# Instability of a quantum particle induced by a randomly varying spring coefficient

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**Abstract.** This paper investigates the evolution of a quantum particle in a harmonic oscillator whose spring coefficient randomly fluctuates around its mean value. The perturbations are small, but they act long enough so that we can solve the problem in the asymptotic framework corresponding to a perturbation amplitude which tends to zero and a perturbation duration which tends to infinity. We describe the effective evolution equation of the state vector which reads as a stochastic partial differential equation. We exhibit a closed-form equation for the transition probabilities, which can be interpreted in terms of a jump process. Using standard probability tools, we are then able to compute explicitly the probabilities for observing the different energy eigenstates and give the exact statistical distribution of the energy of the particle.
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## 1. Introduction

This paper is a contribution to the study of time-dependent perturbations of quantum systems. One can find in the literature a lot of work devoted to special types of perturbations: sudden, adiabatic, periodic, .... [27]. The considered phenomena are described by a Hamiltonian which is the sum $H^0 + H^1$ of a time-independent piece $H^0$ whose eigenvalue problem has been solved, and of a small time-dependent perturbation. The typical question one asks is the following. If at $t = 0$ the system is in the eigenstate $\psi^0$ of $H^0$, what is the probability for it to be observed in a given eigenstate ? Most results that have been obtained follow a scheme in which the answers are computed in a perturbation series in powers of $H^1$ [27, 21]. We shall apply a new method for obtaining answers to the above questions, which is based on the one hand on some rigorous asymptotic theory and on the other hand on a representation of the evolution of the transition probabilities in terms of a Markov jump process. In this paper we shall focus on a quadratic perturbation of the harmonic oscillator, although the method can be applied to more general situations.

The quantum harmonic oscillator has been extensively studied because a lot of systems close to a stable equilibrium can be described by an oscillator or a collection of decoupled harmonic oscillators [21]. Furthermore modifications of this model have been investigated, handling by the perturbation theory. Indeed, even for this simple model it is exceptional to find closed-form expressions, except for very particular types of perturbations [17]. Nevertheless rigorous results have been obtained for time-dependent perturbations of the harmonic oscillator. Most of them concern periodic driven force [8, 9, 16]. Although the problem is far less understood in the case of random perturbations, one can find some results in the literature devoted to systems with randomly time-dependent external driving force. A general class of quantum systems in Markovian potentials has been treated in detail [25, 26]. Under suitable conditions on the dynamics of the random potential, it is shown in Ref. [23] that the spectrum of the quasi-energy operator is continuous. In Ref. [7] the authors study the long-time stability of oscillators driven by time-dependent forces originating from dynamical systems with varying degrees of randomness and focus on the asymptotic energy growth. Recently in Ref. [13] we have studied the energy density of a charged particle in a harmonic oscillator driven by a time-dependent homogeneous electric field. In this paper we consider a particle in a harmonic oscillator whose spring coefficient is not constant but randomly fluctuates around its mean value. We aim at studying this problem by a rigorous and non-perturbative method. Our approach is inspired by the works of Papanicolaou and its co-workers about waves in random media [19, 24]. The first step consists in determining the characteristic scales of the problem at hand: mean oscillation frequency of the harmonic oscillator, amplitude, coherence time and duration of the random perturbations. We then study the asymptotic evolution of the state vector in the asymptotic framework based on the separation of these scales. Our main aim is to exhibit the asymptotic regime which corresponds to the case where the amplitudes of the random fluctuations go to zero and the duration of the external perturbation goes to infinity. We then describe explicitly the effective random evolution of the state vector and the probability transitions. The paper is organized as follows. In Section 2 we review the main features of the harmonic oscillator and we formulate the problem at hand in Section 3. In Section 4 we establish a relation between the Lyapunov exponents of the energy of the quantum harmonic oscillator and that of the classical harmonic oscillator, which has already been studied. In Sections 5-6 we derive effective evolution equations for the modal decomposition and for the state vector of the particle. By exploiting a representation of the evolution of the energy of the particle in terms of a jump process, we are able to give closed-form expressions for the transition probabilities in Section 7. Finally we compare the theoretical results with numerical simulations in Section 8.

## 2. The harmonic oscillator

We consider the quantum oscillator, that is to say, a particle of mass $M$ whose state vector in the coordinate basis obeys the Schrödinger equation [21]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}M\omega^2x^2\psi, \tag{1}$$

where $\omega$ is the oscillation frequency. In order to transform this equation into a standard and dimensionless form, we multiply the spatial coordinate $x$ by $r_0^{-1} := (M\omega/\hbar)^{1/2}$ and the time $t$ by $t_0^{-1} := \omega$, so that (1) now reads:

$$2i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + x^2\psi. \tag{2}$$

The spectrum of the harmonic oscillator is pure point with state energies $(2p+1)/2$ and corresponding eigenstates [21]:

$$f_p(x) = \frac{1}{\sqrt{2^p\sqrt{\pi}p!}}H_p\,(x)\,e^{-x^2/2}, \tag{3}$$

$$H_p(x) = (-1)^p e^{x^2}\frac{d^p}{dx^p}e^{-x^2}. \tag{4}$$

The family $(f_p)_{p\in\mathbb{N}}$ is complete in the following sense [15, Prop. 1.5.7].

**Proposition 2.1.** *1. The $(f_p)_{p\in\mathbb{N}}$ are an orthonormal basis of $L^2(\mathbb{R},\mathbb{C})$.*
*2. $(t,x)\mapsto e^{-i\frac{2p+1}{2}t}f_p(x)$ is a solution of (2) for any $p\in\mathbb{N}$.*

We define the eigenstate decomposition as the map $\Theta : \psi \in L^2(\mathbb{R},\mathbb{C}) \mapsto (c_p)_{p\in\mathbb{N}}$, where $c_p$ are the coefficients of the expansion of $\psi$ in the basis $(f_p)$:

$$\Theta(\psi)_p := c_p = \int_{\mathbb{R}} f_p(x)\psi(x)dx. \tag{5}$$

By Proposition 2.1, $\Theta$ is an isometry from $L^2(\mathbb{R},\mathbb{C})$ onto $l^2$, the space of all the sequences $(c_p)_{p\in\mathbb{N}}$ from $\mathbb{N}$ into $\mathbb{C}$ which are squared integrable. In view of the fundamental postulates of the quantum mechanics, if $\psi$ is the state vector of the particle, then the measurement of the energy will yield the eigenvalue $(2p+1)/2$ with probability $|\Theta(\psi)_p|^2$.

## 3. Evolution driven by time-dependent forces

Suppose that the spring coefficient is shaken so that it fluctuates randomly with respect to its mean value. The problem is to compute the manner in which this randomness distributes th population from the initial (low) energy state into other (high) ones. The perturbed equation which governs the evolution of the state vector is then:

$$2i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + x^2(1+\varepsilon m(t))\psi. \tag{6}$$

We assume that the amplitudes of the fluctuations are of order $\varepsilon \ll 1$. The real-valued function $m$ is assumed to be zero-mean, time-stationary and time-ergodic process. More exactly the forthcoming results require that the random process $t \mapsto m(t)$ fulfills the technical mixing condition "$m$ is $\phi$-mixing, with $\phi \in L^{1/2}(\mathbb{R}^+)$" (see [20, Section 4-6-2]).

The initial state vector at time $t = 0$ is $\psi_0$, which corresponds to the decomposition $c(0) = \Theta(\psi_0)$. Since $\Theta$ is an isometry, it is equivalent to study the evolution of the expansion of $\psi$ in the family of eigenstates $(f_p)_{p \in \mathbb{N}}$, i.e. the corresponding normalized coefficients $c$:

$$c_p(t) = \Theta(\psi(t,.))_p e^{i\frac{2p+1}{2}t}, \tag{7}$$

$$\psi(t,x) = \sum_{p'=0}^{\infty} c_{p'}(t) e^{-i\frac{2p'+1}{2}t} f_{p'}(x). \tag{8}$$

Substituting the expression (8) into Eq. (6) and integrating with respect to $f_p(x)dx$ we get the equation that governs the evolution of $c$:

$$\frac{dc_p}{dt} = \varepsilon \frac{i}{4} m(t) \left( \sqrt{\Gamma_p} c_{p-2} e^{-2it} + (2p+1) c_p + \sqrt{\Gamma_{p+2}} c_{p+2} e^{2it} \right), \tag{9}$$

where the coefficient $\Gamma_p$ is simply:

$$\Gamma_p = p(p-1).$$

The coefficients $\mathbb{E}[|c_p|^2(t)]$ represent the probabilities that the particle driven by the random process $\varepsilon m x^2$ be observed in the state $f_p$ at time $t$. Equivalently, one can say that the measurement of the energy at time $t$:

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} \left( \left| \frac{\partial \psi}{\partial x} \right|^2 + x^2 |\psi|^2 \right) (t,x) dx \tag{10}$$

will yield the eigenvalue $p + 1/2$ with probability $\mathbb{E}[|c_p|^2(t)]$. This implies that the expected value of the energy can be expressed as:

$$\mathbb{E}[E(t)] = \frac{1}{2} + \sum_{p=0}^{\infty} p \mathbb{E}[|c_p|^2(t)]. \tag{11}$$

## 4. Lyapunov exponents of the energy

We mention in the introduction that there is a close relation between the classical and quantum harmonic oscillators. In particular their energy growths may be related by the following way. As pointed out in [18], a state vector $\psi$ can be characterized by its Wigner transform:

$$\mathcal{W}(x,k) = \frac{1}{\pi} \int dy \psi^*(x+y) \psi(x-y) e^{i2ky}.$$

Although $\mathcal{W}$ is not necessarily positive, it possesses some properties of a local density of state and in particular the following identities hold true:

$$\begin{aligned}\int k^2\mathcal{W}(x,k)dk &= \left|\frac{\partial\psi}{\partial x}(x)\right|^2,\\ \int \mathcal{W}(x,k)dk &= |\psi(x)|^2.\end{aligned}$$

The Wigner transform $\mathcal{W}(t,x,k)$ of the state vector $\psi(t)$ satisfies the transport equation:

$$\frac{\partial\mathcal{W}}{\partial t} = -k\frac{\partial\mathcal{W}}{\partial x} + x(1+\varepsilon m(t))\frac{\partial\mathcal{W}}{\partial k}$$

which coincides with the classical Liouville equation. The solution can be expressed as:

$$\mathcal{W}(t,x,k) = \mathcal{W}(0,X(-t,x,k),K(-t,x,k)),$$

where $(X,K)$ is the classical harmonic oscillator whose time-dependent Hamiltonian is $H = \frac{K^2}{2} + (1+\varepsilon m(t))\frac{X^2}{2}$:

$$\begin{aligned}\frac{\partial X}{\partial t} &= K(t), & X(0) &= x, \qquad (12)\\ \frac{\partial K}{\partial t} &= -(1+\varepsilon m(t))X(t), & K(0) &= k. \qquad (13)\end{aligned}$$

$X(-t,x,k), K(-t,x,k)$ are the initial conditions that evolve into $(x,k)$ at time $t$. The energy of the quantum particle is therefore:

$$\begin{aligned}E(t) &= \frac{1}{2}\int dxdk(x^2+k^2)\mathcal{W}(t,x,k)\\ &= \frac{1}{2}\int dxdk\left(X(t,x,k)^2+K(t,x,k)^2\right)\mathcal{W}(0,x,k).\end{aligned}$$

Introducing polar coordinates $R(t,\theta_0)$ and $\theta(t,\theta_0)$:

$$\begin{aligned}X(t,x,k) &= \sqrt{x^2+k^2}R(t,\theta_0)\cos(\theta(t,\theta_0)),\\ K(t,x,k) &= \sqrt{x^2+k^2}R(t,\theta_0)\sin(\theta(t,\theta_0)),\\ \theta_0 &= \arctan(k/x),\end{aligned}$$

the system (4) is equivalent to:

$$\begin{aligned}\frac{\partial R}{\partial t} &= -R\varepsilon m(t)\frac{\sin(2\theta)}{2}, & R(0,\theta_0) &= 1,\\ \frac{\partial\theta}{\partial t} &= -1-(1+\varepsilon m(t))\frac{1+\cos(2\theta)}{2}, & \theta(0,\theta_0) &= \theta_0.\end{aligned}$$

The energy of the quantum particle then writes:

$$E(t) = \frac{1}{2}\int_0^{2\pi} R(t,\theta_0)^2 p(\theta_0)d\theta_0,$$

where $p(\theta_0)$ is the density with respect to the Lebesgue measure over $[-\pi, \pi]$ of $\arctan(k/x)$ under $\mathcal{W}(0, x, k)$:

$$p(\theta) = \int_0^\infty \mathcal{W}(0, r\cos(\theta), r\sin(\theta)) r dr.$$

If the initial state is the fundamental eigenstate, then $p(\theta_0) \equiv 1/(2\pi)$. Under appropriate assumptions on the law of the process $m$, [4, Theorem 4] proves that there exists an analytic function $q \mapsto g(q)$ such that:

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}[R^q(t,\theta)] = g(q), \tag{14}$$

$$\lim_{t\to\infty} \frac{1}{t} \ln R(t,\theta) = g'(0) \text{ almost surely.} \tag{15}$$

Moreover the convergence is uniform over $\theta \in [-\pi, \pi]$. In [4] this theorem is stated for piecewise constant process $m$, various versions exist which yield the same conclusion for many different classes of processes $m$ [12, 28, 6, 2]. Thus the energy of the quantum particle satisfies:

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}[E(t)] = g(2), \tag{16}$$

$$\lim_{t\to\infty} \frac{1}{t} \ln E(t) = 2g'(0) \text{ almost surely.} \tag{17}$$

Nevertheless the Lyapunov exponent of the $q$-th moment ($q > 1$) requires a little more information than (4). For instance the exponential growth rate of $\mathbb{E}[E(t)^2]$ requires the knowledge of the growth rate of $\mathbb{E}[R^2(t,\theta_0)R^2(t,\theta_1)]$ for all pairs $(\theta_0, \theta_1) \in [-\pi, \pi]^2$. Although the expression of $g(q)$ is very intricate, even for a very simple random process $m$, it can be expanded as powers of $\varepsilon$ [3]:

$$g(2q) = \frac{q^2+q}{2}\alpha_2\varepsilon^2 + O(\varepsilon^3), \tag{18}$$

$$2g'(0) = \frac{1}{2}\alpha_2\varepsilon^2 + O(\varepsilon^3), \tag{19}$$

where $\alpha_2$ is the parameter:

$$\alpha_2 = \int_0^\infty \cos(2t)\mathbb{E}[m(0)m(t)]dt.$$

The main feature is that the sample and mean Lyapunov exponents of the system are different. More exactly that the mean Lyapunov energy is twice as large as the sample Lyapunov energy. In the following sections we aim at describing more precisely the energy distribution of the quantum particle. Indeed $E(t)$ is the mean energy, where the averaging is with respect to the quantum distribution. Nevertheless the second quantum theory postulate claims that an observation of the energy can only give a value amongst the discrete set of eigenvalues $1/2+\mathbb{N}$. Consequently a precise description of the energy distribution consists in the determination of the probability transitions.

## 5. Asymptotic behavior of the modal energy distribution

Let us define the process $\bar{c}$ as the solution of the following infinite-dimensional system of linear stochastic differential equations starting from $c(0)$:

$$d\bar{c}_p = \frac{\sqrt{\alpha_2}}{4}\left(\sqrt{\Gamma_{p+2}}\bar{c}_{p+2} - \sqrt{\Gamma_p}\bar{c}_{p-2}\right) \circ dW_t^1 + \frac{i\sqrt{\alpha_2}}{4}\left(\sqrt{\Gamma_{p+2}}\bar{c}_{p+2} + \sqrt{\Gamma_p}\bar{c}_{p-2}\right) \circ dW_t^2 + \frac{i\sqrt{\alpha_0}}{2\sqrt{2}}(2p+1)\bar{c}_p \circ dW_t^3, \quad (20)$$

where $W^j$, $j = 1, ..., 3$ are independent standard Brownian motions, $\circ$ stands for the Stratonovich stochastic integral, and $\alpha_j$ is the real parameter given by:

$$\alpha_j = \int_0^\infty \mathbb{E}[m(0)m(t)]\cos(jt)dt, \quad (21)$$

which is proportional to the power spectral density of the process $m$ (nonnegative by the Wiener-Khintchine theorem [22]).

**Proposition 5.1.** *1. There exists a unique solution $\bar{c}$ of Equation (20).*
*2. The processes $c(./\varepsilon^2)$ converge in distribution as continuous functions from $[0,\infty)$ into $l^2$ to the diffusion Markov process $\bar{c}$ solution of (20) as $\varepsilon \to 0$.*

*Proof.* Apply formally the (unique!) theorem of Ref. [24]. Take care to separate the real and imaginary parts of the process $c(./\varepsilon^2)$: Denoting $\chi_{2p} := \mathrm{Re}c_p$ and $\chi_{2p+1} := \mathrm{Im}c_p$, the process $\chi^\varepsilon(t) := \chi(t/\varepsilon^2)$ satisfies the linear differential equation:

$$\frac{d\chi^\varepsilon(t)}{dt} = \frac{1}{\varepsilon}m\left(\frac{t}{\varepsilon^2}\right)F\left(\chi^\varepsilon, \frac{t}{\varepsilon^2}\right),$$

where

$$\begin{aligned}
F_{2p}(\chi, h) &= \frac{\cos(2h)}{4}\left(\sqrt{\Gamma_{p+2}}\chi_{2p+5} - \sqrt{\Gamma_p}\chi_{2p-3}\right) \\
&\quad + \frac{\sin(2h)}{4}\left(\sqrt{\Gamma_{p+2}}\chi_{2p+4} + \sqrt{\Gamma_p}\chi_{2p-4}\right) - \frac{2p+1}{4}\chi_{2p+1}, \\
F_{2p+1}(\chi, h) &= \frac{\cos(2h)}{4}\left(\sqrt{\Gamma_{p+2}}\chi_{2p+4} + \sqrt{\Gamma_p}\chi_{2p-4}\right) \\
&\quad + \frac{\sin(2h)}{4}\left(-\sqrt{\Gamma_{p+2}}\chi_{2p+5} + \sqrt{\Gamma_p}\chi_{2p-3}\right) + \frac{2p+1}{4}\chi_{2p}.
\end{aligned}$$

Note that we deal with an infinite-dimensional system while only finite-dimensional systems are addressed in Ref. [24]. Nevertheless $c(./\varepsilon^2)$ can be approximated by finite-dimensional processes. The technical developments are proposed in Ref. [13] in the case of a perturbation of the form $2xm(t)$ and are very similar in the present case $x^2m(t)$. This technique based on a martingale approach to some limit theorems in the diffusion-approximation regime is now well-known and extensively reviewed in literature [19, 20]. The processes $\left(c_p(./\varepsilon^2)\right)_{p\in\mathbb{N}}$ converge in distribution in $l^2$ to the diffusion process $(\bar{\chi}_{2p}(.) + i\bar{\chi}_{2p+1}(.))_{p\in\mathbb{N}}$ where $\bar{\chi}$ is the diffusion process with infinitesimal generator $\mathcal{L}$:

$$\mathcal{L} = \sum_{i,j=0}^{\infty} a_{ij}(\bar{\chi})\frac{\partial^2}{\partial\bar{\chi}_i\partial\bar{\chi}_j} + \sum_{j=0}^{\infty} b_j(\bar{\chi})\frac{\partial}{\partial\bar{\chi}_j}.$$

The diffusion and drift coefficients are:

$$\begin{aligned} a_{i,j}(\bar{\chi}) &= \int_0^\infty \mathbb{E}[m(0)m(t)]\left\langle F_i(\bar{\chi},h)F_j(\bar{\chi},h+t)\right\rangle_h dt, \\ b_j(\bar{\chi}) &= \int_0^\infty \sum_{i=0}^{\infty} \mathbb{E}[m(0)m(t)]\left\langle F_i(\bar{\chi},h)\frac{\partial F_j}{\partial\chi_i}(\bar{\chi},h+t)\right\rangle_h dt, \end{aligned}$$

where $\langle . \rangle_h$ stands for an averaging over a period in $h$. The application of Itô's formula to system (20) then yields that the distribution of $\bar{c}$ is the same as $(\bar{\chi}_{2p}(.) + i\bar{\chi}_{2p+1}(.))_{p\in\mathbb{N}}$. □

There exist also technical conditions on the initial condition $c(0)$ so that the above proposition holds true. These conditions require that the initial sequence $(c(0)_p)_{p\in\mathbb{N}}$ decays fast enough and they are fulfilled in particular if the initial state is a pure eigenstate, i.e. $c(0)_p = \delta_{p,p_0}$ for some $p_0$. Thus, in order to avoid unnecessary intricate technical developments, we shall assume throughout the paper that the initial state is a pure eigenstate. If the initial state is $f_{p_0}$, the coefficients

$$C_{p_0,p}(t) := \mathbb{E}[|\bar{c}_p|^2(t)] \tag{22}$$

represent the probabilities that the particle driven by the random process $\varepsilon m x^2$ be observed in the state $f_p$ at time $t/\varepsilon^2$ in the asymptotic framework $\varepsilon \to 0$. Proposition 5.1 is very useful since it allows us to compute efficiently these relevant quantities:

**Proposition 5.2.** *The family* $(C_{p_0,p}(t))_{p\in\mathbb{N}}$ *satisfies a closed-form set of ordinary differential equations:*

$$\frac{dC_{p_0,p}}{dt} = \frac{\alpha_2}{8}\Gamma_{p+2}(C_{p_0,p+2} - C_{p_0,p}) + \frac{\alpha_2}{8}\Gamma_p(C_{p_0,p-2} - C_{p_0,p}), \tag{23}$$

*starting from* $C_{p_0,p}(0) = \delta_{p,p_0}$.

*Proof.* This results from a direct application of Itô's formula. □

The system (23) is one of the most important results of the paper. It shows that the probabilities $C_{p_0,p}$ can be computed theoretically from the coupling coefficients $\Gamma_p$, and that their evolutions are self-consistent in the sense that no other relevant quantities come into. In particular the relative phases between the coefficients $(\bar{c}_p)$ of the expansion of the state vector $\psi$ in the basis $(f_p)$ have no importance in the asymptotic evolution of the probability distribution $(C_{p_0,p})$. This statement is not at all obvious, since it is not satisfied by the original equations (6).

## 6. Asymptotic behavior of the state vector

We denote by $\tilde{\psi}^\varepsilon(x,t)$ the state vector at the regularly spaced instants $2\pi[t/(2\pi\varepsilon^2)]$ defined by:

$$\tilde{\psi}^\varepsilon(t,x) = \psi\left(2\pi\left[\frac{t}{2\pi\varepsilon^2}\right], x\right),$$

where $[\tau]$ stands for the integral part of a real number $\tau$. The process $\tilde{\psi}^\varepsilon$ possesses nice convergence properties in the space of the càd-làg functions $\mathbf{D}$ equipped with the Skorohod topology The following result is a straightforward corollary of Proposition 5.1 since $\Theta$ is an isometry from $L^2$ into $l^2$.

**Proposition 6.1.** *$\tilde{\psi}^\varepsilon$ converges in distribution in $\mathbf{D}([0,\infty), L^2)$ to $\tilde{\psi}$ which is the unique solution of:*

$$\begin{aligned} d\tilde{\psi} &= \frac{\sqrt{\alpha_2}}{4}\left(1 + 2x\frac{\partial}{\partial x}\right)\tilde{\psi}\circ dW_t^1 \\ &+\frac{i\sqrt{\alpha_2}}{4}\left(x^2 + \frac{\partial^2}{\partial x^2}\right)\tilde{\psi}\circ dW_t^2 + \frac{i\sqrt{\alpha_0}}{2\sqrt{2}}\left(-x^2 + \frac{\partial^2}{\partial x^2}\right)\tilde{\psi}\circ dW_t^3, \end{aligned} \tag{24}$$

*starting from $\tilde{\psi}(0,x) = f_{p_0}(x)$.*

Very fortunately the solution of Equation (24) can be written explicitly as:

$$\tilde{\psi}(t,x) = \sqrt{a(t)}e^{-ib(t)x^2} f_{p_0}(a(t)x)e^{-i(2p_0+1)\phi(t)},$$

where the coefficients $a$, $b$, and $\phi$ obey the following system of stochastic differential equations:

$$da = \frac{\sqrt{\alpha_2}}{2}a\circ dW_t^1 + \sqrt{\alpha_2}ab\circ dW_t^2 + 2\sqrt{2}\sqrt{\alpha_0}ab\circ dW_t^3, \tag{25}$$

$$db = \sqrt{\alpha_2}b\circ dW_t^1 - \frac{\sqrt{\alpha_2}}{4}(1+a^4-4b^2)\circ dW_t^2 + \frac{\sqrt{\alpha_0}}{2\sqrt{2}}(1-a^4+4b^2)\circ dW_t^3, \tag{26}$$

$$d\phi = \frac{\sqrt{\alpha_2}}{8}a^2\circ dW_t^2 + \frac{\sqrt{\alpha_0}}{2\sqrt{2}}a^2\circ dW_t^3, \tag{27}$$

starting from $a(0)=1$, $b(0)=0$, $\phi(0)=0$. In particular the spatial distribution of the quantum particle is $|\tilde{\psi}(t,x)|^2 = a(t)|f_{p_0}(a(t)x)|^2$, which is the initial profile rescaled by the factor $a(t)$. Furthermore, the state vector $\tilde{\psi}_\delta^\varepsilon$ in the intermediate planes $\delta + 2\pi[t/(2\pi\varepsilon^2)]$ defined by:

$$\tilde{\psi}_\delta^\varepsilon(t,x) = \psi\left(\delta + 2\pi\left[\frac{t}{2\pi\varepsilon^2}\right], x\right),$$

converges in distribution in $\mathbf{D}([0,\infty), L^2)$ to $\tilde{\psi}_\delta$:

$$\tilde{\psi}_\delta(t,x) = \sum_{p=0}^{\infty}\bar{c}_p(t)e^{-i(p+1/2)\delta}f_p(x).$$

The asymptotic state vector $\tilde{\psi}_\delta$ can be derived from $\tilde{\psi}$ through the following equation, in which $t$ is frozen:

$$2i\frac{\partial \tilde{\psi}_\delta}{\partial \delta} - x^2\tilde{\psi}_\delta + \frac{\partial^2 \tilde{\psi}_\delta}{\partial x^2} = 0, \qquad \tilde{\psi}_\delta(t,x)\Big|_{\delta=0} = \tilde{\psi}(t,x).$$

The general solution $\tilde{\psi}_\delta(t,x)$ is found to be:

$$\tilde{\psi}_\delta(t,x) = \sqrt{a(\delta,t)}e^{-ib(\delta,t)x^2} f_{p_0}(a(\delta,t)x)e^{-i(2p_0+1)\phi(\delta,t)},$$

where

$$\begin{aligned}
a(\delta,t) &= \frac{a(t)}{\sqrt{(\cos(\delta)+2b(t)\sin(\delta))^2 + a(t)^4\sin(\delta)^2}},\\
b(\delta,t) &= \frac{b(t)\cos(2\delta) + (4b(t)^2 + a(t)^4 - 1)\frac{\sin(2\delta)}{4}}{(\cos(\delta)+2b(t)\sin(\delta))^2 + a(t)^4\sin(\delta)^2},\\
\phi(\delta,t) &= \phi(t) - \frac{\delta}{2}.
\end{aligned}$$

We can give simple representations of the distributions of the variables $a(\delta,t)$ and $b(\delta,t)$ in terms of two auxiliary Brownian motions.

**Proposition 6.2.** *1. The processes $(a(\delta,.), b(\delta,.))$ obey the same distribution as $(a(.), b(.))$.*
*2. For any $\delta$ and $t$, the variables $(a^2(\delta,t), b(\delta,t))$ obey the distributions of:*

$$a^2(t) = \exp\left(\sqrt{\alpha_2}W_t^4 - \frac{\alpha_2}{2}t\right), \tag{28}$$

$$b(t) = \frac{\sqrt{\alpha_2}}{2}\exp\left(\sqrt{\alpha_2}W_t^4 - \frac{\alpha_2}{2}t\right)\int_0^t \exp\left(-\sqrt{\alpha_2}W_{t'}^4 + \frac{\alpha_2}{2}t'\right)dW_{t'}^5, \tag{29}$$

*where $W^4$ and $W^5$ are independent standard Brownian motions.*

*Proof.* The proof consists in applying Itô's formula for the system (25-26) on the one hand and for the processes defined by Eqs. (28-29) on the other hand. One can then check that both formulations are identical. □

We can deduce relevant features from the representations (28-29). Let us discuss one of them, which concerns the fundamental difference between the long-range mean behavior and long-range typical behavior of $a(t)$. For large $t$ it is well known that the typical value of $W_t^4$ is of the order of $\sqrt{t}$, so that the typical value of the random variable $a^2(t)$ is of the order of $\exp\left(-\frac{\alpha_2}{2}t \pm \sqrt{\alpha_2 t}\right)$. In this exponential the first term prevails for large $t$, so that $a^2(t)$ is exponentially small. Nevertheless, the mean value $\mathbb{E}[a^2(t)] = 1$, whatever $t$. This means that the *mean* value does not correspond at all to a *typical* value that can be reached by the random variable $a^2(t)$. The mean value is actually imposed by very few events which represent large deviations of the Brownian motion. These exceptional events are enhanced by the exponential, which gives $\mathbb{E}[\exp\sqrt{\alpha_2}W_t^4] = \exp(\frac{\alpha_2}{2}t)$. These observations are of course closely related with the fact that the sample and mean Lyapunov exponents of the system at hand are different, as exhibited in Section 4.

**Proposition 6.3.** *Let us consider the energy $E(t)$ defined by (10). The quantity $E(./\varepsilon^2)$ converges as a continuous function from $[0,\infty)$ into $\mathbb{R}$ to the process $\bar{E}$ given by:*

$$\bar{E}(t) = E_0 \left( \frac{a^2(t)}{2} + \frac{1+4b^2(t)}{2a^2(t)} \right), \tag{30}$$

*where $E_0$ is the energy of the particle in the initial state $f_{p_0}$: $E_0 = p_0 + 1/2$. The moments of the energy satisfy:*

$$\begin{aligned} \mathbb{E}[\bar{E}(t)] &= E_0 e^{\alpha_2 t}, & (31)\\ \mathbb{E}[\bar{E}(t)^2] &= E_0^2 \left( \frac{2}{3} e^{3\alpha_2 t} + \frac{1}{3} \right), & (32)\\ \lim_{t\to\infty} \frac{1}{t} \mathbb{E}[\bar{E}(t)^q] &= \frac{q^2+q}{2}\alpha_2. & (33) \end{aligned}$$

*Proof.* The first moment of $\bar{E}$ can be computed directly since we get from the expression (11) that it satisfies a closed-form differential equation $\frac{d\mathbb{E}[\bar{E}]}{dt} = \alpha_2 \mathbb{E}[\bar{E}]$. The expression of $\bar{E}$ in terms of $a$, $b$ can be obtained from the expression (30). By Itô's formula we get:

$$\frac{d\mathbb{E}[a^n b^q]}{dt} = \alpha_2 \left( \frac{n(n-2)}{8} + \frac{nq+q(q-1)}{2} \right) \mathbb{E}[a^n b^q] + \alpha_2 \frac{q(q-1)}{8} \mathbb{E}[a^n b^{q-2}],$$

so that we can compute $\mathbb{E}[\bar{E}(t)^q]$ for any $q$. □

The energy has a very large variance, much larger than the square of its mean, which shows that it has very large fluctuations with respect to its mean value. This can be also put into evidence by comparing the typical value of the energy with its mean value. As pointed out here above, $a^2(t)$ decays as $\exp(-\frac{\alpha_2}{2}t)$ for $t \gg 1$, since $W_t^4 \sim \sqrt{t} \ll t$ with very high probability. Furthermore $b(t)$ obeys a normal distribution with mean 0 and variance 1/2 for $t \gg 1$. By substituting these approximations into Eq. (30), we get that the typical behavior of the energy $\bar{E}$ is for large $t$:

$$\frac{1}{t} \ln \bar{E}(t) \simeq \frac{\alpha_2}{2} \text{ for } t \gg 1. \tag{34}$$

The typical exponential behavior (34) is different from the exponential behavior of the mean energy (6.3), and the departure originates from large deviations of the Brownian motion $W^4$.

## 7. Interpretation of the limit system in terms of a jump process

We shall show that the transition probabilities $C_{p_0,p}(t)$ can be regarded as the statistical distribution of a jump process. Let $N_t$ be the Markov process with state space $\mathbb{N}$ and infinitesimal generator $\mathcal{K}$:

$$\mathcal{K} = \frac{\alpha_2}{8} \left( \Gamma_{N+2} \nabla_2 + \Gamma_N \nabla_{-2} \right), \tag{35}$$

where $\nabla_{p'} g(N) = g(N+p') - g(N)$. $(N_t)_{t\geq 0}$ is a time-homogeneous jump process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The operator $\nabla_{p'}$ corresponds to a jump from $n$ to $n+p'$. The complete construction of the process $N_t$ is standard [11, Section 7, Theorem 33]. There exists a sequence of integer-valued random variables $(\xi_j)_{j\in\mathbb{N}}$ and positive real-valued variables $(\tau_j)_{j\in\mathbb{N}}$ such that:

$$N_t = \xi_n \text{ if } \sum_{j=0}^{n-1} \tau_j \leq t < \sum_{j=0}^{n} \tau_j .$$

The sequence $(\xi_j)_{j\in\mathbb{N}}$ is a Markov chain with stationary transition matrix $Q$:

$$\mathbf{P}\left(\xi_{n+1} = p + p'/\xi_n = p\right) = Q_{p,p+p'},$$

where $Q_{p,p+p'} = 0$ if $p' \neq \{-2, 2\}$ and:

$$\begin{aligned} Q_{p,p+2} &= \frac{\Gamma_p}{\Gamma_p + \Gamma_{p+2}} = \frac{1}{2} - \frac{2p+1}{2(p^2+p+1)}, \\ Q_{p,p-2} &= \frac{\Gamma_{p+2}}{\Gamma_p + \Gamma_{p+2}} = \frac{1}{2} + \frac{2p+1}{2(p^2+p+1)}. \end{aligned}$$

Given $(\xi_j)_{j\in\mathbb{N}}$, the random variables $(\tau_j)_{j\in\mathbb{N}}$ are conditionally independent and exponentially distributed, with parameters $(q(\xi_j))_{j\in\mathbb{N}}$ :

$$q(p) = \frac{\alpha_2}{8}(\Gamma_p + \Gamma_{p+2}) = \frac{\alpha_2}{4}(p^2+p+1).$$

We denote by $\mathbf{P}_{t_0,p_0}$ the distribution of the paths $(N_t)_{t\geq t_0}$ starting at time $t_0$ with the initial condition $N_{t_0} = p_0$. The Markov process has stationary transition probabilities:

$$P_t(p_0, p) = \mathbf{P}_{t',p_0}(N_{t'+t} = p) \text{ independent of } t',$$

and it is reversible since the infinitesimal generator $\mathcal{K}$ is self-adjoint. Furthermore $P_t$ is the unique probabilistic solution (i.e. $\sum_{p'} P_t(p, p+p') = 1$) of the Kolmogorov's forward equation [10, Section X-3]:

$$\frac{\partial P_t}{\partial t}(p_0, p_1) = -q(p_1) P_t(p_0, p_1) + \sum_p q(p) Q_{p,p_1} P_t(p_0, p).$$

In view of the choices of the kernel $Q$ and parameters $q(.)$, we get that the Kolmogorov's forward equation is equivalent to (23), which implies the relation $C_{p_0,p}(t) = \mathbf{P}_{0,p_0}(N_t = p)$. This interpretation of the transition probabilities is very powerful to solve problems and study the system (23) since it allows us to apply existing results on Markov jump processes which can be found in the literature.

Note that we have only showed that $(N_t)$ gives the correct statistical distribution of the measurement of the energy of the particle at a given time $t$. In terms of probability theory, we have only proved that the one-dimensional distributions of $(N_t)$ and the ones of the energy distribution coincide. We shall see that $(N_t)$ gives the correct statistical distribution of any sequence of measurements. For that

purpose we revisit the postulates of quantum mechanics in terms of the jump process $N_t$.
*Postulate 1.* "The state vector $\psi$ obeys the Schrödinger equation (6)." The time evolution of the process $N_t$ is governed by the Markovian dynamics described by the infinitesimal generator (35).
*Postulate 2.* "The measurement of the energy at (normalized) time $t$ will yield one of the eigenvalues $(2p+1)/2$ with probability $C_{p_0,p}(t)$." The jump process $1/2+N_t$ takes values only in the set $1/2+\mathbb{N}$. At time $t$ one will find the value $1/2+p$ with probability $\mathbf{P}_{0,p_0}(N_t=p)$ which is equal to $C_{p_0,p}(t)$.
*Postulate 3.* "If a measure at time $t$ of the energy gives the result $1/2+p$, then the state of the system will change from $\psi(t)$ to $f_p$ as a result of the measurement." Let us assume that we start from state $f_{p_0}$ at time 0. If we observe the energy of the particle at times $t_1$ and $t_2$, then the probability for measuring first the energy $1/2+p_1$ and then $1/2+p_2$ is equal to the probability for observing $1/2+p_1$ at $t_1$ starting from $f_{p_0}$ multiplied by the probability for observing $1/2+p_2$ at $t_2$ starting from $f_{p_1}$ at $t_1$, because the system is in state $f_{p_1}$ just after the measurement at $t_1$. This product of probabilities also reads as $\mathbf{P}_{0,p_0}(N_{t_1}=p_1)\times\mathbf{P}_{t_1,p_1}(N_{t_2}=p_2)$. From the Markov property of the process $N_t$, this product is exactly equal to $\mathbf{P}_{0,p_0}(N_{t_1}=p_1,N_{t_2}=p_2)$. Of course this statement can be generalized to any sequence of measurements so that we conclude that if we observe the energies of the particle at times $t_1$,..., and $t_n$, then the probability to measure the sequence of energies: $1/2+p_1$,..., $1/2+p_n$ is exactly $\mathbf{P}_{0,p_0}(N_{t_1}=p_1,...,N_{t_n}=p_n)$. This means that the dynamics of the observations is Markovian and exactly described by the jump process $N_t$ (Markovian means that the future is independent from the past conditionally to the present). More remarkable, this property is essentially equivalent to the Postulate 3 of quantum mechanics. Indeed, if instead of Postulate 3 we assume that the dynamics of the observations is Markovian, then just after a measurement the system depends only on the result of the measurement, which means that just after measuring the energy $1/2+p_1$, the system must be in a state with energy $1/2+p_1$ with probability 1. Since there exists a unique eigenstate $f_{p_1}$ with energy $1/2+p_1$, this means that the system must be in state $f_{p_1}$ with probability 1 (here the non-degeneracy of the system plays a primary role). As a conclusion, the Postulate 3 is exactly the right condition under which we can extent the statement "$(N_t)$ describes the statistical distribution of the measurement of the energy at some given time" to the statement "$(N_t)$ describes the statistical distribution of any sequence of measurements of the energy".

By studying the generating function of the jump process $(N_t)$ the system (23) can be solved in the sense that closed-form expressions can be derived for the probabilities $C_{p_0,p}(t)$. We sketch out the method we use to derive these expressions. We first establish the equation that governs the evolution of the generating function $u(t,z):=\mathbb{E}\left[z^{N_t}\right]$ of the jump process:

$$\frac{\partial u}{\partial t}=\frac{\alpha_2}{8}(1-z^2)\frac{\partial^2}{\partial z^2}\left((1-z^2)u\right).$$

We then express $v(t,z) := (1-z^2)u(t,z)$ as the expectation of a functional of a diffusion process $Z$:

$$v(t,z) = \mathbb{E}\left[Z^{p_0}(t)(1-Z^2(t))\right],$$

where the infinitesimal generator of the Markov process $Z$ starting from $z$ is:

$$\mathcal{L}_Z = \frac{\alpha_2}{8}(1-Z^2)^2\frac{\partial^2}{\partial Z^2}.$$

We introduce the auxiliary process $Y_t = \mathrm{argth}(Z_t)$ whose infinitesimal generator is:

$$\mathcal{L}_Y = \frac{\alpha_2}{8}\frac{\partial^2}{\partial Y^2} + \frac{\alpha_2}{4}\tanh(Y)\frac{\partial}{\partial Y}.$$

We can then apply a generalized version [1, Prop.1] of Bougerol's identity which states that, for any $t$, $\sinh(Y_t)$ has the same distribution as $X_t$:

$$X_t = \exp\left(\frac{\sqrt{\alpha_2}}{2}B_t^1 + \frac{\alpha_2}{4}t\right)\left(\frac{z}{\sqrt{1-z^2}} + \frac{\sqrt{\alpha_2}}{2}\int_0^t \exp\left(-\frac{\sqrt{\alpha_2}}{2}B_s^1 - \frac{\alpha_2}{4}s\right)dB_s^2\right),$$

where $B^1$ and $B^2$ are two independent Brownian motions. We then get the expression:

$$v(t,z) = \mathbb{E}\left[X_t^{p_0}\left(1+X_t^2\right)^{-1-p_0/2}\right]. \tag{36}$$

We finally note that for $z=0$, $u(t,0) = v(t,0) = \mathbb{P}_{0,p_0}(N_t = 0)$. Furthermore, still for $z=0$, the distribution of $X_t$ can be represented as:

$$X_t = \frac{\sqrt{\alpha_2}}{2}\int_0^t \exp\left(\frac{\sqrt{\alpha_2}}{2}B_s^1 + \frac{\alpha_2}{4}s\right)dB_s^2. \tag{37}$$

Substituting into Eq. (36) yields in particular that:

$$C_{0,0}(t) = \mathbf{P}_{0,0}(N_t = 0) = \mathbb{E}\left[\left(1+X_t^2\right)^{-1}\right],$$

with $X_t$ as defined by Eq. (37). Corollary 2.3.3 [5] gives the probability density of $X_t$ (after a straightforward application of Girsanov's formula). Some further calculations then establishes:

$$C_{0,0}(t) = \frac{2}{\sqrt{\pi}}\exp\left(-\frac{\alpha_2 t}{8}\right)\int_0^\infty du \frac{e^{-u^2}}{\cosh\left(u\sqrt{\alpha_2 t/2}\right)}.$$

This proves in particular that the decay of the probability $C_{0,0}(t)$ for long $t$ is $\sqrt{2\pi}(\alpha_2 t)^{-1/2}\exp(-\alpha_2 t/8)$. The transition probabilities $C_{0,p}(t)$ in the particular case when the initial state is the fundamental can then be computed by repeated derivations from Eq. (23):

$$C_{0,2p}(t) = \frac{2}{\sqrt{\pi}}\exp\left(-\frac{\alpha_2 t}{8}\right)\int_0^\infty du e^{-u^2}\frac{F_{0,2p}(\alpha_2 t, u\sqrt{\alpha_2 t/2})}{\cosh\left(u\sqrt{\alpha_2 t/2}\right)}, \tag{38}$$

where the functions $F_{0,2p}$ are:

$$F_{0,0}(\bar{t},\bar{u}) = 1, \tag{39}$$

$$F_{0,2}(\bar{t},\bar{u}) = \frac{1}{2} - 2\tanh(\bar{u})\,\frac{\bar{u}}{\bar{t}}, \tag{40}$$

$$F_{0,4}(\bar{t},\bar{u}) = \left(\frac{3}{8} - \frac{2}{3}\frac{\bar{u}^2}{\bar{t}^2}\right) + \tanh(\bar{u})\left(\frac{2}{3}\frac{\bar{u}}{\bar{t}^2} - \frac{7}{3}\frac{\bar{u}}{\bar{t}}\right) + \tanh^2(\bar{u})\frac{4}{3}\frac{\bar{u}^2}{\bar{t}^2}, \tag{41}$$

$$\begin{aligned} F_{0,2p}(\bar{t},\bar{u}) = {} & \frac{4}{\Gamma_{2p}}\left(\frac{\bar{u}}{\bar{t}}\frac{\partial F_{0,2p-2}}{\partial \bar{u}} + 2\frac{\partial F_{0,2p-2}}{\partial \bar{t}}\right) \\ & + \left(1 - \frac{1}{\Gamma_{2p}} - \frac{4}{\Gamma_{2p}}\frac{\bar{u}}{\bar{t}}\tanh(\bar{u})\right) F_{0,2p-2} + \frac{\Gamma_{2p-2}}{\Gamma_{2p}}\left(F_{0,2p-2} - F_{0,2p-4}\right). \end{aligned} \tag{42}$$

We now examine the general case where the initial state is $f_{2p_0}$. The jump process is reversible, so $\mathbf{P}_{0,2p_0}(N_t = 2p) = \mathbf{P}_{0,2p}(N_t = 2p_0)$ which reads:

$$C_{2p_0,2p}(t) = C_{2p,2p_0}(t).$$

In particular $C_{2p_0,0}(t) = C_{0,2p_0}(t)$ has been derived here above, and repeated derivations finally establish the expression of $C_{2p_0,2p}(t)$:

$$C_{2p_0,2p}(t) = \frac{2}{\sqrt{\pi}}\exp\left(-\frac{\alpha_2 t}{8}\right)\int_0^\infty du e^{-u^2}\frac{F_{2p_0,2p}(\alpha_2 t, u\sqrt{\alpha_2 t/2})}{\cosh\left(u\sqrt{\alpha_2 t/2}\right)}, \tag{43}$$

where the functions $F_{2p_0,2p}$ are:

$$\begin{aligned} F_{2p_0,2p}(\bar{t},\bar{u}) = {} & \frac{4}{\Gamma_{2p}}\left(\frac{\bar{u}}{\bar{t}}\frac{\partial F_{2p_0,2p-2}}{\partial \bar{u}} + 2\frac{\partial F_{2p_0,2p-2}}{\partial \bar{t}}\right) \\ & + \left(1 - \frac{1}{\Gamma_{2p}} - \frac{4}{\Gamma_{2p}}\frac{\bar{u}}{\bar{t}}\tanh(\bar{u})\right) F_{2p_0,2p-2} \\ & + \frac{\Gamma_{2p-2}}{\Gamma_{2p}}\left(F_{2p_0,2p-2} - F_{2p_0,2p-4}\right), \end{aligned} \tag{44}$$

$$F_{2p_0,0}(\bar{t},\bar{u}) = F_{0,2p_0}(\bar{t},\bar{u}). \tag{45}$$

As an application the functions $F_{2,2p}$ which govern the probabilities that a particle initially in state $f_2$ be observed in state $f_{2p}$ at normalized time $t$ are:

$$F_{2,0}(\bar{t},\bar{u}) = \frac{1}{2} - 2\tanh(\bar{u})\,\frac{\bar{u}}{\bar{t}}, \tag{46}$$

$$F_{2,2}(\bar{t},\bar{u}) = \left(\frac{1}{4} - 4\frac{\bar{u}^2}{\bar{t}^2}\right) + \tanh(\bar{u})\left(4\frac{\bar{u}}{\bar{t}^2} - 2\frac{\bar{u}}{\bar{t}}\right) + 8\tanh^2(\bar{u})\frac{\bar{u}^2}{\bar{t}^2}, \tag{47}$$

$$\begin{aligned} F_{2,4}(\bar{t},\bar{u}) = {} & \left(\frac{3}{16} + 4\frac{\bar{u}^2}{\bar{t}^3} - 5\frac{\bar{u}^2}{\bar{t}^2}\right) + \tanh(\bar{u})\left(\frac{20}{3}\frac{\bar{u}^3}{\bar{t}^3} - 4\frac{\bar{u}}{\bar{t}^3} + 5\frac{\bar{u}}{\bar{t}^2} - \frac{23}{12}\frac{\bar{u}}{\bar{t}}\right) \\ & + \tanh^2(\bar{u})\left(10\frac{\bar{u}^2}{\bar{t}^2} - 8\frac{\bar{u}^2}{\bar{t}^3}\right) - 8\tanh^3(\bar{u})\frac{\bar{u}^3}{\bar{t}^3}. \end{aligned} \tag{48}$$

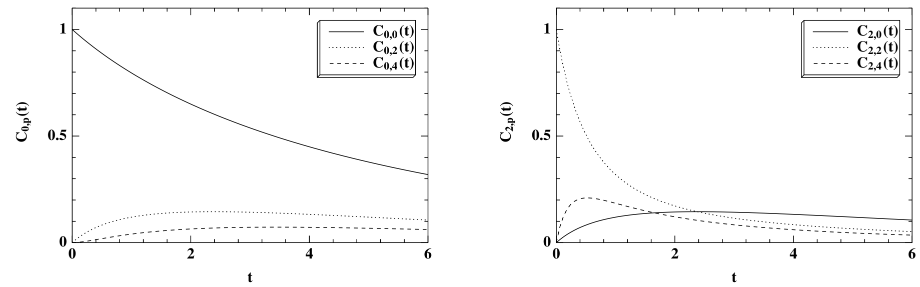


FIGURE 1. Theoretical histograms of the probability distributions $(C_{p_0,p}(t))_{p\in\mathbb{N}}$ of the energy of the particle under time-dependent perturbations with $\alpha_2 = 1$. In the left picture (resp. right picture) the particle is assumed to be at $t = 0$ in the state $f_0$ (resp. $f_2$). Only even states have positive probabilities.

## 8. Numerical simulations

The results in the previous sections are theoretically valid in the limit case $\varepsilon \to 0$, where the amplitude (resp. duration) of the perturbations goes to zero (resp. infinity). In this section we aim at showing that the asymptotic behaviors of the state vector can be easily observed in numerical simulations in the case where the perturbation is small, so that its effect appears after a long time. We use a split-step method to simulate the one-dimensional perturbed linear Schrödinger equations (6). We adopt in this section the following model for the perturbation:

$$m(t) = u_l \text{ if } lt_c \le t < (l+1)t_c,$$

where $(u_l)_{l=0,\dots,M-1}$ is a sequence of $M$ independent and identically distributed variables, which take the value $-1/2$ or $1/2$ with probability $1/2$. $t_c$ is the so-called coherence time of the random process $m$. The power spectral density is:

$$\alpha_j = \frac{1}{4}\frac{1-\cos(jt_c)}{j^2 t_c},$$

and $\alpha_0 = t_c/8$. The quantity $Mt_c$ which is equal to the time duration of the perturbation will be chosen so large that we can observe the effect of the small perturbation $\varepsilon m(t)x^2$. We measure the $L^2$-norm and the energy (10) of the particle that we can compare with the corresponding data of the initial state vector. We present results corresponding to simulations where the initial state at $z = 0$ is the fundamental Gaussian mode $f_0(x)$ or the second mode $f_2(x)$. We have first simulated the homogeneous Schrödinger equation (with $m \equiv 0$) which admits as an exact solution $f_0(x)e^{-it/2}$ and $f_2(x)e^{-5it/2}$ respectively. We can therefore check the accuracy of the numerical method, since we can see that the modulus of the computed solution maintains a very close resemblance to the initial profile (data not shown), while the $L^2$-norm and the energy are almost constant.

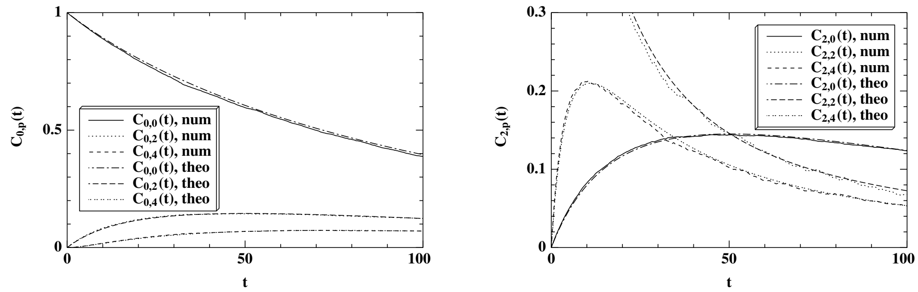


FIGURE 2. Evolution of the modal distribution of the particle computed by simulation of the perturbed Schrödinger equation (6) and averaged over 2000 realizations and comparison with the theoretical formulas given by (39-41) and (46-48). In the left picture (resp. right picture) the initial state vector is the fundamental state $f_0$ (resp. the state $f_2$). The values of the parameters are $t_c = 0.4$ ($\alpha_2 \simeq 0.0474$) and $t_{max} = 100$ ($M = 250$).

The other simulations are carried out with different realizations of the random process $m$ with $\varepsilon = 1$. In Figure 8 we plot the modal distributions of the particle averaged over 2000 realizations and compare them with the theoretical distributions derived in Section 7. It thus appears that the numerical simulations are in very good agreement with the theoretical results. All these observations confirm that our formulas describe with accuracy the evolution of the harmonic oscillator under small perturbations and long times.

## 9. Conclusion

We have analyzed in this paper the effects of random perturbations of the spring coefficient on the evolution of a quantum particle in a quadratic potential. The precise results we have obtained are a consequence of the particular form of the stable system at hand (quantum harmonic oscillator) and of the considered perturbation. Eigenvalues and eigenstates of this system are explicit and tabulated formulas are available, so that we are able to perform exact calculations and compute closed-form expressions for the eigenstates probabilities. Nevertheless the results demonstrated in this paper can be generalized to a large class of systems. In [14] we study the energy distribution of the quantum harmonic oscillator under a very general class of random time-dependent perturbations which satisfy some spatial growth conditions. Nevertheless the quadratic perturbation we address in this paper did not fulfill these conditions, so a specific approach was required. Second we can also expect generalizations to systems that are different from the harmonic oscillator. Indeed the basic assumption which requires the existence of a complete set of normal eigenstates for the unperturbed Hamiltonian holds true for many systems. The remainder of the study then consists in technical developments to exhibit and analyze the coupling mechanisms between the eigenstates.

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