Statistical analysis of the sizes and velocities of laser hot spots of smoothed beams

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This paper presents a precise description of the characteristics of the hot spots of a partially coherent pulse. The average values of the sizes and velocities of the hot spots are computed, as well as the corresponding probability density functions. Applications to the speckle patterns generated by optical smoothing techniques for uniform irradiation in plasma physics are discussed.
Keywords. Speckle patterns, hot spots, statistics.

I. INTRODUCTION

Partially coherent light has become a subject of great interest for many applications such as smoothing techniques for uniform irradiation in plasma physics [1]. This paper is a contribution to the study of optical smoothing for application to Inertial Confinement Fusion (ICF), which requires a high level of irradiation uniformity for both direct and indirect drive. This criterion can be reached by implementing active smoothing methods, such as Induced Spatial Incoherence (ISI) with echelons [2], Smoothing by Spectral Dispersion (SSD) [3], Smoothing by multi-mode Optical Fiber (SOF) [4]. All these methods involve the illumination on the target with an intensity which is a time varying speckle pattern, so that the time integrated intensity averages towards a flat profile. We aim in this paper at studying the statistical properties of the hot spots of a partially coherent pulse. It is relevant to report a precise account of these properties because the hot spots play a primary role in that they can give rise to the growth of laser-plasma instability such as filamentation, Stimulated Brillouin Scattering (SBS), and Stimulated Raman Scattering (SRS) [5].

The future French Laser MegaJoule (LMJ) and the US National Ignition Facility (NIF) [6] are designed for the indirect drive scheme, whose principle is the following. Numerous laser beams are focused into a hohlraum and irradiate the inner gold wall to produce X-rays. However it was found that it is necessary to diminish the wall expansion by filling the cavity by a gas. Unfortunately this means that the laser light has to propagate through an underdense plasma where parametric instabilities such as SRS or SBS may occur. Since experimental evidence of SRS and SBS in hohlraums from randomized laser beams have been observed [7–11], the control of high laser intensities has become of crucial importance. The literature contains a lot of work which deals with self-focusing (SF) and/or parametric instabilities from optically smoothed beams. The theoretical papers develop careful methods to study SBS and/or SF [12–14] from a single speckle and then average the single hot spot reflectivity over the statistics of hot spots. It is thus of great interest to have precise expressions for the joint probability distributions of the intensities of the hot spots of a speckle pattern, their transverse and longitudinal lengths, as well as their velocities.

The statistical distribution of the top intensities of the hot spots can be calculated theoretically for a focal spot in vacuum or homogeneous medium, in the asymptotic framework where the number of spatial modes is large, so that the Gaussian limit is valid [15,16]. The corresponding distribution is then entirely characterized by the Autocorrelation Function (AF) of the field, which can also be computed very precisely for any smoothing method [17]. The theory developed in Refs. [15,16] and refined in Ref. [18] provides accurate expressions for the number of local maxima that exceed a given intensity level. But the statistical description of the radius and velocity of a speckle spot has not yet been performed. We can find preliminary results in Ref. [19] where the mean values of the velocities are computed for some specific configurations. We shall extend these results by addressing more general configurations and by deriving precise expressions for the Probability Density Functions (PDF) of the radius and the velocity of the speckle spots.

This work is triggered by the study of the statistics of the hot spots of the speckle patterns generated by Random Phase Plates (RPP) [20] or Kinoform Phase Plates (KPP) [21]. It also deals with active smoothing methods, such as ISI, SSD, or SOF. Although thorough attention is devoted to these cases, the formulae we derive in our paper can be applied to more general situations where the Gaussian limit is valid. This is the case as soon as the pattern consists of the superposition of many independent modes by the Central Limit Theorem [22].

The paper is organized as follows. We derive the AF of the speckle pattern generated by the different optical smoothing techniques in Sec. III. We describe the local maxima of a generic speckle pattern in terms of its AF in Sec. IV. We apply these results to compute the PDF of the duration, the radius and the velocity of a speckle spot in Secs. V, VI, and VII, respectively.

II. PARTIALLY COHERENT PULSES

We shall denote by A the complex electromagnetic field. The field is normalized so that the laser intensity is simply $|A|^2$. We shall derive our results in the framework where the field A has Gaussian statistics with stationary AF. This corresponds to the asymptotic framework when the field is the superposition of a large number of incoherent modes. It is the case for the standard optical smoothing methods [17].

In Ref. [18] we give closed-form expressions for the density $M(I)$ of hot spots per unit volume with top intensity I. This is important since parametric instabilities are very sensitive to the top intensities of the hot spots. However this description is not sufficient, since the transverse radius r, the length z of the hot spots, as well as their duration t and their velocity $\mathbf{v}$, should be taken into account. We shall show in the following that the quantities $(r, z, t, \mathbf{v})$ are random quantities whose statistical distributions depend on the top intensity I of the corresponding hot spot, and can be described by a PDF $p_I(r, z, t, \mathbf{v})$ indexed by I. Accordingly, the density $M(I, r, z, t, \mathbf{v})$ of hot spots per unit volume with top intensity I, transverse radius r, longitudinal length z, duration t, and velocity $\mathbf{v}$ is given by:

$$M(I, r, z, t, \mathbf{v}) = M(I) p_I(r, z, t, \mathbf{v})$$

In this paper, we shall focus on the PDF $p_I(r, z, t, \mathbf{v})$ since $M(I)$ is already known.

III. OPTICAL SMOOTHING TECHNIQUES

We shall focus our attention to standard smoothing techniques, although the general formulas we shall present can be applied to other configurations. In this section we present the active smoothing methods that are practically relevant. We also compute the four-dimensional (3D in space and 1D in time) AFs of the fields generated by these smoothing techniques, as it will appear in the following that these functions contain all the information required for the derivation of the results.

A. Smoothing by Spectral Dispersion

A straightforward implementation of one-dimensional Smoothing by Spectral Dispersion (1D-SSD) is shown in Figure 1a [3]. The spectral dispersion is obtained by applying a phase modulator then a grating. An auxiliary grating is imposed before the modulator in order to compensate for the temporal skew of the global shape of the field. We assume that the input beam is a monochromatic pulse with wavelength λ_0 and duration T_{pulse} (longer than all the characteristic time scales involved in the problem) and we consider a pure phase modulator. After modulation and dispersion the field is:

$$A_1(x, y, t) = \exp\left(i\phi(t + sx)\right) A_0(t),$$

where ϕ is the phase modulation and s is the temporal skew per unit length generated by the grating in the x-direction.

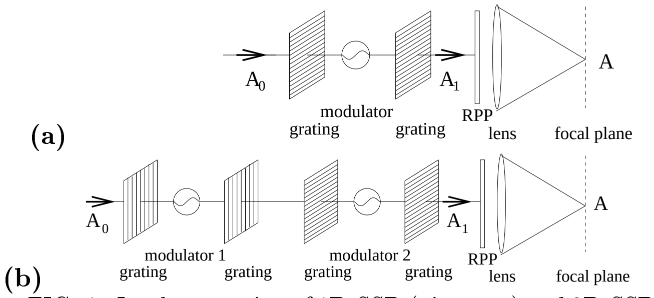


FIG. 1. Implementation of 1D-SSD (picture a) and 2D-SSD (picture b).

A two-dimensional version of this technique is possible (see Figure 1b). It consists in applying a phase modulator between a pair of gratings in each orthogonal direction [23]. After modulation and dispersion the field is:

$$A_1(x, y, t) = \exp i\left(\phi_1(t + s_1 x) + \phi_2(t + s_2 y)\right) A_0(t),$$

where ϕ_j is the phase modulation and s_j is the temporal skew per unit length generated by the gratings in the x-direction for $j = 1$ and in the y-direction for $j = 2$.

In the following we shall neglect the time variations of the overall envelope A_0. In the focal plane of the lens the fields of the beamlets overlap:

$$A(x, y, t) = \sum_{j,l} A_1(jh, lh, t) e^{-ijx\frac{2\pi h}{\lambda_0 f}} e^{-ily\frac{2\pi h}{\lambda_0 f}} e^{i\phi_{j,l}},$$

where D is the near field square beam aperture, f is the lens focal length, and h is the length of the side of a square element of the RPP. We denote by N^2 the number of elements of the square RPP, i.e. $N = D/h$. $\phi_{j,l}$ is the random phase imposed by the j,lth element. We assume that the $\phi_{j,l}$ are independent random variables which take either the value 0 or π with probability 1/2. We consider points in the focal plane such that $x, y < \lambda_0 f/h$, so that the smooth sinc envelope of the diffraction function of a square aperture can be considered as quasi uniform. If N^2 is large, the field A is a stationary, centered, Gaussian process. We normalize the mean intensity so that it is equal to 1. The AF of the field in the focal plane is defined by:

$$\gamma(x, y, t; x', y', t') = \langle A(x, y, t) A^*(x', y', t') \rangle .$$

where the brackets stand for statistical averaging. In the framework $N \gg 1$, it can be shown [17] that the AF is separable in the two orthogonal directions:

$$\gamma = \gamma_1(x,t;x',t')\gamma_2(y,t;y',t'),$$

$$\gamma_j(\eta,t;\eta',t') = \int_{-1/2}^{1/2} \exp i\left(\phi_j(t+T_{dj}u) - \phi_j(t'+T_{dj}u) -2\pi u(\eta-\eta')/\rho_c\right) du,$$

where T_{dj} are the time delays induced by the gratings and ρ_c is the correlation radius of the speckle pattern:

$$T_{dj} = s_j D, \quad \rho_c = \frac{\lambda_0 f}{D}. \tag{1}$$

In case of 1D-SSD, note that γ_2 is simply:

$$\gamma_2(y,t;y',t') = \mathrm{sinc}\left(\frac{y-y'}{\rho_c}\right)$$

As a numerical example we can consider the NIF geometry [24]: $D = 0.35$ m, $f = 8$ m and $\lambda_0/D = 10^{-6}$ (for $\lambda_0 = 351$ nm). The correlation radius ρ_c is then about 8 μm, while the size of the focal spot is of the order of 500 μm. Moreover the typical value for the lateral time delay imposed by the grating is of the order of 100 ps.

We consider in this paper sinusoidal phase modulators

$$\phi_j(t) = \beta_j \sin(2\pi t/T_{mj}), \quad j = 1,2,$$

where β_j and T_{mj} are respectively the modulation depth and the period of the modulator j. In such conditions

$$\gamma_j = \int_{-1/2}^{1/2} \exp i\left(2\beta_j \cos\left(2\pi\frac{T_0+T_{dj}u}{T_{mj}}\right)\sin\left(\pi\frac{\tau}{T_{mj}}\right) - 2\pi u\frac{\eta-\eta'}{\rho_c}\right) du,$$

where $T_0 = (t+t')/2$ and $\tau = t'-t$. Note that $t = T_0 - \tau/2$, $t' = T_0 + \tau/2$ so that T_0 can be referred to as the carrier time. This shows that the statistics of the field is locally stationary with respect to time, since it depends on the carrier time T_0. The computation of the four-dimensional AF:

$$C(\boldsymbol{\rho},\zeta,\tau) = \langle A(\mathbf{r}+\boldsymbol{\rho}, z+\zeta, t+\tau)A^*(\mathbf{r},z,t)\rangle \tag{2}$$

is required for the forthcoming results. The field satisfies the paraxial wave equation:

$$i\frac{\partial A}{\partial z} + \frac{i}{c}\frac{\partial A}{\partial t} + \frac{1}{2k_0}\Delta_{\mathbf{r}} A = 0$$

Accordingly the pulse in the moving reference frame:

$$\breve{A}(\mathbf{r},z,t) := A(\mathbf{r},z,t+z/c)$$

satisfies the standard Schrödinger equation:

$$i\frac{\partial \breve{A}}{\partial z} + \frac{1}{2k_0}\Delta_{\mathbf{r}}\breve{A} = 0$$

Note that the AF of A reads in terms of the AF $\breve{C}$ of $\breve{A}$ as:

$$C(\boldsymbol{\rho},\zeta,\tau) = \breve{C}(\boldsymbol{\rho},\zeta,\tau-\zeta/c) \tag{3}$$

The equation that governs the ζ-evolution of $\breve{C}$ is:

$$i\frac{\partial \breve{C}}{\partial \zeta}(\boldsymbol{\rho},\zeta,\tau) + \frac{1}{2k_0}\Delta_{\boldsymbol{\rho}}\breve{C}(\boldsymbol{\rho},\zeta,\tau) = 0$$

This equation can be solved explicitly by Fourier transform [25]. We get that:

$$\breve{C}(\boldsymbol{\rho},\zeta,\tau) = \frac{1}{(2\pi)^2}\int \hat{f}(\mathbf{k},\tau)\exp i\left(-\mathbf{k}.\boldsymbol{\rho} - \frac{|\mathbf{k}|^2\zeta}{2k_0}\right) d^2\mathbf{k}$$

where $\hat{f}(.,\tau)$ is the Fourier transform of $\breve{C}(.,\zeta=0,\tau)$ with respect to the transverse coordinates:

$$\hat{f}(\mathbf{k},\tau) = \int \breve{C}(\boldsymbol{\rho},\zeta=0,\tau)\exp i\mathbf{k}.\boldsymbol{\rho} d^2\boldsymbol{\rho}$$

Since $\breve{C}(\boldsymbol{\rho},\zeta=0,\tau) = C(\boldsymbol{\rho},\zeta=0,\tau)$, the function $\hat{f}$ can be computed easily from the expressions of γ_1 and γ_2:

$$\hat{f}(\mathbf{k},\tau) = \hat{f}_1(k_x,\tau)\hat{f}_2(k_y,\tau)$$

$$\hat{f}_j(k,\tau) = \rho_c \mathbf{1}_{|k|\leq\pi/\rho_c} \times \exp i\left(2\beta_j\cos(2\pi\frac{T_0+T_{dj}\rho_c k}{T_{mj}})\sin(\frac{\pi\tau}{T_{mj}})\right).$$

We finally substitute this identity into Eq. (3) so as to get a closed form expression of the four-dimensional AF:

$$C(\boldsymbol{\rho},\zeta,\tau) = C_1(\rho_x,\zeta,\tau)C_2(\rho_y,\zeta,\tau)$$

$$C_1 = \int_{-1/2}^{1/2}\exp i\left(2\beta_1\cos\left(2\pi\frac{T_0+T_{d1}u}{T_{m1}}\right)\sin\left(\pi\frac{\tau-\zeta/c}{T_{m1}}\right) - 2\pi u\frac{\rho_x}{\rho_c} - 8u^2\frac{\zeta}{\zeta_c}\right) du, \tag{4}$$

$$C_2 = \int_{-1/2}^{1/2}\exp i\left(2\beta_2\cos\left(2\pi\frac{T_0+T_{d2}v}{T_{m2}}\right)\sin\left(\pi\frac{\tau-\zeta/c}{T_{m2}}\right) - 2\pi v\frac{\rho_y}{\rho_c} - 8v^2\frac{\zeta}{\zeta_c}\right) dv, \tag{5}$$

where ζ_c is the longitudinal correlation length:

$$\zeta_c = \frac{8\lambda_0 f^2}{\pi D^2} \tag{6}$$

B. Smoothing by Optical Fiber

A broadband source illuminates a long multimode fiber with numerical aperture θ [4]. This source is a monochromatic source spectrally broadened by a sinusoidal phase modulator with modulation depth β and period T_m. The incident beam excites many optical modes, which propagate at different velocities. The time delay between the fastest and the slowest mode is assumed to be much longer than the modulation period T_m. Moreover small random fluctuations of the core radius or of the index of refraction affect the propagation by introducing random phases into the modes. If the fiber is long enough, these phases can be considered as independent processes which obey uniform distributions over $[0,2\pi]$. As a consequence the optical modes interfere at the output of the fiber and their overlap produces a speckle pattern [17]. The field at the output of the fiber is a Gaussian field with zero-mean and AF:

$$C(\boldsymbol{\rho},\zeta,\tau) = \int_0^1 ds J_0\left(\frac{\pi\sqrt{s}|\boldsymbol{\rho}|}{\rho_c}\right)\exp\left(-2is\frac{\zeta}{\zeta_c}\right) \times J_0\left(2\beta\sin(\pi\frac{\tau-\zeta/c}{T_m})\right),$$

where the transverse and longitudinal correlation radius are $\rho_c = \lambda_0/\theta$ and $\zeta_c = (8\lambda_0)/(\pi\theta^2)$ respectively.

C. Longitudinal Spectral Dispersion

In LMF design the focusing system consists in a grating that directly disperses the square beam [26]. This configuration was found to offer advantages in terms of color separation as well as frequency conversion efficiency. Furthermore the focal length of the grating strongly depends on the frequency ω:

$$f(\omega) = \frac{\omega+\omega_0}{\omega_0} f_0$$

where f_0 is the focal length corresponding to the carrier frequency ω_0. The grating is tailored so that more than 90 % of the energy is focused in the central spot. The grating can then be regarded as a chromatic lens [27]. Each frequency of the spectrum of the broadband pulse focuses at a specific point along the z-axis. Two different frequencies will give rise to two independent speckle patterns if the difference between their focal lengths is larger than the longitudinal speckle dimension. Smoothing by Longitudinal Spectral Dispersion (SLSD) thus belongs to the SSD category but with a spectral dispersion along the longitudinal axis instead of the transverse direction as in standard 1D or 2D-SSD techniques.

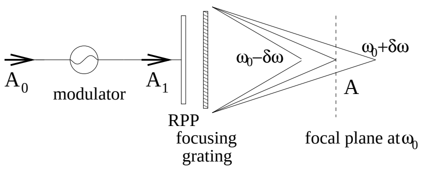


FIG. 2. Implementation of SLSD.

Let us assume that the near field is a phase modulated pulse with a sinusoidal phase modulation with depth β and period T_m. Let us denote by T_d the maximal time delay between the edge and the center of the square beam:

$$T_d = \frac{k_0}{2f_0\omega_0}\left((D/2)^2+(D/2)^2\right) = \frac{D^2}{4f_0 c}$$

The computation of the four-dimensional AF can be carried out as in the SSD configuration. We get:

$$C(\boldsymbol{\rho},\zeta,\tau) = \int_{-1/2}^{1/2} du \int_{-1/2}^{1/2} dv \times \exp i\left(2\beta\cos\left(2\pi\frac{T_0+2T_d(u^2+v^2)}{T_m}\right)\sin\left(\pi\frac{\tau-\zeta/c}{T_m}\right)\right) \times \exp i\left(-2\pi u\frac{\rho_x}{\rho_c} - 2\pi v\frac{\rho_y}{\rho_c} - 8(u^2+v^2)\frac{\zeta}{\zeta_c}\right), \qquad (7)$$

where $\rho_c = \lambda f/D$ and $\zeta_c = (8\lambda_0 f^2)/(\pi D^2)$.

IV. DESCRIPTION OF A GAUSSIAN FIELD AROUND A LOCAL MAXIMUM

In this section we shall consider a field A with Gaussian statistics that depends on a time-space coordinate $\mathbf{x}$. We shall here assume in great generality that $\mathbf{x}\in\mathbb{R}^d$, with $d = 1,2,3$, or 4, so that we shall be able in the following to apply the results to different configurations. The aim of this section is to derive the statistical distribution of this field around a local maximum located at some point $\mathbf{x}_0$ with top intensity $I_0 = |A(\mathbf{x}_0)|^2$. The results can be formulated only in terms of the AF. They can be obtained easily if the AF is real-valued (the so-called orthogonal case). This is the case for instance when a speckle pattern generated by one of the standard smoothing techniques is observed in the focal plane. However the situation is a little more tricky when the speckle patterns are observed in the focal volume since the AF has a nonzero imaginary part for $\zeta\neq 0$ (the so-called unitary case). We shall first address the orthogonal case, and then the unitary case.

A. The distribution of a orthogonal complex-valued Gaussian field around a local maximum

Let $A(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^d$, be a stationary zero-mean complex-valued Gaussian field. We denote by C the stationary AF:

$$C(\mathbf{x}) = \langle A(\mathbf{y}+\mathbf{x})A^*(\mathbf{y})\rangle \qquad (8)$$

The field is normalized so that the mean intensity is $C(\mathbf{0}) = 1$. It is equivalent to assume that the process has *independent real and imaginary parts* and that the AF is real-valued. In such conditions the Gaussian field is said to be orthogonal. Note that the function C is also even since

$$C(-\mathbf{x}) = \langle A(\mathbf{y}-\mathbf{x})A^*(\mathbf{y})\rangle = \langle A(\mathbf{y}')A^*(\mathbf{y}'+\mathbf{x})\rangle = C(\mathbf{x})$$

Assume that there is a local maximum at point $\mathbf{0}$ with top value $a_0\sqrt{I_0}$, $a_0\in\mathbb{C}$, $|a_0| = 1$. Since $\mathbf{0}$ is a local maximum we have $\nabla|A|^2(\mathbf{0}) = \mathbf{0}$, or else $\mathrm{Re}\left(A^*(\mathbf{0})\nabla A(\mathbf{0})\right) = \mathbf{0}$. The computation of the conditional distribution of A

given $A(\mathbf{0}) = a_0\sqrt{I_0}$ and $\mathrm{Re}\,(a_0^*\nabla A(\mathbf{0})) = \mathbf{0}$ is performed as in the real case (see Appendix) by taking care to deal with the real and imaginary parts of A separately. We can then claim that the distribution of the field A *given that there is a local maximum at point* $\mathbf{0}$ *with top value* $a_0\sqrt{I_0}$, $a_0 \in \mathbb{C}$, $|a_0| = 1$ is:

$$A(\mathbf{x}) = \tilde{A}(\mathbf{x}) + C(\mathbf{x})\left(\sqrt{I_0}a_0 - \tilde{A}(\mathbf{0})\right) \\ +a_0\nabla C(\mathbf{x}).\Lambda^{-1}\mathrm{Re}\left(\nabla\tilde{A}(\mathbf{0})\right) \qquad (9)$$

where $\tilde{A}$ obeys stationary Gaussian statistics with AF C, and Λ is the matrix of the second-order spectral moments:

$$\Lambda_{ij} = \left\langle \frac{\partial A}{\partial x_i}(\mathbf{0})\frac{\partial A^*}{\partial x_j}(\mathbf{0})\right\rangle = -\frac{\partial^2 C}{\partial x_i \partial x_j}(\mathbf{0}). \qquad (10)$$

If the top intensity I_0 of the hot spot is far above the mean intensity 1, then there is a dominant term in the right-hand side of Eq. (9):

$$A(\mathbf{x}) \simeq C(\mathbf{x})\sqrt{I_0}a_0 + O(1)$$

This means that the local shape of a strong maximum of a speckle pattern is roughly deterministic and given by the AF. This result is well-known [12,29], and the exact description (9) gives the precise expression of the random part.

B. The distribution of a unitary complex-valued Gaussian field around a local maximum

Let $A(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, be a stationary zero-mean unitary complex-valued Gaussian field. Unitary means that the distribution of A is invariant with respect to any rotation: $\mathcal{L}(A(.)) = \mathcal{L}(a_0A(.))$ for any $a_0 \in \mathbb{C}$, $|a_0| = 1$. This implies that:

1) The real and imaginary parts A_r and A_i are identically distributed (but *not* necessarily independent).

2) The AF C_r of A_r (or A_i) is stationary:

$$C_r(\mathbf{x}) = \langle A_r(\mathbf{y})A_r(\mathbf{y}+\mathbf{x})\rangle = \langle A_i(\mathbf{y})A_i(\mathbf{y}+\mathbf{x})\rangle$$

and it is an even function.

3) The cross-correlation function C_i of A_r and A_i is stationary:

$$C_i(\mathbf{x}) = \langle A_r(\mathbf{y})A_i(\mathbf{y}+\mathbf{x})\rangle = -\langle A_i(\mathbf{y})A_r(\mathbf{y}+\mathbf{x})\rangle$$

and it is an odd function.

We denote by C the stationary AF (8) of A. Note that we have $C(\mathbf{x}) = 2C_r(\mathbf{x}) + 2iC_i(\mathbf{x})$, and that $C(-\mathbf{x}) = C^*(\mathbf{x})$. We further normalize the field so that $C(\mathbf{0}) = 1$.

In the case $\nabla C(\mathbf{0}) = \mathbf{0}$ it is easy to check that the results of the previous section still hold true. The hypothesis $\nabla C(\mathbf{0}) = \mathbf{0}$ is important, otherwise the distributions of $A(\mathbf{0})$ and $\nabla A(\mathbf{0})$ would not be independent, and the conditional distribution of A given $A(\mathbf{0}) = \sqrt{I_0}a_0$ and $\mathrm{Re}\,(a_0^*\nabla A(\mathbf{0})) = \mathbf{0}$ would be more complicated.

In the case $\nabla C(\mathbf{0}) \neq \mathbf{0}$, then this vector has purely imaginary coordinates of the form $-i\mathbf{C}_1$, $\mathbf{C}_1 \in \mathbb{R}^d$. If you consider the auxiliary field $\check{A}(\mathbf{x}) := A(\mathbf{x})\exp(i\mathbf{C}_1.\mathbf{x})$, then $\check{A}$ is a unitary complex-valued Gaussian field whose AF fulfills the condition $\nabla\check{C}(\mathbf{0}) = \mathbf{0}$. It is therefore more comfortable to study the local maxima of A by considering them as local maxima of $\check{A}$.

V. THE DURATION OF A HOT SPOT

We shall derive in this section the distribution of the duration of a speckle spot defined as a local maximum of a orthogonal complex-valued Gaussian field. We consider a Gaussian field with mean intensity 1. The AF:

$$C(\tau) = \langle A(t+\tau)A^*(\tau)\rangle$$

can be seen as the inverse Fourier transform of the spectral intensity by the Wiener-Khintchine theorem [28].

Assume that there is a local maximum at $t = 0$ with top intensity I_0. We aim at deriving an expression of the PDF of the full width at level $I_0 - I_1$ of this maximum. The important parameters in this section are T_0 and α_t defined by:

$$T_0 = \frac{1}{\sqrt{-C_{tt}}}, \quad \alpha_t = \frac{1}{2}\left(\frac{C_{tttt}}{C_{tt}^2} - 1\right) \qquad (11)$$

where C_{tt} (resp. C_{tttt}) is the second derivative (resp. forth derivative) of C with respect to time evaluated at 0. Note that the time T_0 may also be seen as the inverse of the bandwidth of the field A while α_t is dimensionless and depends only on the shape of the AF around 0.

The derivation of the expression of the full width of a speckle spot is based on the quadratic expansion of the field around the local maximum located at 0:

$$|A(t)|^2 = I_0 + I_0C_{tt}t^2 + \left|\partial_t\tilde{A}_i(0)\right|^2 t^2 \\ +\sqrt{I_0}\left(\partial_{tt}\tilde{A}_r(0) - \tilde{A}_r(0)C_{tt}\right)t^2$$

where $\tilde{A}_r = \mathrm{Re}\left(a_0^*\tilde{A}\right)$ and $\tilde{A}_i = \mathrm{Im}\left(a_0^*\tilde{A}\right)$. Note that $\partial_{tt}\tilde{A}_r(0) - \tilde{A}_r(0)C_{tt}$ obeys a normal distribution with zero mean and variance α_t/T_0^2. Accordingly, the quadratic expansion of the intensity of the field around 0 is:

$$|A(t)|^2 = I_0(1 - t^2/T_0^2) + \sqrt{I_0\alpha_t}Z_1t^2/T_0^2 + \frac{1}{2}Z_2^2t^2/T_0^2$$

where Z_1 and Z_2 are random Gaussian variables with zero-mean and variance 1. We get that the full width t_{FW} at level $I_0 - I_1$ is:

$$t_{FW}^2 = \frac{4I_1T_0^2}{I_0 - \sqrt{I_0\alpha_t}Z_1 - \frac{1}{2}Z_2^2} \qquad (12)$$

Neglecting the term Z_2^2 which is of order 1 with respect to the top intensity I_0 allows us to compute the PDF of this random variable. This establishes the following result: The PDF of of the full width t_{FW} at level $I_0 - I_1$ of a local maximum with top intensity I_0 is described by the random variable T:

$$T = \frac{t_{FW}}{T_0}$$

whose PDF is:

$$p_{I_0,I_1}(T) = \frac{8 I_1 C_{I_0,I_1}}{\sqrt{2\pi I_0 \alpha_t} T^3} \exp\left(- \frac{\left(I_0 - 4 I_1 T^{-2}\right)^2}{2\alpha_t I_0} \right) \quad (13)$$

and C_{I_0,I_1} is a normalization constant that is chosen such that $\int_0^\infty p_{I_0,I_1}(T) dT = 1$.

This expression is all the more valid as the top intensity I_0 is large, for a given level I_1. Note also that C_{I_0,I_1} is very close to 1 as $I_0 \gg 1$. We could also get a more precise expression that would take into account Z_2, but would be much more intricate.

As expected, the typical value of the duration of a hot spot is proportional to T_0. Indeed, at first order the local shape of a hot spot is deterministic and imposed by the AF: $|A(t)|^2 \simeq I_0 C^2(t) \simeq I_0(1 - t^2/T_0^2)$. Note also that the width of the distribution of t_{FW} is governed by the parameter α_t. The PDF is all the wider as α_t is larger.

The tails of the PDF of the duration are worth studying since they characterizes the probability that there exists a hot spot with a very large or a very small duration. The right tail can be approximated by:

$$p_{I_0,I_1}(T) \overset{T \gg \sqrt{2I_1/I_0}}{\simeq} \frac{8 I_1}{\sqrt{2\pi I_0 \alpha_t} T^3} \exp\left(- \frac{I_0}{2\alpha_t} \right)$$

This demonstrates that the tail of the PDF of the duration corresponding to the large values is quite long, so it is likely that some of the hot spots of a speckle pattern may have a duration much longer than the mean value given by the inverse of the spectral width. On the other hand, the tail of the PDF of the duration corresponding to the small values is very small:

$$p_{I_0,I_1}(T) \overset{T \ll \sqrt{2I_1/I_0}}{\simeq} \frac{8 I_1}{\sqrt{2\pi I_0 \alpha_t} T^3} \exp\left(- \frac{8 I_1^2}{\alpha_t I_0 T^4} \right)$$

So it is very unlikely that there exists a hot spot whose duration is much smaller than the mean value.

We can compare these results with full numerical simulations. We simulate a random field with Gaussian statistics, Gaussian AF: $C(\tau) = \exp\left(-\tau^2/(2\tau_c^2)\right)$ and mean intensity 1. We detect all local maxima whose top intensities are above $I_0 - 0.1$ and below $I_0 + 0.1$. We measure the full widths of these maxima at level PI_0, $P = 75\%$. Once these results are collected, we plot the histograms of the widths and compare with the theoretical PDF. The results are shown in Figure 3.

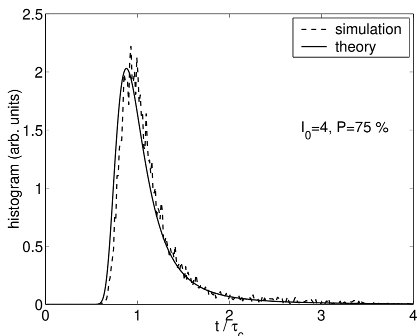


FIG. 3. Probability distributions of the full widths at level PI_0, $P = 75\%$ for local maxima with top intensities $I_0 = 4$. The widths are normalized in multiples of τ_c.

VI. THE SPATIAL RADIUS OF A HOT SPOT

A. Transverse radius

The study of the statistical distribution of the radius of a hot spot is formally identical to the study of the duration, since it consists in analyzing the width of a local maximum of the field A. We get that the radius r_{FW} at level $I_0 - I_1$ (say in the x-direction) of a hot spot with top intensity I_0 is described by the random variable R:

$$R = \frac{r_{FW}}{R_0}, \qquad R_0 = \frac{1}{\sqrt{-C_{xx}}}$$

whose PDF is:

$$p_{I_0,I_1}(R) = \frac{8 I_1 C_{I_0,I_1}}{\sqrt{2\pi I_0 \alpha_x} R^3} \exp\left(- \frac{\left(I_0 - 4 I_1 R^{-2}\right)^2}{2\alpha_x I_0} \right), \quad (14)$$

and $\alpha_x = \left(C_{xxxx}/C_{xx}^2 - 1\right)/2$. This result holds true for a orthogonal Gaussian field. This is the case when we consider speckle patterns generated by the standard optical smoothing methods in the focal plane since $\boldsymbol{\rho} \mapsto C(\boldsymbol{\rho}, \zeta = 0)$ is real-valued (see Section III). However this is not the case when we consider these patterns in the focal volume since $\zeta \mapsto C(\boldsymbol{\rho} = 0, \zeta)$ has a nonzero imaginary part. This case deserves a specific study.

B. Longitudinal length

We shall now compute the longitudinal length of a hot spot. Let a speckle field A be given in the plane $z = 0$ with AF $\langle A(\mathbf{r} + \boldsymbol{\rho}, z = 0) A^*(\mathbf{r}, z = 0)\rangle = C(\boldsymbol{\rho}, \zeta = 0)$. As shown in Section III the field A in the whole space $\mathbb{R}^3$ obeys the distribution of a Gaussian process with AF:

$$C(\boldsymbol{\rho}, \zeta) = \langle A(\mathbf{r} + \boldsymbol{\rho}, z + \zeta) A^*(\mathbf{r}, z)\rangle$$

given by:

$$C(\boldsymbol{\rho}, \zeta) = \frac{1}{(2\pi)^2} \int \hat{f}(\mathbf{k}) \exp i \left(-\mathbf{k}.\boldsymbol{\rho} - \frac{|\mathbf{k}|^2 \zeta}{2k_0} \right) d^2\mathbf{k}$$

where $\hat{f}$ is the Fourier transform of $C(\boldsymbol{\rho}, \zeta = 0)$:

$$\hat{f}(\mathbf{k}) = \int C(\boldsymbol{\rho}, \zeta = 0) \exp i\mathbf{k}.\boldsymbol{\rho} d^2\boldsymbol{\rho}$$

We are interested in the longitudinal forms of the hot spots, which are characterized by the longitudinal correlation function $\zeta \mapsto C(\boldsymbol{\rho} = \mathbf{0}, \zeta)$ which is such that $C(0) = 1$, $C_z(0) = -ic_1$, $C_{zz}(0) = -c_2$, $C_{zzz}(0) = ic_3$, and $C_{zzzz}(0) = c_4$, where c_j are given by:

$$c_j = \frac{1}{(2\pi)^2 (2k_0)^j} \int \hat{f}(\mathbf{k}) |\mathbf{k}|^{2j} d^2\mathbf{k}$$

Since $C_z(0) \neq 0$ we should consider a modified version of A so as to be able to apply the general results we derive here above. We should consider a process whose correlation function has a first derivative that vanishes at 0. We thus introduce $\check{A}(\mathbf{r}, z) = A(\mathbf{r}, z) \exp(ic_1 z)$ whose correlation function is $\check{C}(\zeta) = C(\zeta)\exp(ic_1\zeta)$. The second derivative is $\check{C}_{zz}(0) = -c_2 + c_1^2$ and the fourth derivative is $\check{C}_{zzzz}(0) = c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4$. As a consequence, the probability distribution of the length z_{FW} of a hot spot is described by the random variable Z:

$$Z = \frac{z_{FW}}{Z_0}, \quad Z_0 = \frac{1}{\sqrt{c_2 - c_1^2}}$$

whose PDF is:

$$p_{I_0,I_1}(Z) = \frac{8 I_1 C_{I_0,I_1}}{\sqrt{2\pi I_0 \check{\alpha}_z} Z^3} \exp\left(-\frac{\left(I_0 - 4 I_1 Z^{-2}\right)^2}{2\check{\alpha}_z I_0} \right), \quad (15)$$

and the coefficient $\check{\alpha}_z$ is given by $(1/2)(c_4 - c_2^2 - 4c_3c_1 + 8c_2c_1^2 - 4c_1^4)/(c_2 - c_1^2)^2$.

C. Applications

Example 1. Speckle pattern generated by a circular Random Phase Plate. A circular beam with diameter D and wavelength λ_0 irradiates a Random Phase Plate and is focused by a lens with focal length f. The field A in the focal plane obeys Gaussian statistics with zero-mean and AF:

$$C(\boldsymbol{\rho}, \zeta = 0) = 2\frac{J_1(\pi|\boldsymbol{\rho}|/\rho_c)}{(\pi|\boldsymbol{\rho}|/\rho_c)}$$

where $\rho_c = \lambda_0 f/D$. Accordingly the transverse radius of a hot spot with top intensity I_0 is $r_{FW} = 2\rho_c R/\pi$ where R admits the PDF (14) with $\alpha_x = 1/2$.

The transverse AF $C(\boldsymbol{\rho})$ corresponds to a flat spectrum of the form $\hat{f}(\mathbf{k}) \sim \mathbf{1}_{|\mathbf{k}| \leq k_c}$, with $k_c = \pi/\rho_c$. Accordingly the longitudinal form of the AF is:

$$C(\boldsymbol{\rho} = \mathbf{0}, \zeta) = \exp\left(-i\frac{k_c^2\zeta}{4k_0}\right) \operatorname{sinc}\left(\frac{k_c^2\zeta}{4k_0}\right)$$

where $\operatorname{sinc}(s) = \sin(s)/s$. The field $\check{A}(\mathbf{r}, z) = A(\mathbf{r}, z)\exp(ik_c^2 z/(4k_0))$ has also Gaussian statistics with real-valued AF:

$$\check{C}(\boldsymbol{\rho} = \mathbf{0}, \zeta) = \operatorname{sinc}\left(\frac{\zeta}{\zeta_c}\right)$$

where $\zeta_c = 4k_0/(k_c^2) = (8\lambda_0 f^2)/(\pi D^2)$. Thus a hot spot with top intensity I_0 has a length that is described by $\sqrt{3}\zeta_c Z$ where Z obeys the distribution (15) with $\check{\alpha}_z = 2/5$.

Example 2. Smoothing by Optical Fiber. A monochromatic source illuminates a long multimode fiber with numerical aperture θ. The field at the output of the fiber is a Gaussian field with zero-mean and AF:

$$C(\boldsymbol{\rho}, \zeta) = \int_0^1 ds J_0\left(\frac{\pi\sqrt{s}|\boldsymbol{\rho}|}{\rho_c}\right) \exp\left(-2is\frac{\zeta}{\zeta_c}\right) ds$$

where $\rho_c = \lambda_0/\theta$ and $\zeta_c = 8\lambda_0/(\pi\theta^2)$. This is exactly the same configuration as in Example 1.

Example 3. Speckle pattern generated by a square Random Phase Plate. Consider the very same implementation as in Example 1, but assume here that the RPP as well as the beam are square shaped with side length D. The field A in the focal plane obeys Gaussian statistics with zero-mean and AF:

$$C(\boldsymbol{\rho}, \zeta = 0) = \operatorname{sinc}(\pi|\rho_x|/\rho_c)\operatorname{sinc}(\pi|\rho_y|/\rho_c)$$

where $\rho_c = \lambda_0 f/D$. Accordingly the transverse radius of a hot spot with top intensity I_0 is $r_{FW} = \sqrt{3}\rho_c R/\pi$ where R admits the PDF (14) with $\alpha_x = 2/5$.

The AF f here corresponds to a flat spectrum of the form $\hat{f}(\mathbf{k}) \sim \mathbf{1}_{|k_x| \leq k_c, |k_y| \leq k_c}$, with $k_c = \pi/\rho_c$. Accordingly the longitudinal form of the AF is:

$$C(\boldsymbol{\rho} = \mathbf{0}, \zeta) = \frac{\pi}{4} \frac{\left(C_i(2/\sqrt{\pi}\sqrt{\zeta/\zeta_c}) - iS_i(2/\sqrt{\pi}\sqrt{\zeta/\zeta_c})\right)^2}{(\zeta/\zeta_c)}$$

where C_i and S_i are the Fresnel cosine and sine:

$$C_i(t) = \int_0^t \cos\left(\frac{\pi}{2}s^2\right) ds, \quad S_i(t) = \int_0^t \sin\left(\frac{\pi}{2}s^2\right) ds,$$

and ζ_c is given by (6). Note that $C(0) = 1$, $C_z(0) = -4i/(3\zeta_c)$. Introducing $\check{A} = A\exp(4iz/(3\zeta_c))$, we get that the hot spots of the field have lengths that are described by $\sqrt{45/32}\zeta_c Z$ where Z obeys the distribution (15) with $\check{\alpha}_z = 11/14$.

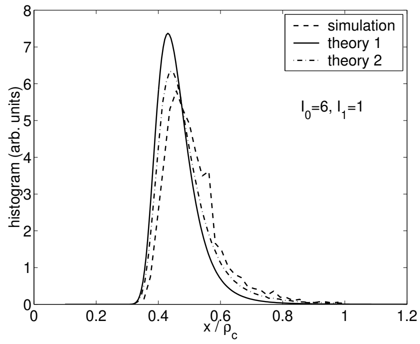
We can compare these results with full numerical simulations. We simulate the field in the x coordinate as the superposition of $4N^2$ modes:

$$A(x, y, z) = \frac{1}{2N} \sum_{j,l=-N}^{N-1} A_{j,l}(x, y, z) \exp\left(i\phi_{j,l}\right),$$

$$A_{j,l}(x, y, z) = \exp i\left(-\pi\frac{jx + ly}{N\rho_c} - 2\frac{j^2 + l^2}{N^2}\frac{z}{\zeta_c}\right),$$

where $\phi_{j,l}$, $j,l=-N,...,N-1$ are independent random variables which take either the value 0 or π with probability 1/2. We adopt the following dimensionless values: $2N=50$, ρ_c and $\zeta_c=1$. The numerical histogram is computed from a set of 5000 hot spots with the desired top intensity. In Figure 4a we plot the theoretical histograms as well as the numerical histogram for the full widths in the x-direction at level I_0-I_1 of hot spots with top intensities I_0 (remember that the mean intensity is 1). The theoretical histogram 1 corresponds to the PDF (14). The theoretical histogram 2 corresponds to the representation (12). Indeed, if α_x is small or when I_0 is not very large, then the last term in the denominator of Eq. (12) may play a role. This is the case in this configuration. In Figure 4b the theoretical and numerical histograms of the longitudinal length (full width in the z-direction) at level I_0-I_1 of hot spots with top intensities I_0 are compared.

(a)



(b)

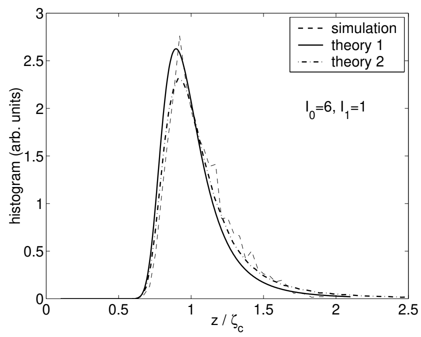


FIG. 4. Histograms of the full widths at level I_0-I_1 in the x-direction (picture a) and z-direction (picture b) of hot spots with top intensities I_0. The field is a speckle pattern generated by a square RPP. The widths are normalized in multiples of ρ_c (picture a) and ζ_c (picture b).

VII. THE VELOCITY OF A HOT SPOT

We consider a complex-valued Gaussian field $A(.)$ that depends on a space coordinate $\mathbf{x}\in\mathbb{R}^3$ and on a time coordinate $t\in\mathbb{R}$. Consider that there is a local maximum at some point $(\mathbf{x}_0,t_0)$ with value $\sqrt{I_0}a_0$, $|a_0|=1$. We would like to compute the distribution of the velocity of this local maximum. The velocity is defined as follows. For t in a neighborhood of t_0, plot the maximum of the field:

$$\mathbf{x}(t) := \mathrm{argmax}\left\{|A|^2(\mathbf{x},t),\ \mathbf{x}\text{ close to }\mathbf{x}_0\right\}$$

The velocity of the local maximum is:

$$\mathbf{v} := \frac{d\mathbf{x}(t)}{dt}\,|_{t=t_0}$$

Note that $\mathbf{v}$ is a random variable, since it depends on the realization $A(.)$. Remember the field around the local maximum is of the form:

$$\begin{aligned}A(\mathbf{x}_0+\mathbf{x},t_0+t) &= \tilde{A}(\mathbf{x}_0+\mathbf{x},t_0+t)\\ &\quad +C(\mathbf{x},t)(\sqrt{I_0}a_0-\tilde{A}(\mathbf{x}_0,t_0))\\ &\quad +a_0\nabla C(\mathbf{x},t).\Lambda^{-1}\mathrm{Re}\left(\nabla\tilde{A}(\mathbf{x}_0,t_0)\right)\end{aligned}$$

where ∇ is the row operator $(\partial_{x_1},...,\partial_{x_3},\partial_t)$ while Λ is the symmetric positive 4×4 matrix defined by:

$$\Lambda_{ij} = -\frac{\partial^2 C}{\partial x_i\partial x_j}(\mathbf{0},0),\quad i,j=1,...,4,$$

where by convention $x_4=t$. Expanding $|A|^2$ around $(\mathbf{x}_0,t_0)$ establishes the identity:

$$\mathbf{v} = \underset{\mathbf{u}\in\mathbb{R}^3,\,\mathbf{w}=(\mathbf{u},1)}{\mathrm{argmax}}\left\{|\mathbf{w}.\nabla A|^2+\mathrm{Re}\left(A^*(\mathbf{w}.\nabla)^2A\right)\right\}(\mathbf{x}_0,t_0)$$

It shows that only the first and second derivatives of A in the point $(\mathbf{x}_0,t_0)$ are coming into the definition of the velocity. From this identity we can get a closed form expression of $\mathbf{v}$ in terms of the first and second derivatives of A in the point $(\mathbf{x}_0,t_0)$, since $\mathbf{v}$ reads as the localtion of the maximum of a quadratic function parametrized by the derivatives of A. The distribution of $(A,\nabla A,\nabla\otimes\nabla A)(\mathbf{x}_0,t_0)$ given that there is a local maximum at $(\mathbf{x}_0,t_0)$ is a Gaussian random vector with known mean and covariance (see Section IV). The calculation of the distribution of the velocity is then a matter of long but straightforward algebra. The result can be formulated as follows. Once again we assume that $\nabla C(\mathbf{0})=\mathbf{0}$. We introduce the random symmetric 4×4 matrix Z whose entries are Gaussian normal variables with zero-mean and covariance:

$$\langle Z_{jl}Z_{j'l'}\rangle = \frac{\partial^4 C_r}{\partial x_j\partial x_l\partial x_{j'}\partial x_{l'}} - \frac{\partial^2 C_r}{\partial x_j\partial x_l}\frac{\partial^2 C_r}{\partial x_{j'}\partial x_{l'}}$$

where C_r is the real part of $\frac{1}{2}C$ and derivatives are evaluated at $(\mathbf{0},0)$. The velocity $\mathbf{v}$ of a local maximum of the field A with top intensity I_0 obeys the distribution of:

$$\mathbf{v} = -M^{-1}\mathbf{N} \qquad (16)$$

where M is the 3×3 random matrix:

$$M_{ij} = \sqrt{I_0}\Lambda_{ij}+Z_{ij},\quad i,j=1,2,3,$$

and $\mathbf{N}$ is the vector:

$$N_i = \sqrt{I_0}\Lambda_{i4}+Z_{i4},\quad i=1,2,3.$$

Note that, if we consider a local maximum with a very high top intensity I_0, or else if we average over many maxima, the expression of the velocity becomes simpler, independent of I_0, and deterministic:

$$\mathbf{v} = -\left(\left(\frac{\partial^2 C}{\partial x_i \partial x_j}\right)_{i,j=1,2,3}\right)^{-1} \left(\frac{\partial^2 C}{\partial x_i \partial t}\right)_{i=1,2,3} \quad (17)$$

This identity was derived in [18]. The new expression (16) takes into account the fluctuations around this mean value.

Remark 1. The PDF corresponding to Eq. (16) has a very complicated form, so we shall not write it down. Note however that it is easy to plot this PDF by straightforward Monte-Carlo simulations. Indeed a zero-mean Gaussian vector $(Z_i)_{i=1,..,n}$ with covariance matrix $K := \langle Z_i Z_j \rangle_{i,j=1,...,n}$ can be simulated by multiplying a vector of independent Gaussian random variables with variance 1 by a square root of the matrix K.

Remark 2. There exists one case for which we can give a closed-form expression of the PDF of the velocity. If the AF is separable $C(x,y,z,t) = C_1(x)C_2(y)C_3(z)C_4(t)$, then the component of the velocity in the x-direction is described by the PDF:

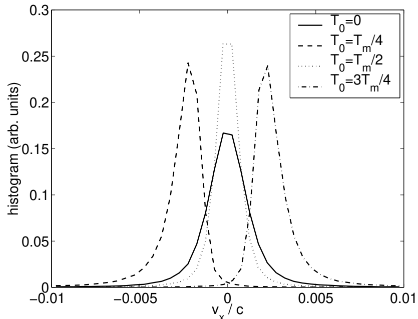
$$p_{I_0}(v) = \frac{\sqrt{I_0} v_0^2}{\sqrt{\pi}\,(2\alpha_x v^2 + v_0^2)^{3/2}} \exp\left(-\frac{I_0 v^2}{2\alpha_x v^2 + v_0^2}\right) \quad (18)$$

where $v_0^2 = |C_{tt}/C_{xx}|$ and $\alpha_x = (C_{xxxx}/C_{xx}^2 - 1)/2$.

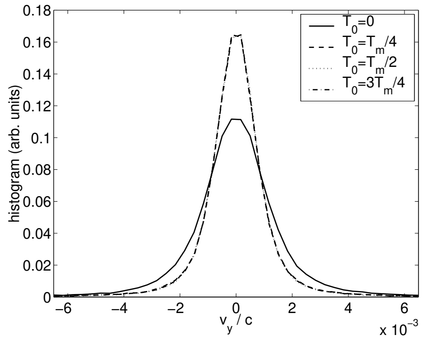
A. Application to 1D-SSD

To simplify the presentation of the forthcoming results we shall assume that $T_d = N_c T_m$, where N_c is a positive integer (the so-called "number of cycles"). Note that we must take care to consider the modified version of the field A such that the first derivatives of the AF vanish: $\check{A}(\mathbf{r}, z, t) = A(\mathbf{r}, z, t) \exp(i4\zeta/(3\zeta_c))$ whose AF is $\check{C} = C \exp(i4\zeta/(3\zeta_c))$.

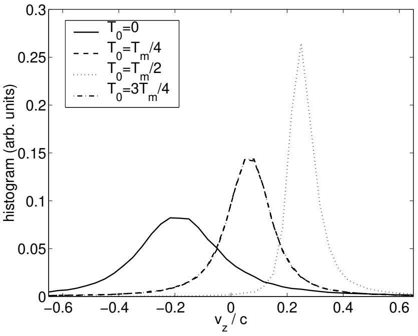
We shall not give the complete and awkward expression of the statistical distribution of the velocity. But it can plotted by Monte-Carlo simulations of Eq. (16). Figure 5 plots the PDF of the three components of the velocity. It can be seen that the means and the variances of these distributions depend on the carrier time T_0.



(a)



(b)



(c)

FIG. 5. PDF of the velocity of a hot spot with top intensity $I_0 = 6$ as a function of the carrier time T_0. Here $T_m^{-1} = 10$ GHz, $\beta = 5$, $\lambda = 350$ nm, $f = 8$ m, and $D = 0.35$ m. The velocity is normalized in multiples of the light velocity c.

Furthermore we can give a closed-form expression of the mean velocity defined by (17). The component of the mean velocity along the transverse direction y is 0, the component along the dispersion direction x is:

$$\langle v_x \rangle = \frac{6\rho_c \beta}{T_m} \frac{(-1)^{N_c}}{\pi N_c} \frac{\sin(2\pi \frac{T_0}{T_m})\left(\frac{1}{45} + \kappa C_1(T_0)\right)}{\frac{1}{45} + 2\kappa C_1(T_0) + \kappa^2\left(1 - C_2(T_0)\right)} \quad (19)$$

while the component along the longitudinal direction:

$$\langle v_z \rangle = c \frac{\kappa C_1(T_0) + \kappa^2\left(1 - C_2(T_0)\right)}{\frac{1}{45} + 2\kappa C_1(T_0) + \kappa^2\left(1 - C_2(T_0)\right)} \quad (20)$$

where

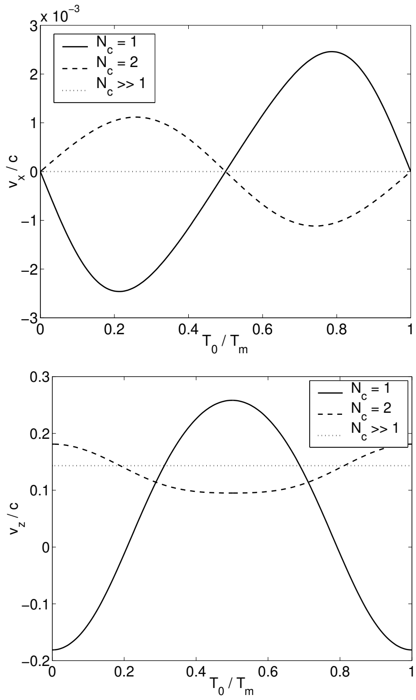
$$\kappa = \frac{\pi \beta \zeta_c}{4 c T_m}$$

$$C_1(T_0) = \cos(2\pi \frac{T_0}{T_m}) \frac{(-1)^{N_c}}{(\pi N_c)^2}$$

$$C_2(T_0) = 6 \sin^2(2\pi \frac{T_0}{T_m}) \frac{1}{(\pi N_c)^2}$$

Note that, when $N_c \gg 1$, the component of the mean velocity along the dispersion direction is 0 while the component along the longitudinal direction is simply:

$$\langle v_z \rangle = c \frac{45\kappa^2}{1 + 45\kappa^2}$$



(a)

(b)

FIG. 6. Mean velocity of a hot spot with top intensity $I_0 = 6$ as a function of the carrier time. Here $T_m^{-1} = 10$ GHz, $\beta = 5$, $\lambda = 350$ nm, $f = 8$ m, and $D = 0.35$ m. The velocity is normalized in multiples of the light velocity c. Only the x and z components of the mean velocity are plotted since $\langle v_y \rangle = 0$.

B. Application to SOF

We must take care to consider the modified version of the field A such that the first derivatives vanish: $\check{A}(\mathbf{r}, z, t) = A(\mathbf{r}, z, t)\exp(i\zeta/\zeta_c)$ whose AF is $\check{C} = C\exp(i\zeta/\zeta_c)$. The mean velocity has a rather simple expression. The only component of the mean velocity which is not zero is along the longitudinal axis:

$$\langle v_z \rangle = c\frac{96\kappa^2}{1 + 96\kappa^2}$$

where $\kappa = (\pi\beta\zeta_c)/(4cT_m)$.

C. Numerical simulations

It is not possible to compare the theoretical results with full numerical simulations. Indeed it would require the simulation of a four-dimensional field with sufficient accuracy to compute the velocities of local maxima with a given top intensity, which is out of the computational range of our computers. We have considered the auxiliary problem which consists in observing the speckle pattern generated by 1D-SSD along the x-axis of the focal plane. The $(t - x)$-AF is given by (4) with $\zeta = 0$. The computation of the velocity v_x can be performed as above, the only difference is that the spatial coordinate is here one-dimensional. It is found that the statistical distribution of the velocity v_x is described by

$$v_x = -\frac{\sqrt{2I_0}C_{xt} + \mu}{\sqrt{2I_0}C_{xx} + \nu} \tag{21}$$

where

$$C_{xt} = \sqrt{2I_0}\frac{(-1)^{N_c+1}}{N_c}\frac{2\pi\beta}{\rho_c T_m}\sin(2\pi\frac{T_0}{T_m})$$
$$C_{xx} = \sqrt{2I_0}\frac{\pi^2}{3\rho_c^2}$$

and (μ, ν) is a zero-mean Gaussian random vector with covariance matrix K:

$$\begin{aligned} K_{\mu\mu} &= C_{xxtt} - C_{xt}^2 \\ &= \frac{\pi^2\beta^2}{\rho_c^2 T_m^2}\left(\frac{2\pi^2}{3} + \frac{1}{N_c^2}\left(1 - 6\sin^2(2\pi\frac{T_0}{T_m})\right)\right) \\ K_{\nu\nu} &= C_{xxxx} - C_{xx}^2 = \frac{4\pi^4}{45\rho_c^4} \\ K_{\mu\nu} &= C_{xxxt} - C_{xt}C_{xx} \\ &= (-1)^{N_c+1}\frac{4\pi\beta}{3T_m N_c\rho_c^3}\left(\pi^2 - \frac{9}{N_c^2}\right)\sin(2\pi\frac{T_0}{T_m}) \end{aligned}$$

Note that v_x and $\langle v_x \rangle$ do not coincide with the x-projections of Eqs. (16) and (19) respectively. Indeed Eq. (16) describes the velocities of the true hot spots which are local maxima with respect to x, y, and z. Here we observe the motions of spots which are local maxima with respect to x, but not with respect to y and z.

When $N_c \gg 1$, the dependence with respect to T_0 vanishes, and the vector (μ, ν) becomes two independent random normal variables with variances $\langle \mu^2 \rangle = 2\pi^4\beta^2/(3\rho_c^2 T_m^2)$ and $\langle \nu^2 \rangle = 4\pi^4/(45\rho_c^4)$ respectively. This means that the velocity admits the PDF (18) with $v_0 = \sqrt{6}\beta\rho_c/T_m$ and $\alpha_x = 2/5$.

We have performed numerical simulations to compare with the theoretical predictions. We simulate the field in the $x - t$ coordinates as the superposition of $2N$ modes:

$$A(t, x) = \frac{1}{\sqrt{2N}}\sum_{j=-N}^{N-1} A_j(t, x)\exp(i\phi_j),$$
$$A_j(t, x) = \exp\left(-i\pi\frac{jx}{N\rho_c} + i\beta\sin(2\pi(\frac{t}{T_m} + \frac{jT_d}{2NT_m}))\right),$$

where ϕ_j, $j = -N, ..., N-1$ are independent random variables which take either the value 0 or π with probability $1/2$. In Figure 7 we plot the theoretical PDF corresponding to Eq. (21) as well as the numerical histograms for hot spots with top intensity $I_0 = 6$ (remember that the mean intensity is 1). The numerical histograms are computed from a set of 1000 hot spots with the desired top intensity. For $T_0 = 0$ the velocity has mean 0. For $T_0 = 3T_m/4$

the mean velocity is $6\beta\rho_c/(\pi T_d)$. The support of the distribution lies in the positive half-line, which means that all spots are moving in the right-direction. The FWHM of the distribution is smaller than in the configuration $T_0 = 0$. This is a general feature: When the mean velocity is maximal (at times $T_0 = T_m/2 \bmod T_m$), the FWHM of the distribution is minimal, and vice-versa.

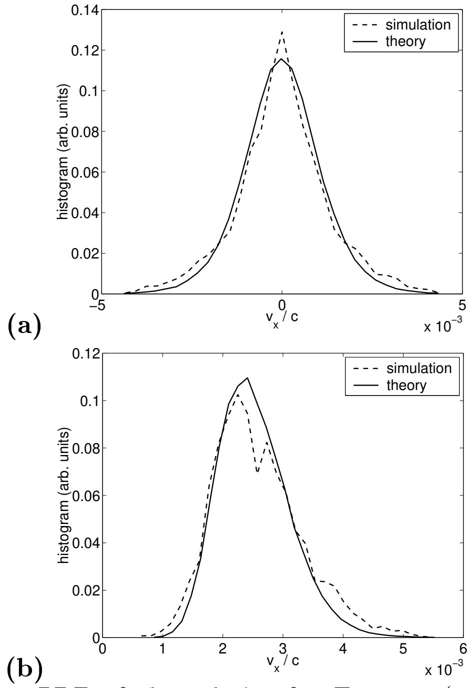


FIG. 7. PDF of the velocity for $T_0 = 0$ (picture a) and $T_0 = 3T_m/4$ (picture b). 1D-SSD with $2N = 100$, $T_d^{-1} = T_m^{-1} = 10$ GHz, $\beta = 5$, and $\rho_c = 8$ μm.

VIII. CONCLUSION

We have derived closed-form expressions for the probability distributions of the characteristic parameters of hot spots of speckle patterns such as the duration, the radius, and the velocity. These expressions depend only on the four-dimensional AF of the field. The analysis of the results exhibit the following statements. The distribution of the radius or the velocity of a hot spot depends on the top intensity of the hot spot. The histograms are not symmetric with respect to the mean values. We have demonstrated the existence of heavy tails for the probability distributions, which means that it is likely that some of the hot spots in a speckle pattern have a size (resp. a duration, a velocity) larger than the corresponding mean value.

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APPENDIX A: GAUSSIAN RANDOM FIELDS

We first list the basic results of the probabilistic theory of Gaussian vectors [29]. A random $\mathbb{R}^n$-valued vector $(Z_i)_{i=1,..,n}$ is said to be a Gaussian vector if for any $\mathbf{u} \in \mathbb{R}^n$ the real-valued variable $\mathbf{u}.\mathbf{Z}$ obeys a real Gaussian distribution. A Gaussian vector is fully characterized by its mean M_Z and covariance matrix $K_{Z,Z}$:

$$M_Z(i) = \langle Z_i \rangle ,$$
$$K_{Z,Z}(i,j) = \langle (Z_i - M_Z(i))(Z_j - M_Z(j)) \rangle .$$

If $K_{Z,Z}$ is invertible then the distribution of the zero-mean vector $\mathbf{Z} - M_Z$ admits a PDF given by:

$$p(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n \det K_{Z,Z}}} \exp\left(-\frac{\mathbf{z}.K_{Z,Z}^{-1}\mathbf{z}}{2} \right)$$

If $(\mathbf{Y}, \mathbf{Z})$ is a Gaussian vector, and $K_{Y,Y}$ is invertible, then the distribution of $\mathbf{Z}$ given $\mathbf{Y}$ is a Gaussian vector with mean:

$$M_{Z|Y} = M_Z + K_{Z,Y} K_{Y,Y}^{-1} (\mathbf{Y} - M_Y) \tag{A1}$$

and covariance matrix:

$$K_{Z,Z|Y} = K_{Z,Z} - K_{Z,Y} K_{Y,Y}^{-1} K_{Y,Z} \tag{A2}$$

Let $A(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, be a real-valued Gaussian field with zero-mean and stationary AF C:

$$C(\mathbf{x}) = \langle A(\mathbf{y} + \mathbf{x}) A(\mathbf{y}) \rangle$$

The field is normalized so that $C(\mathbf{0}) = 1$. Assume that there is a local maximum at point $\mathbf{0}$ with top value $A_0 \in \mathbb{R}$. Applying Eqs. (A1-A2) establishes that the conditional distribution of A given $A(\mathbf{0}) = A_0$ and $\nabla A(\mathbf{0}) = \mathbf{0}$ is a Gaussian field with non-stationary mean:

$$M(\mathbf{x}) = C(\mathbf{x}) A_0 \tag{A3}$$

and non-stationary correlation function:

$$K(\mathbf{x}, \mathbf{y}) = C(\mathbf{y} - \mathbf{x}) - C(\mathbf{x})C(\mathbf{y}) - \nabla C(\mathbf{x}).\Lambda^{-1} \nabla C(\mathbf{y}), \tag{A4}$$

where Λ is the matrix of the second-order spectral moments (10). We can then claim that the distribution of the field A *given that there is a local maximum at point* $\mathbf{0}$ *with top value* $A_0 \in \mathbb{R}$ is:

$$A(\mathbf{x}) = \tilde{A}(\mathbf{x}) + C(\mathbf{x})(A_0 - \tilde{A}(\mathbf{0})) + \nabla C(\mathbf{x}).\Lambda^{-1} \nabla \tilde{A}(\mathbf{0}) \tag{A5}$$

where $\tilde{A}$ obeys Gaussian statistics with AF C. Indeed, we can compute the mean and correlation function of the process defined by (A5). We find the very same expressions as Eqs. (A3-A4). Since a Gaussian process is fully characterized by its mean and correlation function the identity (A5) is proved.

APPENDIX: REFERENCES

[1] R. H. Lehmberg and S. P. Obenschain, Opt. Commun. **46**, 27 (1983).

[2] R. H. Lehmberg, A. J. Schmitt, and S. E. Bodner, J. Appl. Phys. **62**, 2680 (1987).

[3] S. Skupsky, R. W. Short, T. Kessler, R. S. Craxton, S. Letzring, and J. M. Soures, J. Appl. Phys. **66**, 3456 (1989).

[4] D. Véron, H. Ayral, C. Gouédard, D. Husson, J. Lauriou, O. Martin, B. Meyer, M. Rostaing, and C. Sauteret, Opt. Commun. **65**, 42 (1988).

[5] R. P. Drake, E. A. Williams, P. E. Young, K. Estabrook, W. L. Kruer, H. A. Baldis, and T. W. Johnston, Phys. Rev. Lett. **60**, 1018 (1988).

[6] M. André, in *Technical Committee Meeting on Drivers and Ignition facilities for Inertial Fusion*, J. Coutant, ed., Proceedings of the International Atomic Energy Agency (CEA/DAM Publications, Limeil-Valenton, France, 1995).

[7] R. W. Short, W. Seka, and R. Bahr, Phys. Fluids **30**, 3245 (1987).

[8] A. N. Mostovych, S. P. Obenschain, J. H. Gardner, J. Grun, K. J. Kearney, C. K. Manka, E. A. McLean, and C. J. Pawley, Phys. Rev. Lett. **59**, 1193 (1987).

[9] S. Baton, C. Labaune, G. Mathieussant, and W. Seka, Opt. Commun. **70**, 50 (1989).

[10] O. Willi, T. Afshar-rad, S. Coe, and A. Giulietti, Phys. Fluids B **2**, 1318 (1990).

[11] C. Labaune, S. Baton, T. Jalinaud, H. Baldis, and D. Pesme, Phys. Fluids B **4**, 2224 (1992).

[12] H. A. Rose, Phys. Plasmas **2**, 2216 (1995).

[13] V. T. Tikhonchuk, P. Mounaix, and D. Pesme, Phys. Plasmas **4**, 2670 (1997).

[14] V. T. Tikhonchuk, S. Hüller, and P. Mounaix, Phys. Plasmas **4**, 4369 (1997).

[15] H. A. Rose and D. F. DuBois, Phys. Fluids B **5**, 590 (1993).

[16] H. A. Rose and D. F. DuBois, Phys. Rev. Lett. **72**, 2883 (1994).

[17] J. Garnier, C. Gouédard, L. Videau, and A. Migus, J. Opt. Soc. Am. A **14**, 1928 (1997).

[18] J. Garnier, Phys. Plasmas **6**, 1601 (1999).

[19] L. Videau, C. Rouyer, J. Garnier, and A. Migus, J. Opt. Soc. Am. A **16**, 1672 (1999).

[20] Y. Kato, K. Mima, N. Miyanaga, S. Arinaga, Y. Kitagawa, M. Nakatsuka, and C. Yamanaka, Phys. Rev. Lett. **53**, 1057 (1984).

[21] S. N. Dixit, M. D. Feit, M. D. Perry, and H. T. Powell, Opt. Lett. **21**, 1715 (1996)

[22] J. W. Goodman, Statistical properties of laser speckle patterns. In *Laser speckle and related phenomena*, edited by J. C. Dainty (Topics in Applied physics **9**, Springer-Verlag, Berlin, 1984), pp. 9-75.

[23] R. S. Craxton and S. Skupsky, Bull. Am. Phys. Soc. **40**, 1826 (1995).

[24] J. E. Rothenberg, J. Opt. Soc Am. B **14**, 1664 (1997).

[25] J. Garnier, C. Gouédard, and L. Videau, Opt. Commun. **176**, 281 (2000).

[26] A. C. L. Boscheron, E. Journot, A. Dulac, and A. Adolf, in *Conference on Lasers and Electro-Optics*, Vol. 6 of OSA Technical Digest Series (optical Society of America, Wasington D.C., 1998), pp. 342-343.

[27] J. Strong, *Concepts of classical optics* (Freeman, New York, 1958).

[28] D. Middleton, *Introduction to statistical communication theory* (Mc Graw Hill, New York, 1960), p. 141.

[29] J. Adler, *The geometry of random fields* (Wiley, New York, 1981).