Energy distribution of the quantum harmonic oscillator under random time-dependent perturbations

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This paper investigates the evolution of a quantum particle in a harmonic oscillator driven by time-dependent forces. The perturbations are small, but they act long enough so that we can solve the problem in the asymptotic framework corresponding to a perturbation amplitude that tends to zero and a perturbation duration that tends to infinity. We describe the effective evolution equation of the state vector, which reads as a stochastic partial differential equation. We exhibit a closed-form equation for the transition probabilities, which can be interpreted in terms of a jump process. Using standard probability tools, we are then able to compute explicitly the probabilities for observing the different energy eigenstates and give the exact statistical distribution of the energy of the particle. [S1063-651X(99)04810-2]

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I. INTRODUCTION

This paper is devoted to the study of time-dependent perturbations of quantum systems. Literature contains a lot of applications and discussions of special types of perturbations: sudden, adiabatic, periodic, . . . [1]. The considered phenomena are described by the Hamiltonian

$$H(t)=H^0+H^1(t), \tag{1}$$

where H^0 is the time-independent piece whose eigenvalue problem has been solved, and H^1 is a small time-dependent perturbation. The typical question one asks is the following. If at $t=0$ the system is in the eigenstate ψ^0 of H^0, what is the probability for it to be observed in a given eigenstate? Most results that have been obtained follow a scheme in which the answers are computed in a perturbation series in powers of H^1 [1,2]. We shall present a method for obtaining answers to the above questions, which is based on the one hand on some rigorous asymptotic theory and on the other hand on a representation of the evolution of the transition probabilities in terms of a jump process. In this paper we shall focus on perturbations of the harmonic oscillator, although the method could be applied to more general situations.

The quantum harmonic oscillator has been extensively studied, not only because it is a system that can be exactly solved and a great pedagogical tool, but it is also a very relevant system [2]. Indeed a lot of systems close to a stable equilibrium can be described by an oscillator or a collection of decoupled harmonic oscillators. Furthermore, time-independent and time-dependent modifications of this model have been investigated, being handled by the perturbation theory. Indeed, even for this simple model it is exceptional to find closed-form expressions, except for very particular types of perturbations [3]. Nevertheless, rigorous results have been obtained for time-dependent perturbations of the harmonic oscillator. Most of them concern periodic driven force [4–6]. Although the problem is far less understood in the case of random perturbations, literature contains results about systems with randomly time-dependent external driving force. A general class of quantum systems in Markovian potentials has been treated in detail [7,8]. Under suitable conditions on the dynamics of the random potential, it is shown in Ref. [9] that the spectrum of the quasienergy operator is continuous. In Ref. [10] the authors study the long-time stability of oscillators driven by time-dependent forces originating from dynamical systems with varying degrees of randomness and focus on the asymptotic energy growth. Recently [11] we have studied the energy density of a charged particle in a harmonic oscillator driven by a time-dependent homogeneous electric field. In this paper we consider a particle in a harmonic oscillator that is driven by an external force, which derives from a weak random time-dependent external potential. We aim at studying this problem by a rigorous and nonperturbative method. Our approach is inspired by the works of Papanicolaou and co-workers about waves in random media [12,13]. The first step consists in determining the characteristic scales of the problem at hand: oscillation frequency of the harmonic oscillator, amplitude, coherence time, and duration of the random perturbations. We then study the asymptotic evolution of the state vector in the asymptotic framework based on the separation of these scales. Our main aim is to exhibit the asymptotic regime, which corresponds to the case where the amplitudes of the random fluctuations go to zero and the duration of the external perturbation goes to infinity. We then describe explicitly the effective random evolution of the state vector and the probability transitions. The paper is organized as follows. In Sec. II we review the main features of the harmonic oscillator, while we state our main convergence result about the effective evolution of the particle in Sec. III. By exploiting a representation of the evolution of the energy of the particle in terms of a jump process, we give general results on the long-time behavior of the particle in Sec. IV, which we apply to a couple of examples

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in Secs. V–VII. Finally, we compare the theoretical results with numerical simulations in Sec. VIII.

II. THE HARMONIC OSCILLATOR

We consider the quantum oscillator, that is to say, a particle of mass M whose state vector in the coordinate basis obeys the Schrödinger equation [2]:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}M\omega^2x^2\psi, \tag{2}$$

where ω is the oscillation frequency. In order to transform this equation into a standard and dimensionless form, we multiply the spatial coordinate x by $r_0^{-1}:=(M\omega/\hbar)^{1/2}$ and the time t by $t_0^{-1}:=\omega$, so that Eq. (2) now reads

$$2i\frac{\partial\psi}{\partial t}=-\frac{\partial^2\psi}{\partial x^2}+x^2\psi. \tag{3}$$

The spectrum of the harmonic oscillator is pure point with state energies $(2p+1)/2$ and corresponding eigenstates [2]

$$f_p(x)=\frac{1}{\sqrt{2^p\sqrt{\pi}p!}}H_p(x)e^{-x^2/2}, \tag{4}$$

$$H_p(x)=(-1)^pe^{x^2}\frac{d^p}{dx^p}e^{-x^2}. \tag{5}$$

The family $(f_p)_{p\in\mathbb{N}}$ is complete in the following sense [14 Prop. 1.5.7].

Proposition II.1. (1) The $(f_p)_{p\in\mathbb{N}}$ are an orthonormal and complete set in $L^2(\mathbb{R},\mathbb{C})$:

$$\int_R f_p(x)f_{p'}(x)dx=\delta_{p,p'}, \tag{6}$$

where δ stands for the Kronecker's symbol.

(2) $(t,x)\mapsto e^{-i(p+1/2)t}f_p(x)$ is a solution of Eq. (3) for any $p\in\mathbb{N}$.

We define the eigenstate decomposition as the map $\Theta:\psi\in L^2(\mathbb{R},\mathbb{C})\mapsto(c_p)_{p\in\mathbb{N}}$, where c_p are the coefficients of the expansion of ψ in the basis (f_p):

$$\Theta(\psi)_p:=c_p=\int_R f_p(x)\psi(x)dx. \tag{7}$$

By Proposition II.1, Θ is an isometry from $L^2(\mathbb{R},\mathbb{C})$ onto l^2, the space of all the sequences $(c_p)_{p\in\mathbb{N}}$ from $\mathbb{N}$ into $\mathbb{C}$, which are squared integrable. In view of the fundamental postulates of the quantum mechanics, if ψ is the state vector of the particle, then the measurement of the energy will yield the eigenvalue $(2p+1)/2$ with probability $|\Theta(\psi)_p|^2$.

III. EVOLUTION DRIVEN BY TIME-DEPENDENT FORCES

Suppose that the particle is also subjected to external time-dependent forces, which originate from the potential $\varepsilon V(t,x)$. The dimensionless quantity ε is a parameter that characterizes the amplitude of the perturbation. The perturbed equation, which governs the evolution of the state vector, is then

$$2i\frac{\partial\psi}{\partial t}=-\frac{\partial^2\psi}{\partial x^2}+x^2\psi+\varepsilon V(t,x)\psi. \tag{8}$$

We assume that the amplitudes of the fluctuations are of the order $\varepsilon\ll1$. The real-valued function V is assumed to be a zero-mean, time-stationary, and time-ergodic process. The proof actually requires that the random process $t\mapsto V(t,\cdot)$ has ''enough decorrelation,'' more exactly, that it fulfills the technical mixing condition ''V is ϕ mixing, with $\phi\in L^{1/2}(\mathbb{R}^+)$'' (see [15, Sec. 4-6-2]). We shall give more detail in the following.

We introduce the normalized process $\psi^\varepsilon(x,t):=\psi(x,t/\varepsilon^2)$. We aim at studying the evolution of the state vector ψ^ε of the particle. The initial state vector at time $t=0$ is ψ_0, which corresponds to the decomposition $c(0)=\Theta(\psi_0)$. Since Θ is an isometry, it is equivalent to study the evolution of the expansion of ψ^ε in the family of eigenstates $(f_p)_{p\in\mathbb{N}}$, i.e., the corresponding normalized coefficients c^ε:

$$c_p^\varepsilon(t)=\Theta(\psi^\varepsilon(t,\cdot))_p e^{i(p+1/2)(t/\varepsilon^2)}, \tag{9}$$

$$\psi^\varepsilon(t,x)=\sum_{p'=0}^{\infty}c_{p'}^\varepsilon(t)e^{-i(p'+1/2)(t/\varepsilon^2)}f_{p'}(x). \tag{10}$$

Substituting expression (10) into Eq. (8) and integrating with respect to $f_p(x)dx$ we get the equation that governs the evolution of c^ε:

$$\frac{dc_p^\varepsilon}{dt}=-\frac{i}{2\varepsilon}\sum_{p'=-\infty}^{\infty}v_{p,p+p'}\left(\frac{t}{\varepsilon^2}\right)c_{p+p'}^\varepsilon e^{-ip't/\varepsilon^2}, \tag{11}$$

where the coupling coefficients are given by

$$v_{p,p+p'}(t):=\int_R f_p(x)V(t,x)f_{p+p'}(x)dx. \tag{12}$$

We adopt the convention $v_{p,p+p'}\equiv0$ if $p+p'<0$. In order to be allowed to apply the diffusion-approximation theorems, we have to take care to consider separately the real and imaginary parts of the coefficients c_p^ε. Denoting $X_{2p}^\varepsilon:=\mathrm{Re}\,c_p^\varepsilon$ and $X_{2p+1}^\varepsilon:=\mathrm{Im}\,c_p^\varepsilon$, the process $X^\varepsilon(t)$ satisfies the linear differential equation

$$\frac{dX^\varepsilon(t)}{dt}=\frac{1}{\varepsilon}F\left(X^\varepsilon,\frac{t}{\varepsilon^2},\frac{t}{\varepsilon^2}\right),$$

where

$$F_{2p}(X,h,t):=\frac{1}{2}\sum_{p'=-\infty}^{\infty}v_{p,p+p'}(t)\times[X_{2p+2p'+1}\cos(p'h)-X_{2p+2p'}\sin(p'h)],$$

$$F_{2p+1}(X,h,t) := \frac{1}{2} \sum_{p'=-\infty}^{\infty} v_{p,p+p'}(t) \times [-X_{2p+2p'} \cos(p'h) - X_{2p+2p'+1} \sin(p'h)].$$

To describe the limit diffusion process we introduce diffusion and drift coefficients:

$$a_{i,j}(X) := \int_0^\infty \langle \mathbb{E}[F_i(X,h,0)F_j(X,h+t,t)] \rangle_h dt,$$

$$b_j(X) := \int_0^\infty \sum_{i=0}^{\infty} \left\langle \mathbb{E}\left[F_i(X,h,0) \frac{\partial F_j}{\partial x_i}(X,h+t,t) \right] \right\rangle_h dt,$$

where $\langle \cdot \rangle_h$ stands for an averaging over a period in h. Let $\mathcal{L}$ be the differential operator:

$$\mathcal{L} := \sum_{i,j=0}^{\infty} a_{ij}(X) \frac{\partial^2}{\partial X_i \partial X_j} + \sum_{j=0}^{\infty} b_j(X) \frac{\partial}{\partial X_j}.$$

In particular, the diagonal terms of the diffusion matrix a are

$$a_{2p,2p} := \frac{1}{8} \sum_{p'=-\infty}^{\infty} \Gamma_{p,p+p'}(X_{2p+2p'}^2 + X_{2p+2p'+1}^2) + \frac{1}{8} \sum_{p'=-\infty}^{\infty} \Xi_{p,p+p'}(X_{2p+2p'+1}X_{2p-2p'+1} - X_{2p+2p'}X_{2p-2p'}),$$

$$a_{2p+1,2p+1} := \frac{1}{8} \sum_{p'=-\infty}^{\infty} \Gamma_{p,p+p'}(X_{2p+2p'}^2 + X_{2p+2p'+1}^2) + \frac{1}{8} \sum_{p'=-\infty}^{\infty} \Xi_{p,p+p'}(-X_{2p+2p'+1}X_{2p-2p'+1} + X_{2p+2p'}X_{2p-2p'}),$$

while the drift b is simply

$$b_{2p} := -\frac{1}{4} \sum_{p'=-\infty}^{\infty} \Gamma_{p,p+p'} X_{2p},$$

$$b_{2p+1} := -\frac{1}{4} \sum_{p'=-\infty}^{\infty} \Gamma_{p,p+p'} X_{2p+1},$$

where

$$\Gamma_{p,p+p'} := \int_0^\infty \mathbb{E}[v_{p,p+p'}(0) v_{p,p+p'}(t)] \cos(p't) dt, \tag{13}$$

$$\Xi_{p,p+p'} := \int_0^\infty \mathbb{E}[v_{p,p+p'}(0) v_{p,p-p'}(t)] \cos(p't) dt. \tag{14}$$

We shall assume that the following conditions are fulfilled:

H1: $\forall\ p,p',\ \exists M_{p,p+p'}$ such that $\forall\ t, |v_{p,p+p'}(t)| \leqslant M_{p,p+p'}$, almost surely,

H2: $\exists\ \phi \in L^{1/2}$ such that the process $t \mapsto v_{\cdot,\cdot}(t)$ is ϕ mixing,

H3: a_{ij} has a symmetric square root,

H4: $\forall\ p, \forall n \geqslant 1, \exists\ K_n$ such that $\sum_{p'=-\infty}^{\infty} p'^n M_{p,p+p'} \leqslant K_n (1+p)^n$.

Note that conditions H1 and H2 imply that the coefficients $\Gamma_{p,p+p'}$ and $\Xi_{p,p+p'}$ are well-defined and finite. Furthermore, $\Gamma_{p,p+p'}$ is non-negative because it is proportional to the p'-frequency evaluation of the spectral density function of the time-stationary random process $v_{p,p+p'}(.)$ by the Wiener-Khintchine theorem [16].

Proposition III.1. Under conditions H1–H4, the processes c^ε converge in distribution to the diffusion process c defined by $c_p(t) := X_{2p}(t) + iX_{2p+1}(t)$, where X is the diffusion process with infinitesimal generator $\mathcal{L}$.

Proof. Apply formally the (unique) theorem of Ref. [13]. Conditions H1–H3 seem to be sufficient for applying this theorem, but we actually deal with an infinite-dimensional system while only finite-dimensional systems are addressed in Ref. [13]. That is why supplementary condition H4 should be fulfilled. It insures that we can approximate both X^ε and X by finite-dimensional processes. The complete proof of Proposition III.1 can be found in Ref. [11] in the case $V(t,x) := 2xm(t)$. The technique based on a martingale approach to some limit theorems in the diffusion-approximation regime is now well-known and extensively reviewed in literature [12,15]. □

We can give explicit sufficient conditions in the case $V(t,x) = \sum_{k=1}^{M} m_k(t) V_k(x)$, where m_k, $k=1,\ldots,M$, are real-valued zero-mean, independent, stationary and ergodic processes, and $V_k(x)$ are deterministic functions. Let us define the process c as the unique solution of the following infinite-dimensional system of linear stochastic differential equations starting from $c(0)$:

$$dc_p = \frac{i}{2} \sum_{k=1}^{M} \sum_{p'=1}^{\infty} \sqrt{\alpha_{k,p'}} (\gamma_{k,p,p+p'} c_{p+p'} + \gamma_{k,p,p-p'} c_{p-p'}) \circ dW_{k,p',1_t} + \frac{1}{2} \sum_{k=1}^{M} \sum_{p'=1}^{\infty} \sqrt{\alpha_{k,p'}} \times (\gamma_{k,p,p+p'} c_{p+p'} - \gamma_{k,p,p-p'} c_{p-p'}) \circ dW_{k,p',2_t} + \frac{i}{2} \sum_{k=1}^{M} \sqrt{2\alpha_{k,0}} \gamma_{k,p,p} c_p \circ dW_{k,0_t}, \tag{15}$$

where $W_{k,0}$, $W_{k,p',j}$, $k=1,\ldots,M$, $p'=1,\ldots,\infty$, and $j=1,2$, are independent standard Brownian motions, $\circ$ stands for the Stratonovich stochastic integral, and $\gamma_{k,p,p+p'}$ is the coefficient:

$$\gamma_{k,p,p+p'} := \int_R f_p(x) V_k(x) f_{p+p'}(x) dx, \tag{16}$$

and $\alpha_{k,p'}$ is the real (non-negative by the Wiener-Khintchine theorem [16]) given by

$$\alpha_{k,p'} = \int_0^\infty \mathbb{E}[m_k(0) m_k(t)] \cos(p' t) dt. \tag{17}$$

Under such conditions, the coefficients $\Gamma_{p,p+p'}$ defined by Eq. (13) are equal to

$$\Gamma_{p,p+p'} = \sum_{k=1}^{M} \gamma_{k,p,p+p'}^2 \alpha_{k,p'}.$$

The general result of Proposition III.1 can then be rewritten more explicitly.

Proposition III.2. Let us assume that for every positive integer n there exists a constant K_n such that the coefficients $\gamma_{k,p,p+p'}$ satisfy for any k:

$$\sum_{p'=-\infty}^{\infty} p'^n \gamma_{k,p,p+p'}^2 \leqslant K_n (1+p)^n, \tag{18}$$

and that m_k is almost surely bounded and ϕ_k mixing with $\phi_k \in L^{1/2}(\mathbb{R}^+)$.

(1) There exists a unique solution c of Eq. (15).

(2) The processes c^ε converge in distribution to the continuous Markov process c solution of Eq. (15) as $\varepsilon \to 0$.

There exist also technical conditions on the initial condition $c(0)$ so that the above proposition holds true. These conditions require that the initial sequence $(c(0)_p)_{p \in \mathbb{N}}$ decays fast enough and they are fulfilled in particular if the initial state is a pure eigenstate, i.e., $c(0)_p = \delta_{p,p_0}$ for some p_0. Thus, in order to avoid unnecessary intricate technical developments, we shall assume throughout the paper that the initial state is a pure eigenstate.

The coefficients $\mathbb{E}[|c_p|^2(t)]$ represent the probabilities that the particle driven by the random potential εV be observed in the state f_p at time t/ε^2 in the asymptotic framework $\varepsilon \to 0$. Equivalently, one can say that the measurement of the energy at time t/ε^2,

$$E^\varepsilon(t) = \frac{1}{2} \int_R \left(\left| \frac{\partial \psi^\varepsilon}{\partial x} \right|^2 + x^2 |\psi^\varepsilon|^2 \right)(t,x) dx, \tag{19}$$

will yield the eigenvalue $p + 1/2$ with probability

$$C_p(t) = \mathbb{E}[|c_p|^2(t)].$$

This implies that the expected value of the energy can be expressed as

$$E(t) = \frac{1}{2} + \sum_{p=0}^{\infty} p C_p(t). \tag{20}$$

Proposition III.1 is very useful since it allows us to compute efficiently these relevant quantities:

Proposition III.3. The family $[C_p(t)]_{p \in \mathbb{N}}$ satisfies a closed-form set of ordinary differential equations:

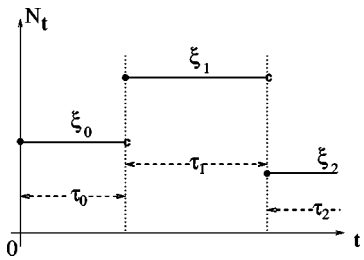

FIG. 1. Description of the jump process (N_t) in terms of the variables (ξ_j) and (τ_j).

$$\frac{dC_p}{dt} = \frac{1}{2} \sum_{p'=-\infty}^{\infty} \Gamma_{p,p+p'} (C_{p+p'} - C_p), \tag{21}$$

starting from $C_p(0) = \delta_{p,p_0}$.

Proof. Applying the infinitesimal generator $\mathcal{L}$ to $|c_p|^2 = X_{2p}^2 + X_{2p+1}^2$, we have

$$\frac{dC_p}{dt} = \mathbb{E}[2a_{2p,2p}(X) + 2a_{2p+1,2p+1}(X) + 2b_{2p}(X) X_{2p} + 2b_{2p+1}(X) X_{2p+1}].$$

Substituting the corresponding expressions of $a_{2p,2p}, \ldots$ into this equation readily yields the result. □

System (21) is one of the most important results of the paper. It shows that the probabilities C_p can be computed theoretically from the coupling coefficients $\Gamma_{p,p+p'}$, and that their evolutions are self-consistent in the sense that no other relevant quantities come into play. In particular, the relative phases between the coefficients (c_p) of the expansion of the state vector ψ in the basis (f_p) have no importance in the asymptotic evolution of the probability distribution (C_p). This statement is not at all obvious, since it is not satisfied by original Eq. (8) or Eq. (11).

IV. INTERPRETATION OF THE LIMIT SYSTEM IN TERMS OF A JUMP PROCESS

We shall show that the transition probabilities $C_p(t)$ can be regarded as the statistical distribution of a jump process. Let N_t be the Markov process with state space $\mathbb{N}$ (i.e., the set of all non-negative integer numbers) and infinitesimal generator $\mathcal{K}$:

$$\mathcal{K} = \frac{1}{2} \sum_{p'} \Gamma_{N,N+p'} \nabla_{p'}, \tag{22}$$

where $\nabla_{p'} g(N) = g(N+p') - g(N)$. $(N_t)_{t \geqslant 0}$ is a time-homogeneous jump process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If at time t the process is in state n, then the probability that between t and $t+h$ the transition $n \mapsto n+p'$ occurs equals $\Gamma_{n,n+p'} h + o(h)$. The probability that more than one transition occurs is $o(h)$. The operator $\nabla_{p'}$ corresponds to a jump from n to $n+p'$. The complete construction of the process N_t is the following [17, Sec. 7, Theorem 33]. There exists a sequence of integer-valued random variables $(\xi_j)_{j \in \mathbb{N}}$ and positive real-valued variables $(\tau_j)_{j \in \mathbb{N}}$ such that (see Fig. 1)

$$N_t=\xi_n \text{ if } \sum_{j=0}^{n-1}\tau_j\leqslant t<\sum_{j=0}^{n}\tau_j .$$

The sequence $(\xi_j)_{j\in\mathbb{N}}$ is a Markov chain with stationary transition matrix Q:

$$\mathbf{P}(\xi_{n+1}=p+p'/\xi_n=p)=Q_{p,p+p'},$$

$$Q_{p,p+p'}:=\begin{cases}\frac{1}{2}\dfrac{\Gamma_{p,p+p'}}{q(p)} & \text{if } p'\neq 0\\ 0 & \text{if } p'=0,\end{cases}$$

$$q(p):=\frac{1}{2}\sum_{p''\neq 0}\Gamma_{p,p+p''}.$$

Given $(\xi_j)_{j\in\mathbb{N}}$, the random variables $(\tau_j)_{j\in\mathbb{N}}$ are conditionally independent and exponentially distributed, with parameters $[q(\xi_j)]_{j\in\mathbb{N}}$, that is,

$$\mathbf{P}(\tau_j\geqslant t)=\exp[-q(\xi_j)t].$$

We denote by $\mathbf{P}_{t_0,p_0}$ the distribution of the paths $(N_t)_{t\geqslant t_0}$ starting at time t_0 with the initial condition $N_{t_0}=p_0$. The Markov process has stationary transition probabilities

$$P_t(p_0,p)=\mathbf{P}_{t',p_0}(N_{t'+t}=p), \quad \text{independent of } t'.$$

Furthermore, P_t is the unique probabilistic solution [i.e., $\Sigma_{p'}P_t(p,p+p')=1$] of the Kolmogorov's forward equation [18, Sec. X-3]:

$$\frac{\partial P_t}{\partial t}(p_0,p_1)=-q(p_1)P_t(p_0,p_1)+\sum_p q(p)Q_{p,p_1}P_t(p_0,p). \tag{23}$$

Since $q(p)Q_{p,p_1}=\frac{1}{2}\Gamma_{p,p_1}=\frac{1}{2}\Gamma_{p_1,p}$ we get that Eq. (23) is equivalent to Eq. (21), which implies the relation $C_p(t)=\mathbf{P}_{0,p_0}(N_t=p)$. This interpretation of the transition probabilities is very powerful to solve problems and study system (21) since it allows us to apply existing results on Markov jump processes, which can be found in the mathematical literature.

Note that we have only showed that (N_t) gives the correct statistical distribution of the measurement of the energy of the particle at a given time t. In terms of probability theory, we have only proved that the one-dimensional distributions of (N_t) and the ones of the energy distribution coincide. We shall see that in fact (N_t) gives the correct statistical distribution of any sequence of measurements. For that purpose we revisit the postulates of quantum mechanics in terms of the jump process N_t.

Postulate 1. The state vector ψ obeys Schrödinger equation (8). The time evolution of the process N_t is governed by the Markovian dynamics described by infinitesimal generator (22).

Postulate 2. The measurement of the energy at (normalized) time t will yield one of the eigenvalues $(2p+1)/2$ with probability $C_p(t)$. The jump process $1/2+N_t$ takes values only in the set $1/2+\mathbb{N}$. At time t one will find the value $1/2+p$ with probability $\mathbf{P}_{0,p_0}(N_t=p)$, which is equal to $C_p(t)$.

Postulate 3. If a measure at time t of the energy gives the result $1/2+p$, then the state of the system will change from $\psi(t)$ to f_p as a result of the measurement. Let us assume that we start from state f_{p_0} at time 0. If we observe the energy of the particle at times t_1 and t_2, then the probability for measuring first the energy $1/2+p_1$ and then $1/2+p_2$ is equal to the probability for observing $1/2+p_1$ at t_1 starting from f_{p_0} multiplied by the probability for observing $1/2+p_2$ at t_2 starting from f_{p_1} at t_1, because the system is in state f_{p_1} just after the measurement at t_1. This product of probabilities also reads as $\mathbf{P}_{0,p_0}(N_{t_1}=p_1)\times\mathbf{P}_{t_1,p_1}(N_{t_2}=p_2)$. From the Markov property of the process N_t, this product is exactly equal to $\mathbf{P}_{0,p_0}(N_{t_1}=p_1,N_{t_2}=p_2)$. Of course, this statement can be generalized to any sequence of measurements so that we conclude that if we observe the energies of the particle at times $t_1,\ldots,$ and t_n, then the probability to measure the sequence of energies $1/2+p_1,\ldots,1/2+p_n$ is exactly $\mathbf{P}_{0,p_0}(N_{t_1}=p_1,\ldots,N_{t_n}=p_n)$. This means that the dynamics of the observations is Markovian and exactly described by the jump process N_t (Markovian means that the future is independent from the past conditionally to the present). More remarkable, this property is essentially equivalent to Postulate 3 of quantum mechanics. Indeed, if instead of Postulate 3 we assume that the dynamics of the observations is Markovian, then just after a measurement the system depends only on the result of the measurement that means that just after measuring the energy $1/2+p_1$, the system must be in a state with energy $1/2+p_1$ with probability 1. Since there exists a unique eigenstate f_{p_1} with energy $1/2+p_1$, this means that the system must be in state f_{p_1} with probability 1 (here the nondegeneracy of the system plays a primary role). As a conclusion, Postulate 3 is exactly the right condition under which we can extend the statement, ''(N_t) describes the statistical distribution of the measurement of the energy at some given time'' to the statement, ''(N_t) describes the statistical distribution of any sequence of measurements of the energy.''

Interpretation of the technical conditions Hi in terms of the jump process. In order that the jump process be well-defined, it is necessary and sufficient to assume that $\Sigma_{p'}\Gamma_{N,N+p'}<\infty$ for every N, which means that the first jump starting from N does not occur instantaneously. The requirements $\Sigma_{p'}p'^n\Gamma_{N,N+p'}<\infty$ mean that the statistical distribution of the first jump starting from N has finite moments. Finally, the conditions $\Sigma_{p'}p'^n\Gamma_{N,N+p'}\leqslant K_n(1+N)^n$ prevent the jump process from going to infinity in finite time.

General long-time behavior. Let us denote by $\mathcal{A}$ the set of the accessible states, that is to say:

$$\mathcal{A}:=\{p\in\mathbb{N},\exists \text{ a sequence } p_0,\ldots,p_n=p \text{ such that } \Gamma_{p_i,p_{i+1}}>0\}.$$

In probability theory $\mathcal{A}$ is the so-called communicating class, which contains the initial state p_0. The number of elements of the set is denoted by $|\mathcal{A}|$. One can divide the possible evolutions of the process (N_t) into two different cases.

Proposition IV.1. If $|\mathcal{A}|=\infty$, then for every $p\in\mathbb{N}$, we

have $C_p(t)\to 0$ as $t\to\infty$. If $|\mathcal{A}|<\infty$, then the probability distributions $(C_p(t))_{p\in N}$ converge as $t\to\infty$:

$$C_p(t) \xrightarrow[t\to\infty]{} \begin{cases} \dfrac{1}{|\mathcal{A}|} & \text{if } p\in\mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is a universal feature that $P_{0,p_0}(N_t=p)$ converges as $t\to\infty$ as soon as (N_t) is a Markov process with denumerable state space satisfying the continuity condition $\mathbf{P}_{0,p_0}(N_t=p)\to\delta_{p,p_0}$ as $t\to 0^+$ [19], which is the case in our configuration. What remains is to find the values of the limits. The proof is based on elementary tools of Markov processes theory [17]. The process is irreducible within the class $\mathcal{A}$. It is either recurrent or transient. In the transient case $\Sigma_n Q^n<+\infty$ in $\mathcal{A}\times\mathcal{A}$. From the relation

$$\int_0^\infty \mathbf{P}_{0,p_0}(N_t=p)dt=\sum_n (Q^n)_{p_0,p}/q(p),$$

we deduce that $\int_0^\infty C_p(t)dt<\infty$ and necessarily $C_p(t)\to 0$ as $t\to\infty$.

In the recurrent case $\Sigma Q^n\equiv+\infty$ in $\mathcal{A}\times\mathcal{A}$ and there exists a positive measure over $\mathcal{A}$, unique up to a multiplicative constant, which is invariant with respect to Q: $\mu Q=\mu$. More exactly, from the definition of the matrix Q and coefficients $q(p)$:

$$\mu Q=\mu \Leftrightarrow \sum_{p'} \Gamma_{p+p',p}\mu(p+p')=\sum_{p'} \Gamma_{p,p+p'}\mu(p).$$

Since $\Gamma_{p,p+p'}=\Gamma_{p+p',p}$, we get that μ must satisfy for every integer p:

$$\sum_{p'} \Gamma_{p+p',p}(\mu(p+p')-\mu(p))=0.$$

Thus the invariant measure is simply the uniform measure over $\mathcal{A}$. Consequently, if $|\mathcal{A}|<\infty$, there exists an invariant probability measure (i.e., with total mass 1), which is $\mu(p)=1/|\mathcal{A}|$, and $C_p(t)$ converges to $\mu(p)$ by the ergodic theorem for Markov chains. If $|\mathcal{A}|=\infty$, then the ergodic theorem implies that $C_p(t)$ converges to 0. □

The second case of the above proposition is a very special case, which occurs only for very particular configurations. We shall see an example in Sec. VII. The general configuration is indeed the first one, and we can be much more precise under complementary assumptions. We warn the reader that the hypotheses of the following proposition may seem very strange and restrictive, but they are actually fulfilled in many examples, in particular, in the configurations that will be examined in the further sections.

Proposition IV.2. Let us assume that $|\mathcal{A}|=\infty$ and that there exists $c<1$ such that for every positive integer n, the following limits exist:

$$\frac{\sum_{p'=-\infty}^{\infty} p'^n\Gamma_{p,p+p'}}{p^{c+n-1}} \xrightarrow[p\to\infty]{} a_n.$$

If $a_n=0$ for $n\geqslant 3$, and a_1 or a_2 is different from 0 and satisfy $ca_2\leqslant 2a_1$, then

$$\frac{N_t}{t^{1/(1-c)}} \xrightarrow[t\to\infty]{} \begin{cases} \left(\dfrac{(1-c)a_1}{2}\right)^{1/(1-c)} & \text{if } a_2=0 \\ \left(\dfrac{(1-c)^2a_2}{4}\right)^{1/(1-c)} Z & \text{if } a_2>0, \end{cases}$$

where Z is a random variable whose distribution has a density with respect to the Lebesgue measure over $(0,\infty)$:

$$p_Z(z)=\frac{1-c}{\Gamma\left(\dfrac{2a_1-ca_2}{(1-c)a_2}\right)} z^{-1+(2a_1-ca_2)/a_2}\exp(-z^{1-c}),$$

where Γ is the so-called Euler's Gamma function $\Gamma(r):=\int_0^\infty s^{r-1}e^{-s}ds$.

Proof. Since $C_p(t)\to 0$ for any fixed p, the long-time behavior of N_t will depend on the coefficients $\Gamma_{p,p+p'}$ with large indices p. Let r be positive real. Since $\mathbb{E}[N_t^r]=\sum_{p=0}^\infty p^r C_p(t)$, we have

$$\frac{d\mathbb{E}[N_t^r]}{dt}=\frac{1}{2}\sum_{p=0}^\infty p^r\sum_{p'=-\infty}^\infty \Gamma_{p,p+p'}(C_{p+p'}-C_p)$$

$$=\frac{1}{2}\sum_{p=0}^\infty\sum_{p'=-\infty}^\infty (p-p')^r\Gamma_{p-p',p}C_p-p^r\Gamma_{p,p+p'}C_p.$$

Changing p' into $-p'$ in the first term of the double sum, and using the fact that $\Gamma_{p+p',p}=\Gamma_{p,p+p'}$, we get that

$$\frac{d\mathbb{E}[N_t^r]}{dt}=\frac{1}{2}\sum_{p=0}^\infty\sum_{p'=-\infty}^\infty [(p+p')^r-p^r]\Gamma_{p,p+p'}C_p$$

$$\simeq\frac{r^2a_2+r(2a_1-a_2)}{4}\sum_{p=0}^\infty p^{r-1+c}C_p.$$

Choosing $r=n(1-c)$, $n\in\mathbb{N}$ and integrating, we get that the moments of the process N_t^{1-c} satisfy

$$\mathbb{E}\left[\left(\frac{N_t^{1-c}}{t}\right)^n\right] \xrightarrow[t\to\infty]{} \prod_{j=1}^n \frac{j(1-c)^2a_2+(1-c)(2a_1-a_2)}{4},$$

from which we can deduce the result at hand. □

In the following sections we apply the general results derived in this paper to some particular and relevant situations.

V. A LINEAR PERTURBATION

This kind of perturbation corresponds to the evolution of a charged particle driven by a randomly time-dependent electric field. More exactly, let us assume that the particle possesses a charge q. Suppose that we apply an external, homogeneous, and time-dependent electric field $\mathcal{E}_0 m(t)$. The dimensionless function m describes the time fluctuations of the field. This corresponds to an electrostatic potential $\mathcal{E}_0 xm(t)$ and a potential energy $-q\mathcal{E}_0 xm(t)$. The dimensionless quantity ϵ is a parameter, which characterizes the amplitudes of the fluctuations and is defined by

$$\varepsilon := \frac{q\mathcal{E}_0}{\hbar^{1/2} M^{1/2} \omega^{3/2}}.$$

This situation is a particular case of the general framework discussed in Sec. III with $V(t,x)=2xm(t)$. In such conditions, $\Gamma_{p,p+p'}=\gamma^2_{p,p+p'}\alpha'_p$, where $\alpha_{p'}$ is given by Eq. (17) and

$$\gamma^2_{p,p+p'} = \begin{cases} 2(p+1) & \text{if } p'=1 \\ 2p & \text{if } p'=-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Sigma_{p'} p' \Gamma_{p,p+p'}=2$ and $\Sigma_{p'} p'^2 \Gamma_{p,p+p'}=4p+2$. By applying Proposition IV.2 with $c=0$, $a_1=2\alpha_1$, and $a_2=4\alpha_1$, we get that $N_t/(\alpha_1 t)$ converges as $t\to\infty$ to a random variable with exponential density and mean 1, which reads in terms of the family of probabilities $[C_p(t)]_{p\in\mathbb{N}}$ as

$$C_p(t) \underset{t\gg 1}{\simeq} \frac{1}{\alpha_1 t}\exp\left(-\frac{p}{\alpha_1 t}\right). \tag{24}$$

Besides, in this particular case, we are able to solve explicitly homogeneous linear system (21) with any initial condition. If the initial state is f_{p_0}, then the distribution C_p is for every time t [11, Prop. 4.3]:

$$C_p(t) = \sum_{j=p_m}^{p_0} \frac{p_0! p!}{(p_0-j)!^2 j! (p-p_0+j)!} \frac{(\alpha_1 t)^{p+2j-p_0}}{(1+\alpha_1 t)^{p+p_0+1}},$$

where $p_m := \max(p-p_0,0)$. We can then check that these expressions are consistent with long-time behavior (24). Furthermore, the energy growth is linear:

$$E(t) = E(0) + \alpha_1 t,$$

which was first established in Ref. [10].

VI. A GAUSSIAN PERTURBATION

We consider in this section the case of a spatially Gaussian potential $V(t,x)=m(t)\exp(-x^2)$. Then $\Gamma_{p,p+p'}=\gamma^2_{p,p+p'}\alpha'_p$, where $\alpha_{p'}$ is given by Eq. (17) and, from tabulated formulas [20, formula 7.374.2], we get

$$\gamma^2_{p,p+2p'} = \frac{(2p+2p'-1)!^2 2^{-4p-4p'+1}}{p!(p+p'-1)!^2(p+2p')!},$$

$$\gamma^2_{p,p+2p'+1} = 0.$$

Since $\exp(-x^2)$ is an even function, odd transitions corresponding to $\gamma^2_{p,p+2p'+1}$ are forbidden. As a consequence, if the initial state is an even (respectively, odd) eigenstate, then only even (respectively, odd) eigenstates can be observed. We shall assume in what follows that the initial state is even.

A. Narrow-band coupling

We assume in this section that the power spectrum of the perturbation m has compact support so that $\alpha_2>0$ and $\alpha_j=0$ for $j\geqslant 4$. After some algebra one can establish that

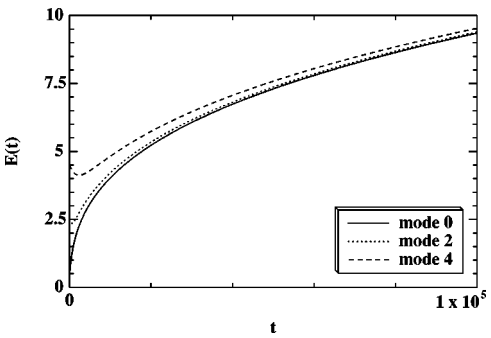

FIG. 2. Expected energy $E(t)$ of the particle under time-dependent perturbations with spatial Gaussian shape in the narrow-band case with $\alpha_2=1/(12\pi)$. The particle is assumed to be at $t=0$ in the state 0, 2, or 4. Remember that all quantities are dimensionless by the transformations explained in Sec. II.

$$\sum_{p'=-\infty}^{\infty} p' \Gamma_{p,p+p'} = 2\alpha_2(\gamma^2_{p,p+2} - \gamma^2_{p,p-2}) \underset{p\gg 1}{\simeq} -\frac{2\alpha_2}{\pi p^2},$$

$$\sum_{p'=-\infty}^{\infty} p'^2 \Gamma_{p,p+p'} = 4\alpha_2(\gamma^2_{p,p+2} + \gamma^2_{p,p-2}) \underset{p\gg 1}{\simeq} \frac{4\alpha_2}{\pi p}.$$

Applying Proposition IV.2 with $c=-2$, $a_1=-2\alpha_2/\pi$, and $a_2=4\alpha_2/\pi$, we get that the jump process N_t associated with this configuration satisfies

$$\frac{N_t}{t^{1/3}} \xrightarrow[t\to\infty]{} z_{nc} Z_{nc},$$

where $z_{nc} := (9\alpha_2/\pi)^{1/3}$ and Z_{nc} is the positive-real-valued random variable whose distribution is characterized by the probability density

$$p_{nc}(z) = \frac{3}{\Gamma(1/3)} e^{-z^3},$$

where $\Gamma(1/3)\simeq 2.679$. Computing $\mathbb{E}[Z_{nc}] = \sqrt{3}\Gamma(2/3)^2/(2\pi) \simeq 0.505$, the expected energy grows as $t^{1/3}$ for large t:

$$E(t) \underset{t\gg 1}{\simeq} z_{nc}\mathbb{E}[Z_{nc}] t^{1/3},$$

and the histogram $[C_p(t)]_{p\in\mathbb{N}}$ is asymptotically

$$C_{2p}(t) \underset{t\gg 1}{\simeq} \frac{6}{\Gamma(1/3) z_{nc} t^{1/3}} \exp\left(-\frac{8p^3}{z_{nc}^3 t}\right). \tag{25}$$

The resolution of Eq. (21) by a standard numerical routine (see Fig. 2) confirms the $t^{1/3}$ long-time behavior of the expected energy, but it also reveals that the short-time behavior of the expected energy strongly depends on the initial state. If the initial state is a low-order eigenstate, then the energy first grows quite rapidly to reach the asymptotic $t^{1/3}$ rate. If the initial state is a high-order eigenstate, then the energy first decreases, so that the long-time curves corresponding to the different initial states are quite close to each other.

This dynamics is made more transparent when analyzing the evolutions of the probability distributions $[C_p(t)]_p$ of the energy. In Figs. 3–5 one can observe the different steps of

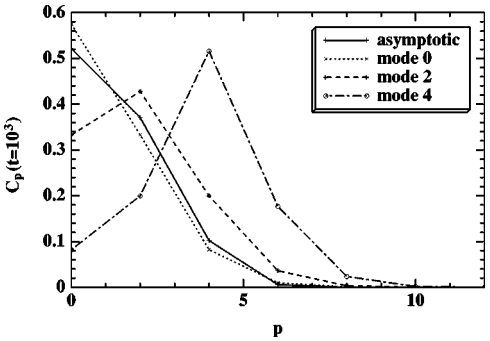

FIG. 3. Theoretical histograms of the probability distributions $[C_p(t)]_{p\in\mathbb{N}}$ of the state of the particle under time-dependent perturbations with spatial Gaussian shape in the narrow-band case with $\alpha_2=1/(12\pi)$. The particle is assumed to be at $t=0$ in the state 0, 2, or 4. Here $t=10^3$. Only even states have positive probabilities. The solid line corresponds to approximate formula (25). It is clear that the histograms have still the memory of the initial state.

the reorganizations of the probability distributions C_p for different initial conditions. It appears that if the initial state is a low-order eigenstate [i.e., $C_p(0)=\delta_{p,p_0}$ for small p_0], then the short-time dynamics consists in filling high-order probabilities in order to get asymptotic shape (25). Conversely, if the initial state is a high-order eigenstate [$C_p(0)=\delta_{p,p_0}$ for large p_0], then the short-time dynamics consists in filling the low-order probabilities in order to get the asymptotic shape. That is why the expected energy either grows or decreases if the initial state is either low or high order. Furthermore, the convergence to an asymptotic shape whatever the initial state guarantees that the expected energy will have a long-time behavior independent of the initial condition.

B. Broadband coupling

We assume here that the power spectrum of the perturbation m is flat so that $\alpha_{p'}\equiv\alpha$ for every p'. Since

$$\sum_{p'=-\infty}^{\infty} p'\Gamma_{p,p+p'} \underset{p\gg 1}{\simeq} \frac{\alpha}{4\sqrt{\pi}\sqrt{p}},$$

$$\sum_{p'=-\infty}^{\infty} p'^2\Gamma_{p,p+p'} \underset{p\gg 1}{\simeq} \frac{\alpha\sqrt{p}}{\sqrt{\pi}},$$

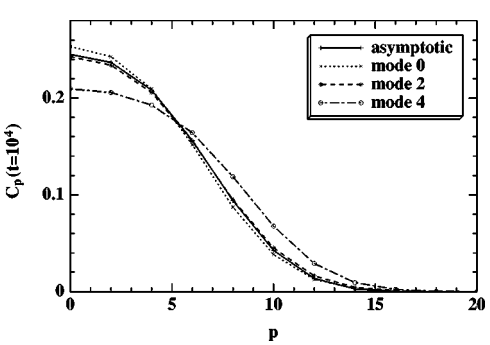

FIG. 4. The same as in Fig. 3, but at $t=10^4$. The histograms corresponding to the cases where the initial state is the fundamental or the second mode are identical and coincide with asymptotic formula (25), while the histogram corresponding to the initial state f_4 is still different.

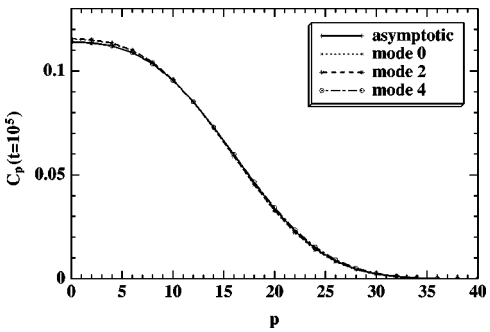

FIG. 5. The same as in Fig. 3, but at $t=10^5$. All histograms are identical and coincide with asymptotic formula (25).

we get by applying Proposition IV.2 with $c=-1/2$, $a_1=\alpha/(4\sqrt{\pi})$, and $a_2=\alpha/\sqrt{\pi}$, that the jump process N_t associated with this configuration satisfies

$$\frac{N_t}{t^{2/3}} \xrightarrow[t\to\infty]{} z_{bc}Z_{bc},$$

where $z_{bc}:=[9\alpha/(16\sqrt{\pi})]^{2/3}$ and Z_{bc} is the positive-real-valued random variable whose distribution is characterized by the probability density

$$p_{bc}(z)=\frac{3}{2\Gamma(2/3)}e^{-z^{3/2}},$$

where $\Gamma(2/3)\simeq 1.354$. Computing $\mathbb{E}[Z_{bc}]=2\sqrt{3}\pi/[9\Gamma(2/3)]^2\simeq 0.659$, the expected energy grows as $t^{2/3}$ for long t:

$$E(t) \underset{t\gg 1}{\simeq} z_{bc}\mathbb{E}[Z_{bc}]t^{2/3},$$

and the histogram $[C_p(t)]_{p\in\mathbb{N}}$ is asymptotically

$$C_{2p}(t) \underset{t\gg 1}{\simeq} \frac{3}{\Gamma(2/3)z_{bc}t^{2/3}}\exp\left(-\frac{(2p)^{3/2}}{z_{bc}^{3/2}t}\right). \tag{26}$$

Figure 6 plots the probability distributions $[C_p(t)]_{p\in\mathbb{N}}$ at a given time for different initial conditions. Comparing with Figs. 3–5, it clearly appears that the coupling mechanisms, which tend to strengthen the probabilities corresponding to

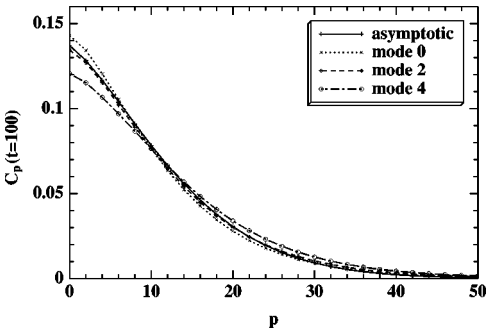

FIG. 6. Theoretical histograms of the probability distributions $[C_p(t)]_{p\in\mathbb{N}}$ of the state of the particle under time-dependent perturbations with spatial Gaussian shape in the broadband case with $\alpha=1/(12\pi)$. The particle is assumed to be at $t=0$ in the state f_0, f_2, or f_4. Here $t=100$. Only even states have positive probabilities. The solid line corresponds to approximate formula (26).

the high-order eigenstates, are quicker than in the narrow-band case. Indeed, the flat power spectrum of process m makes possible long-range jumps of the process N_t, while a narrow-band spectrum only authorizes jumps to the nearest neighbors. That is why the energy grows much quicker ($\sim t^{2/3}$) in the broadband case than in the narrow-band case ($\sim t^{1/3}$). In the general case, that is to say whatever the spectrum of the perturbation, the energy grows at a rate between the above regimes, i.e., between $t^{1/3}$ and $t^{2/3}$.

VII. A SPATIALLY PERIODIC PERTURBATION

We consider in this section the case of a spatially periodic potential $V(t,x)=m(t)\cos(bx)$. Then $\Gamma_{p,p+p'}=\gamma^2_{p,p+p'}\alpha'_p$, where $\alpha_{p'}$ is given by Eq. (17) and, using tabulated formulas [20, formula 7.388.7]:

$$\gamma^2_{p,p+2p'}=\left(\frac{b^2}{2}\right)^{2p'}\frac{p!}{(p+2p')!}e^{-b^2/2}L_p^{2p'}\left(\frac{b^2}{2}\right)^2,$$

$$\gamma^2_{p,p+2p'+1}=0,$$

where the L_n^m are the Laguerre polynomials defined by [20, formula 8.970]

$$L_n^m(x):=\frac{1}{n!}e^x x^{-m}\frac{d^n}{dx^n}(e^{-x}x^{n+m}).$$

For large p and even p' smaller than p, we have [20, formula 8.978.3]

$$\gamma^2_{p,p+p'}\simeq\frac{1}{\pi}\left(\frac{b^2p}{2}\right)^{-1/2}\cos^2\left(\sqrt{2pb^2}-\frac{\pi}{4}\right). \qquad (27)$$

We shall assume that the initial state is an eigenstate of even index p_0. Only the probabilities corresponding to even eigenstates are weighted by the effective evolution, as shown by the fact that $\gamma_{p,p+p'}$ is zero if p' is odd. This is, of course, also readable on the spatial form of the perturbation, which is the even function cosine.

A. Narrow-band coupling

In order to have a clear and simple presentation, we shall first assume that the process m has a power spectral density with compact support and that only α_2 is different from 0, so that the set of evolution equations reduces as

$$\frac{dC_{2p}}{dt}=\frac{\alpha_2}{2}[\gamma^2_{2p,2p+2}(C_{2p+2}-C_{2p}) + \gamma^2_{2p-2,2p}(C_{2p-2}-C_{2p})]. \qquad (28)$$

The structure of the coupling coefficients $\gamma^2_{p,p+2}$ is shown on Fig. 7. As demonstrated also by formula (27) the coefficients get almost zero at some indices $2p_1,2p_2,\ldots$. Evolution (28) strongly depends on the coefficients, which are zero or very close to 0. Let us assume first that there exists an even index such that $\gamma_{p,p+2}=0$ and denote by $2p_1$ (respectively, $2p_2$) the largest (respectively, smallest) index smaller (respectively, larger) than p_0, which satisfies this vanishing condition. If all coefficients with indices smaller than p_0 are positive, then $p_1=0$. If all coefficients with indices larger than p_0 are positive, then $p_2=\infty$. Then no transition towards eigenstates of indices larger than $2p_2$ and smaller than $2p_1$ is possible. Applying Proposition IV.1 yields that the family $C_p(t)$ converges as $t\to\infty$ to

$$C_{2p}(\infty)=\begin{cases}\dfrac{1}{p_2-p_1} & \text{if } p_1+1\leqslant p\leqslant p_2\\ 0 & \text{if } p>p_2 \text{ or } p\leqslant p_1.\end{cases}$$

In terms of the expected energy $E(t)=1/2+\Sigma_p pC_p(t)$, this corresponds to the asymptotic value

$$E(\infty)=p_1+p_2+3/2.$$

If there exists no vanishing coupling coefficients, there exists nevertheless coefficients whose values are very close to zero as shown by Eq. (27). The transitions through such indices are very slow and will actually be the ones that will impose the evolution rate of the probabilities (C_p).

Figure 8 plots the expected energy for different initial conditions. It appears that the behavior of the energy depends on the initial state f_{p_0}. The mechanism behind this statement can be made more transparent when regarding the evolutions of the probability distributions (C_p): Figure 9 puts into evidence that the histograms strongly depend on the initial state. To understand the picture it is necessary to remember from Fig. 7 that there exists two indices, namely,

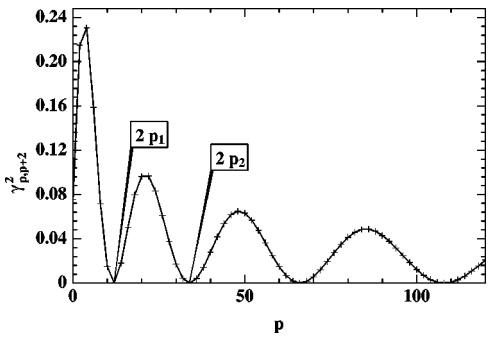

FIG. 7. Coupling coefficients $\gamma^2_{p,p+2}$ for even p for a spatially periodic potential with period $b^{-1}=1$. Note in particular that the first indices, which correspond to very small values, are $2p_1=12$ and $2p_2=34$.

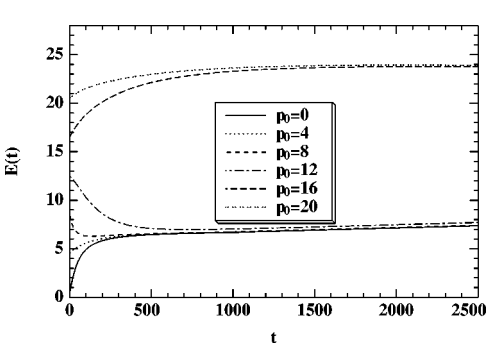

FIG. 8. Expected energy $E(t)$ of the particle under time-dependent perturbations with spatially periodic shape ($b=1$) in the narrow-band case ($\alpha_2=1$). The particle is assumed to be at $t=0$ in the state $f_0,f_4,f_8,\ldots,$ or f_{20}.

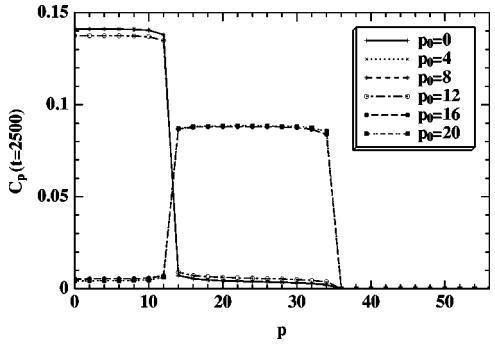

FIG. 9. Theoretical histograms of the probability distributions $[C_p(t)]_{p\in\mathbb{N}}$ of the state of the particle under time-dependent perturbations with spatially periodic shape ($b=1$) in the narrow-band case ($\alpha_2=1$). The particle is assumed to be at $t=0$ in the state f_0, f_4, f_8, . . . , or f_{20}. Here $t=2.5\times10^3$.

$2p_1=12$ and $2p_2=34$, which correspond to very small values of the coupling coefficients $\gamma^2_{2p,2p+2}$. Consequently, if $p_0\leqslant 2p_1$, then the histograms first tend to a uniform distribution over $[0,2p_1]$, but for times longer than a few hundreds there is also some leakage to the second area $[2p_1+2,2p_2]$ (see Fig. 10). In terms of the expected energy, this implies that $E(t)\simeq p_1+1/2=6.5$ for times of the order of a few hundreds. Nevertheless, due to the slight leakage towards high-order states, the energy then grows beyond this value at a slow rate. If $2p_1+2\leqslant p_0\leqslant 2p_2$, then the histograms first tend to a uniform distribution over $[2p_1+2,2p_2]$, but there is also some leakage to the first area $[0,2p_1]$ for times of the order of a few hundreds. One can also notice that there is almost no leakage towards $p>2p_2$, and this is related to the fact that the value $\gamma_{2p_2,2p_2+2}\simeq 9.2\times10^{-5}$ is much closer to 0 than $\gamma_{2p_1,2p_1+2}\simeq 2.9\times10^{-2}$. In terms of the expected energy, this implies that $E(t)\simeq p_1+p_2+3/2=24.5$ for times of the order of a few hundreds. For longer times, due to the leakage towards the eigenstates of indices smaller than $2p_1$, the energy decreases below the value 24.5 at a slow rate. Finally, note in Fig. 11 that for very long times (of the order 10^5), the small leakages through $2p_1$ lead to a uniform distribution of the probabilities C_{2p} over $[0,2p_2]$ for all initial configurations f_{p_0} with $p_0\leqslant 2p_2$. If one waits even longer, one will be able to detect

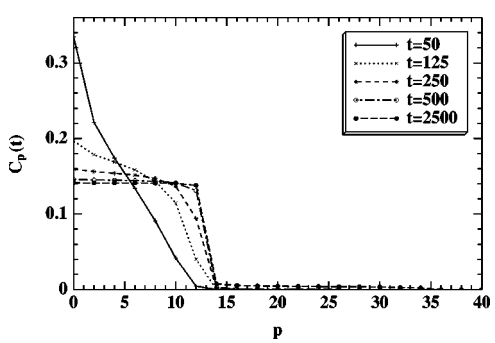

FIG. 10. Theoretical histograms of the probability distributions $[C_p(t)]_{p\in\mathbb{N}}$ of the state of the particle under time-dependent perturbations with spatially periodic shape ($b=1$) in the narrow-band case ($\alpha_2=1$). The particle is assumed to be at $t=0$ in the fundamental state 0.

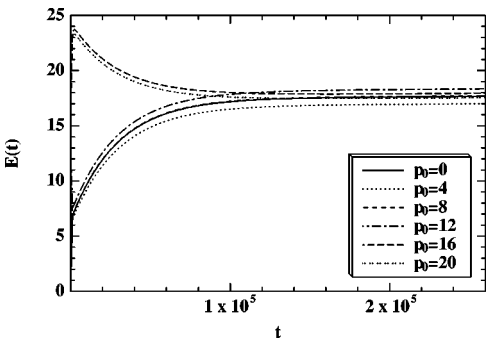

FIG. 11. The same as in Fig. 10, but the time evolution of the expected energy $E(t)$ is plotted for long times.

a small leakage through $2p_2$ towards eigenstates of indices larger than $2p_2$ and so on.

B. Broadband coupling

Let us now analyze the case of a broadband perturbation, which has a large number of positive coefficients $\alpha_{2p'}$. We can expect that such a broadband coupling allows jumps with a range larger than 2 and makes it possible to bypass the difficult points exhibited here above. This argument is partially true. Indeed, Fig. 12 plots the coefficients $\gamma^2_{p,p+p'}$ as functions of p for different p'. It clearly shows that all curves have similar shapes, but the slow points are shifted by a quantity, which depends on p'. Unfortunately, for large p the indices for which $\gamma_{p,p+p'}$ are very small do not depend on p', as shown by formula (27). Let us assume there exists a large integer p_s, the so-called bandwidth, such that $\alpha_{p'}=\alpha$ if $|p'|\leqslant p_s$ and 0 otherwise. In such conditions, an eigenstate with energy $p+1/2$ will be called a slow point if the values of the coefficients $\gamma^2_{p,p+p'}$ are very small for all $|p'|\leqslant p_s$, in the sense that they are much smaller than the typical value $p^{-1/2}$ exhibited by formula (27). For large indices p, the coefficients $\gamma_{p,p+p'}$ can be approximated by Eq. (27), so that it appears that the slow points are grouped around particular points p_k that are indexed by an integer k and expressed as the integral parts of $\pi^2/(2b^2)(k-1/4)^2$. More exactly, around p_k, the coupling coefficients can be estimated by

$$\sup_{|q_0|,|q_1|\leqslant p'} \gamma^2_{p_k+q_0,p_k+q_1}\sim \frac{bp'^2}{\sqrt{2}\pi p_k^{3/2}} \quad \text{for } p'\ll p_k .$$

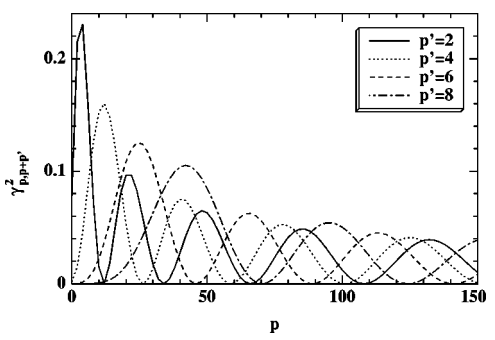

FIG. 12. Coupling coefficients $\gamma^2_{p,p+p'}$ for even p and p' for a spatially periodic potential with period $b^{-1}=1$.

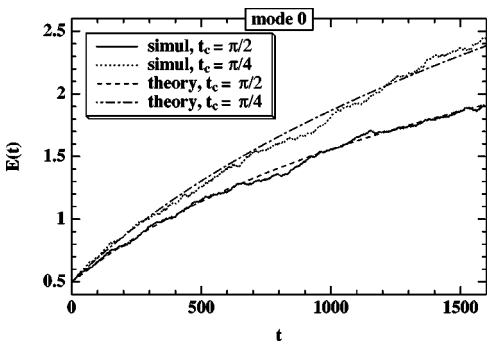

FIG. 13. Evolution of the energy of the particle computed by simulation of perturbed Schrödinger equation (8) and averaged over 200 realizations and in comparison with the theoretical formulas given by Eqs. (21) and (20). The initial state vector is the fundamental state f_0.

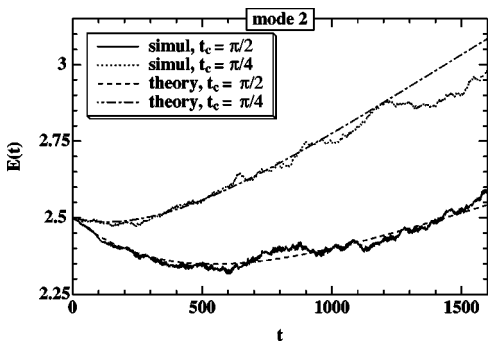

FIG. 14. The same as in Fig. 13, but the initial state vector is the second mode f_2.

Consequently, if $p \ll p_s^2$, then there exists $p' < p_s$ such that $\gamma^2_{p,p+p'} \sim p^{-1/2}$, which means that the broadband coupling is sufficient for eliminating the slow points. But, for $p \gg p_s^2$, there are intervals around p_k of length larger than $2p_s$, which consist of successive states that are slow points; the jump process cannot avoid the slow points anymore; the corresponding coupling coefficients have small values and $\sup_{|p'| \leqslant p_s} \gamma^2_{p,p+p'}$ is of the order $p_s^2 p^{-3/2} \ll p^{-1/2}$. Therefore, there are always slow points, whatever the spectrum of the random process m, and the discussion we have presented in the narrow-band case still holds true qualitatively.

VIII. NUMERICAL SIMULATIONS

The results in the previous sections are theoretically valid in the limit case $\varepsilon \to 0$, where the amplitude (respectively, duration) of the perturbations goes to zero (respectively, infinity). In this section we aim at showing that the asymptotic behaviors of the state vector can be easily observed in numerical simulations in the case where the perturbation is small, so that its effect appears after a long time. We use a split-step method to simulate the one-dimensional perturbed linear Schrödinger equations (8), and more exactly a fourth-order variant [21], so that we obtain a reliable numerical algorithm, which provides accurate solutions even for a long computational time domain. We adopt in this section the following model for the perturbation:

$$V(t,x) = m(t)\exp(-x^2),$$

$$m(t) = u_l \text{ if } l + a_0 \leqslant \frac{t}{t_c} < l + a_0 + 1,$$

where $(u_l)_{l=0,\ldots,M-1}$ is a sequence of M-independent and identically distributed variables, which obey uniform distributions over the interval $[-1/2,1/2]$, and a_0 is a random variable independent of u, which also obeys an uniform distribution over $[-1/2,1/2]$. t_c is the so-called coherence time of the random process m. The autocorrelation function of the ergodic process m is equal to $\mathbb{E}[m(0)m(t)] = \frac{1}{12}(1 - t/t_c)\mathbb{I}_{t \leqslant t_c}$, so that the power spectral density defined by Eq. (17) is

$$\alpha_{p'} = \frac{1}{12} \frac{1 - \cos(p' t_c)}{p'^2 t_c},$$

and $\alpha_0 = t_c/24$. The quantity $M t_c$, which is equal to the time duration of the perturbation, will be chosen so large that we can observe the effect of the small perturbation εV. We measure the L^2 norm and energy (19) of the particle that we can compare with the corresponding data of the incident state vector. We present results corresponding to simulations where the initial state at $z = 0$ is the fundamental Gaussian mode $f_0(x)$ or the second mode $f_2(x)$. We have first simulated the homogeneous Schrödinger equation (with $V \equiv 0$), which admits as an exact solution $f_0(x)e^{-it/2}$ and $f_2(x)e^{-5it/2}$, respectively. We can, therefore, check the accuracy of the numerical method, since we can see that the modulus of the computed solution maintains a very close resemblance to the initial field (data not shown), while the L^2 norm and the energy are almost constant. The other simulations are carried out with different realizations of the random process m with $\varepsilon = 1$. In Figs. 13 and 14 we present the simulated evolutions of the energy of the particle averaged over 200 realizations and compare them with the mean theoretical evolutions given by Eq. (21) in the scale t/ε^2. It thus appears that the numerical simulations are in very good agreement with the theoretical results. All these observations confirm that system (21) describes with accuracy the evolution of the harmonic oscillator under small perturbations and long times.

IX. CONCLUSION

We have analyzed in this paper the effects of very general types of random perturbations on the evolution of a quantum particle. The precise results we have got are limited to the quantum harmonic oscillator. For such a system the eigenvalues and eigenstates are explicit and tabulated formulas are available, so that we are able to perform exact calculations and compute closed-form expressions for the eigenstates probabilities. Nevertheless, the results demonstrated in this paper can be generalized to a large class of systems. Indeed, the basic assumption, which requires the existence of a complete set of normal eigenstates for the unperturbed Hamiltonian, holds true for most of them. The only—but important—restriction to insure that our results can be applied is the nondegeneracy of the energy levels. The remain-

der of the study then consists in technical developments to exhibit and analyze the coupling mechanisms between the eigenstates. Furthermore, we feel that the problem corresponding to a degenerate system could be addressed with the above techniques with some more work, although it will involve a mechanism more complicated than the jump process exhibited in this paper. This will be the subject of a further work.

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