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Solitons in media with random dispersive perturbations

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Abstract

A statistical approach of the propagation of solitons in media with spatially random dispersive perturbations is developed. Applying the inverse scattering transform several regimes are put into evidence which are determined by the mass and the velocity of the incoming soliton and also by the correlation length of the perturbation. Namely, the mass of the soliton is almost conserved if it is initially large. If the initial mass is too small, then the mass decays with the length of the system. The decay rate is exponential in case of a white noise perturbation, but the mass will decrease as the inverse of the square root of the length if the central wave number of the soliton lies in the tail of the spectrum of the perturbation. ©1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the last decade a lot of papers have been devoted to the study of propagation of solitons through random media. Most of them concern a perturbation of the potential or the nonlinear coefficient [1–5]. Numerical modeling of the nonlinear Schrödinger (NLS) equation with random linear potential confirms this theory [6,7]. There are also experimental investigations of the propagation of nonlinear waves in random media [8,9], describing the Korteweg–deVries type wave propagation, but they have small relevance to the analytical work where the NLS case is addressed. Only recently random dispersive perturbations have attracted attention because of their importance for nonlinear optics and long Josephson junctions applications [10]. In Ref. [11] the authors find the probability distribution for the effective potential which is equivalent at the first order to the dispersive perturbation for the

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trapped soliton. As a first motivation we think of the continuum limit of a nonlinear Anderson model [1], but the same model can also be derived from the continuum limit corresponding to randomly dispersed impurities [12]. Our contribution consists in the study of the long scale behavior of these systems in the asymptotic framework where both the amplitudes of the perturbations tend to small values and the size of the system L tends to large values. The basic tool of this approach relies on a perturbed inverse scattering transform which has already been successfully applied to the study of different configurations and random perturbations [13]. Our main aim is to demonstrate that original and unexpected behaviors can be exhibited in the sense that they are not encountered when studying perturbations of the potential and the nonlinear coefficient. We shall show that random perturbations of the dispersion are sometimes equivalent to a perturbation of the potential or else to a perturbation of the nonlinear coefficient. The first case corresponds to an exponential decay of the mass of the transmitted soliton, and the second one to a logarithmic decay of the velocity. But we shall besides prove that a random perturbation of the dispersion may also give rise to an anomalous decay of the mass of the transmitted soliton as a power law $L^{-1/2}$.

2. Physical motivations

Spatial random dispersive perturbations naturally arise from many real physical situations. The first one concerns the electromagnetic beam propagation in planar waveguides with randomly varying tunnel-coupling terms. More precisely we shall investigate the evolution of a spatial soliton in an inhomogeneous system of tunnel-coupled nonlinear planar waveguides. The nonlinearity is assumed to be of Kerr type. The spacings between the waveguides of the array are assumed to depend on the transverse coordinate. The waveguides are weakly coupled and so the coupled modes theory is valid. Then the evolution of nth complex electromagnetic field amplitude u_n is described by the equation [14,15]

$$-\mathrm{i}\partial_z u_n = V_{n,n+1}u_{n+1} + V_{n,n-1}u_{n-1} + |u_n|^2 u_n. \tag{1}$$

Here $V_{n,n\pm1}$ are the linear tunnel-coupling coefficients. The constant coupling has first been considered in [16]. If the envelope occupies a sufficient number of waveguides, typically of the order of 10, then we can apply a continuum limit and further the effects of higher order terms in the Taylor expansion can be neglected:

$$\begin{aligned} V_{n,n\pm1} = V\left(x \pm \frac{h}{2}\right) &= V(x) \pm \frac{h}{2}V_x + \frac{h^2}{8}V_{xx} + \dots \\ \times u_{n,n\pm1} = u(x \pm h) &= u(x) \pm hu_x + \frac{h^2}{2}u_{xx} + \dots \end{aligned} \tag{2}$$

Taking into account these expressions, we obtain that for (1) the continuum limit is

$$-\mathrm{i}u_z = 2Vu + h^2\left(V_x u_x + \frac{1}{4}V_{xx}u + Vu_{xx}\right) + |u|^2 u + \mathrm{O}(h^4). \tag{3}$$

By introducing the new variables $x = x/\sqrt{2}h$, $v = u\exp(-4\mathrm{i}z)$, and $t = z/2$ we obtain [11]:

$$\mathrm{i}v_t + v_{xx} + 2|v|^2 v = (1-V)v_{xx} - V_x v_x - \tfrac{1}{4}V_{xx}v - 4(V-1)v. \tag{4}$$

The second example is the propagation of a fluxon in the long Josephson junction with a randomly distributed current. The governing equation is the randomly perturbed sine-Gordon equation [10]:

$$u_{tt} - (J(x)u_x)_x + \sin u = 0. \tag{5}$$

If we consider the case of a small amplitude breather, i.e. when $u = e^{it}\phi + \text{c.c.}$ we obtain

$$2i\phi_t - (J(x)\phi_x)_x - \tfrac{1}{2}|\phi|^2\phi = 0. \tag{6}$$

Considering that these two examples are relevant motivations, we address in this work the nonlinear Schrödinger equation with a random perturbation of the dispersion.

3. Perturbed Schrödinger equation

We consider a perturbed Schrödinger equation with a nonzero right-hand side:

$$iu_t + u_{xx} + 2|u|^2u = \varepsilon R(u)(t, x). \tag{7}$$

The small parameter $\varepsilon \in (0, 1)$ characterizes the amplitude of the perturbation. The model of the perturbation is taken to be

$$R(u)(t, x) = \alpha V(x)u_{xx}(t, x) + \beta V_x(x)u_x(t, x) + \gamma V_{xx}(x)u(t, x),$$

where α, β and γ are constant numbers. V is assumed to be a zero-mean, stationary and ergodic process with $\mathcal{C}^2$-paths. The jth derivative of V will be denoted either by $V_{x\cdots x}$ or by $V^{(j)}$. For $j = 0, 1, 2$, the Fourier transform of the autocorrelation function of the jth derivative of the process V is

$$d_j(k) = 2\int_0^\infty \mathbb{E}[V^{(j)}(0)V^{(j)}(x)]\cos(kx)\,dx, \tag{8}$$

which is nonnegative since it is proportional to the k-frequency evaluation of the power spectral density of the stationary process $V^{(j)}$ (Wiener–Khintchine theorem [17]). By integrating by parts, we get that

$$d_j(k) = k^{2j}d_0(k). \tag{9}$$

For instance, if the autocorrelation function of the process V has Gaussian shape with correlation length l_c:

$$\mathbb{E}[V(0)V(x)] = \sigma^2\exp\left(-\frac{x^2}{l_c^2}\right), \tag{10}$$

then $d_0(k) = \sqrt{\pi}\sigma^2 l_c \exp\left(-k^2l_c^2/4\right)$. Since V has $\mathcal{C}^2$-paths, the following quantity is finite, but random:

$$M^\varepsilon := \sup_{x\in[0,L/\varepsilon^2]} \left(|V(x)| + |V_x(x)| + |V_{xx}(x)|\right). \tag{11}$$

The following lemma shows that we can only consider a sub-class of the general problem (7) if we impose the very natural condition that the total mass is preserved by the perturbation. If the perturbation does not fulfill this condition, then the problem becomes unstable and the method we develop below cannot be applied.

Lemma 3.1. *If $\alpha = \beta$, then the total mass $N_{\text{tot}} = \int |u|^2 dx$ is preserved by the perturbed Schrödinger equation* (7).

Proof. Differentiating the mass, we get

$$\frac{dN_{\text{tot}}}{dt} = 2(\alpha - \beta)\text{Im}\left(\int V_x u_x^* u\,dx\right)$$

from which we can deduce the conservation of the total mass in the case $\alpha = \beta$. □

From now on we shall assume that $\alpha = \beta$, but we shall keep both notations in Sections 5–7 in order to track the influence of both terms. Eq. (7) can be derived from the Hamiltonian density $H(x) = H_0(x) + \varepsilon H_1(x)$:

$$H_0(x) = |u_x|^2 - |u|^4, \tag{12}$$

$$H_1(x) = -\alpha V|u_x|^2 + \gamma V_{xx}(x)|u|^2. \tag{13}$$

We can now state a priori estimates of the solution u of Eq. (7).

Lemma 3.2.

1. *The following quantities (mass and energy) are preserved by the perturbed Schrödinger equation* (7):
$$N_{\text{tot}} = \int |u|^2 \mathrm{d}x, \quad E_{\text{tot}} = \int H_0(x)\mathrm{d}x + \varepsilon \int H_1(x)\mathrm{d}x. \tag{14}$$
2. *If $\varepsilon M^\varepsilon \to 0$ as $\varepsilon \to 0$, then the H^1-norm, the L^4-norm and the L^∞-norm of $u(t,.)$ are uniformly bounded with respect to $t \in \mathbb{R}$ and $\varepsilon \in (0,1)$.*
3. *$\varepsilon \int H_1(x)\mathrm{d}x$ can be bounded uniformly with respect to $t \in \mathbb{R}$ by $K(N_{\text{tot}}, E_{\text{tot}})\varepsilon M^\varepsilon$.*

Proof. The conservation of the mass has been established in Lemma 3.1. The conservation of the energy is straightforward since it is equal to the Hamiltonian density integrated over the full space. Besides, from the energy conservation we have

$$\int (1 - \varepsilon\alpha V)|u_x|^2 \mathrm{d}x = E_{\text{tot}} + \int |u|^4 - \varepsilon\gamma V_{xx}|u|^2 \mathrm{d}x.$$

Consequently, considering ε small enough so that $|\alpha|\varepsilon M^\varepsilon < 1$:

$$\|u_x(t,.)\|_{L^2}^2 \leq \frac{|E_{\text{tot}}| + \|u(t,.)\|_{L^4}^4 + |\gamma|\varepsilon M^\varepsilon N_{\text{tot}}}{1 - |\alpha|\varepsilon M^\varepsilon}. \tag{15}$$

Further, substituting the Sobolev's inequality $\|u\|_{L^\infty}^2 \leq 2\|u\|_{L^2}\|u_x\|_{L^2}$ into the obvious estimate $\|u(t,.)\|_{L^4}^4 \leq \|u(t,.)\|_{L^2}^2 \|u(t,.)\|_{L^\infty}^2$, we can deduce from the mass conservation that, for any $\eta > 0$,

$$\|u(t,.)\|_{L^4}^4 \leq N_{\text{tot}} \left(\eta^{-1} N_{\text{tot}} + \eta \|u_x(t,.)\|_{L^2}^2\right). \tag{16}$$

Substituting (16) into (15) and choosing $\eta = (1/2)N_{\text{tot}}^{-1}(1 - |\alpha|\varepsilon M^\varepsilon)$, we find that the L^2-norm of the derivative u_x is uniformly bounded with respect to $\varepsilon \in (0,1)$ and $t \in \mathbb{R}$. Since the mass is conserved we get the estimate of the H^1-norm. The Sobolev's inequality then yields the estimate of the L^∞-norm and (16) provides the estimate of the L^4-norm. □

Throughout the paper we shall assume that, for any $\delta > 0$,

$$\mathbb{P}\left(\varepsilon M^\varepsilon \leq \delta\right) \underset{\varepsilon \to 0}{\to} 1, \tag{17}$$

so that we are allowed to apply the estimates of Lemma 3.2. This condition is readily fulfilled in realistic configurations, as shown by the following result due to Adler [18]:

Lemma 3.3. *If V is a stationary Gaussian process with a smooth autocorrelation function (say of class $\mathcal{C}^8$), then the expectation of the supremum M^ε of $|V(x)|$, $|V_x(x)|$, and $|V_{xx}(x)|$ over $x \in [0, L/\varepsilon^2]$ is of order $|\ln\varepsilon|^{1/2}$ as $\varepsilon \to 0$.*

Thus, in the framework of a perturbation with Gaussian statistics, the condition (17) is satisfied since $\mathbb{P}(\varepsilon M^\varepsilon \geq \delta) \leq \frac{\varepsilon}{\delta}\mathbb{E}[M^\varepsilon] \leq \frac{K\varepsilon|\ln\varepsilon|^{1/2}}{\delta} \to 0$ as $\varepsilon \to 0$.

4. The inverse scattering transform

We shall use the inverse scattering transform to study our problem. Indeed the random perturbation induces variations of the spectral data. Calculating these changes we are able to find the asymptotic evolution of the field and calculate the characteristic parameters of the wave. We shall be interested in the asymptotic dynamics of the soliton propagating over the long distance L/ε^2. The total mass and energy are conserved but the discrete and continuous components evolve during the propagation. The evolution of the continuous component corresponding to the radiation will be found from the evolution equations of the Jost coefficients. The evolutions of the coefficients of the soliton will then be derived from the conservation of the total mass and energy.

4.1. The incident soliton

We assume that a pure soliton is incident from the left and propagates through the slab $[0, L/\varepsilon^2]$:

$$u_0(t,x) = 2\nu_0 \frac{\exp[\mathrm{i}(2\mu_0(x-4\mu_0 t) + 4(\nu_0^2+\mu_0^2)t)]}{\cosh(2\nu_0(x-4\mu_0 t))}. \tag{18}$$

The mass and the velocity of the soliton are respectively $N_0 = 4\nu_0$ and $V_0 = 4\mu_0$, while its energy $E_0 := \int H_0(x)\mathrm{d}x$ is equal to $E_0 = 16(\nu_0\mu_0^2 - \nu_0^3/3)$. The width of the envelop of the soliton is conversely proportional to its mass. The soliton solution (18) is associated with the following scattering data [19]:

$$a_0(\lambda) = \frac{\lambda - (\mu_0 + \mathrm{i}\nu_0)}{\lambda - (\mu_0 - \mathrm{i}\nu_0)}, \quad b_0(\lambda) = 0. \tag{19}$$

a_0 admits a unique zero in the upper complex half-plane denoted by $\lambda_0 = \mu_0 + \mathrm{i}\nu_0$.

4.2. Expressions of the mass and energy in terms of the scattering data

Introducing $n(\lambda) = -\pi^{-1}\ln|a(\lambda)|^2$, and denoting by λ_r the zeros of the coefficient a in the upper complex half-plane, which correspond to solitons, the total mass N_{tot} of the wave is also given by

$$N_{\mathrm{tot}} = \sum_r 2\mathrm{i}(\lambda_r^* - \lambda_r) + \int n(\lambda)\mathrm{d}\lambda. \tag{20}$$

The unperturbed energy $E_0 := \int H_0(x)\mathrm{d}x$ can also be expressed as

$$E_0 = \sum_r \frac{8\mathrm{i}}{3}(\lambda_r^{*3} - \lambda_r^3) + 4\int \lambda^2 n(\lambda)\mathrm{d}\lambda. \tag{21}$$

4.3. Evolutions of the scattering data

We now describe the evolutions of the Jost coefficients a and b during the propagation through a slab contained in the region $[0, L/\varepsilon^2]$. The initial scattering data $a(t=-\infty,\lambda)$ and $b(t=-\infty,\lambda)$ are given by (19). They satisfy the following exact equations [20]:

$$\begin{aligned}
\frac{\partial a(t,\lambda)}{\partial t} &= -\varepsilon\left(a(t,\lambda)\bar{\gamma}(t,\lambda) + b(t,\lambda)\gamma(t,\lambda)\right),\\
\frac{\partial b(t,\lambda)}{\partial t} &= -4\mathrm{i}\lambda^2 b(t,\lambda) + \varepsilon\left(a(t,\lambda)\gamma^*(t,\lambda) + b(t,\lambda)\bar{\gamma}(t,\lambda)\right),
\end{aligned} \tag{22}$$

where the functions γ and $\bar{\gamma}$ are defined by

$$\gamma(t,\lambda) = \int \mathrm{d}x\, R(u)^* f_2^2 + R(u) f_1^2, \quad \bar{\gamma}(t,\lambda) = \int \mathrm{d}x\, R(u) f_1 f_2^* - R(u)^* f_1^* f_2, \tag{23}$$

and f_1, f_2 are the so-called Jost functions. It thus appears that the perturbations $R(u)$ couple the time evolution equations of the Jost coefficients.

4.4. The adiabatic approximation

The adiabatic approximation consists in assuming a priori that, while the soliton exists, its evolution and that of the radiated wave do not interact. More precisely, we assume that the time evolutions of the Jost coefficients a and b given by (22) depend only on the components of the functions γ and $\bar{\gamma}$ defined by (23) which are associated with the soliton. We are first going to carry out the calculations under this approximation, which provide an expression of the total wave u. A posteriori we check for consistency that this approximation is actually justified in the asymptotic framework $\varepsilon \to 0$. More exactly we show that the components of the functions γ and $\bar{\gamma}$ which correspond to the interplay between the computed radiation and the soliton, or else which originate from the sole effect of the radiation, can be considered as negligible terms for the soliton evolution.

5. Convergence results

Let $L > 0$. We denote by $\mathcal{C}^0$ the space $\mathcal{C}^0([0, L], \mathbb{R}^2)$ of all the $\mathbb{R}^2$-valued continuous functions equipped with the topology associated to the uniform norm. $(\nu_l(x), \mu_l(x))_{x\in[0,L]}$ is the element of $\mathcal{C}^0$ defined as the unique solution of the system of ordinary differential equations:

$$\frac{\mathrm{d}\nu_l}{\mathrm{d}x} = F(\nu_l, \mu_l), \quad \nu_l(0) = \nu_0, \qquad \frac{\mathrm{d}\mu_l}{\mathrm{d}x} = G(\nu_l, \mu_l), \quad \mu_l(0) = \mu_0. \tag{24}$$

The functions F and G are equal to

$$\begin{aligned}
F(\nu,\mu) &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} C(\nu,\mu,\lambda) d_0(k(\nu,\mu,\lambda))\mathrm{d}\lambda,\\
G(\nu,\mu) &= -\frac{1}{8\pi}\int_{-\infty}^{\infty} \left(\frac{\lambda^2}{\mu\nu} + \frac{\nu}{\mu} - \frac{\mu}{\nu}\right) C(\nu,\mu,\lambda) d_0(k(\nu,\mu,\lambda))\mathrm{d}\lambda,
\end{aligned} \tag{25}$$

$$C(\nu,\mu,\lambda) = \Big(\gamma c_0 k^2 - \alpha(4(\nu^2-\mu^2)c_0 - 4\mu c_1 - 2c_2)\Big)^2 + (\beta(2\mu c_0 + c_1)k)^2\,, \tag{26}$$

where the functions c_j and k are defined by

$$\begin{aligned}
c_0(\nu,\mu,\lambda) &= \frac{\pi}{2^4\mu^3}\frac{(\lambda-\mu+\mathrm{i}\nu)^2}{\cosh\left(\pi(\mu^2-\nu^2-\lambda^2)/(4\mu\nu)\right)},\\
c_1(\nu,\mu,\lambda) &= \frac{\pi}{3\times 2^4\mu^4}\frac{(\lambda-\mu+\mathrm{i}\nu)^2\left(\left((\lambda-\mu)^2+\nu^2\right)^2 + 8\mu^2(-\nu^2+\mu\lambda-\mu^2)\right)}{\left((\lambda-\mu)^2+\nu^2\right)\cosh\left(\pi(\mu^2-\nu^2-\lambda^2)/(4\mu\nu)\right)},\\
c_2(\nu,\mu,\lambda) &= \frac{\pi}{3\times 2^6\mu^5}\frac{(\lambda-\mu+\mathrm{i}\nu)^2\left((\lambda+\mu)^2+\nu^2\right)\left(\nu^2+17\mu^2-6\lambda\mu+\lambda^2\right)}{\cosh\left(\pi(\mu^2-\nu^2-\lambda^2)/(4\mu\nu)\right)},\\
k(\nu,\mu,\lambda) &= \frac{(\lambda-\mu)^2+\nu^2}{\mu}
\end{aligned} \tag{27}$$

Let us denote by Ω_L^ε the measurable set of realizations of the process V such that the transmitted wave consists of one soliton plus some radiation. We denote by ν^ε and μ^ε the re-scaled processes defined on Ω_L^ε by $\nu^\varepsilon(x) = \nu(x/\varepsilon^2)$ and $\mu^\varepsilon(x) = \mu(x/\varepsilon^2)$ (i.e. the coefficients of the transmitted soliton in position x/ε^2), and on $\Omega_L^{\varepsilon c}$ by $\nu^\varepsilon(x) = 0$ and $\mu^\varepsilon(x) = 0$. We can now state our main convergence result.

Proposition 5.1. *Under the adiabatic approximation, the following assertions hold true for any $L > 0$.*

1. $\liminf_{\varepsilon\to 0}\mathbb{P}\left(\Omega_L^\varepsilon\right) = 1$.
2. *The $\mathbb{R}^2$-valued process $(\nu^\varepsilon(x), \mu^\varepsilon(x))_{x\in[0,L]}$ converges in probability in $\mathcal{C}^0$ to the $\mathbb{R}^2$-valued deterministic function $(\nu_l(x), \mu_l(x))_{x\in[0,L]}$ which satisfies the system* (24).

The first point means that the event "the transmitted wave consists of one soliton plus some radiation" occurs with very high probability for small ε, while the second statement gives the effective evolution equation of the coefficients of the transmitted soliton in the asymptotic framework $\varepsilon \to 0$.

6. Strategy of the proof

We shall only give the main steps of the proof which follows closely the strategy developed in [4] and we shall underline the key-points.

(1) *Prove the stability of the zero of the Jost coefficient a*. The zero corresponds to the soliton. This part strongly relies on the analytical properties of a in the upper complex half-plane, and is very similar to the corresponding part in [4].

(2) *Compute the amount of radiation*. Under the adiabatic approximation, we solve the evolution equation (22) so that we get a closed-form expression of the ratio b/a.

Lemma 6.1. *Under the adiabatic approximation, the scattering data $\bar{b}/a(t,\lambda) = b/a(t,\lambda)\mathrm{e}^{4\mathrm{i}\lambda^2 t}$ at time t/ε^2 is given by*

$$\frac{\bar{b}}{a}\left(\frac{t}{\varepsilon^2},\lambda\right) = -\mathrm{i}\varepsilon\int_0^{x_s(t/\varepsilon^2)} \mathrm{e}^{\mathrm{i}\psi_s(x,\lambda)}(\gamma c_0(\lambda,\mu(x),\nu(x))V_{xx}(x) + \mathrm{i}\beta(2\mu c_0 + c_1)V_x(x) + \alpha(4(\nu^2-\mu^2)c_0 - 4\mu c_1 - 2c_2)V(x))\mathrm{d}x, \tag{28}$$

$$\psi_s(x,\lambda) = \phi_s(x) - 2\lambda x + 4\lambda^2 t_s(x), \tag{29}$$

where $x_s(t)$ is the position of the center of the soliton at time t defined by (31), *$t_s(x)$ is the arrival time of the soliton at point x, $\phi_s(x)$ is the phase defined by* (31) *of the soliton when its center is at x, and the coefficients c_j are given by* (27).

From Lemma 6.1 we can estimate the amount of radiation which is emitted during some time interval in terms of mass and energy thanks to (20) and (21). We are then able to deduce the evolution equations of the coefficients of the soliton by using the conservations of the total mass and energy. For times of order O(1), since N_{tot} and E_{tot} are conserved, the variations $\Delta(..)$ of the relevant quantities are linked together by the relations

$$0 = 4\Delta\nu + \int \Delta n(\lambda)\mathrm{d}\lambda, \qquad 0 = 16\Delta\left(\nu\mu^2 - \nu^3/3\right) + 4\int \lambda^2\Delta n(\lambda)\,\mathrm{d}\lambda + \varepsilon\Delta\left(\int_{\mathbb{R}} H_1(x)\mathrm{d}x\right).$$

$\Delta n(\lambda)$ is of order ε^2, but the last term in the expression of the total energy is of order ε. Thus our strategy is not efficient for estimating the variations of the coefficients of the soliton for times of order O(1). Let us now consider times of order $\mathrm{O}(\varepsilon^{-2})$. $\Delta n(\lambda)$ is now of order 1, while the last term in the expression of the total energy is of order

$\varepsilon M^{\varepsilon}$ by Lemmas 3.1 and 3.3. Thus we can efficiently compute the long-time behavior of the coefficients of the soliton in the asymptotic framework $\varepsilon \to 0$, when the last term in the expression of the total energy is uniformly negligible. Using probabilistic limit theorems, we then find that the coefficients of the soliton converge in probability to nonrandom functions which satisfy the system (24).

(3) *Compute the form of the transmitted wave*.

Lemma 6.2. *Under the adiabatic approximation, neglecting the terms of higher order, the total wave is given by the sum* $u(t/\varepsilon^2, x) = u_S(t/\varepsilon^2, x) + u_L(t/\varepsilon^2, x)$, *where* u_S *is a soliton of mass* $4\nu(t/\varepsilon^2)$ *and velocity* $4\mu(t/\varepsilon^2)$:

$$u_S\left(\frac{t}{\varepsilon^2}, x\right) = -2\mathrm{i}\nu \frac{\exp[\mathrm{i}(2\mu(x - x_s) + \phi_s)]}{\cosh(2\nu(x - x_s))}, \tag{30}$$

x_s *and* ϕ_s *are respectively the position and the phase of the soliton at time* t/ε^2:

$$x_s = \frac{1}{2\nu} \ln\left(\frac{1}{2\nu}\left|\frac{\rho_r(t/\varepsilon^2)}{a'(t/\varepsilon^2, \lambda_s)}\right|\right), \quad \phi_s = \arg\left(-\mathrm{i}\frac{\rho_r(t/\varepsilon^2)}{a'(t/\varepsilon^2, \lambda_s)}\right) + 2\mu x_s, \tag{31}$$

$\lambda_s = \mu(t/\varepsilon^2) + \mathrm{i}\nu(t/\varepsilon^2)$ *and* u_L *admits the following expression:*

$$\begin{aligned} u_L\left(\frac{t}{\varepsilon^2}, x\right) = {} & \frac{1}{\mathrm{i}\pi}\int_{-\infty}^{\infty} \frac{b}{a}(\lambda) \frac{(\lambda - \mu + \mathrm{i}\nu \tanh(2\nu(x - x_s)))^2}{(\lambda - \mu + \mathrm{i}\nu)^2} \mathrm{e}^{2\mathrm{i}\lambda x} d\lambda \\ & + \frac{\nu^2}{\mathrm{i}\pi} \frac{\exp[2\mathrm{i}(2\mu(x - x_s) + \phi_s)]}{\cosh^2(2\nu(x - x_s))} \int_{-\infty}^{\infty} \frac{b^*}{a^*}(\lambda) \frac{1}{(\lambda - \mu - \mathrm{i}\nu)^2} \mathrm{e}^{-2\mathrm{i}\lambda x} \mathrm{d}\lambda. \end{aligned} \tag{32}$$

u_S is the soliton part of the total wave. The first component of u_L represents the scattered wavepacket, with a correction in the neighborhood of the soliton $x \sim x_s(t/\varepsilon^2)$. The second component of u_L represents the interaction between the soliton and the scattered wavepacket, which is only noticeable in the neighborhood of the soliton. This result is not surprising. Roughly speaking, the support of the scattered wavepacket lies in an interval with length of order ε^{-2}. Since the L^2-norm is bounded by the conservation of the total mass, we can expect that the amplitude of the radiation is of order ε. More exactly, using the same arguments as in Lemma 4.2 [4], it can be rigorously proved that the amplitude of the radiated wavepacket can be bounded above by $K\varepsilon|\ln\varepsilon|$.

(4) *Check a posteriori the adiabatic approximation*. The final part of the proof consists in checking a posteriori the adiabatic hypothesis, that is to say proving that the radiated wavepacket which has been determined here above has actually no noticeable influence on the evolutions (22) of the Jost coefficients a and b. One must estimate the components of the functions γ and $\bar{\gamma}$ given by (23) which have been neglected until now and which are related to the interplay between the soliton and the radiation on the one hand, and which are due to the sole effect of the radiation on the other hand. These are technical calculations which are based upon the mixing properties of the process V.

7. Analysis of the asymptotic behavior of the transmitted soliton

This section is devoted to the study of the asymptotic evolutions of the coefficients of the transmitted soliton as a function of the macroscopic length L of the random slab, i.e. L/ε^2 in the microscopic scale. By Proposition 5.1 these evolutions are given by (24). We aim at exhibiting the relevant characteristics of this deterministic system of ordinary differential equations.

7.1. Limit behavior in the approximation $v_0 \ll \mu_0$ *(linear regime)*

The system (24) can be simplified to a first approximation:

$$\frac{\mathrm{d}\nu}{\mathrm{d}L} = -d_0(4\mu)\nu\mu^2\left((\alpha+4\gamma)^2+4\beta^2\right), \quad \nu(0)=\nu_0, \tag{33}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}L} = -\frac{5\nu}{3\mu}d_0(4\mu)\nu\mu^2\left((\alpha+4\gamma)^2+4\beta^2\right), \quad \mu(0)=\mu_0. \tag{34}$$

It appears that $\left(1-5/3\,(\nu_0/\mu_0)^2\right)^{1/2} \leq \mu(L)/\mu_0 \leq 1$, which means that the velocity of the soliton is almost constant during the propagation, while the mass (equal to 4ν) decreases. More exactly the coefficient ν decreases exponentially with L:

$$\nu(L) \simeq \nu_0 \exp\left(-\frac{L}{L_1}\right), \quad L_1^{-1} = \mu_0^2 d_0(4\mu_0)\left((\alpha+4\gamma)^2+4\beta^2\right). \tag{35}$$

It should be emphasized that the three terms of the right-hand member of the perturbed Schrödinger equation (7) give rise to a perturbation of similar order in this regime. We can give a simple physical explanation of this exponential regime. In the limit case $\nu_0/\mu_0 \to 0$, the incoming soliton can be approximated by a linear wavepacket:

$$u_0(t,x) \simeq \int_{-\infty}^{+\infty} \mathrm{d}k\hat{\phi}_0(k)e^{\mathrm{i}kx-\mathrm{i}k^2t}, \quad \text{with } \hat{\phi}_0(k) = \frac{1}{2}\cosh^{-1}\left(\frac{\pi}{4}\left(\frac{k-2\mu_0}{\nu_0}\right)\right), \tag{36}$$

whose spectrum $\hat{\phi}_0$ is sharply peaked about the wave number $k_0 := 2\mu_0$. Eq. (35) is therefore in agreement with the linear approximation, where it is well-know that a random potential gives rise to an exponential decay of the transmittivity of a linear wavepacket.

We can also analyze the spectrum of the radiation. This spectrum exhibits two peaks. The main peak is about the wave number $k_{r,1} = -2\mu_0$ and its mass is proportional to $\nu\mu^2 d_0(4\mu)$. The term $d_0(4\mu)$ is the spectral power density of the perturbation evaluated at wave number $k_{r,1} - k_0 = -4\mu_0$. When taking into account only this main peak, the simplified system (33) and (34) is obtained. There exists also a secondary peak about $k_{r,2} = +2\mu_0$ whose mass is proportional to $\nu^3 d_0(0)$. The term $d_0(0)$ is the spectral power density of the perturbation evaluated at wave number $k_{r,2} - k_0 = 0$. The second peak is in general weaker than the first one, but not necessarily, since it actually gives rise to corrective terms in Eqs. (33) and (34):

$$\frac{\mathrm{d}\nu}{\mathrm{d}L} = -d_0(4\mu)\nu\mu^2\left((\alpha+4\gamma)^2+4\beta^2\right) - \frac{4}{27}d_0(0)\nu^3\alpha^2, \tag{37}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}L} = -\frac{5\nu}{3\mu}d_0(4\mu)\nu\mu^2\left((\alpha+4\gamma)^2+4\beta^2\right) + \frac{8\nu}{135\mu}d_0(0)\nu^3\alpha^2. \tag{38}$$

If $d_0(4\mu_0)$ is of the same order as $d_0(0)$, then the supplementary terms are very negligible compared to the first ones, so that the system (33) and (34) and the corresponding solution (35) hold true. If the autocorrelation function of V has the Gaussian shape (10), this corresponds to the case $\mu_0 l_c \leq 1$, which means that the central wave number $2\mu_0$ the soliton lies in the spectrum of the perturbation whose bandwidth is $\sim 1/l_c$. Nevertheless, in the converse case $\mu_0 l_c \gg 1$ (i.e. if the central wave number of the soliton lies in the tail of the spectrum of the perturbation), then $d_0(4\mu_0) = \sqrt{\pi}\sigma^2 l_c \exp - \mu_0^2 l_c^2/4$ is much smaller than $d_0(0) = \sqrt{\pi}\sigma^2 l_c$, so that the supplementary terms have to be taken into account. If $(\mu_0 l_c)^2 \gg \ln(\mu_0/\nu_0)$, then the second terms actually prevail over the first ones, and the decay rate of the coefficient ν and the mass of the soliton obeys a power law:

$$\nu(L) \simeq \nu_0\left(1+\frac{L}{L_2}\right)^{-1/2}, \quad L_2^{-1} = \frac{8}{27}d_0(0)\nu_0^2\alpha^2. \tag{39}$$

Note that only the first term in the right-hand member of the perturbed Schrödinger equation (7) corresponding to the perturbation of the diffusive component gives rise to the supplementary terms of Eqs. (37) and (38). That is why the

$L^{-1/2}$ decay has now been exhibited in the study of the soliton propagation in a random linear potential. Note also that the value of μ does not come into Eq. (39), which means that the decay (39) holds true whenever $\mu_0 l_c$ is large enough, and does not depend on the exact value of μ_0. Finally, since ν is decaying, the ratio $(\mu_0 l_c)^2/\ln(\mu_0/\nu)$ also decays, so that when ν becomes small enough, the decrease of the mass obeys again the exponential law (35). The crossover between the power decay and the exponential decay occurs when L reaches the value $L_3 \sim \exp(8\mu_0^2 l_c^2)$.

7.2. Limit behavior in the approximation $\mu_0 \ll \nu_0$ (nonlinear regime)

The system (24) can then be simplified:

$$\frac{\mathrm{d}\nu}{\mathrm{d}L} = -\frac{\pi\sqrt{2}d_0(\nu^2\mu^{-1})}{2^8}\left(\frac{\nu}{\mu}\right)^{25/2}\mu^3 \mathrm{e}^{-(\pi\nu/2\mu)}\left(\left(\frac{\alpha}{6}+\gamma\right)^2+\left(\frac{\beta}{3}\right)^2\right), \quad \nu(0)=\nu_0, \tag{40}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}L} = -\frac{\pi\sqrt{2}d_0(\nu^2\mu^{-1})}{2^9}\left(\frac{\nu}{\mu}\right)^{27/2}\mu^3 \mathrm{e}^{-(\pi\nu/2\mu)}\left(\left(\frac{\alpha}{6}+\gamma\right)^2+\left(\frac{\beta}{3}\right)^2\right), \quad \mu(0)=\mu_0. \tag{41}$$

It can be readily checked that $\left(1-2\,(\mu_0/\nu_0)^2\right)^{1/2} \le \nu(L)/\nu_0 \le 1$, which means that the mass of the soliton is almost constant during the propagation, while the velocity of the soliton decreases. The limit behavior for large L of the coefficient μ depends on the function d_0, more exactly on the high frequency behavior of the Fourier transform of the autocorrelation function of the process V. The exact decay rate of the velocity results from the competition between the terms $d_0(\nu^2\mu^{-1})$ and $\exp(-(\pi\nu/2\mu))$ in Eqs. (40) and (41). Let us assume that the autocorrelation of V has Gaussian shape so that $d_0(\nu^2\mu^{-1}) = \sqrt{\pi} l_c \sigma^2 \exp(-\nu^4 l_c^2/4\mu^2)$. If $(\nu_0 l_c)^2 \times \nu_0/\mu_0 \ll 1$, then the exponential term is the smallest one and consequently imposes the decay rate of μ:

$$\mu(L) \sim \frac{\pi\nu_0}{2\ln L}. \tag{42}$$

This logarithmic rate actually represents the maximal decay of the velocity. Whatever the process V, the terms of the right-hand sides of (40) and (41) have at least an exponential decay of the type $\exp(-(\pi\nu/2\mu))$, which implies $\liminf_{L\to\infty}\mu(L) \times \ln(L) \ge \pi\nu_0/2$. However the decay rate may be much slower. If $(\nu_0 l_c)^2 \times \nu_0/\mu_0 \gg 1$, then $d_0(\nu^2\mu^{-1})$ is much smaller than $\exp(-(\pi\nu/2\mu))$, so that the velocity decreases as the square root of the logarithm of L:

$$\mu(L) \sim \frac{\nu_0^2 l_c}{2\sqrt{\ln L}}. \tag{43}$$

Note that, since μ is decaying, the ratio $(\nu_0 l_c)^2\nu_0/\mu$ goes to infinity, so that the second regime (43) is reached when L exceeds values of order $\exp(\pi^2/(\nu_0 l_c)^2)$.

The decay rate $(\ln L)^{-1/2}$ is imposed by the shape of the tail of the spectrum of the random process V. If the spectrum decays faster than a Gaussian, then the regime corresponding to Eq. (43) will be slower. Conversely, if the spectrum decays slower than any exponential, then one can only observe the regime (42).

As in the linear and exponential regime one can notice that the three terms of the right-hand member of the perturbed Schrödinger equation (7) give rise to a perturbation of similar order.

7.3. Numerical integration of the asymptotic system

In the above paragraphs, we have exhibited two domains which are stable with respect to the evolutions of the coefficients of the transmitted soliton. We aim at showing here that these regimes are not only stable, but also

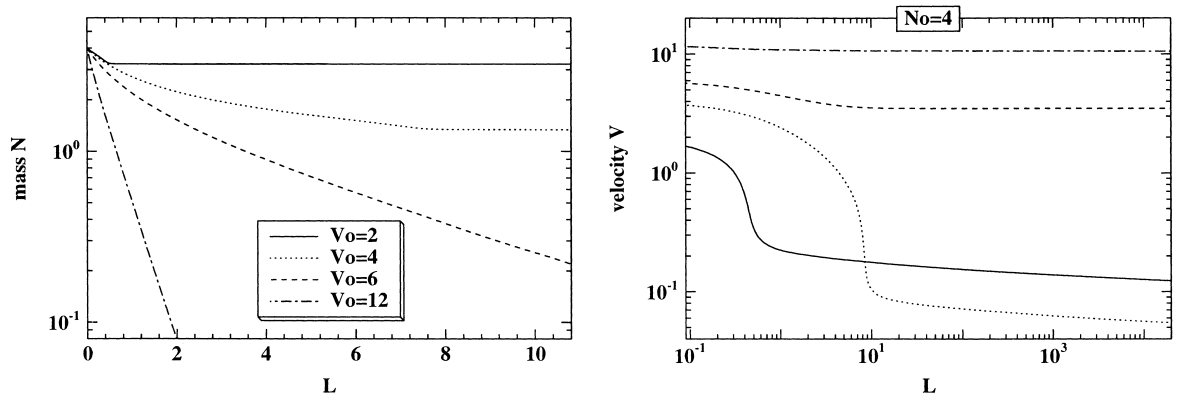

Fig. 1. Coefficients of the transmitted soliton, with the initial mass $N_0 = 4$ and different initial velocities. Here $l_c = 0.1$ and $\sigma^2 = 1/(2\sqrt{\pi})$.

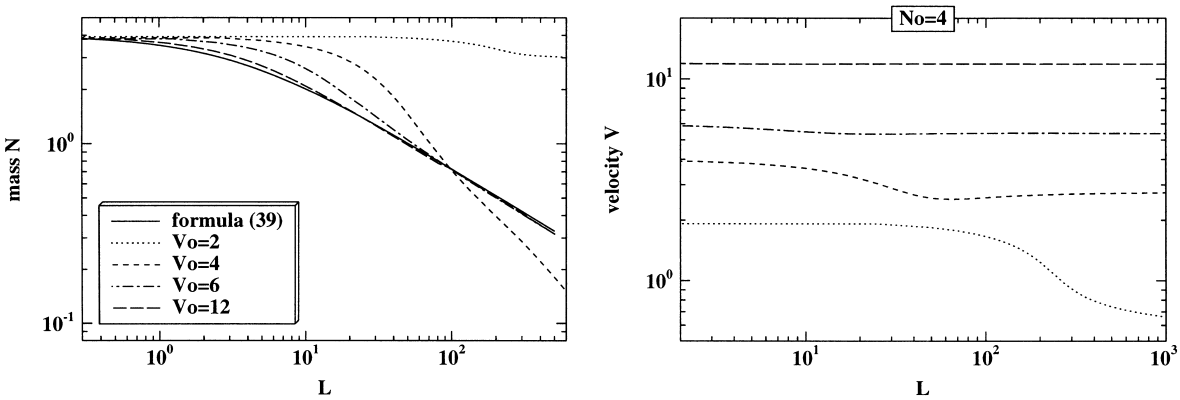

Fig. 2. Coefficients of the transmitted soliton, with the initial mass $N_0 = 4$ and different initial velocities. Here $l_c = 2$ and $\sigma^2 = 1/(2\sqrt{\pi})$.

attractive in the sense that, up to transitory regimes, either the mass tends to a constant positive value while the velocity decays towards 0 (this is the so-called nonlinear regime) or the velocity tends to a constant positive value while the mass decays towards 0 (this is the so-called linear regime). In order to prove this statement, we are going to solve numerically the system (24) for different values of the coefficients of the incoming soliton μ_0 and ν_0. For the sake of simplicity we choose to analyze the case where $\gamma = 0$ and $\alpha = \beta = 1$.

Figs. 1 and 2 plot the evolutions of the coefficients of the transmitted soliton as functions of the length of the random slab. The mass N_0 is chosen at a fixed value for all figures, equal to 4, but the initial velocity V_0 varies from 2 to 12 and two different correlation lengths l_c are considered.

Let us first consider the case $l_c = 0.1$ and $\sigma^2 = 1/(2\sqrt{\pi})$. In these conditions $d_0(4\mu_0) \simeq d_0(0) = 0.05$. Comparing the terms in Eq. (37) then yields that the first term which gives rise to the exponential decay prevails over the second term. Therefore two different behaviors can be put into evidence, and they are separated from each other by a critical value V_c of the initial velocity V_0 (see Fig. 1).

If $V_0 = 6, 12$, then the velocity goes to a constant value, while the mass decays exponentially with a localization length with depends on the limit value of the velocity. The exponential decay is noticeable in the (L − log mass) scale.

If $V_0 = 2, 4$, then the mass goes to a constant value while the velocity decays very slowly to 0.

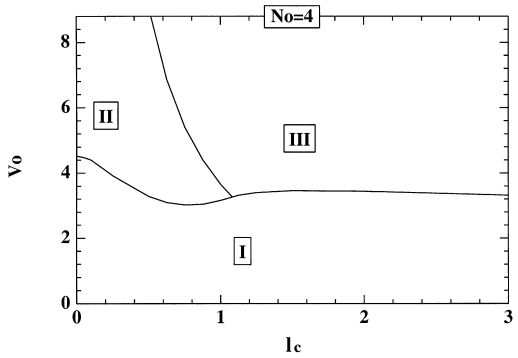

Fig. 3. The three possible regimes for an incoming soliton with mass $N_0 = 4$ as a function of the initial velocity V_0 and the correlation length l_c (the correlation function is assumed to have Gaussian shape). Region I corresponds to an almost constant mass and a decaying velocity. Region II corresponds to an almost constant velocity and an exponentially decaying mass. Region III corresponds to an almost constant velocity and a $L^{-1/2}$-decaying mass. If $V_0 \gg 1$, then the boundary between regions II and III lies about $l_c \sim 2\sqrt{2}\sqrt{\ln(V_0)}/V_0$. If $l_c \gg 1$, then the boundary between regions I and III lies about $V_0 \sim 3.3$.

Let us now consider the case $l_c = 2$ and $\sigma^2 = 1/(2\sqrt{\pi})$. In these conditions $d_0(4\mu_0) = \exp(-16\mu_0^2)$ and $d_0(0) = 1$. Comparing the terms in Eq. (37) it appears that, if $\mu_0 > 1$, the first term is dominated by the second term which gives rise to the $L^{-1/2}$ decay of the mass. In Fig. 2 the solid line corresponds to formula (39) with $d_0(0) = 1$ and $\alpha = 1$. If $V_0 = 2$, then the mass decays very slowly and tends to a positive constant, while the velocity converges to 0 at logarithmic rate, as shown in the ($\log L$–log velocity) scale.

If $V_0 = 4, 6, 12$, then the velocity is almost constant, while the mass decays as a power law because the second term in Eq. (37) prevails. The slopes of the lines in the ($\log L$–log mass) scale confirm the $L^{-1/2}$-decay for $V_0 = 6, 12$. Further the figure clearly puts into evidence that, if the initial velocity is large enough, then the decay of the mass is almost exactly described by formula (39).

In Fig. 3 we represent the three different regimes that can be encountered. The locations of the boundaries depend on N_0 and also on the precise shape of the correlation function of the process V, but the qualitative picture remains unchanged if one modifies the value of the initial mass or the shape of the correlation function. As a final remark one can notice that the $L^{-1/2}$ decay of the mass cannot be observed in case of a white noise ($l_c \to 0$). This anomalous decay of the mass is involved by the colored property of the dispersive perturbation.

8. Conclusion

We have studied the propagation of a soliton in a nonlinear dispersive medium with random dispersive perturbations by applying the powerful inverse scattering transform. Although a similar behavior as in the case of a perturbation of the nonlinear coefficient was expected because of the natural balance between dispersion and Kerr nonlinearity, we have put into evidence that a spatial random dispersion gives rise to more complicated regimes.

In case of large amplitude, we have shown that the mass of the soliton is almost constant, as in the case of a perturbation of the linear potential or the nonlinear coefficient. Furthermore the velocity is found to decrease at a slow rate (at most logarithmic) which depends on the high-frequency behavior of the power spectrum of the spatial dispersive random perturbation.

In case of small amplitude, we have proved that the amplitude of the soliton decays exponentially with the length of the system if $\mu_0 l_c < 1$. These results are similar to a perturbation of the linear potential, but only if the initial velocity of the soliton $4\mu_0$ is smaller than the width $1/l_c$ of the spectrum of the perturbation. If $\mu_0 l_c > 1$, then

a power decay as $L^{-1/2}$ of the amplitude is shown. This regime is original and differs on the one hand from the regime exhibited in case of a perturbation of the nonlinear coefficient (decay as $L^{-1/4}$ [4]) and on the other hand from the regime corresponding to a perturbation of the linear coefficient (exponential decay [3]). There exists a critical value of the velocity $4\mu_0$ of the soliton which depends only on the spectrum of the perturbation above which we observe the $L^{-1/2}$ decay of the amplitude, and below which we get back the usual exponential decay. Thus a random dispersive perturbation presents more complexity than both a linear and nonlinear perturbation.

We feel that it should be of great interest to consider other integrable systems with a different type of dispersion. For instance the Korteweg–de Vries equation, with a third order dispersion, is worth studying. Besides the interaction of solitons in random dispersive media represents also a challenge for practical applications in telecommunication for instance.

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