EXPONENTIAL LOCALIZATION VERSUS SOLITON PROPAGATION

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Abstract The scattering of a wavepacket by a random nonlinear medium is analyzed. In the linear limit strong localization occurs, which means that the transmission coefficient decays exponentially with a characteristic localization length. In some nonlinear homogeneous media solitons propagate without changes in their shape or velocity. Solitons are therefore candidates to test the robustness of the exponential localization in random nonlinear media. Using the inverse scattering transform for the nonlinear Schrödinger equation different typical behaviors can be exhibited depending on the amplitude of the incoming soliton.

Keywords: Nonlinear Schrödinger equation, soliton, random media.

1. INTRODUCTION

It is well-known that in one-dimensional random linear media strong localization occurs, which means in particular that the transmitted intensity decays exponentially as a function of the size of the medium [1]. The purpose of this paper is to give a short review about the competition between nonlinearity and localization. There exist some results for the stationary case. It has been shown that the presence of a slight nonlinearity competes with the effects which lead to exponential localization and allows some probability of transmission [2], even though there must be some decay [3]. In [4] a power law decay of the transmittivity is demonstrated by the use of the embedding method. These results are for steady waves and do not apply to a time-dependent problem since the superposition principle fails due to nonlinearity. Kivshar [5] was the first one to show that the nonlinearity does have an effect in a time-dependent problem, by studying the propagation of a soliton through a slab of random nonlinear medium. We shall consider in this paper a random Schrödinger equation with cubic nonlinearity, and discuss the ability of a soliton to propagate without distortion.

2. RANDOM LINEAR MEDIA

Wave propagation in random linear media has been studied for a long time by perturbation techniques when the random inhomogeneities are small. One finds that the mean amplitude decreases with distance traveled, since wave energy is converted to incoherent fluctuations. The fluctuating part of the field intensity is calculated approximatively from a linear radiative transport equation. This theory is well-known [6], although a complete mathematical theory is still lacking (for the most recent developments, see for instance [7]). However this theory is known to be false in one-dimensional random media. This was first noted by Anderson [8], who claimed that random inhomogeneities trap wave energy in a finite region and do not allow it to spread as it would normally. This is the so-called wave localization phenomenon. Random media behave then like periodic media that have band gap spectra, allowing wave propagation in some frequency ranges but not in others. It is remarkable that this happens for random media that are not close to periodic ones at all. A lot of work was then devoted to the thorough analysis of this problem, in particular by Furstenberg [9], Rytov, Tatarski, Klyatskin [10], and by Papanicolaou and its co-workers [11]. The tools for the quantitative analysis are limit theorems for stochastic equations developed by Khasminskii [12] and Kushner [13].

There are three basic length scales in wave propagation phenomena [11]: the typical wavelength λ, the typical propagation distance L, and the typical size of the inhomogeneities l_c. There is also a typical order of magnitude ε that characterizes the standard deviation of the relative fluctuations of the parameters of the medium. It is not always so easy to identify the scale l_c, but we may think of l_c as a typical correlation length. If $\varepsilon \ll 1$, then the most effective interaction of the waves with the random medium will occur when $l_c \sim \lambda$, that is, the wavelength is comparable to the correlation length. And this interaction will be observable when the propagation distance L is large ($L \sim \lambda\varepsilon^{-2}$).

We first consider the propagation of monochromatic waves. This is the natural approach since any wave can be described as the superposition of such elementary wavetrains by Fourier transform. Let $\hat{u}(x)$ be the amplitude at $x \in \mathbb{R}$ of a monochromatic wave $u(t,x) = \exp(-ik^2 t)\hat{u}(x)$ traveling in the one-dimensional medium described in Fig. 1. The medium is homogeneous ($V = 0$) outside the slab $[0, L]$, so that $\hat{u}$ has the form $\hat{u}(x) = e^{ikx} + R^\varepsilon e^{-ikx}$ for $x \leq 0$ and $\hat{u}(x) = T^\varepsilon e^{ikx}$ for $x \geq L$. The complex-valued random variables R^ε and T^ε are the reflection and transmission coefficients, respectively. Inside the slab $[0, L]$ the potential is nonzero. It is the realization of a random, stationary, ergodic, and zero-mean process V. The dimensionless parameter $\varepsilon > 0$ characterizes the amplitude of the random potential.

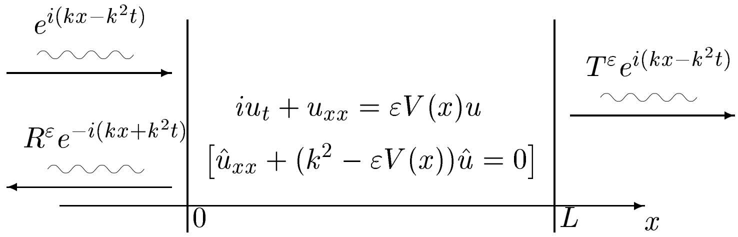

Figure 1. Scattering of a monochromatic pulse.

Proposition 1 *There exists a length L^ε_{loc} such that, with probability one:*

$$\lim_{L\to\infty} \frac{1}{L} \ln |T^\varepsilon|^2(k, L) = -\frac{1}{L^\varepsilon_{loc}}. \quad (1)$$

This length can be expanded as powers of ε:

$$\frac{1}{L^\varepsilon_{loc}} = \frac{\alpha(2k)}{2k^2}\varepsilon^2 + O(\varepsilon^3), \qquad \alpha(2k) := \int_0^\infty du \mathbb{E}[V(0)V(u)] \cos(2ku). \quad (2)$$

Proof. The study of the exponential behavior of the transmittivity $|T^\varepsilon|^2$ can be divided into two steps. First one shows that the localization length is equal to the inverse of the Lyapunov exponent associated to the random harmonic oscillator $v_{xx} + (k^2 - \varepsilon V(x))v = 0$ by using the fundamental solution matrix to obtain a suitable expression for $|T^\varepsilon|^2$ [14]. Second one computes the expansion of the Lyapunov exponent of the random oscillator [14, 15]. $\square$

Note that $\alpha(2k)$ is a nonnegative real number since it is proportional to the power spectral density of the stationary random process V (Wiener-Khintchine theorem [16, p. 141]). The existence and positivity of the exponent $1/L^\varepsilon_{loc}$ can be obtained with minimal hypotheses. Kotani [17] established that a sufficient condition is that V is a stationary, ergodic process that is bounded with probability one and is nondeterministic. The expansion of the localization length requires some technical hypotheses about the mixing properties of V [15, 18].

Proposition 2 *The expectation (i.e. the mean value) $\mathbb{E}[|T^\varepsilon(k, L/\varepsilon^2)|^2]$ of the transmittivity for a slab of size L/ε^2 converges as $\varepsilon \to 0$ to:*

$$\bar{T}(L, k) = \frac{4}{\sqrt{\pi}} \exp\left(-\frac{\alpha(2k)L}{8k^2}\right) \int_0^\infty dx \frac{x^2 e^{-x^2}}{\cosh(\sqrt{\alpha(2k)k^{-2}L/2x})}. \quad (3)$$

Proof. The transmission coefficient T^ε can be expressed in terms of a random variable that is the solution of a Ricatti equation. The application of a diffusion-approximation theorem then establishes that the square modulus $|T^\varepsilon(k, L/\varepsilon^2)|^2$ converges in distribution to a diffusion Markov process. The computation of the expectation is then a simple matter of calculation [10]. $\square$

This proposition shows that:

$$\frac{1}{L}\ln \bar{T}(L,k) \overset{L\gg 1}{\simeq} -\frac{\alpha(2k)}{8k^2}.$$

Note that the exponential behavior of the *mean* transmittivity is very different from the *typical* transmittivity. Actually the value of the mean transmittivity is imposed by a very small set of realizations of the potential for which the medium is much more transparent than the typical behavior. It thus seems very unlikely to observe this mean value. However it will appear that this does not hold true, because this very small set of realizations strongly depends on the frequency k, so that the transmittivity in the time-domain will be qualitatively very different. We now turn to the problem of the propagation of a wavepacket.

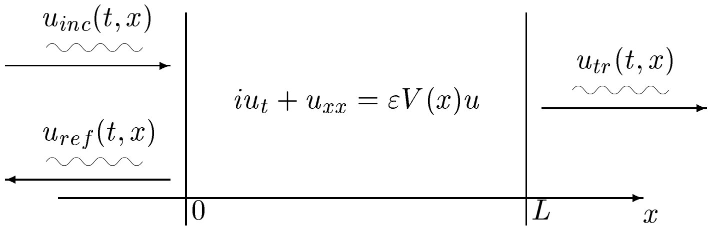

Figure 2. Scattering of a wavepacket.

We consider an incoming wave from the left:

$$u_{inc}(t,x) = \frac{1}{2\pi}\int_0^\infty f(k)\exp i\left(kx - k^2 t\right)dk, \quad x \le 0, \tag{4}$$

where f is a squared integrable function ($f \in L^2$). The field in the region $x \ge L$ consists of the transmitted wave that is right going:

$$u_{tr}(t,x) = \frac{1}{2\pi}\int_0^\infty f(k)T^\varepsilon(k,L)\exp i\left(kx - k^2 t\right)dk, \quad x \ge L, \tag{5}$$

where $T^\varepsilon(k,L)$ is the transmission coefficient. The total transmitted energy is:

$$\mathcal{T}^\varepsilon(L) = \frac{1}{2\pi}\int_0^\infty |f(k)T^\varepsilon(k,L)|^2 dk.$$

Proposition 3 *The transmittivity $\mathcal{T}^\varepsilon(L/\varepsilon^2)$ converges in probability as $\varepsilon \to 0$ to the deterministic value:*

$$\mathcal{T}(L) = \frac{1}{2\pi}\int_0^\infty |f(k)|^2 \bar{T}(k,L)dk,$$

where $\bar{T}(k,L)$ is the asymptotic value (3) of the expectation of the square modulus of the transmission coefficient $T^\varepsilon(k, L/\varepsilon^2)$.

Proof. Proposition 2 gives the limit value of the expectation of $|T^\varepsilon(k, L/\varepsilon^2)|$ for one frequency k, so that we immediately get that $\mathbb{E}\left[\mathcal{T}^\varepsilon(L/\varepsilon^2)\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \mathcal{T}(L)$. Then one considers the second moment:

$$\mathbb{E}\left[\mathcal{T}^\varepsilon(L/\varepsilon^2)^2\right] = \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty |f(k)|^2|f(k')|^2\mathbb{E}\left[|T^\varepsilon(k,L/\varepsilon^2)|^2|T^\varepsilon(k',L/\varepsilon^2)|^2\right]dkdk'.$$

The computation of this expectation requires to study the two-frequency process $(|T^\varepsilon(k,L/\varepsilon^2)|, |T^\varepsilon(k',L/\varepsilon^2)|)$ for $k \neq k'$. Applying a diffusion-approximation theorem one finds that $|T^\varepsilon(k,L/\varepsilon^2)|$ and $|T^\varepsilon(k',L/\varepsilon^2)|$ are asymptotically uncorrelated, so that $\mathbb{E}\left[\mathcal{T}^\varepsilon(L/\varepsilon^2)^2\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \mathcal{T}(L)^2$. Thus $\mathbb{E}\left[\left(\mathcal{T}^\varepsilon(L/\varepsilon^2) - \mathcal{T}(L)\right)^2\right] \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ which proves the convergence of $\mathcal{T}^\varepsilon(L/\varepsilon^2)$ to $\mathcal{T}(L)$ in probability. □

Let us assume that the incoming wave is narrowband, that is to say that the spectrum f is concentrated around the carrier wavenumber k_0 and has narrow bandwidth (smaller than 1, but larger than ε^2). Then $\mathcal{T}(L)$ decays exponentially with the width of the slab as:

$$\frac{1}{L}\ln\mathcal{T}(L) \stackrel{L\gg 1}{\simeq} -\frac{\alpha(2k_0)}{8k_0^2}.$$

Note that this is the typical behavior of the *mean* transmittivity of a monochromatic wave with wavenumber k_0. In the time domain the localization process is self-averaging ! This self-averaging is a consequence of the asymptotic decorrelation of the moduli of the transmission coefficients at different frequencies.

Remark 4 (The O'Doherty-Anstey theory) This theory describes the deformation of a pulse traveling in a slab of random medium. It is well-known in geophysics literature [19]. It predicts that in proper conditions the transmitted pulse can be divided into two parts. The front part has a deterministic shape which is the result of a deterministic convolution of the initial pulse. Behind this front part emerges the "coda" which is incoherent, but may contains most of the transmitted energy. A rather convincing heuristic explanation can be found in [20, Section 2]. The theory is analyzed in detail in [20] for a special case of stepwise media, and further results can be found in [11]. Let us assume that the frequency content of the initial pulse is concentrated around the carrier wavenumber k_0. If one analyzes the energy content of the front part of the wave in the framework introduced here above, then one finds that it decays exponentially with the size of the slab with a characteristic length which is the *sample* localization length $2k_0^2/\alpha(2k_0)$. However, as shown here above, the total transmitted energy decays exponentially with a characteristic length which is the *mean* localization length $8k_0^2/\alpha(2k_0)$.

3. SOLITARY WAVES

A solitary wave is a wave that propagates without change of form or diminution of speed. The study of solitary waves began in 1838 with the observation by J. Scott Russel of such a water wave while riding on horseback along a channel. However no mathematical theory available at the time predicted a solitary wave. The problem was resolved in 1895 by Korteweg and de Vries who derived an equation (now known as the KdV equation) which governs small shallow-water waves [21]. Despite this early work no further application was discovered until the 1960's. In 1967 Gardner, Green, Kruskal, and Miura first discovered an original method of solution of KdV by applying an implicit linearization of the equation: the so-called inverse scattering transform [22]. Lax (1968) considerably generalized these ideas [23], and Zakharov and Shabat (1972) showed that the method worked for the nonlinear Schrödinger (NLS) equation [24] (see the appendix for a review). At this time it was known that the NLS equation describes the propagation of short pulses in mono-mode optical fibers [25]. Hasegawa (1973) then claimed that the soliton was the ideal candidate to be the information bit for the next generation of optical fibers [26]. In order to confirm this hope, it is relevant to study the behavior of a soliton when it propagates through weakly perturbed media over very large distances. In this paper we shall restrict ourselves to the case of the NLS equation with spatially random coefficients, but the method can be generalized to other types of random perturbations (time-dependent) and other completely integrable systems. Note that the NLS equation with a random potential has been studied by many different authors both theoretically [5, 27, 28, 29] and numerically [30, 31]. We shall present in the following a rigorous application of the inverse scattering transform that allows us to derive relevant results for a large class of perturbations.

4. SOLITONS IN RANDOM MEDIA

We consider a perturbed Schrödinger equation with a non-zero right-hand side:

$$iu_t + u_{xx} + 2|u|^2 u = \varepsilon R(u)(t,x). \tag{6}$$

The small parameter $\varepsilon \in (0,1)$ characterizes the amplitude of the perturbation. The model of the perturbation is taken to be:

$$R(u)(t,x) = m_1(x)u(t,x) + m_2(x)|u|^2 u(t,x) + \left(m_3(x)u_x\right)_x .$$

The initial condition is a soliton incoming from the left:

$$u_0(t,x) = 2\nu_0 \frac{\exp i\left(2\mu_0(x-4\mu_0 t) + 4(\nu_0^2+\mu_0^2)t\right)}{\cosh\left(2\nu_0(x-4\mu_0 t)\right)}, \tag{7}$$

whose mass and velocity are respectively $N_0 = 4\nu_0$ and $V_0 = 4\mu_0$. We also assume that m_1, m_2, and m_3 are random, stationary, ergodic, zero-mean, and independent processes. We aim at showing that, for a slab of size L/ε^2, the two following statements hold true in the limit $\varepsilon \to 0$. First the transmitted wave consists of a soliton plus some radiation. Second the soliton dynamics for almost every realization is described by non-random evolution equations, where only the Fourier transforms of the autocorrelation functions of the random processes appear. From the physical point of view this fact was first established by Doucot and Rammal [32]. These theoretical predictions were then clearly stated by Kivshar [5] and extended by Bronski [33] and the author [34, 35]. The proof of the following proposition is sketched out in the appendix.

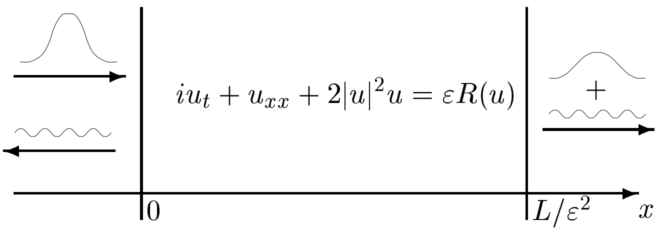

Figure 3. Scattering of a soliton.

Proposition 5 *Let us consider a slab of size L/ε^2.*
1. With probability that goes to 1 *as $\varepsilon \to 0$, the scattered wave consists of one transmitted soliton plus some radiation.*
2. The coefficients of the transmitted soliton converge in probability to the deterministic functions $\nu_l(L)$, $\mu_l(L)$ which satisfy the system of ordinary differential equations:

$$\left\{\begin{array}{l} \dfrac{d\nu_l}{dx} = F(\nu_l,\mu_l), \quad \nu_l(0) = \nu_0, \\ \dfrac{d\mu_l}{dx} = G(\nu_l,\mu_l), \quad \mu_l(0) = \mu_0, \end{array}\right. \tag{8}$$

where F and G are deterministic functions where only the Fourier transforms of the autocorrelation functions appear:

$$F(\nu,\mu) = -\frac{1}{2\pi}\sum_{j=1}^{3}\int_{-\infty}^{\infty}|c_j|^2(\nu,\mu,\lambda)\alpha_j(k(\nu,\mu,\lambda))d\lambda,$$

$$G(\nu,\mu) = -\frac{1}{4\pi}\sum_{j=1}^{3}\int_{-\infty}^{\infty}\left(\frac{\lambda^2}{\mu\nu}+\frac{\nu}{\mu}-\frac{\mu}{\nu}\right)|c_j|^2(\nu,\mu,\lambda)\alpha_j(k(\nu,\mu,\lambda))d\lambda.$$

The coefficients α_j and k are defined by:

$$\alpha_j(k) = \int_0^{\infty}\mathbb{E}[m_j(0)m_j(x)]\cos(kx)dx, \qquad k(\nu,\mu,\lambda) = \frac{(\lambda-\mu)^2+\nu^2}{\mu}.$$

The functions c_j can be computed explicitly:

$$c_1(\nu,\mu,\lambda) = \frac{\pi}{2^4\mu^3}\frac{(\lambda-\mu+i\nu)^2}{\cosh\left(\pi(\mu^2-\nu^2-\lambda^2)/(4\mu\nu)\right)},$$

$$c_2(\nu,\mu,\lambda) = \frac{\left((\lambda+\mu)^2+\nu^2\right)\left(\nu^2+17\mu^2-6\lambda\mu+\lambda^2\right)}{12\mu^2}c_1(\nu,\mu,\lambda),$$

$$\tilde{c}(\nu,\mu,\lambda) = \frac{\left((\lambda-\mu)^2+\nu^2\right)^2+8\mu^2(-\nu^2+\mu\lambda-\mu^2)}{3\mu\left((\lambda-\mu)^2+\nu^2\right)}c_1(\nu,\mu,\lambda),$$

$$c_3(\nu,\mu,\lambda) = (4(\nu^2-\mu^2)c_1-4\mu\tilde{c}-2c_2)+i(2\mu c_1+\tilde{c})k(\nu,\mu,\lambda).$$

The analysis of the effective system (8) exhibits several possible regimes. Two of them are stable and attractive, the first one being characterized by a logarithmic decay of the velocity for solitons of sufficiently large mass, and the second one by an exponential decay of the mass for solitons of small mass. There are also intermediate regimes where both the mass and the velocity decrease at a polynomial rate. More exactly [34, 35]:

1) If the mass of the incoming soliton is small (below a critical value depending on $\alpha_j(.)$ and μ_0), then the velocity of the soliton is almost constant, while its mass decreases to 0 as a function of the length of the slab L:
- as $\exp(-L/L_{loc})$, where $L_{loc} = 32\mu_0^2/\alpha_1(4\mu_0)$ (perturbation of the linear potential),
- as $L^{-1/4}$ (perturbation of the nonlinear coefficient),
- as $L^{-1/2}$, then as $\exp(-L/L'_{loc})$ (dispersive perturbation).

2) If the mass of the soliton is initially large enough, then it will be almost constant during propagation, while the velocity of the soliton slowly decreases to 0. The decay rate depends on the tail of the spectrum of the perturbation, but we can state in great generality that it is at most logarithmic.

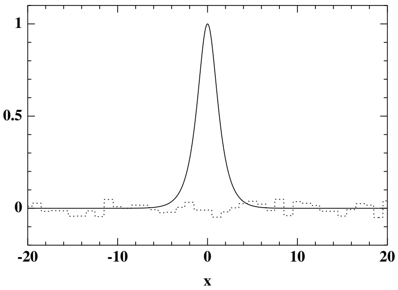

Figure 4a. Envelop of the initial soliton (solid line) with mass $N_0 = 2$ and velocity $V_0 = 1.6$. In dashed line is plotted a realization of the random potential εm_1 with $\varepsilon = 0.05$ and $l_c = 0.4$.

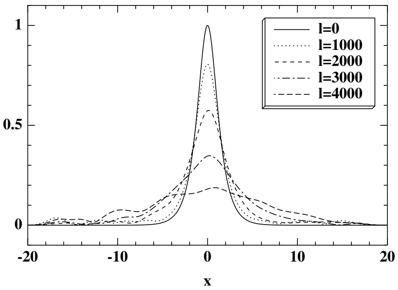

Figure 4b. Envelops of the soliton when its center crosses different depth lines l for one of the realization of the random potential. The coordinate x is normalized around the depth line l.

It can be noted that, in the limit case $\nu_0/\mu_0 \to 0$, the incoming soliton can be approximated by a linear wavepacket:

$$u_0(t,x) \simeq \int_{-\infty}^{+\infty} dk f(k) e^{ikx - ik^2 t}, \text{ with } f(k) = \frac{1}{2\cosh\left(\frac{\pi}{4}\left(\frac{k-2\mu_0}{\nu_0}\right)\right)},$$

whose spectrum is narrow around the carrier wavenumber $k_0 = 2\mu_0$. The result is in agreement with the linear approximation, since the localization length L_{loc} corresponding to a perturbation of the linear potential can be written in terms of the carrier wavenumber as $L_{loc} = 8k_0^2/\alpha_1(2k_0)$. It is equal to the localization length of a monochromatic wave with wavenumber k_0 scattered by a slab of random linear medium (see Section 2).

5. NUMERICAL SIMULATIONS

The results in the previous sections are theoretically valid in the limit case $\varepsilon \to 0$, where the amplitudes of the perturbations go to zero and the length of the random slab goes to infinity. In this section we aim at showing that the asymptotic behaviors of the soliton can be observed in numerical simulations in the case where ε is small, more precisely smaller than any other characteristic scale of the problem. We use a fourth-order split-step method to simulate the perturbed nonlinear Schrödinger equation (6). This numerical algorithm provides accurate and stable solutions [30]. For the sake of simplicity we only consider perturbations of the linear potential m_1 and take $m_2 = m_3 = 0$.

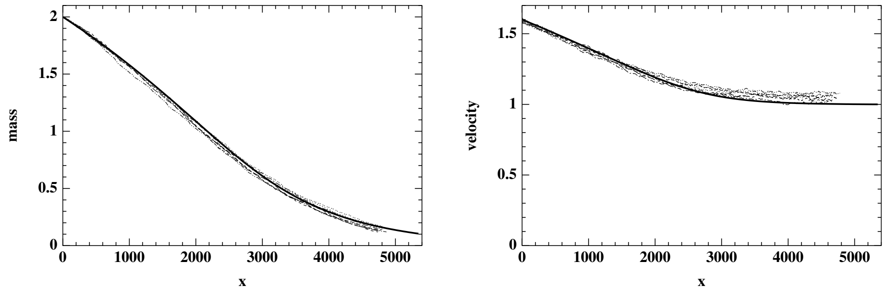

Figure 5. Coefficients of the transmitted soliton whose initial coefficients are $N_0 = 2$, $V_0 = 1.6$ with a random potential whose amplitude is $\varepsilon = 0.05$ and correlation length $l_c = 0.4$. The left (resp. right) figure is devoted to the mass (resp. velocity). The thick solid lines represent the theoretical coefficients of the transmitted soliton. In thin dashed and dotted lines are plotted the simulated masses and velocities of the transmitted solitons for 7 different realizations of the random potential.

We assume in this section that the potential is constant over elementary intervals of length l_c and take independent random values over each interval which obey uniform distributions over $[-1, 1]$. We present simulations where the initial wave at time $t = 0$ is a soliton with mass $N_0 = 2$ and velocity $V_0 = 1.6$ centered at $x = 0$ (see Fig. 4a). The simulated evolutions of the coefficients of the soliton are presented in Fig. 5 for seven different realizations of the random potential with $\varepsilon = 0.05$ and $l_c = 0.4$. They are compared with the theoretical evolutions given by (8) in the scale x/ε^2. It thus appears that the numerical simulations are in very good agreement with the theoretical results. Fig. 4b plots the envelops of the solution at different depths corresponding to one of the simulations, which shows that the wave keeps the basic form of a soliton although it looses some mass. All these results confirm that system (8) describes with accuracy the transmission of a soliton through a random slab for small perturbations and long slab length.

6. CONCLUSION

We have studied the propagation of a soliton in a nonlinear dispersive medium with spatially random perturbations by applying the powerful inverse scattering transform. If the incoming soliton has small mass, then the mass of the soliton decays to zero exponentially or algebraically with the length of the system. In case of large mass, the mass of the soliton is almost constant. Furthermore the velocity is found to decrease at a slow rate (at most logarithmic) which depends on the high-frequency behavior of the power spectrum of the random perturbation.

Appendix: An introduction to the inverse scattering transform

For more detail about the following statements and their proofs we refer to Refs. [36, 37]. Let us consider the NLS equation:

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{A.1}$$

The inverse scattering transform consists in a linearization of this nonlinear equation. It is based on the fact that $u(t,.)$ can be characterized by a set of spectral data of an operator $L(u(t,.))$ in which u plays the role of a potential:

$$L(u) = iP\frac{\partial}{\partial x} + Q(u), \text{ with } P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } Q(u) = \begin{pmatrix} 0 & u^* \\ -u & 0 \end{pmatrix}.$$

The domain of $L(u)$ is the space $\mathbb{H}^1(\mathbb{R})$,

$$\mathbb{H}^1(\mathbb{R}) = \left\{\psi \text{ such that } \psi \in \mathbb{L}^2(\mathbb{R}), \psi_x \in \mathbb{L}^2(\mathbb{R})\right\},$$

which is a dense subset of the Hilbert space $\mathbb{L}^2(\mathbb{R})$:

$$\mathbb{L}^2(\mathbb{R}) = \left\{\psi = \psi_1\mathbf{e_1} + \psi_2\mathbf{e_2}, \psi_j \in L^2(\mathbb{R})\right\}, \quad \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Operator $L(0)$. $L(0)$ is self-adjoint. The real axis constitutes its essential spectrum. The eigenspace associated with the eigenvalue $\lambda \in \mathbb{R}$ has dimension 2 and admits the couple $\left(\mathbf{e_1}e^{-i\lambda x}, \mathbf{e_2}e^{i\lambda x}\right)$ as a base. Besides the point spectrum of $L(0)$ is empty, because the non-trivial solutions of $v_x = i\lambda v$ are not in $L^2(\mathbb{R})$.

Essential spectrum of the operator $L(u(t=t_0,.))$. Let us consider the spectral problem associated with the operator $L(u) = L(0) + Q(u)$:

$$L(u(t,x))\psi(t,x) = \lambda(t)\psi(t,x), \quad \psi = \psi_1\mathbf{e_1} + \psi_2\mathbf{e_2}. \tag{A.2}$$

If $u(t=t_0,.) \in L^1(\mathbb{R})$, then $Q(u)$ is $L(0)$-compact. As a consequence of the Weyl theorem, the essential spectrum of $L(u)$ is equal to the real axis. Eq. (A.2) actually admits two linearly independent solutions when λ is real. We introduce the so-called Jost functions f and g, defined as the eigenfunctions of $L(u)$ associated with the real eigenvalue λ which satisfy the following boundary conditions:

$$f(x,\lambda) \overset{x\to+\infty}{\longrightarrow} \mathbf{e_2}e^{i\lambda x}, \quad g(x,\lambda) \overset{x\to-\infty}{\longrightarrow} \mathbf{e_1}e^{-i\lambda x}.$$

If we denote by $\bar{\psi}$ the vector $(\psi_2^*, -\psi_1^*)$ associated with a vector ψ solution of (A.2), then $\bar{\psi}$ is a solution of $L\bar{\psi} = \lambda^*\bar{\psi}$. In the case of a real eigenvalue, ψ and $\bar{\psi}$ are linearly independent and form a base of the space of the solutions of (A.2). It can then be proved that the Jost functions are related by:

$$g(x,\lambda) = a(\lambda)\bar{f}(x,\lambda) + b(\lambda)f(x,\lambda), \quad f(x,\lambda) = -a(\lambda)\bar{g}(x,\lambda) + b^*(\lambda)g(x,\lambda).$$

Multiplying the first identity by the vector $\bar{f}^*$, we get an explicit representation of the coefficient a as the Wronskian of f and g:

$$a(\lambda) = g_1(x,\lambda)f_2(x,\lambda) - g_2(x,\lambda)f_1(x,\lambda).$$

Point spectrum of the operator $L(u(t=t_0,.))$**.** It is possible to define an analytic continuation of $a(\lambda)$ over the upper complex half-plane. A noticeable feature then appears. If λ_r is a zero of $a(\lambda)$, then f and g are linearly dependent, so there exists a coefficient ρ_r such that $g(x,\lambda_r)=\rho_r f(x,\lambda_r)$. The corresponding eigenfunction is bounded and decays exponentially as $x\to+\infty$ (because $|f|\sim e^{-\mathrm{Im}\lambda_r x}$) and as $x\to-\infty$ (because $|g|\sim e^{+\mathrm{Im}\lambda_r x}$). Thus λ_r is an element of the point spectrum of $L(u)$. It can then be proved that the set $(a(\lambda),b(\lambda),\lambda_r,\rho_r,a'(\lambda_r))$ characterizes the Jost functions f and g as well as the solution u. The inverse transform is essentially based on the resolution of the linear integro-differential Gelfand-Levitan-Marchenko equation, whose entries are constituted by the set $(a,b,\lambda_r,\rho_r,a'(\lambda_r))$:

$$K_1(x,y)=\Phi^*(x+y)-\int_x^\infty K_1(x,y'')\int_x^\infty \Phi^*(y+y')\Phi(y'+y'')dy'dy'',$$
$$\text{where } \Phi(y)=-\sum_r \frac{i\rho_r}{a'(\lambda_r)}e^{i\lambda_r y}+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{b(\lambda)}{a(\lambda)}e^{i\lambda y}d\lambda.$$

We then obtain u by the formula $u(x)=-2iK_1^*(x,x)$. The study of the inverse problem associated with the operator $L(u)$ has not yet been completely achieved. In particular the precise characterization of the spectral data which lead to well-defined potentials u has not yet been completed. However, in the case where the initial condition u_0 is rapidly decaying so that it satisfies $x\mapsto |x|^n|u_0|(x)\in L^1$ for any n, the inverse scattering can be rigorously achieved [36].

The great advantage of the method is that the evolution equations of the scattering data are uncoupled:

$$a(t,\lambda)=a(t_0,\lambda),\quad b(t,\lambda)=b(t_0,\lambda)e^{-4i\lambda^2(t-t_0)},\quad \rho_r(t)=\rho_r(t_0)e^{-4i\lambda_r^2(t-t_0)}.$$

To sum up, the scattering transform involves the following operations:

$$\begin{array}{ccl} u(t_0,x) & \stackrel{\text{direct scatt.}}{\longrightarrow} & (a,b,\lambda_r,\rho_r,a'(\lambda_r))\,(t_0) \\ \text{NLS}\downarrow & & \downarrow \text{uncoupled evolution equations} \\ u(t,x) & \stackrel{\text{inverse scatt.}}{\longleftarrow} & (a,b,\lambda_r,\rho_r,a'(\lambda_r))\,(t) \end{array}$$

There exists an infinite number of quantities which are preserved by the homogeneous NLS equation [37]. They can be represented as functionals of the solution u or in terms of the scattering data. We shall present here only two of them.

• The mass of the wave $N=\int|u|^2dx$. Denoting $n(\lambda)=-\pi^{-1}\ln|a(\lambda)|^2$, the mass is also given by

$$N=\sum_r 2i(\lambda_r^*-\lambda_r)+\int n(\lambda)d\lambda. \tag{A.3}$$

• The Hamiltonian or energy $H=\int|u_x|^2-|u|^4dx$, which can also be expressed as

$$H=\sum_r \frac{8i}{3}({\lambda_r^*}^3-{\lambda_r}^3)+4\int\lambda^2 n(\lambda)d\lambda. \tag{A.4}$$

Appendix: Proof of Proposition 5

We list the main steps of the proof [34, 35].

a. A priori estimates.
The following quantities (mass and energy) are preserved by the perturbed Schrödinger equation (6):

$$N_{tot} = \int |u|^2 dx, \quad E_{tot} = \int |u_x|^2 - |u|^4 dx + \varepsilon \int H_1(x) dx,$$
$$H_1(x) := m_1(x)|u|^2 + \frac{1}{2} m_2(x)|u|^4 - m_3(x)|u_x|^2.$$

Assume that the m_j are bounded processes. Sobolev inequalities then prove that the H^1-norm, the L^4-norm and the L^∞-norm of $u(t,.)$ are uniformly bounded with respect to $t \in \mathbb{R}$ and $\varepsilon \in (0,1)$. Furthermore $\varepsilon \int H_1(x)dx$ can be bounded uniformly with respect to $t \in \mathbb{R}$ by $K(N_{tot}, E_{tot})\varepsilon$.

b. Prove the stability of the zero of the Jost coefficient a.
The zero corresponds to the soliton. This part strongly relies on the analytical properties of a in the upper complex half plane. Basically we apply Rouché's theorem so as to prove that the number of zeros is constant. This method is efficient to prove that the zero is preserved, but it does not bring control on its precise location in the upper half plane. This step is not sufficient to compute the variations of the soliton parameters.

c. Compute the radiation.
The Jost coefficients a and b satisfy coupled equations [38]:

$$\left\{ \begin{array}{rclr} \dfrac{\partial a(\lambda,t)}{\partial t} & = & 0 & +\varepsilon \left(a(\lambda,t)\bar{\gamma}(\lambda,t) + b(\lambda,t)\gamma(\lambda,t) \right), \\ \dfrac{\partial b(\lambda,t)}{\partial t} & = & -4i\lambda^2 b(\lambda,t) & -\varepsilon \left(a(\lambda,t)\gamma^*(\lambda,t) + b(\lambda,t)\bar{\gamma}(\lambda,t) \right), \end{array} \right.$$

where $\gamma(\lambda,t) = -\int dx R(u) f_2^2 + R(u)^* f_1^2$ and $\bar{\gamma}(\lambda,t) = -\int dx R(u)^* f_1^* f_2 - R(u) f_1 f_2^*$. From these equations we can estimate the amount of radiation which is emitted during some time interval in terms of mass and energy thanks to (A.3) and (A.4). We are then able to deduce the evolution equations of the coefficients of the soliton by using the conservations of the total mass and energy. For times of order $O(1)$, since N_{tot} and E_{tot} are conserved, the variations $\Delta(..)$ of the relevant quantities are linked together by the relations:

$$\begin{array}{rcl} 0 & = & 4\Delta\nu + \int \Delta n(\lambda) d\lambda, \\ 0 & = & 16\Delta\left(\nu\mu^2 - \nu^3/3\right) + 4\int \lambda^2 \Delta n(\lambda) d\lambda + \varepsilon\Delta\left(\int_{\mathbb{R}} H_1(x) dx \right). \end{array}$$

$\Delta n(\lambda)$ is of order ε^2, but the last term in the expression of the total energy is of order ε. Thus our strategy is not efficient for estimating the variations of the coefficients of the soliton for times of order $O(1)$. Let us now consider times of order $O(\varepsilon^{-2})$. $\Delta n(\lambda)$ is now of order 1, while the last term in the expression of the total energy is of order ε by the a priori estimates. Thus we can efficiently compute the long-time behavior of the coefficients of the soliton in the asymptotic framework $\varepsilon \to 0$, when the last term in the expression of the total energy is uniformly negligible. Applying probabilistic limit theorems (approximation-diffusion), establishes that the coefficients of the soliton converge in probability to non-random functions which satisfy the system (8).

d. Compute the form of the scattered wave.
By applying the inverse scattering transform one finds that the total wave is given by the sum of a soliton and of radiation. Radiation looks like any linear dispersive wave far from the soliton, but it has complex structure in the neighborhood of the soliton. Roughly speaking, the support of the radiation lies in an interval with length of order ε^{-2}. Since the L^2-norm is bounded by the conservation of the total mass, we can expect that the amplitude of the radiation is of order ε. It can be rigorously proved that the amplitude of the radiation can be bounded above by $K\varepsilon|\ln\varepsilon|$ [34].

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