \ldots, d\). The set \(\mathcal{A}^C_t\) of \(C\)-admissible strategies on \([t, T]\) consists of all predictable \(\mathbb{R}^d\)-valued processes \(\pi = (\pi_s)_{t \leq s \leq T}\) which satisfy a.s., \(\pi_s \in C\) for all \(s \in [t, T]\), \(E[\int_t^T |m_s \pi_s|^2 dD_s] < \infty\) and are such that \(\exp(-\gamma X^x_{s t,\pi}, t \leq s \leq T\), is of class \((D)\). We define \(V_t^{H,C}(x_t)\) by
\[
V_t^{H,C}(x_t) := \text{ess sup}_{\pi \in \mathcal{A}^C_t} E[U(X^x_{T t,\pi} + H)|\mathcal{F}_t]
\]
so that \(V_t^{H,C}(x_t)\) is the maximal expected utility the investor can achieve by starting at time \(t\) with initial capital \(x_t\), using some \(C\)-admissible strategy \(\pi\), and receiving \(H\) at time \(T\). For ease of notation, we write

\[
V_t^{H,C}(x_t) = e^{-\gamma x_t} V_t^{H,C}(0) = e^{-\gamma x_t} V_t^{H,C}.
\]

Viewed over time \(t\), \(V^{H,C}\) is then the dynamic value process for the stochastic control problem associated to exponential utility maximisation.

The time \(t\) indifference (buyer) value \(h_t^{H,C}(x_t)\) for \(H\) is implicitly defined by
\[
V_t^{0,C}(x_t) = V_t^{H,C}(x_t - h_t^{H,C}(x_t)).
\]

This says that the investor is indifferent between solely trading with initial capital \(x_t\), versus trading with initial capital \(x_t - h_t^{H,C}(x_t)\) but receiving \(H\) at \(T\). By (3.2),
\[
h_t^{H,C}(x_t) = h_t^{H,C} = \frac{1}{\gamma} \log \frac{V_t^{0,C}}{V_t^{H,C}}
\]
does not depend on \(x_t\).

The following proposition can be seen as a kind of continuity result for \(V_t^{H,C}\) and \(h_t^{H,C}\) in \((H, C)\). Its proof is based on Theorem 2.1.
Proposition 3.1. Let $H_n, n = 1, 2, \ldots, \infty,$ be uniformly bounded random variables with $\lim_{n \to \infty} H_n = H_\infty$ a.s., and let $C_n, n = 1, 2, \ldots, \infty,$ be closed subsets of $\mathbb{R}^d$ which contain zero and are such that $(P \otimes D)$-a.e.,

$$\lim_{n \to \infty} \inf_{y \in C^n} |m(y - z)| = \inf_{y \in C^\infty} |m(y - z)| \quad \text{for all } z \in \mathbb{R}^d. \tag{3.5}$$

Then $\lim_{n \to \infty} V_t^{H_n, C_n} = V_t^{H_\infty, C_\infty}$ and $\lim_{n \to \infty} h_t^{H_n, C_n} = h_t^{H_\infty, C_\infty}$ a.s., and there exist continuous versions $V_t^{H_n, C_n}$ and $h_t^{H_n, C_n}, n = 1, \ldots, \infty,$ such that

$$\sup_{n \to \infty} \sup_{s \in [t, T]} |V_s^{H_n, C_n} - V_s^{H_\infty, C_\infty}| = 0 \quad \text{in probab. and in } L_p, 1 \leq p < \infty,$$

$$\sup_{n \to \infty} \sup_{s \in [t, T]} |h_s^{H_n, C_n} - h_s^{H_\infty, C_\infty}| = 0 \quad \text{in probab. and in } L_p, 1 \leq p < \infty. \tag{3.6}$$

Proof. Fix $n \in \{1, \ldots, \infty\}.$ By Theorem 4.1 of Morlais [11], there is a version $V_t^{H_n, C_n}$ such that $V_t^{H_n, C_n} = -\exp(\gamma t^n)$, where $(\Gamma^n, Z^n)$ is the solution of (2.1) with $\beta := \gamma,$ $H$ replaced by $-H^n$ and with generator $f^n$ given by

$$f^n(s, z) := \inf_{y \in C^n} \left(\frac{\gamma}{2} m_s \left(y - z - \frac{1}{\gamma} \lambda_s\right)^2\right) - \langle m_s z \rangle (m_s \lambda_s) - \frac{1}{2\gamma} |m_s \lambda_s|^2$$

for $s \in [0, T]$ and $z \in \mathbb{R}^d.$ Remarks 2.3 and 2.4 of Morlais [11] and (3.5) imply that the assumptions (i)-(iii) of Theorem 2.1 are satisfied, which yields

$$\lim_{n \to \infty} \Gamma^n_t = \Gamma^\infty_t \quad \text{a.s. and} \quad \lim_{n \to \infty} \sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| = 0 \quad \text{in probab. and in } L_p. \tag{3.7}$$

Therefore, we obtain $\lim_{n \to \infty} V_t^{H_n, C_n} = \lim_{n \to \infty} -\exp(\gamma t^n) = -\exp(\gamma t^\infty) = V_t^{H_\infty, C_\infty}$ a.s. and analogously $\lim_{n \to \infty} V_t^{0, C_n} = V_t^{0, C_\infty}$ a.s., so $\lim_{n \to \infty} h_t^{H_n, C_n} = h_t^{H_\infty, C_\infty}$ a.s. by (3.4). Because we have

$$\sup_{s \in [t, T]} |h_s^{H_n, C_n} - h_s^{H_\infty, C_\infty}| = \frac{1}{\gamma} \sup_{s \in [t, T]} \left|\log \frac{V_s^{0, C_n}}{V_s^{0, C_\infty}} - \log \frac{V_s^{H_n, C_n}}{V_s^{H_\infty, C_\infty}}\right| \\
\leq \frac{1}{\gamma} \sup_{s \in [t, T]} \left|\log \frac{V_s^{0, C_n}}{V_s^{0, C_\infty}}\right| + \sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \\
\leq \gamma \exp(\gamma) \sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \quad \text{in probab. and in } L_p.$$

we obtain (3.6) from (3.7) and its analogue with $(H^n, C^n)$ replaced by $(0, C^n).$ We also have $\lim_{n \to \infty} \sup_{s \in [t, T]} |\exp(\gamma t^n) - \exp(\gamma t^\infty)| = 0$ in probab. and in $L_p,$ since

$$\sup_{s \in [t, T]} |\exp(\gamma t^n) - \exp(\gamma t^\infty)| \leq \gamma \exp(\gamma) \sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s| \tag{3.8}$$

and $\Gamma^n$ is uniformly bounded by Lemma 3.1 of Morlais [11]. This concludes the proof because $V_t^{H_n, C_n} = -\exp(\gamma t^n)$ for a version $V_t^{H_n, C_n}.$
Remark 3.2. 1) The condition (3.5) can be rephrased as follows: Define a
time-dependent random inner product $\langle \cdot, \cdot \rangle_m$ by
$\langle x,y \rangle_m := x'm'ym$ for $x,y$ in $\mathbb{R}^d$ and denote by $d_m$ the induced metric, i.e., $d_m(x,y) := \langle x - y, x - y \rangle_m$
for $x,y \in \mathbb{R}^d$. Then $\langle \cdot, \cdot \rangle_m$ is the standard scalar product on $\mathbb{R}^d$ after a basis
transformation by $m^{-1}$. Defining $d_m(x,C) := \inf_{y \in C} d_m(x,y)$ for a closed set $C \subseteq \mathbb{R}^d$, the condition (3.5) is equivalent to $\lim_{n \to \infty} d_m(x,C^n) = d_m(x,C^\infty)$
for all $x \in \mathbb{R}^d$. This means that the sets $(C^n)_{n \in \mathbb{N}}$ are Wijsman convergent to
$C^\infty$ with respect to the metric $d_m$. Beer [1] gives a survey on Wijsman con-
vergence, which is a weaker notion than convergence in the Hausdorff metric.

2) We have used an exponential utility function $U(x) = -\exp(-\gamma x)$,
x $\in \mathbb{R}$, for a fixed $\gamma > 0$. By applying Theorems 4.4 and 4.7 of Mor-
lais [11], analogous results can be derived for the value process related to
power utility $U(x) = x^\gamma / \gamma$, $x > 0$, for a fixed $\gamma \in ]0,1[$, and to logarithmic
utility $U(x) = \log x$, $x > 0$, when there is no claim, i.e., $H = 0$.

4 Indifference valuation in a Brownian setting

We now apply the convergence Theorem 2.1 to the indifference valuation in a
Brownian setting with variable correlation. We first introduce in Section 4.1
the model and explain the problem. We then apply Theorem 2.1 in Sections
4.2 and 4.3 to give convergence results for the indifference value and the
dynamic value process.

4.1 Model setup and preliminary results

We work on a finite time interval $[0,T]$ for a fixed $T > 0$, and we fix
t $\in [0,T]$ throughout this section. On a complete filtered probability space
$(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_s)_{0 \leq s \leq T}, \mathbb{P})$, we have two independent one-dimensional $(\mathbb{G}, \mathbb{P})$-
Brownian motions $Y$ and $Y \perp$. We denote by $\mathbb{Y} = (\mathcal{Y}_s)_{0 \leq s \leq T}$ the $\mathbb{P}$-augmented filtration generated by $Y$. Let $W$ be a $(\mathbb{G}, \mathbb{P})$-Brownian motion with instan-
taneous correlation $\rho$ to $Y$ so that

$$dW_s = \rho_s dY_s + \sqrt{1 - \rho^2_s} dY_s^\perp, \quad 0 \leq s \leq T. \quad (4.1)$$

Our financial market consists of a risk-free bank account yielding zero
interest and a traded risky asset $S$ with dynamics

$$dS_s = S_s \mu_s ds + S_s \sigma_s dW_s, \quad 0 \leq s \leq T, \quad S_0 > 0;$$
the drift $\mu$ and the (positive) volatility $\sigma$ are $\mathcal{G}$-predictable. We set $\lambda := \frac{\mu}{\sigma}$ and assume that $\int_0^T \lambda_s^2 \, ds$ is bounded. The processes

$$\hat{W} := W + \int \lambda \, ds \quad \text{and} \quad \hat{Y} := Y + \int \rho \lambda \, ds$$

are Brownian motions under the minimal martingale measure $\hat{P}$ given by

$$\frac{d\hat{P}}{dP} := \mathcal{E}\left(-\int \lambda \, dW\right)_T := \exp\left(-\int_0^T \lambda_s \, dW_s - \frac{1}{2} \int_0^T \lambda_s^2 \, ds\right). \quad (4.2)$$

In contrast to Section 3, the investor here can trade in $S$ without constraints. He starts at time $t$ with bounded $\mathcal{G}_t$-measurable capital $x_t$ and runs a self-financing strategy $\pi = (\pi_s)_{t \leq s \leq T}$ so that his wealth at time $s \in [t, T]$ is

$$X_{s,\pi} = x_t + \int_t^s \frac{\pi_r}{S_r} \, dS_r \quad \text{for} \quad t \leq s \leq T \quad (4.3)$$

where $\pi$ represents the amount invested in $S$. For a bounded random variable $H$, we define $V_t^H(x_t)$ like $V_t^{H,C}(x_t)$ in (3.1) with $\mathcal{G}_t$ instead of $\mathcal{F}_t$ and $\mathcal{A}_t^C$ replaced by $\mathcal{A}_t$ which consists of all $\mathcal{G}$-predictable real-valued processes $\pi = (\pi_s)_{t \leq s \leq T}$ which satisfy $\int_t^T \pi_s^2 \sigma_s^2 \, ds < \infty$ a.s. and are such that

$$\exp(-\gamma X_{s,\pi}), \ t \leq s \leq T, \ \text{is of class (D) on} \ (\Omega, \mathcal{G}_T, \mathcal{G}, \mathbb{P}). \quad (4.4)$$

The dynamic value process $V^H$ and the indifference value $h_t^H$ are defined analogously to (3.2) and (3.3). From (3.4), we see that once we can calculate $V_t^H$ and $V_t^0$, we also know $h_t^H$. So our focus lies on studying $V_t^H$.

We always impose without further mention the standing assumption that

$$H \in L^\infty(\mathcal{Y}_T, \mathbb{P}) \quad \text{and} \quad \lambda, \rho \text{ are } \mathcal{Y}\text{-predictable}. \quad (4.5)$$

This reflects a situation where the payoff $H$ is driven by $Y$, whereas hedging can only be done in $S$ which is in general imperfectly correlated with $Y$. We refer to Sections 4.1 and 4.2 of Frei and Schweizer [4] for an overview of the related literature and a thorough explanation of the standing assumption (4.5), which corresponds to case (I) in [4]. (For case (II) in [4], results analogous to those in Section 4.2 can be derived if $\rho$ is predictable for the filtration generated by $\hat{Y}$.)

The standing assumption (4.5) allows us to give a BSDE characterisation for $V_t^H$ in the $\mathcal{Y}$-filtration. This BSDE is a special case of the BSDE (2.1).

**Lemma 4.1.** The BSDE

$$\Gamma_s = H - \int_s^T \left(\frac{1}{2} \gamma (1 - \rho_s^2) Z_r^2 - Z_r \rho_s \lambda_r - \frac{\lambda_r^2}{2 \gamma}\right) \, dr + \int_s^T Z_r \, dY_r \quad (4.6)$$
for $0 \leq s \leq T$ has a unique solution $(\Gamma, Z)$ where $\Gamma$ is a real-valued bounded continuous $(\mathbb{Y}, P)$-semimartingale and $Z$ is a $\mathbb{Y}$-predictable process such that $E_P \left[ \int_0^T Z_s^2 \, ds \right] < \infty$. Moreover, there exists a continuous version $V^H$ (which we always use in the sequel) such that $V^H = -\exp(-\gamma \Gamma)$, and its optimal strategy denoted by $\pi^*$ is given by $\pi^* = \frac{\rho}{\sigma} Z + \frac{\lambda}{\gamma \sigma}$.

Lemma 4.1 is essentially known. In particular, Proposition 5.5 of Frei et al. [5] gives a multidimensional version. However, two assumptions of that proposition are not satisfied in our setting; firstly, $\mathbb{G}$ is not necessarily generated by $W$ and a Brownian motion orthogonal to $W$, and secondly, $|\rho|$ is not bounded away from 1. Instead of adapting the proof of Proposition 5.5 of Frei et al. [5], we give the complete argument in the following.

Proof of Lemma 4.1. Existence and uniqueness of a solution $(\Gamma, Z)$ of (4.6) follow from Theorem 2.1 with $\mathbb{F} := \mathbb{Y}$, $\mathbb{M} := -Y$ and

$$f(s, z) := -\frac{1}{2} \gamma (1 - \rho^2) z^2 + \frac{\lambda}{\gamma} s + \frac{\lambda^2}{2 \gamma} \quad \text{for } s \in [0, T] \text{ and } z \in \mathbb{R}.$$ (Since any $\mathbb{Y}$-martingale orthogonal to $Y$ is constant, we can choose in (2.1) $\beta \in \mathbb{R}$ arbitrarily.) Moreover, Proposition 7 of Mania and Schweizer [10] and its proof yield that $\int Z \, dY$ is in both $\text{BMO}(\mathbb{Y}, P)$ and $\text{BMO}(\mathbb{G}, P)$.

To establish the result, it remains to show $V^H_t = -\exp(-\gamma \Gamma_t)$. A simple calculation based on (4.3) and (4.6) yields for $\pi \in \mathcal{A}_t$ that

$$\exp(-\gamma X^0,\pi^0) = \frac{\mathcal{E}(\int \gamma Z \, dY - \int \gamma \pi \sigma \, dW)}{\mathcal{E}(\int \gamma Z \, dY - \int \gamma \pi \sigma \, dW)^{\frac{1}{2}}} \times \exp \left( \frac{1}{2} \int_t^s (\gamma \rho_r Z_r + \lambda_r - \gamma \pi_r \sigma_r)^2 \, dr \right),$$

(4.7)

$$\geq \frac{\mathcal{E}(\int \gamma Z \, dY - \int \gamma \pi \sigma \, dW)}{\mathcal{E}(\int \gamma Z \, dY - \int \gamma \pi \sigma \, dW)^{\frac{1}{2}}}, \quad t \leq s \leq T.$$

Therefore, if $\int \pi \sigma \, dW \in \text{BMO}(\mathbb{G}, P)$, we obtain

$$E_P \left[ \exp(-\gamma X^{0,\pi} - \gamma H) \mid \mathcal{G}_t \right] \geq \exp(-\gamma \Gamma_t),$$

(4.8)

since the stochastic exponential of a continuous $\text{BMO}$-martingale is a true martingale by Theorem 2.3 of Kazamaki [6]. By a localisation argument and (4.4), we have (4.8) for every $\pi \in \mathcal{A}_t$, which implies $V^H_t \leq -\exp(-\gamma \Gamma_t)$. Equality in (4.8) holds for $\pi = \pi^* := \frac{\rho}{\sigma} Z + \frac{\lambda}{\gamma \sigma}$. Since $\exp(-\gamma X^{0,\pi^*})$ is by (4.7) the product of a bounded process and a $(\mathbb{G}, P)$-martingale, it is of class $(D)$ on $(\Omega, \mathcal{G}_T, \mathbb{G}, P)$; hence $\pi^* \in \mathcal{A}_t$ and $V^H_t = -\exp(-\gamma \Gamma_t)$. \qed
Although $V^H$ is given in terms of the solution of (4.6), there is no fully explicit formula available for $V^n_t$ unless $\rho$ is deterministic and constant in time. While the methods in Frei and Schweizer [4] and Frei et al. [5] give bounds for $V^n_t$, we approximate in the subsequent sections $V^n_t$ by approaching $\rho$ by piecewise constant processes.

**Definition 4.2.** We denote by $\Xi$ the set of all processes $q$ of the form

$$q = q^1\mathbb{1}_{\{\tau_0\}} + \sum_{j=1}^n q^j\mathbb{1}_{[\tau_{j-1}, \tau_j]}, \quad \text{for } t = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = T,$$

where $\tau_j$ is a $\mathcal{Y}$-stopping time and $q^j$ is a $\mathcal{Y}_{\tau_{j-1}}$-measurable random variable valued in $]-1, 1[$. We call $(q^j, \tau_j)_{j=1, \ldots, n}$ a characterising sequence of $q$.

**Proposition 4.3.** Let $q$ be a bounded $\mathcal{Y}$-predictable process. The BSDE

$$\Gamma^q_s = H - \int_s^T \left( \frac{1}{2}\gamma(1 - q^2_t)|Z^q_t|^2 - Z^q_t \rho \lambda - \frac{\lambda^2}{2\gamma} \right) dr + \int_s^T Z^q_t dY_r \tag{4.9}$$

for $0 \leq s \leq T$ has a unique solution $(\Gamma^q, Z^q)$ (in the sense of Lemma 4.1).

1) If $q \in \Xi$ with characterising sequence $(q^j, \tau_j)_{j=1, \ldots, n}$, then

$$e^{-\gamma \Gamma^q_s} = E^\rho \left[ \cdots E^\rho \left[ E^\rho \left[ e^{\hat{H}(1 - |q^n|^2)} \mathcal{Y}_{\tau_{n-1}} \right] \right] \cdots \right] \mathcal{Y}_{\tau_0}^{-\frac{1}{1 - |q|^2}}, \tag{4.10}$$

where $\hat{H} := -\gamma H - \frac{1}{2} \int_s^T \lambda^2_t ds$.

2) If $|q| \geq |\rho|$ $(P \otimes \text{Leb})$-almost everywhere, then $V^H \leq -\exp(-\gamma \Gamma^q)$.

3) If $|q| \leq |\rho|$ $(P \otimes \text{Leb})$-almost everywhere, then $V^H \geq -\exp(-\gamma \Gamma^q)$.

If $\rho$ itself is in $\Xi$, Proposition 4.3 gives explicit formulas for $V^n_t$ and $h^n_t$ by choosing $q = \rho$ and using (3.4). For general $\rho$, the idea is to find a sequence $(q^n)_{n \in \mathbb{N}}$ in $\Xi$ which converges pointwise to $\rho$. The solutions $\Gamma^n_t$ of (4.9) with $q = q^n$ have the explicit form (4.10) and converge a.s. to the solution $\Gamma_t = \Gamma^n_t$ of (4.6) by Theorem 2.1. We thus obtain an explicitly known sequence converging a.s. to $V^n_t$. The only open point, which we treat in Section 4.2, is whether we can find a sequence $(q^n)_{n \in \mathbb{N}}$ in $\Xi$ which converges pointwise to $\rho$.

Note that the right-hand side of (4.10) is not the value of $V^n_t$ in a model with correlation $q$ instead of $\rho$. Comparing (4.9) with (4.6), we see that only the $\rho$ in front of $|Z|^2$ is replaced by $q$; the $\rho$ in the term linear in $Z$ is kept. This implies that the measure used in the iterated expectations in (4.10) is $\hat{P}$, defined in (4.2). It does not depend on $q$ — a property desired for the above-mentioned approximation of $V^n_t$, since we prefer to take always the
same explicitly known measure in calculating the conditional expectations. If we replace \( \rho \) in (4.9) by \( q \), the solution of the BSDE is linked to the value of \( V_t^H \) when \( \rho \) is replaced by \( q \). In Section 4.3, we deduce from this a continuity property of \( V_t^H \) in \( \rho \).

Parts 2) and 3) of Proposition 4.3 can be seen as a monotonicity property of \( V_t^H \). However, since \( \rho \) still appears in (4.9), we cannot simply say that \( V_t^H \) is monotonic in \( |\rho| \). This has already been pointed out in Section 5 of Frei and Schweizer [4] by saying that \( V_t^H \) is monotonic in \( |\rho| \) only when the measure \( \hat{P} \) from (4.2), which depends via \( W \) on \( \rho \), is kept fixed. Proposition 3 of Frei and Schweizer [4] gives a result analogous to parts 2) and 3) of Proposition 4.3 when \( |q| \) and \( |\rho| \) can be separated by a constant. Proposition 4.3 shows that this additional assumption is superfluous and generalises Proposition 3 of [4].

**Proof of Proposition 4.3.** Like in the proof of Lemma 4.1, (4.9) has a unique solution \((\Gamma^q, Z^q)\) and \( \int Z^q \, dY \in \text{BMO}(Y, P) \). Theorem 3.6 of Kazamaki [6] yields \( \int Z \, dY, \int Z^q \, dY \in \text{BMO}(Y, \hat{P}) \) for the solution \((\Gamma, Z)\) of (4.6), and as a consequence, their stochastic exponentials are true martingales.

To prove 1), we fix \( j \in \{1, \ldots, n\} \) and write (4.9), for \( \tau_{j-1} \leq s \leq \tau_j \) as

\[
\Gamma^q_s = \Gamma^q_{\tau_j} + \frac{1}{2\gamma} \int_{\tau_j}^{\tau_s} \lambda^2_r \, dr - \frac{1}{2\gamma} (1 - |q|^2) \int_{\tau_j}^{\tau_s} |Z^q_r|^2 \, dr + \int_{\tau_j}^{\tau_s} Z^q_r \, d\hat{Y}_r,
\]

which implies

\[
e^{-\gamma(1-|q|^2)\Gamma^q_{\tau_{j-1}} - \frac{1}{2\gamma} (1 - |q|^2) \int_{\tau_{j-1}}^{\tau_j} \lambda^2_r \, dr} \exp \left( \gamma (1 - |q|^2) \int_{\tau_{j-1}}^{\tau_j} Z^q_r \, d\hat{Y}_r - \frac{1}{2\gamma} (1 - |q|^2)^2 \int_{\tau_{j-1}}^{\tau_j} |Z^q_r|^2 \, dr \right).
\]

Taking \((\mathcal{Y}_{\tau_{j-1}}, \hat{P})\)-conditional expectations and logarithms yields

\[
\Gamma^q_{\tau_{j-1}} = \frac{-1}{\gamma(1 - |q|^2)} \log \hat{E}_P \left[ \exp \left( -\gamma \Gamma^q_{\tau_j} - \frac{1}{2\gamma} \int_{\tau_{j-1}}^{\tau_j} \lambda^2_r \, dr \right)^{1 - |q|^2} \mathcal{Y}_{\tau_{j-1}} \right].
\]

Using this argument iteratively for \( j = n, \ldots, 1 \) results in (4.10).

To prove 2), we subtract (4.6) from (4.9), which gives

\[
\Gamma^q_s - \Gamma_s = \frac{1}{2\gamma} \int_s^T \left( (1 - \rho^2_r) |Z_r|^2 - (1 - \rho^2_r) |Z^q_r|^2 \right) \, dr + \int_s^T (Z^q_r - Z_r) \, d\hat{Y}_r
\geq \frac{1}{2\gamma} \int_s^T (1 - \rho^2_r) \left( |Z_r|^2 - |Z^q_r|^2 \right) \, dr + \int_s^T (Z^q_r - Z_r) \, d\hat{Y}_r
= \int_s^T (Z^q_r - Z_r) \, (d\hat{Y}_r - \kappa_r \, dr), \quad 0 \leq s \leq T,
\]

(4.11)
with $\kappa := \frac{1}{2}\gamma(1 - \rho^2)(Z^q + Z)$. The $BMO(\mathbb{Y}, \hat{P})$-property of $\int Z \, d\hat{Y}$ and $\int Z^q \, d\hat{Y}$ implies that $\int \kappa \, d\hat{Y}$ is in $BMO(\mathbb{Y}, \hat{P})$, and by Theorem 3.6 of Kazamaki [6], the process $\int (Z^q - Z)(d\hat{Y} - \kappa \, dr)$ is thus also a $BMO(\mathbb{Y}, \hat{P})$-martingale for the probability measure $\hat{P}'$ given by $d\hat{P}' := E(\int \kappa \, d\hat{Y}) \, d\hat{P}$. Taking $(\mathbb{Y}_s, \hat{P}')$-conditional expectations in (4.11) yields $\Gamma^q_s \geq \Gamma_s$ for any $s \in [0, T]$, which gives $V^H = -\exp(-\gamma \Gamma) \leq -\exp(-\gamma \Gamma^q)$ by Lemma 4.1 and the continuity of $\Gamma$ and $\Gamma^q$. The proof of 3) goes analogously to 2).

### 4.2 Approximating the indifference value

As explained after Proposition 4.3, the question whether $V^H_t$ is the a.s. limit of an explicitly known sequence boils down to whether it is possible to find a sequence $(q^n)_{n \in \mathbb{N}}$ in $\Xi$ which converges pointwise to $\rho$. In Section 4.2.1, we show that this is possible in a general way for every deterministic $\rho$ except for "pathological examples". We then give in Section 4.2.2 such a counterexample where the approximation of $V^H_t$ indeed fails. Section 4.2.3 presents the approximation of $V^H_t$ for general (stochastic) $\rho$ with left-continuous paths.

#### 4.2.1 Deterministic correlation

The approximation of $\rho$ with piecewise constant processes is reminiscent of the construction of the Riemann integral. We recall that a bounded function $g : [t, T] \to \mathbb{R}$ is called Riemann integrable if there exists $J \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| J - \sum_{j=1}^n g(s_j)(t_j - t_{j-1}) \right| < \epsilon$$

for every partition $(t_0, \ldots, t_n)$ of $[t, T]$ with $\max_{1 \leq j \leq n}(t_j - t_{j-1}) < \delta$ and every choice of $s_j \in [t_{j-1}, t_j]$.

The following result, which is shown on page 29 of Lebesgue [9], is known as Lebesgue’s theorem.

**Lemma 4.4.** A bounded function $g : [t, T] \to \mathbb{R}$ is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous on $[t, T]$.

We now come to the convergence result for $V^H_t$ and its optimal strategy $\pi^*$ when $\rho$ is deterministic.

**Theorem 4.5.** Assume that $\rho$ is deterministic, Riemann integrable and valued in $[-1, 1]$, and recall $H = -\gamma H - \frac{1}{2} \int_T^T \lambda_s^2 \, ds$. Then for every sequence $(t^n_0, \ldots, t^n_{n_0})_{n \in \mathbb{N}}$ of partitions of $[t, T]$ with $\lim_{n \to \infty}(\max_{1 \leq j \leq n}(t^n_j - t^n_{j-1})) = 0$
and every choice of $s^j \in [t^n_{j-1}, t^n_j]$ (the dependence of $s^j$ on $n$ is omitted for notational reasons),

$$-E_P \left[ \cdots E_P \left[ E_P \left[ e^{H(1-\rho^2_s)}(\gamma_{t^n_{j-1}} - 1) \gamma_{t^n_{j-2}} \cdots \gamma_{t^n_1} - 1 \gamma_{t^n_s} \right] \gamma_{t^n_{j-2}} \cdots \gamma_{t^n_1} - 1 \gamma_{t^n_s} \right] \right] (4.12)$$

converges to $V^H_t$ a.s. Suppose $\sigma$ is uniformly bounded away from zero, and denote by $(\Gamma^n, Z^n)$ the solution of (4.9) with $q = q^n := \sum_{j=1}^n \rho_{n,j} 1_{[t^n_{j-1}, t^n_j]}$ for any $n \in \mathbb{N}$. Then $\lim_{n \to \infty} E_P \left[ \int_{t}^{T} |\frac{p_s}{\sigma_s} Z^n_s + \frac{\lambda_s}{\sigma_s} - \pi^*_s|^2 \, ds \right] = 0$, where $\pi^*$ is the optimiser for $V^H_t$. If $(\nu_n)_{n \in \mathbb{N}} = (t^n_0, \ldots, t^n_n)_{n \in \mathbb{N}}$ is a sequence of partitions of $[t, T]$ with $\nu_n \subseteq \nu_{n+1}$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} (\max_{1 \leq j \leq n} (t^n_j - t^n_{j-1})) = 0$, then

$$-E_P \left[ \cdots E_P \left[ E_P \left[ e^{H(1-\rho^2_s)}(\gamma_{t^n_{j-1}} - 1) \gamma_{t^n_{j-2}} \cdots \gamma_{t^n_1} - 1 \gamma_{t^n_s} \right] \right] \right] (4.12)$$

with $\rho_{n,j} := \inf_{s \in [t^n_{j-1}, t^n_j]} |\rho_s|$ (or $\rho_{n,j} := \sup_{s \in [t^n_{j-1}, t^n_j]} |\rho_s|$) is a nondecreasing (or nonincreasing) sequence which converges to $V^H_t$ a.s.

Proof. Fix $n \in \mathbb{N}$ and let $(\Gamma^n, Z^n)$ be the solution of the BSDE (4.9) with $q = q^n := \sum_{j=1}^n \rho_{n,j} 1_{[t^n_{j-1}, t^n_j]}$. By 1) of Proposition 4.3, $-\exp(-\gamma \Gamma^n_t)$ equals (4.12), and we show that this converges to $V^H_t$ a.s. Because $\rho$ is Riemann integrable, Lemma 4.4 yields $\lim_{n \to \infty} |q^n_s| = |\rho_s|$ for a.a. $s \in [t, T]$. From Lemma 4.1 and Theorem 2.1 follows that $-\exp(-\gamma \Gamma^n_t)$ converges to $V^H_t$ a.s. and also the convergence result for the optimiser $\pi^*$ for $V^H_t$ is implied.

The last part of Theorem 4.5 follows analogously, with $q^n$ replaced by $\sum_{j=1}^n \rho_{n,j} 1_{[t^n_{j-1}, t^n_j]}$, using additionally parts 2) and 3) of Proposition 4.3. $\square$

Let us mention two straightforward generalisations of Theorem 4.5. The convergence still works if $\rho$ itself is not Riemann integrable, but $\rho$ equals Lebesgue-almost everywhere a Riemann integrable function $\hat{\rho}$. One simply replaces $\rho$ by $\hat{\rho}$ in Theorem 4.5, and uses $V^H_t = -\exp(-\gamma \Gamma^\hat{\rho}_t) = -\exp(-\gamma \Gamma^\hat{\rho}_t)$ a.s. for the solutions $(\Gamma^\rho, Z^\rho)$ and $(\Gamma^\hat{\rho}, Z^\hat{\rho})$ of the BSDE (4.9) with $q = \rho$ and $q = \hat{\rho}$, respectively. An example for such a pair of $\rho$ and $\hat{\rho}$ is $\rho = \frac{1}{2} I_{[t,T]}$ and $\hat{\rho} = 0$.

In the first part of Theorem 4.5, one can easily get rid of the restriction that $\rho$ is valued in $[-1, 1]$. To this end, one replaces $|\rho_s|$ by $|\rho_s| \wedge (1 - 1/n)$ in (4.12), and uses for the proof that $\sum_{j=1}^n |\rho_{n,j}| \wedge (1 - 1/n) 1_{[t^n_{j-1}, t^n_j]}$ converges pointwise to $|\rho|$ since the correlation $\rho$ is valued in $[-1, 1]$. The same procedure works for the last part of Theorem 4.5, but the sequence
of iterated expectations with \( \rho_{n,j} := \sup_{s \in [t^1_{j-1}, t^1_j]} |\rho_s| \wedge (1 - 1/n) \) instead of \( \rho_{n,j} := \sup_{s \in [t^1_{j-1}, t^1_j]} |\rho_s| \) is no longer nonincreasing.

Further comments on Theorem 4.5 are given in the next remark.

**Remark 4.6.** 1) One can show that the a.s. convergence of (4.12) to \( V^H_t \) holds uniformly with respect to the partitions. In more detail, we denote by \( a_t(\Delta^n, s^n) \) the random variable given by the iterated conditional expectation in (4.12), where the pair \( (\Delta^n, s^n) = ((t^0_n, \ldots, t^n_n), (s^1, \ldots, s^n)) \) is called a tagged partition of \( [t, T] \) with mesh \( |\Delta^n| \). The first part of Theorem 4.5 yields \( \lim_{n \to \infty} a_t(\Delta^n, s^n) = V^H_t \) a.s. In the Appendix, we sketch the proof of the more general result

\[
\lim_{n \to \infty} \text{ess sup}_{(\Delta, s); |\Delta| < \epsilon} \left| a_t(\Delta, s) - V^H_t \right| = 0 \quad \text{a.s.,} 
\]

where the essential supremum is taken over all tagged partitions \( (\Delta, s) \) of \( [t, T] \) with mesh \( |\Delta| < \epsilon \).

2) For \( q \) valued in \([-1, 1] \), the generator of (4.9) is concave in \( Z^q_s \), and we can apply to \((-\Gamma^q, Z^q)\) the convergence result of Briand and Hu [2]. Recall \((\Gamma^q, Z^q)\) from the proof of Theorem 4.5, and let \((\Gamma, Z)\) be the solution of (4.6). Proposition 7 of [2] implies that, for every \( p \geq 1 \),

\[
\lim_{n \to \infty} E_P \left[ \sup_{s \in [t, T]} \left| \Gamma^q_s - \Gamma_s \right|^p + \left( \int_t^T \left| Z^q_s - Z_s \right|^2 ds \right)^{p/2} \right] = 0.
\]

If \( \sigma \) is uniformly bounded away from zero, this yields similarly to Theorem 4.5 that, for every \( p \geq 1 \), \( \lim_{n \to \infty} E_P \left[ \left( \int_t^T \left| \frac{\pi_s Z^q_s + \lambda_s}{\gamma_s} - \pi_s^* \right|^2 ds \right)^{p/2} \right] = 0 \). \( \Diamond \)

Under a slightly more restrictive assumption on \( \rho \), we can prove a stronger convergence result than Theorem 4.5.

**Proposition 4.7.** Assume that \( \rho \) is deterministic and the one-sided limits \( \lim_{t \nearrow s} \rho_t \) for all \( s \in [t, T] \) and \( \lim_{t \searrow s} \rho_t \) for all \( s \in [t, T] \) exist. Then there exist partitions \((t^n_0, \ldots, t^n_n)_{n \in \mathbb{N}}\) of \([t, T]\) with \( \lim_{n \to \infty} (\max_{1 \leq j \leq \ell_n} (t^n_j - t^n_{j-1})) = 0 \) such that for every choice of \( s^j \in [t^n_{j-1}, t^n_j] \), \((\Gamma^q, Z^q)\) given for \( r \in [t^n_{j-1}, t^n_j] \) by

\[
b_r^q := -E_P \left[ \cdots E_P \left[ E_P \left[ e^{B(1-\rho^2_{s^n})} Y^n_{t^n_{\ell_n-1}} \right] \left| Y^n_{t^n_{\ell_n-2}} \right| \cdots \left| Y^n_{t^n_1} \right| \left| \frac{1}{1-\rho^2_{s^n}} \right| \right] \right],
\]

satisfies \( \lim_{n \to \infty} \left\| \sup_{s \in [t, T]} \left| b^n_s - V^H_s \right| \right\|_{L^\infty(P)} = 0 \). For \( \sigma \) uniformly bounded away from zero, we have \( \lim_{n \to \infty} \sup_{\tau} \left\| E_P \left[ \int_{\tau}^T \frac{\pi_s Z^q_s + \lambda_s}{\gamma_s} - \pi_s^* \right|^2 ds | \mathcal{G}_\tau \right\|_{L^\infty(P)} = 0 \), where the supremum is taken over all \( \mathcal{G}\text{-stopping times} \tau \) valued in \([t, T]\), and \((\Gamma^q, Z^q)\) is the solution of (4.9) with \( q = q^n := \sum_{j=1}^{\ell_n} \rho_{s^n_j} \mathbf{1}_{[t^n_{j-1}, t^n_j]} \) for \( n \in \mathbb{N} \).
Proof. Fix \( n \in \mathbb{N} \) and define by

\[
t^n_0 := t, \quad t^n_j := \inf \left\{ s > t^n_{j-1} : \left| \rho_s - \lim_{\tau \downarrow t^n_{j-1}} \rho_\tau \right| > 1/n \right\} \wedge T, \quad j \in \mathbb{N},
\]
a partition of \( [t, T] \), noting that there is \( \ell_n \in \mathbb{N} \) such that \( t^n_{\ell_n} = T \) by a compactness argument. For every \( s^j \in [t^n_{j-1}, t^n_j] \), \( q^n = \sum_{j=1}^{\ell_n} \rho_{s^j} 1_{[t^n_{j-1}, t^n_j]} \) converges to \( \rho \) in \( L^\infty(\text{Leb}, [t, T]) \) as \( n \to \infty \). Except for the point mentioned next, the remainder of the proof goes like in Theorem 4.5, using Corollary 2.2 instead of Theorem 2.1 and additionally the idea of (3.8). Corollary 2.2 only yields

\[
\limsup_{n \to \infty} \sup_{\tau} \left\| E_P \left[ \int_\tau^T \left| \frac{\partial}{\partial s} Z_s^\rho + \frac{\partial \lambda}{\partial \gamma} - \pi_\gamma^\star \right|^2 ds \right] \right\|_{L^\infty(P)} = 0,
\]
where the supremum is taken over all \( \mathcal{Y} \)-stopping times \( \tau \). But the proof of Corollary 2.2 shows that we also have

\[
\limsup_{n \to \infty} \sup_{\tau} \left\| E_P \left[ \int_\tau^T \left| \frac{\partial}{\partial s} Z_s^\rho + \frac{\partial \lambda}{\partial \gamma} - \pi_\gamma^\star \right|^2 ds \right] \right\|_{L^\infty(P)} = 0
\]
in this situation, where the supremum is taken over all \( \mathcal{G} \)-stopping times \( \tau \).

4.2.2 A counterexample

We have seen in Theorem 4.5 that \( V_t^H \) is the a.s. limit of an explicitly known sequence if \( \rho \) equals almost everywhere a Riemann integrable function. In particular, the choice of a nondecreasing sequence of partitions in Theorem 4.5 allows us to approximate \( V_t^H \) from above and below. We give here an example of a correlation process which is not almost everywhere equal to a Riemann integrable function and where indeed the approximations of \( V_t^H \) in the sense of Theorem 4.5 from above and below are not possible.

We take for simplicity \( t = 0, T = 1, \gamma = 1, \mu \equiv 0 \) and \( \sigma \equiv 1 \). Let \( C \subseteq [0,1] \) be the “fat” Cantor set with Lebesgue measure 1/2. This set, which is also known as Smith-Volterra-Cantor set, is constructed iteratively as follows: Start by removing \( [3/8, 5/8] \) from the interval \([0,1]\); in the \( n \)-th step, remove subintervals of width \( 1/2^{n} \) from the middle of each of the \( 2^{n-1} \) intervals. If we continue like this, \( C \) consists of all points in \([0,1]\) that are never removed. Because \( C \) is the complement of a countable union of open intervals, it is Borel measurable. Moreover, it is well known that \( C \) is nowhere dense, yet has Lebesgue measure 1/2. We assume that the correlation \( \rho \) is given by \( \rho = \frac{1}{2} 1_{C \cap [0,1/2]} + \frac{1}{2} 1_{C \cap [1/2,1]} \) and \( H := Y_1 \). Since \( Y_1 \) is not bounded, we have to adjust slightly the definition of admissible strategies: Instead of (4.4), we impose on \( \pi \in \mathcal{A}_0 \) that \( \left( \exp(-X_0^0, \pi - Y_1) \right)_{0 \leq s \leq T} \) is of class (D).

(Alternatively, one could approximate \( Y_1 \) by bounded random variables like in the example in Section 5 of Frei and Schweizer [4].) We claim that

\[
\sup_{q \in \Xi, |q| \leq \rho} \Gamma^q_0 \leq -15/32 < -\log(-V_0^H) = -7/16 < -13/32 \leq \inf_{q \in \Xi, |q| \geq \rho} \Gamma^q_0,
\]

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where $\Gamma^\varrho$ is the solution of (4.9). This means that $\rho$ cannot be approximated by piecewise constant processes from above and below such that the corresponding values converge to $V_0^H$. We first show $V_0^H = -\exp(7/16)$. For any $\pi \in \mathcal{A}_0$ with bounded $\int_0^T \pi_s^2 \, ds$, we have

$$E_P[U(X_1^0, \pi + H)] = -E_P\left[\exp\left(-\int_0^1 \pi_s \, dW_s - Y_1\right)\right]$$

$$= -E_P\left[\exp\left(-\int_0^1 \pi_s \, dW_s - Y_1 - \frac{1}{2}\left<\int \pi \, dW + Y\right>_t\right) + \frac{1}{2}\int_0^1 ((\pi_s + \rho_s)^2 + 1 - \rho_s^2) \, ds\right]$$

$$\leq -E_P\left[\exp\left(-\int_0^1 \pi_s \, dW_s - Y_1 - \frac{1}{2}\left<\int \pi \, dW + Y\right>_t\right)\right]$$

$$\times \exp\left(\frac{1}{2}\int_0^1 (1 - \rho_s^2) \, ds\right)$$

$$= -\exp\left(\frac{1}{2}\int_0^1 (1 - \rho_s^2) \, ds\right) = -\exp(7/16) \quad (4.14)$$

since $\text{Leb}(C \cap [0, 1/2]) = \text{Leb}(C^c \cap [1/2, 1]) = 1/4$. Equality in (4.14) holds for $\pi = -\rho \in \mathcal{A}_0$. Because of the class (D) condition on $(\exp(-X_s^{0,\pi} - Y_1))_{0 \leq s \leq T}$ for any $\pi \in \mathcal{A}_0$, we obtain $V_0^H = -\exp(7/16)$ by a localisation argument. To prove $\sup_{q \in \Xi, |q| \leq \rho} \Gamma^q_0 \geq -15/32$, we note that $q \in \Xi, |q| \leq \rho$ implies $q \equiv 0$ on $[0, 1/2]$ since $C$ does not contain any nontrivial intervals. By 3) of Proposition 4.3 with $\rho$ replaced by $\check{\rho} := \rho \mathbb{1}_{[1/2, 1]} = \frac{1}{2} \mathbb{1}_{C \cap [1/2, 1]}$, we have

$$\sup_{q \in \Xi, |q| \leq \rho} \Gamma^q_0 \leq \Gamma^\check{\rho}_0,$$

and a calculation similar to (4.14) shows $\check{\Gamma}^\check{\rho}_0 = -15/32$, using that by Lemma 4.1, $-\exp(-\check{\Gamma}^\check{\rho}_0)$ equals $V_0^H$ with $\rho$ replaced by $\check{\rho}$. Similarly, we obtain

$$\inf_{q \in \Xi, |q| \geq \rho} \Gamma^q_0 \geq \check{\Gamma}^\check{\rho}_0 = -13/32,$$

where $\check{\rho} := \rho \mathbb{1}_{[0, 1/2]} + \frac{1}{2} \mathbb{1}_{[1/2, 1]} = \frac{1}{2} \mathbb{1}_{C \cap [0, 1/2]} + \frac{1}{2} \mathbb{1}_{[1/2, 1]}$.

### 4.2.3 Stochastic correlation

When $\rho$ is stochastic, we cannot approximate $V_t^H$ from above and below like in Theorem 4.5. However, we still have a convergence result for $V_t^H$ if $\rho$ is left-continuous.
Theorem 4.8. Assume that $\rho$ is on $[t, T]$ left-continuous and valued in $]-1, 1[$. Then for every sequence $(t = \tau_{n}^{0} \leq \cdots \leq \tau_{n}^{N} = T)_{n \in \mathbb{N}}$ of $[t,T]$-valued $\mathbb{Y}$-stopping times with $\lim_{n \to \infty} \left( \max_{1 \leq j \leq n} (\tau_{j}^{n} - \tau_{j-1}^{n}) \right) = 0$ a.s.,

$$-E_{\hat{\rho}} \left[ \cdots E_{\hat{\rho}} \left[ E_{\hat{\rho}} \left[ e^{H(1-\rho_{n}^{2})^{2} \tau_{n-1}^{2}}} \left| \mathcal{Y}_{\tau_{n-1}^{n}} \right|^{1-\rho_{n}^{2}} \mathcal{Y}_{\tau_{n-2}^{n}} \cdots \right] \mathcal{Y}_{t} \right] \right]^{\frac{1}{1-\rho_{t}^{2}}} \tag{4.15}$$

converges to $V_{t}^{H}$ a.s. Suppose $\sigma$ is uniformly bounded away from zero, and denote by $(\Gamma^{\sigma}, Z^{\sigma})$ the solution of (4.9) with $q = q^{n} := \sum_{j=1}^{n} \rho_{\tau_{j-1}^{n}} \mathbb{1}_{[\tau_{j-1}^{n}, \tau_{j}^{n}]}$ for any $n \in \mathbb{N}$. Then $\lim_{n \to \infty} E_{P} \left[ \int_{t}^{T} \left| \frac{\partial}{\partial s} \mathcal{Y}_{s}^{\sigma} + \frac{\lambda}{\sigma} - \pi_{s}^{*} \right|^{2} ds \right] = 0$, where $\pi^{*}$ is the optimiser for $V_{t}^{H}$.

Proof. Fix $n \in \mathbb{N}$ and let $(\Gamma^{\sigma}, Z^{\sigma})$ be the solution of the BSDE (4.9) with $q = q^{n} := \sum_{j=1}^{n} \rho_{\tau_{j-1}^{n}} \mathbb{1}_{[\tau_{j-1}^{n}, \tau_{j}^{n}]}$, which is $\mathbb{Y}$-predictable. By 1) of Proposition 4.3, $-\exp\left(-\gamma \Gamma^{\sigma}_{t} \right)$ equals (4.15). We have $\lim_{n \to \infty} q^{n} (\omega) = \rho_{s}(\omega)$ for a.a. $(\omega, s) \in [t, T]$ by the left-continuity of $\rho$, and the result follows from Lemma 4.1 and Theorem 2.1.

In the same way as Theorem 4.5, one can slightly generalise Theorem 4.8 to the case where $\rho$ equals $(P \otimes \text{Leb})$-a.e. a $(P \otimes \text{Leb})$-a.e. left-continuous process, and one can get rid of the assumption that $\rho$ is valued in $]-1, 1[$.

Remark 4.9. The assumption from Section 4.1 that $\int_{0}^{T} \lambda_{s}^{2} ds$ is bounded can be slightly weakened. Theorem 4.8 still holds if $\int \lambda dW \in BMO(\mathbb{G}, P)$ and

$$\sup_{s \in [0, T]} \left\| E_{P} \left[ \exp \left( \int_{s}^{T} (1 + \rho^{2}) \lambda_{r}^{2} dr \right) \right] \mathcal{G}_{s} \right\|_{L^{\infty}} < \infty. \tag{4.16}$$

By the John-Nirenberg inequality (see Theorem 2.2 of Kazamaki [6]), (4.16) is satisfied if, for example, the $BMO_{2}(\mathbb{G}, P)$-norm of $\int \lambda dW$ is less than $1/\sqrt{2}$. In the Appendix, we sketch the proof of this slight generalisation of Theorem 4.8.

4.3 Continuity of the value process in the correlation

This short section exploits the convergence Theorem 2.1 to show a continuity property of $V^{H}$ in $\rho$.

Let us introduce more precise notations by writing (4.1) as

$$dW_{s}(\hat{\rho}) = \hat{\rho}_{s} dY_{s} + \sqrt{1 - \hat{\rho}_{s}^{2}} dY_{s}^{L}, \quad 0 \leq s \leq T$$
for a $\mathcal{G}$-predictable process $\tilde{\rho}$ denoting the instantaneous correlation between the $(\mathcal{G}, P)$-Brownian motions $W(\tilde{\rho})$ and $Y$ so that $W = W(\tilde{\rho})$. We replace in all definitions $W$ by $W(\tilde{\rho})$, $V^H(\tilde{\rho})$, etc. $V^H(\tilde{\rho})$ is then the dynamic value process for a stochastic control problem when the correlation between the underlying Brownian motions $W(\tilde{\rho})$ and $Y$ is $\tilde{\rho}$. Note that if we change $\tilde{\rho}$, only $W(\tilde{\rho})$ and all expressions depending on it will change. This is reasonable; clearly $H$ and $Y$ should not be affected.

**Proposition 4.10.** Let $(\rho^n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{Y}$-predictable $[-1, 1]$-valued processes which converge pointwise to $\rho$ on $[t, T]$. Then $V^H_t(\rho^n)$ converges to $V^H_t(\rho)$ $P$-a.s. as $n \to \infty$. Moreover, $\sup_{s \in [t, T]} |V^H_s(\rho^n) - V^H_s| \to 0$ as $n \to \infty$ in $P$-probability and in $L^p(P)$, $1 \leq p < \infty$.

**Proof.** This follows from Lemma 4.1 and Theorem 2.1, using additionally the same argument as in (3.8) to show the second statement. □

Proposition 4.10 can be generalised to a multidimensional setting where $W$ and $Y$ are stochastically correlated multidimensional Brownian motions. But we give no details since this provides no essential new insights.

### A Appendix: Proofs of the convergence results

**Proof of Theorem 2.1.** By Theorems 2.5 and 2.6 of Morlais [11], there exist unique solutions $(\Gamma^n, Z^n, N^n)$ to (2.1) with parameters $(f^n, \beta^n, H^n)$ for $n = 1, \ldots, \infty$. Moreover, Lemma 3.1 of Morlais [11] implies that $\Gamma^n$ and the $\text{BMO}(P)$-norms of $\int Z^n dM$ and $N^n$ are bounded uniformly in $n = 1, \ldots, \infty$. (Theorems 2.5, 2.6 and Lemma 3.1 of [11] do not use the assumption in Section 2.1 of [11] that a.s., the matrix $m_s m'_s$ is invertible for every $s \in [0, T]$.)

We now subtract (2.1) with parameters $(\beta^n, H^n, f^n)$ from that with parameters $(\beta, H^n, f^n)$ for a fixed $n \in \mathbb{N}$ to obtain, for $0 \leq s \leq T$,

$$\Gamma^n_s - \Gamma^\infty_s = H^n - H^\infty + \int_s^T (f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r)) \, dD_r - \int_s^T (Z^n_r - Z^\infty_r) \, dM_r$$

$$+ \frac{\beta^n}{2} (\langle N^n \rangle_T - \langle N^n \rangle_s) - \frac{\beta^\infty}{2} (\langle N^\infty \rangle_T - \langle N^\infty \rangle_s) - \int_s^T d(N^n - N^\infty)_r.$$
which implies

\[
\Gamma^n_s - \Gamma^\infty_s = \frac{\beta^n - \beta^\infty}{2} (\langle N^\infty \rangle_T - \langle N^\infty \rangle_s) - \int_s^T (Z^n_r - Z^\infty_r) \left( dM_r - d\langle M \rangle_r g^n_r \right) \\
- \int_s^T (d(N^n - N^\infty)_r - \frac{\beta^n}{2} d\langle N^n - N^\infty, N^n + N^\infty \rangle_r) \\
+ H^n - H^\infty + \int_s^T \left( f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r) \right) \, dD_r, \\
\text{(A.2)}
\]

where \( g^n \) is defined for \( 0 \leq s \leq T \) by

\[
g^n_s := \begin{cases} 
    \frac{f^n(s, Z^n_s) - f^n(s, Z^\infty_s)}{|m(Z^n_s - Z^\infty_s)|^2} (Z^n_s - Z^\infty_s) & \text{if } |m(Z^n_s - Z^\infty_s)| \neq 0, \\
    0 & \text{otherwise.}
\end{cases}
\]

Due to the assumption (ii) of the theorem, \( \int g^n \, dM \) is in \( BMO(P) \) and its \( BMO(P) \)-norm is uniformly bounded since the \( BMO(P) \)-norm of \( \int Z^n \, dM \) is bounded uniformly in \( n = 1, \ldots, \infty \). Therefore, taking conditional expectations in (A.2) under the probability measure \( Q^n \) given by

\[
\frac{dQ^n}{dP} := \mathcal{E} \left( \int g^n \, dM + \frac{\beta^n}{2} (N^n + N^\infty) \right)_T
\]

yields

\[
\Gamma^n_s - \Gamma^\infty_s = \frac{\beta^n - \beta^\infty}{2} E_{Q^n} \left[ (\langle N^\infty \rangle_T - \langle N^\infty \rangle_s) | \mathcal{F}_s \right] + E_{Q^n}[H^n - H^\infty | \mathcal{F}_s] \\
+ E_{Q^n} \left[ \int_s^T \left( f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r) \right) \, dD_r \bigg| \mathcal{F}_s \right]. \\
\text{(A.3)}
\]

Because the convergent sequence \( (\beta^n)_{n=1,\ldots,\infty} \) is bounded, the \( BMO(P) \)-norm of \( M^n := \int g^n \, dM + \frac{\beta^n}{2} (N^n + N^\infty) \), which equals the stochastic logarithm of the \( P \)-density process of \( Q^n \), is uniformly bounded in \( n = 1, \ldots, \infty \). Therefore, Theorem 3.6 of Kazamaki [6] and continuity in \( s \) of \( E_{Q^n}[\langle N^\infty \rangle_T | \mathcal{F}_s] \) and \( \langle N^\infty \rangle_s \) imply

\[
\left\| \sup_{n=1,\ldots,\infty} \sup_{s \in [0,T]} E_{Q^n}[\langle N^\infty \rangle_T - \langle N^\infty \rangle_s | \mathcal{F}_s] \right\|_{L^\infty(P)} < \infty, \\
\text{(A.4)}
\]

and hence

\[
\lim_{n \to \infty} \sup_{s \in [0,T]} \left| \frac{\beta^n - \beta^\infty}{2} E_{Q^n}[\langle N^\infty \rangle_T - \langle N^\infty \rangle_s | \mathcal{F}_s] \right| = 0 \text{ in } L^\infty(P). \\
\text{(A.5)}
\]
Since the $BMO(P)$-norm of $\tilde{M}_n$ is uniformly bounded in $n$, there exist by Theorem 3.1 of Kazamaki [6] $p > 1$ and a constant $C_p$, both independent of $n$, such that for all $n = 1, \ldots, \infty$

$$E\mathbb{Q}_n \left[ \left( \frac{\mathcal{E}(\tilde{M}_n)_T}{\mathcal{E}(M)_s} \right)^{1/(p-1)} \right| \mathcal{F}_s \right] = E_p \left[ \left( \frac{\mathcal{E}(\tilde{M}_n)_T}{\mathcal{E}(M)_s} \right)^{p/(p-1)} \right| \mathcal{F}_s \right] \leq C_p. \quad (A.6)$$

Recall the constant $c_1$ from the assumption (i) of the theorem and set

$$\alpha := \frac{1}{2pc_1\|Z\|_{BMO}(P) + 1}. \quad (A.7)$$

Applying $\alpha x \leq e^{\alpha x} - 1$ for $x \in \mathbb{R}$, the Hölder inequality (with exponents $p$ and $p/(p-1)$) and (A.6) yields

$$E\mathbb{Q}_n \left[ \int_s^T \left| f^n(r, Z_r) - f^\infty(r, Z_r) \right| dD_r \right| \mathcal{F}_s \right] \leq \frac{1}{\alpha} \left( e^{\alpha \int_s^T \left| f^n(r, Z_r) - f^\infty(r, Z_r) \right| dD_r} - 1 \right)^{1/p} \left| \mathcal{F}_s \right|, \quad (A.8)$$

By the assumption (i), we have

$$\left( e^{\alpha \int_s^T \left| f^n(r, Z_r) - f^\infty(r, Z_r) \right| dD_r} - 1 \right)^p \leq \exp \left( p\alpha \int_s^T \left| f^n(r, Z_r) - f^\infty(r, Z_r) \right| dD_r \right) \leq \exp \left( 2p\alpha \left\| \int_0^T \kappa_r dD_r \right\|_{L^\infty(P)} \right) \exp \left( 2pc_1 \alpha \int_0^T |m_r Z_r| \|dD_r\| \right).$$

Using the definition (A.7) of $\alpha$, the last expression is $P$-integrable by the John-Nirenberg inequality; see Theorem 2.2 of Kazamaki [6]. Therefore, dominated convergence and (A.8) imply that for $s \in [t, T]$,

$$\lim_{n \to \infty} E\mathbb{Q}_n \left[ \int_s^T \left| f^n(r, Z_r) - f^\infty(r, Z_r) \right| dD_r \right| \mathcal{F}_s \right] = 0 \quad P\text{-a.s.} \quad (A.9)$$

Similarly to (A.8), we obtain

$$E\mathbb{Q}_n \left[ \left| H^n - H^\infty \right| \right| \mathcal{F}_s \right] \leq \left| C_p \right|^{(p-1)/p} E_p \left[ \left( e^{\left| H^n - H^\infty \right|} - 1 \right)^{1/p} \right| \mathcal{F}_s \right], \quad (A.10)$$
which again converges $P$-a.s. to zero by dominated convergence. Therefore, (A.3), (A.5) and (A.9) give \( \lim_{n \to \infty} |\Gamma^n_s - \Gamma^\infty_s| = 0 \) $P$-a.s. for every $s \in [t, T]$.

We now prove $\sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s|$ converges in $P$-probability to zero. From (A.8) and the martingale maximum inequality we obtain that, for any $\epsilon > 0$,

\[
e^p P \left[ \sup_{s \in [t, T]} E_{Q^n} \left[ \int_s^T \left| f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r) \right| \, dD_r \bigg| \mathcal{F}_s \right] \right] \geq \frac{1}{\alpha} |C_p|^{(p-1)/p} \epsilon \]

\[
\leq e^p P \left[ \sup_{s \in [t, T]} E_P \left[ \left( e^{\alpha f^n_t} |f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r)| dD_r - 1 \right)^p \right]^{1/p} \geq \epsilon \right] \]

\[
\leq e^p P \left[ \sup_{s \in [t, T]} E_P \left[ \left( e^{\alpha f^n_t} |f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r)| dD_r - 1 \right)^p \right] \right] \geq e^p \]

\[
\leq E_P \left[ \left( e^{\alpha f^n_t} |f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r)| dD_r - 1 \right)^p \right] \right], \quad \text{(A.11)}
\]

which converges to zero as $n \to \infty$ by dominated convergence. Analogously, we obtain from (A.10) that

\[
e^p P \left[ \sup_{s \in [t, T]} E_{Q^n} \left[ |H^n - H^\infty| \bigg| \mathcal{F}_s \right] \right] \geq |C_p|^{(p-1)/p} \epsilon \leq E_P \left[ \left( e^{|H^n - H^\infty|} - 1 \right)^p \right], \quad \text{(A.12)}
\]

which also converges to zero as $n \to \infty$ by dominated convergence. All in all, (A.3), (A.5), (A.11) and (A.12) show that $\sup_{s \in [t, T]} |\Gamma^n_s - \Gamma^\infty_s|$ converges in $P$-probability to zero, and also convergence in $L^p(P)$, $1 \leq p < \infty$, follows since $\Gamma^n$ is bounded uniformly in $n$.

To prove the convergence statements for $Z^n$ and $N^n$, we apply Itô’s formula between $t$ and $T$, and use (A.1) to obtain

\[
(H^n - H^\infty)^2 - (\Gamma^n_t - \Gamma^\infty_t)^2 \\
= \int_t^T 2(\Gamma^n_t - \Gamma^\infty_t)(Z^n_s - Z^\infty_s) \, dM_s + \int_t^T 2(\Gamma^n_s - \Gamma^\infty_s) \, d(N^n - N^\infty)_s \\
- \int_t^T (\Gamma^n_s - \Gamma^\infty_s)(\beta^n \, d\langle N^n \rangle_s - \beta^\infty \, d\langle N^\infty \rangle_s) \\
- \int_t^T 2(\Gamma^n_s - \Gamma^\infty_s)(f^n(s, Z^n_s) - f^\infty(s, Z^\infty_s)) \, dD_s \\
+ \int_t^T (|m_s(Z^n_s - Z^\infty_s)|^2 \, dD_s + d\langle N^n - N^\infty \rangle_s),
\]

which implies by taking expectations that
Because \((H^n)_{n=1,\ldots,\infty}\) and \((\Gamma^n_t)_{n=1,\ldots,\infty}\) are a.s.-convergent bounded sequences, we have \(\lim_{n \to \infty} E_P[(H^n - H^\infty)^2] = 0\) and \(\lim_{n \to \infty} E_P[(\Gamma^n_t - \Gamma^\infty_t)^2] = 0\) by dominated convergence. Hölder’s inequality and the assumption (i) of the theorem imply

\[
E_P \left[ \sup_{s \in [t,T]} |\Gamma^n_s - \Gamma^\infty_s| \left( \int_t^T |f^n(s, Z^n_s) - f^\infty(s, Z^\infty_s)| \, dD_s \right)^2 \right] \leq E_P \left[ \left( \int_t^T \left( 2\kappa^1_s + c^1 |m_s Z^n_s|^2 + c^1 |m_s Z^\infty_s|^2 \right) \, dD_s \right)^2 \right].
\]

This converges to zero since \(\sup_{s \in [t,T]} |\Gamma^n_s - \Gamma^\infty_s| \to 0\) in \(L^2(P)\) and the second term is bounded uniformly in \(n\), which can be seen as follows: We have

\[
E_P \left[ \left( \int_t^T \left( 2\kappa^1_s + c^1 |m_s Z^n_s|^2 + c^1 |m_s Z^\infty_s|^2 \right) \, dD_s \right)^2 \right] \leq E_P \left[ 3 \left( \int_t^T 2\kappa^1_s \, dD_s \right)^2 + 3|c^1|^2 \langle \int Z^n \, dM \rangle_T^2 + 3|c^1|^2 \langle \int Z^\infty \, dM \rangle_T^2 \right],
\]

and this is bounded uniformly in \(n\) because \(\| \int_0^T \kappa^1_s \, dD_s \|_{L^\infty(P)} < \infty\), the \(BMO(P)\)-norms of \(\int Z^n \, dM\) are uniformly bounded and we can take \(j = 2\) in the energy inequalities

\[
E_P \left[ \int Z^n \, dM \right] \leq j! \left( \int Z^n \, dM \right)_{BMO_2(P)}^{2j} < \infty, \quad j \in \mathbb{N};
\]

see the corollary to Theorem 4 of Kikuchi [7]. Analogously, the term

\[
E_P \left[ \sup_{s \in [t,T]} |\Gamma^n_s - \Gamma^\infty_s| \left( |\beta^n|(|\langle N^n \rangle_T - \langle N^n \rangle_t|) + |\beta^\infty|(|\langle N^\infty \rangle_T - \langle N^\infty \rangle_t|) \right) \right]
\]

converges to zero. By (A.13), we obtain that

\[
\lim_{n \to \infty} E_P \left[ \int_t^T (|m_s(Z^n_s - Z^\infty_s)|^2 \, dD_s + d\langle N^n - N^\infty \rangle_s) \right] = 0,
\]

which concludes the proof. \(\square\)
Remark A.1. To prove (A.9), one can also apply directly the energy inequalities instead of using the John-Nirenberg inequality. In fact, taking $\ell \in \mathbb{N}$ with $\ell \geq p$, we obtain from (A.6) and the Hölder inequality that

$$
E_{Q^n} \left[ \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| \, dD_r \bigg| F_s \right] \\
\leq |C_p|^{(p-1)/p} \mathbb{E}_P \left[ \left( \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| \, dD_r \right)^\ell \bigg| F_s \right]^{1/\ell}.
$$

(A.15)

By the assumption (i), we have

$$
\left( \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| \, dD_r \right)^\ell \\
\leq \left( 2 \int_0^T \kappa_r^1 \, dD_r + 2c^1 \int_0^T |m_r Z_r^\infty|^2 \, dD_r \right)^\ell \\
= 2^\ell \sum_{j=0}^\ell \binom{\ell}{j} \left( \int_0^T \kappa_r^1 \, dD_r \right)^{\ell-j} |c^1|^j \left( \int Z_r^\infty \, dM \right)_T^j,
$$

which is $P$-integrable because of (A.14) and $\| \int_0^T \kappa_r^1 \, dD_r \|_{L^\infty(P)} < \infty$. Dominated convergence and (A.15) now imply (A.9).

Proof of Corollary 2.2. To show (2.3), it is by (A.3) and (A.4) enough to prove the existence of a constant $K > 0$ such that

$$
\sup_{s \in [t, T]} E_{Q^n} \left[ \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| \, dD_r \bigg| F_s \right] \\
\leq K(\| a^n - 1 \|_{L^\infty(P \otimes D)} + \| \overline{a}^n - 1 \|_{L^\infty(P \otimes D)}), \quad n \in \mathbb{N}.
$$

But the assumption (v) implies

$$
\sup_{s \in [t, T]} E_{Q^n} \left[ \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| \, dD_r \bigg| F_s \right] \\
\leq \| a^n - 1 \|_{L^\infty(P \otimes D)} \sup_{s \in [t, T]} E_{Q^n} \left[ \int_s^T |f(r, Z_r^\infty)| \, dD_r \bigg| F_s \right] \\
+ \| \overline{a}^n - 1 \|_{L^\infty(P \otimes D)} \sup_{s \in [t, T]} E_{Q^n} \left[ \int_s^T |\overline{f}(r, Z_r^\infty)| \, dD_r \bigg| F_s \right],
$$

and the conditional expectations are bounded in $L^\infty(P)$ uniformly in $n \in \mathbb{N}$ and $s \in [t, T]$ by an argument similar to (A.4). So (2.3) is established, and
similarity to (A.8) by dominated convergence. Now one deduces (4.13) from uniformly in $(\Delta, \vec{s})$ and Proposition 7 and Theorem 8 of Mania and Schweizer [10], one deduces from Lemma 4.4 that $\lim_{\epsilon \to 0} \sup_{s \in [t,T]} |\Gamma_s^n - \Gamma_s| \to 0$ in $L^\infty(P)$.

To show that $\int Z^n dM \to \int Z_\infty dM$, $N^n \to N_\infty$ on $[t, T]$ in $BMO(P)$, we derive similarly to (A.13), using that the right-hand side converges to zero by the assumptions (iii)–(v), we have $\sup_{s \in [t,T]} |\Gamma_s^n - \Gamma_s| \to 0$ in $L^\infty(P)$.

Sketch of the proof of (4.13) in Remark 4.6.1. To check (4.13), one first deduces from Proposition 3 of Briand and Hu [2] in the sense that this uniform convergence of $\Gamma_t^q$ in $(\Delta, \vec{s})$ implies that the corresponding solutions $\Gamma_t^q$ of (4.9) converge a.s. to $\Gamma_t^q = -\frac{1}{\gamma} \log(-V_t^H)$ uniformly in $(\Delta, \vec{s})$. In fact, one needs only to generalise (A.9), which goes similarly to (A.8) by dominated convergence. Now one deduces (4.13) from

$$lim_{\epsilon \to 0} \sup_{(\Delta, \vec{s}) : |\Delta| < \epsilon} \left| -\frac{1}{\gamma} \log(-a_t(\Delta, \vec{s})) + \frac{1}{\gamma} \log(-V_t^H) \right| = 0 \quad a.s.$$
has a unique solution \( (\Gamma^q, Z^q) \) where \( \Gamma^q \) is a real-valued bounded continuous \((\mathcal{Y}, P)\)-semimartingale and \( Z^q \) is a \( \mathcal{Y} \)-predictable process such that 

\[ E_P \left[ \int_0^T |Z^q_s|^2 \, ds \right] < \infty. \]

Furthermore, \( \int Z^q \, dY \) is in both \( \text{BMO}(\mathcal{Y}, P) \) and \( \text{BMO}(\mathcal{G}, P) \), and the \( \text{BMO} \)-norms are bounded uniformly with respect to the \([-1, 1]\)-valued \( q \). Now one can proceed like in Lemma 4.1 and Proposition 4.3 to obtain \( V^H = -\exp(-\gamma \Gamma^q) \) and (4.10). The argument is finished by applying Theorem 2.1, using that, under the assumption of uniform boundedness of the \( \text{BMO}(\mathcal{F}, P) \)-norms of \( \int Z^q \, dM \) and \( N^n \), the convergence result can also be shown if in the assumptions (i) and (ii), one only has

\[ \sup_{\tau} \| E_P \left[ \int_\tau^T \kappa_1^q \, dD_s | \mathcal{F}_\tau \right] \|_{L^\infty} < \infty \quad \text{and} \quad \sup_{\tau} \| E_P \left[ \int_\tau^T |\kappa_2^q|^2 \, dD_s | \mathcal{F}_\tau \right] \|_{L^\infty} < \infty \]

instead of

\[ \| \int_0^T \kappa_1^q \, dD_s \|_{L^\infty} < \infty \quad \text{and} \quad \| \int_0^T |\kappa_2^q|^2 \, dD_s \|_{L^\infty} < \infty, \]

where the suprema are taken over all \( \mathcal{F} \)-stopping times \( \tau \).

\[ \square \]

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**References**


