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A BIRKHOFF CONTRACTION FORMULA WITH APPLICATIONS TO RICCATI EQUATIONS*

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Abstract. In this paper we show that the symplectic Hamiltonian operators on a Hilbert space give rise to linear fractional transformations on the open convex cone of positive definite operators that contract a natural invariant Finsler metric, the Thompson or part metric, on the convex cone. More precisely, the constants of contraction for the Hamiltonian operators satisfy the classical Birkhoff formula: the Lipschitz constant for the corresponding linear fractional transformations on the cone of positive definite operators is equal to the hyperbolic tangent of one fourth the diameter of the image. By means of the close connections between Hamiltonian operators and Riccati equations, this result and the associated machinery are applied to obtain convergence results for discrete algebraic Riccati equations and Riccati differential equations.

Key words. Riccati equation, Birkhoff formula, contraction, symplectic group, control theory, Lie semigroup, Hamiltonian operator, positive definite operator

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1. Introduction.

Connections between linear control theory, the Riccati equation, and the symplectic group are well known; see, for example, Hermann [13], Shayman [22], Jurdjevic [14, Chapter 8], and [23], and the references cited in those sources. In this paper we focus particularly on connections to the symplectic subsemigroup, which consists of those symplectic transformations that are sometimes called Hamiltonian. In [15] we studied in some detail this subsemigroup of symplectic operators in the infinite dimensional setting and its close connection to Riccati differential equations arising in linear control systems. The canonical triple factorization of symplectic Hamiltonian operators and their action via linear fractional transformation on the open convex cone $\mathcal{P}_0$ of positive definite operators on a Hilbert space have played key roles in the study of Riccati equations via Lie semigroup theory. In this paper we study the contraction property of symplectic Hamiltonian operators acting on the convex cone $\mathcal{P}_0$ for the natural *invariant* Finsler metric (Thompson's part metric), and apply it to finite- and infinite dimensional discrete algebraic Riccati equations and Riccati differential equations.

One of our main results is the Birkhoff theorem (section 5) for symplectic Hamiltonian operators with respect to Thompson's metric $p(X, Y)$: each symplectic Hamiltonian $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, regarded as the self map on $\mathcal{P}_0$ given by the linear fractional transformation

$$g.X = (AX + B)(CX + D)^{-1},$$

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satisfies the contraction formula

$$\sup_{\substack{X,Y\in\mathcal{P}_0\\ X\neq Y}} \frac{p(g(X),g(Y))}{p(X,Y)} = \tanh\left(\frac{\mathrm{diam}(g)}{4}\right),$$

where $\mathrm{diam}(g)$ denotes the diameter of the image $g(\mathcal{P}_0)$ for the Thompson's metric. The diameter is completely determined by $\mathrm{diam}(g) = p(BD^{-1}, AC^{-1})$ when both BD^{-1} and AC^{-1} are positive definite; otherwise $\mathrm{diam}(g) = \infty$ (Theorem 5.8). This beautiful and important formula had its origin with Birkhoff [4] for Möbius transformations with positive entries with respect to the Riemannian metric $p(a,b) = |\log a - \log b|$ on the positive reals. Liverani and Wojtkowski [18] and Lim [16] have generalized it to fractional transformations on the symmetric cone of positive definite matrices and on symmetric cones arising from Euclidean Jordan algebras with respect to the invariant Finsler metric associated with the spectral norm. In the linear setting, the Birkhoff formula for positive linear maps on Banach spaces for Hilbert's projective (pseudo)metric [5] is well known, with many applications in analysis [4], [8], [17]; see also [20], [21] and the references therein. It has also found applications in control theory, primarily in filtering theory; see, e.g., [3], [7].

In the connections between linear control theory, the Riccati equation, and symplectic Hamiltonians, the contraction property of symplectic Hamiltonians with explicitly given contraction coefficient is applied to the iterative method of solution for discrete algebraic Riccati equations,

$$X = A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA + H,$$

and to the asymptotic behavior of solutions of the Riccati differential equation,

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t),$$

on an arbitrary Hilbert space. Bougerol [6] has proved that symplectic Hamiltonian matrices are contractions for the standard Riemannian metric on the symmetric space of positive definite matrices and given applications to Kalman filtering theory (cf. [12], [9]). However, in the Riemannian metric case, there is no explicit formula for the contraction coefficient of Hamiltonian matrices. In section 7, we prove that under the invertiblity condition of A and $BR^{-1}B^*$, the discrete Riccati equation has a unique positive solution X_∞ approached by any iteration $X_n \in \mathcal{P}_0$ with the rate of convergence determined by the *computable* Birkhoff constant with respect to Thompson's part metric. Using the *best estimation* given by the Birkhoff constant on the Lie wedge of the symplectic Hamiltonian semigroup studied in section 6, we prove in section 8 that, under the uniform boundedness condition $S^{1/2}(t)R(t)S^{1/2}(t) \geq \mu I$, the solution $K(t)$ of the Riccati differential equation with $K(0) \in \mathcal{P}_0$ is exponentially attracting for Thompson's part metric. These results, obtained mainly from the Birkhoff formula and the invariant Finsler metric, provide new techniques for study of Riccati equations, even for the finite dimensional case, where illustrative numerical experiments can be calculated.

2. Symplectic Hamiltonian operators. In this section we review some basic material on the algebraic structure of the symplectic Lie group and the associated symplectic Hamiltonian semigroup from [15].

Let E be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle\colon E\times E \to \mathbb{R}$, and let $V_E = E \oplus E$. We denote members of V_E by column vectors $\left[\begin{smallmatrix} x\\ y\end{smallmatrix}\right]$, where $x, y \in E$. The standard symplectic form Q on V_E is defined by

$$Q\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) := \langle x_1, y_2\rangle - \langle y_1, x_2\rangle.$$

We denote by $\mathrm{End}(V_E)$ (resp., $\mathrm{End}(E)$) the set of bounded linear operators on V_E (resp., E), and by $\mathrm{GL}(V_E)$ (resp., $\mathrm{GL}(E)$) those that are invertible. We shall always assume that the topology is generated by the operator norm. For a bounded linear transformation A on E, let A^* denote the unique linear operator such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all x, y in E. We call A^* the *adjoint* of A. We say that A is *symmetric* if $A^* = A$. A bounded symmetric operator A on E is *positive semidefinite* if $\langle x, Ax\rangle \geq 0$ for all $x \in E$. We denote by $\mathcal{P}$ (resp., $\mathcal{P}_0$) all positive semidefinite (resp., positive semidefinite invertible) bounded operators on E.

For (V_E, Q) a standard sympletic space, the symplectic Lie group is defined by

$$\mathrm{Sp}(V_E) := \{M \in \mathrm{GL}(V_E) \colon \forall x, y \in V_E,\ Q(Mx, My) = Q(x, y)\}$$

and has the following characterizations.

Proposition 2.1 (see Proposition 2.5 of [15]). *Let $M \in GL(V_E)$. The following are equivalent:*

1. *$M \in Sp(V_E)$; i.e., M preserves $Q(\cdot,\cdot)$.*
2. *$M^*JM = J$, where $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right] \in End(V_E)$.*
3. *If M has block matrix form $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then*
   (a) *A^*C, B^*D are symmetric;*
   (b) *$A^*D - C^*B = I$.*

Members of $\mathrm{Sp}(V_E)$ viewed as linear operators on V_E are called *linear sympletic maps*.

Recall that the symplectic Lie algebra $\mathfrak{sp}(V_E)$ consists of all $X \in \mathrm{End}(V_E)$ such that $\exp(tX) \in \mathrm{Sp}(V_E)$ for all $t \in \mathbb{R}$.

Proposition 2.2. *Let $X \in End(V_E)$. The following are equivalent:*

1. *$X \in \mathfrak{sp}(V_E)$.*
2. *$X^*J + JX = 0$.*
3. *If $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then*
   (a) *B and C are symmetric;*
   (b) *$D = -A^*$.*

We consider four subsets of $\mathrm{Sp}(V_E)$:

$$\mathcal{S} = \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible},\ B^*D \in \mathcal{P},\ CD^* \in \mathcal{P}\right\},$$
$$\mathcal{S}_1 = \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible},\ B^*D \in \mathcal{P}_0,\ CD^* \in \mathcal{P}\right\},$$
$$\mathcal{S}_2 = \left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible},\ B^*D \in \mathcal{P},\ CD^* \in \mathcal{P}_0\right\},$$
$$\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2.$$

We define

$$\Gamma^U = \left\{\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} : B \in \mathcal{P}\right\}, \qquad \Gamma_0^U = \left\{\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} : B \in \mathcal{P}_0\right\},$$

$$\Gamma^L = \left\{ \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} : C \in \mathcal{P} \right\}, \qquad \Gamma_0^L = \left\{ \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} : C \in \mathcal{P}_0 \right\}.$$

We further define a group H of block diagonal matrices by

$$H = \left\{ \begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix} : A \in \mathrm{GL}(E) \right\}.$$

THEOREM 2.3. *We have that $\mathcal{S}$ is a subsemigroup of $Sp(V_E)$ and $\mathcal{S}\mathcal{S}_i\mathcal{S} \subseteq \mathcal{S}_i$ for $i = 0, 1, 2$; i.e., $\mathcal{S}_i$ is a semigroup ideal. We alternatively have that $\mathcal{S} = \Gamma^U H \Gamma^L$, $\mathcal{S}_1 = \Gamma_0^U H \Gamma^L$, $\mathcal{S}_2 = \Gamma^U H \Gamma_0^L$, and $\mathcal{S}_0 = \Gamma_0^U H \Gamma_0^L$. Furthermore these "triple decompositions" are unique: the multiplication mapping from $\Gamma^U \times H \times \Gamma^L$ to $\mathcal{S}$ is a homeomorphism.*

Proof. The proof follows from Theorem 6.7 and [15, Lemmas 6.4, 6.5]. See also [6], [9]. ◻

The unique triple factorization of a symplectic Hamiltonian $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}$ is given by

$$M = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (D^{-1})^* & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}.$$

This factorization occurs more generally for any member $M \in \mathrm{Sp}(V_E)$ with invertible $(2, 2)$-entry. The semigroup $\mathcal{S}$ of the preceding theorem is called the *sympletic semigroup*, and members of $\mathcal{S}$ are sometimes called *Hamiltonian* operators of $\mathrm{Sp}(V_E)$.

3. Fractional transformations and compressions. In this section we show that Hamiltonians arise exactly as compressions of the open convex cone of positive definite operators under the canonical fractional transformation action.

We consider the lower block triangular subgroup $\mathbf{P}$ of $\mathrm{Sp}(V_E)$ given by

$$\mathbf{P} := \left\{ \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : A, C, D \in \mathrm{End}(E) \right\}.$$

We note from Proposition 2.1 that such a lower triangular block matrix is in $\mathrm{Sp}(V_E)$ if and only if $A^* = D^{-1}$ and $A^*C = D^{-1}C$ is symmetric. We denote by $\mathcal{M}$ the homogeneous space

$$\mathcal{M} := \mathrm{Sp}(V_E)/\mathbf{P}.$$

In the finite dimensional setting, $\mathbf{P}$ is a parabolic subgroup and the homogeneous space is a flag manifold of $\mathrm{Sp}(V_E)$. The set $\mathrm{Sym}(E)$ of symmetric operators in $\mathrm{End}(E)$ is embedded into $\mathcal{M}$ as a dense open subset (see Lemma 9.2 of [15]):

$$X \mapsto \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right] \mathbf{P} \in \mathcal{M}.$$

If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E)$ and $X \in \mathrm{Sym}(E)$ such that $CX + D$ is invertible, then

$$M \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \mathbf{P} = \begin{bmatrix} I & (AX + B)(CX + D)^{-1} \\ 0 & I \end{bmatrix} \mathbf{P}.$$

This defines the (partial) action by fractional transformations of $\mathrm{Sp}(V_E)$ on $\mathrm{Sym}(E) \subseteq \mathcal{M}$:

$$M.X = (AX + B)(CX + D)^{-1} \quad \text{if } (CX + D)^{-1} \text{ exists.} \tag{3.1}$$

For $X, Y \in \mathrm{Sym}(E)$, we define

$$X < Y :\Longleftrightarrow Y - X \in \mathcal{P}_0,$$
$$X \leq Y :\Longleftrightarrow Y - X \in \mathcal{P}.$$

The order $\leq$ is sometimes called the *Loewner order*. For $X \leq Y$ (resp., $X < Y$) we define the order intervals

$$[X, Y] = \{Z \in \mathrm{Sym}(E) : X \leq Z \leq Y\},$$
$$(X, Y) = \{Z \in \mathrm{Sym}(E) : X < Z < Y\},$$

respectively.

PROPOSITION 3.1 (see Propositions 9.6 and 9.7 of [15]). *The sets* $\{(-(1/n)A, (1/n)A) : n \in \mathbb{N}\}$ *form a basis of open sets at* 0 *in Sym*(E) *for any* $A \in \mathcal{P}_0$. *For an element* $A \in$ *Sym*(E), *the following are equivalent:*

1. $A \in \mathcal{P}$;
2. $A + X$ *is invertible for all* $X \in \mathcal{P}_0$;
3. $A + rI$ *is invertible for all* $r > 0$.

PROPOSITION 3.2 (see Propositions 9.6 and 9.9 of [15]). *Each order interval* $[A, B] = \{X \in$ *Sym*$(E) : A \leq X \leq B\}$ *for* $A \leq B$ *is closed in* $\mathcal{M}$, *the interior of* $[A, B]$ *is equal to* (A, B), *and the closure* $\overline{\mathcal{P}}$ *of* $\mathcal{P}$ *in* $\mathcal{M}$ *has interior* $\mathcal{P}_0$.

Let us call a member of $\mathrm{Sp}(V_E)$ a *compression* if it carries $\mathcal{P}_0$ into itself under the action of fractional transformation (3.1).

LEMMA 3.3. *If* $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in$ *Sp*(V_E) *is a compression and the image of* $0_E \in \mathrm{Sym}(E) \subseteq \mathcal{M}$ *under* M *is in* $\mathcal{P}$, *then* M *belongs to the sympletic semigroup* $\mathcal{S}$.

Proof. The image of 0_E under M is BD^{-1}. This means that D is invertible, and hence M has a triple decomposition in $\mathrm{Sp}(V_E)$ of the form

$$M = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (D^{-1})^* & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}.$$

Since $0_E \in \mathcal{M}$ corresponds to $\mathbf{P}$ in $\mathrm{Sp}(V_E)/\mathbf{P}$, we conclude that the last two factors of M applied to it return 0_E. Thus, by (3.1),

$$M.0_E = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix}.0 = BD^{-1}.$$

Since the latter is in $\mathcal{P}$ by hypothesis, we conclude that BD^{-1} is positive semidefinite, and hence that the first factor of M is in $\mathcal{S}$. The second factor is trivially in $\mathcal{S}$.

Let $X \in \mathcal{P}_0$. Then

$$\begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}.X = \begin{bmatrix} D^* & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} M.X,$$

where the right-hand side must be in $\mathrm{Sym}(E) \subseteq \mathcal{M}$. It follows that $D^{-1}CX + I$ is invertible for all $X \in \mathcal{P}_0$. Since X is invertible, $(D^{-1}CX + I)X^{-1} = D^{-1}C + X^{-1}$ is invertible for all $X \in \mathcal{P}_0$. It then follows from Proposition 3.1 that $D^{-1}C$ is in $\mathcal{P}$. Thus the third factor of M is also in $\mathcal{S}$. $\square$

THEOREM 3.4. *Let* $M \in$ *Sp*(V_E). *The following are equivalent:*

1. $M.\mathcal{P}_0 \subseteq \overline{\mathcal{P}}$;
2. M *is a compression;*

3. $M \in \mathcal{S}$.

Proof. If $M.\mathcal{P}_0 \subseteq \overline{\mathcal{P}}$, then since M is a homeomorphism, it must carry $\mathcal{P}_0$ into $\mathrm{int}\overline{\mathcal{P}}$, which by Proposition 3.2 is $\mathcal{P}_0$. Thus M is a compression. The converse is immediate. Hence items 1 and 2 are equivalent.

It turns out that elements of $\mathcal{S}$ carry $\mathcal{P}_0$ into itself (Proposition 7.1 of [15]). Conversely suppose that $M.\mathcal{P}_0 \subseteq \mathcal{P}_0$. Define $M_n = M \circ t_n$, where t_n has matrix representation $\left[\begin{smallmatrix} I & (1/n)I \\ 0 & I \end{smallmatrix}\right]$. Since $t_n.0 = (1/n)I$, we conclude that $M_n.0 \in \mathcal{P}_0$. Hence by the preceding lemma, $M_n \in \mathcal{S}$.

Let $A := M.I$. By hypothesis we may write the result in this form with $A \in \mathcal{P}_0$. Since A is in the open order interval $(0, 2A)$, we have for n large enough that $M_n.I \in (0, 2A)$. Since M_n is order-preserving (Proposition 3.5), we have

$$0 \le M_n.0 \le M_n.I \le 2A.$$

Since the interval $[0, 2A]$ is closed in $\mathcal{M}$ (Proposition 3.2), we conclude that

$$M.0 = \lim_n M_n.0 \in [0, 2A].$$

We can now apply the preceding lemma to M to conclude that $M \in \mathcal{S}$. ∎

Proposition 3.5 (see Proposition 9.10 of [15]). *Members of the symplectic semigroup $\mathcal{S}$ satisfy the following monotonicity properties:*

1. *For $g \in \mathcal{S}$ and $X, Y \in \mathcal{P}_0$, $X \le Y$ if and only if $g(X) \le g(Y)$.*
2. *For $g \in \mathcal{S}$ and $X, Y \in \mathcal{P}$, $X \le Y$ implies $g(X) \le g(Y)$.*

4. Hamiltonian operators and the standard sector. There is an alternative context in which Hamiltonian operators arise naturally. We consider the quadratic form $\mathcal{Q}$ on the symplectic space (V_E, Q),

$$\mathcal{Q}(w) = \langle x, y \rangle, \qquad w = \begin{bmatrix} x \\ y \end{bmatrix} \in V_E,$$

and the *standard sector* of the symplectic space (V_E, Q), which is defined by

$$\mathcal{C} = \{w \in V_E : \mathcal{Q}(w) \ge 0\}.$$

By $\mathcal{C}^\circ$ we denote the interior of $\mathcal{C}$:

$$\mathcal{C}^\circ = \{w \in V_E : \mathcal{Q}(w) > 0\}.$$

By continuity $g(\mathcal{C}^\circ) \subset \mathcal{C}^\circ$ for any $\mathcal{Q}$-monotone g. Each member of H, the subgroup of block diagonals in $\mathrm{Sp}(V_E)$, acts as an $\mathcal{Q}$-isometry.

The following is immediate from the triple decompositions of $\mathcal{S}$ and $\mathcal{S}_0$ (Theorem 2.3) and the preservation of (strict) $\mathcal{Q}$-monotonicity under composition.

Theorem 4.1. *Each member of $\mathcal{S}$ (resp., $\mathcal{S}_0$) is $\mathcal{Q}$-monotone (resp., strictly $\mathcal{Q}$-monotone). Furthermore, each member of $\mathcal{S}$ (resp., $\mathcal{S}_0$) increases (resp., strictly increases) the quadratic form.*

Remark. The quadratic form $\mathcal{Q}$ and the associated sector $\mathcal{C}$ define two natural closed subsemigroups in $\mathrm{Sp}(V_E)$ containing the symplectic semigroup $\mathcal{S}$: the subsemigroup of (strictly) monotone maps and the subsemigroup of (strictly) symplectic maps (strictly) increasing the quadratic form. It turns out that these are all the same in the finite dimensional case [19].

We derive an explicit relationship between the action of symplectic Hamiltonians on the sector $\mathcal{C}^\circ$ and the Möbius action of fractional transformation on the positive definite cone $\mathcal{P}_0$.

LEMMA 4.2. *Let* $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{C}^\circ$. *Then there exists a positive definite operator* P *on* E *such that* $y = Px$. *In particular,*

$$\mathcal{C}^\circ = \left\{ \left[\begin{array}{c} x \\ Px \end{array} \right] : x \neq 0, P \in \mathcal{P}_0 \right\}.$$

Proof. Let W be the subspace generated by x and y. If x and y are linearly dependent, then $y = \lambda x$ for some $\lambda > 0$, so we may take $P = \lambda I$. Suppose that W is two-dimensional. Then it is enough to construct a positive definite operator A on W sending x into y by observing that $P := \left[\begin{array}{cc} A & 0 \\ 0 & I_{W^\perp} \end{array} \right]$ is positive definite.

Suppose that $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ such that $l := x_1y_1 + x_2y_2 > 0$. We will solve the equations $ax_1 + bx_2 = y_1, bx_1 + dx_2 = y_2$ with $a > 0, ad > b^2$.

Case 1. If $x_1 = 0$, then take $b = y_1/x_2, d = y_2/x_2 > 0$, and a (positive) large enough. If $x_2 = 0$, then take $b = y_2/x_1, a = y_1/x_1 > 0$, and d large enough.

Case 2. $x_1 \neq 0$ and $x_2 \neq 0$: If $x_1x_2 > 0$, then take

$$b < \min\left\{\frac{y_1}{x_2}, \frac{y_1y_2}{l}\right\}, \quad a = \frac{(y_1 - bx_2)}{x_1}, \quad d = \frac{(y_2 - bx_1)}{x_2}.$$

If $x_1x_2 < 0$, then take $b > \max\{y_1/x_2, y_1y_2/l\}$, $a = (y_1 - bx_2)/x_1$, and $d = (y_2 - bx_1)/x_2$. $\square$

A *slice* of the sector $\mathcal{C}^\circ$ consists of sets of the form

$$\mathcal{P}_x = \left\{ P_x := \left[\begin{array}{c} Px \\ x \end{array} \right] : P \in \mathcal{P}_0 \right\}.$$

The preceding lemma shows that the sector $\mathcal{C}^\circ$ is the disjoint union of slices.

PROPOSITION 4.3. *Let* $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}$. *Then for* $P > 0$,

$$g(P_x) = (g.P)_{(CP+D)x}.$$

Proof. We calculate that

$$\left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \left[\begin{array}{c} Px \\ x \end{array} \right] = \left[\begin{array}{c} (AP+B)x \\ (CP+D)x \end{array} \right] = \left[\begin{array}{c} (AP+B)(CP+D)^{-1}y \\ y \end{array} \right] = \left[\begin{array}{c} (g.P)y \\ y \end{array} \right],$$

where $y = (CP + D)x$. $\square$

5. Contractions and the Birkhoff formula. In this section, we show that each element of $\mathcal{S}$ (resp., $\mathcal{S}_0$) is a contraction (resp., strict contraction) of $\mathcal{P}_0$ for a natural invariant metric on it, with an explicit contraction constant given by the Birkhoff formula.

For $A, B \in \mathcal{P}_0$, we define

$$M(A/B) := \inf\{t > 0 : A \leq tB\},$$
$$m(A/B) := \sup\{t > 0 : tB \leq A\}.$$

Then $M(A/B) = m(B/A)^{-1}$. *Thompson's metric* (sometimes called the *part metric*) on $\mathcal{P}_0$ is defined by

$$p(A, B) = \log(\max\{M(A/B), M(B/A)\});$$

see, e.g., [24], [25], [20].

LEMMA 5.1. *The set $\mathcal{P}_0$ becomes a complete metric space with respect to the metric p, and the metric p induces the topology of $\mathcal{P}_0$.*

Proof. The space $\mathrm{Sym}(E)$ of symmetric operators equipped with the operator norm is a Banach space satisfying that $0 \le A \le B$ implies $\|A\| \le \|B\|$. It follows from Proposition 3.1 that for $A, B \in \mathcal{P}_0$ there exists $t \in \mathbb{R}$ such that $A \le tB$. It then follows from Lemma 3 of [24] that the Thompson metric p is indeed a metric and is complete on $\mathcal{P}_0$, and by Proposition 1.1 of [21] that the Thompson metric induces the same topology. $\square$

LEMMA 5.2. *The metric p is invariant under the block diagonal group H and inversion $j(A) = A^{-1}$.*

Proof. The lemma follows directly from the observations

$$\forall D \in \mathrm{GL}(E), M(D^*AD/D^*BD) = M(A/B), \quad M(A^{-1}/B^{-1}) = M(B/A),$$

where the last equality follows from the fact that inversion on $\mathcal{P}_0$ is order-reversing (cf. Proposition 9.8 of [15]). $\square$

A map $\gamma : [0, 1] \to \mathcal{P}_0$ is said to be a *minimal geodesic* for the metric p if, whenever $0 \le t_1 \le t_2 \le 1$, we have

$$p(\gamma(t_1), \gamma(t_2)) = (t_2 - t_1)p(\gamma(0), \gamma(1)).$$

PROPOSITION 5.3 (see Proposition 1.10 of [20]). *Let $A, B \in \mathcal{P}_0$. Then*

$$\gamma(t) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

is a minimal geodesic curve from A to B with respect to p.

For $X \in \mathrm{Sym}(E)$, the *order unit norm* for the order unit I is given by

$$\|X\| = \inf\{t > 0 : -tI \le X \le tI\}.$$

LEMMA 5.4. *On Sym(E) we have the following:*

1. *The order unit norm agrees with the operator norm on Sym(E).*
2. *For $X \in \mathcal{P}_0$, $m(X/I) = \|X^{-1}\|^{-1}$.*
3. *The map $X \mapsto m(X/I)$ is continuous on $\mathcal{P}$.*

Proof. Part 1. Let us temporarily denote the order unit norm by $\|X\|_{or}$. Then

$$|\langle x, Xx\rangle| \le \|x\|(\|X\|\,\|x\|) = \|X\|\langle x, Ix\rangle = \langle x, \|X\|Ix\rangle$$

implies that $\|X\|_{or} \le \|X\|$. For $\|x\| = 1$ and $X \ge 0$, we have

$$\|X^{1/2}x\|^2 = \langle X^{1/2}x, X^{1/2}x\rangle = \langle x, Xx\rangle \le \langle x, \|X\|_{or}Ix\rangle = \|X\|_{or}.$$

It follows that $\|X\| \le \|X^{1/2}\|^2 \le \|X\|_{or}$. We then have for arbitrary symmetric X

$$\|X\|^2 = \|X^*X\| = \|X^2\| = \|X^2\|_{or} \le \|X\|_{or}^2,$$

since $-tI \le X \le tI$ implies that $t^2I - X^2 = (tI + X)^{1/2}(tI - X)(tI + X)^{1/2} \ge 0$.

2. For $X \in \mathcal{P}_0$, we have directly that

$$\begin{aligned} m(X/I) &= \sup\{t > 0 : tI \le X\} = \sup\{t > 0 : (1/t)I \ge X^{-1}\} \\ &= \sup\{(1/s) > 0 : X^{-1} \le sI\} = \|X^{-1}\|^{-1}. \end{aligned}$$

3. It follows from part 2 that the function $X \mapsto m(X/I)$ is continuous on $\mathcal{P}_0$. For $X \in \mathcal{P}$, let $0 \le X_n \to X$. Then for $\varepsilon > 0$, $X_n + \varepsilon I \to X + \varepsilon I$ in $\mathcal{P}_0$, which in turn implies $m(X_n + \varepsilon I/I) \to m(X + \varepsilon I/I)$. Since $m(A + \varepsilon I/I) = m(A/I) + \varepsilon$ for $A \in \mathcal{P}$, the desired conclusion follows. $\square$

Remarks. (1) The map $X \mapsto m(X/I)$ on $\mathcal{P}_0$ is one of special interest; it agrees with the smallest eigenvalue function in the finite dimensional case.

(2) For $X \in \mathcal{P}_0$, $||X|| = M(X/I)$, and hence $p(I, X) = \max \log\{||X||, ||X^{-1}||\}$. Thus for $X, Y \in \mathcal{P}_0$,

$$p(X, Y) = p(I, X^{-1/2}YX^{-1/2}) = \log \max\{||X^{-1/2}YX^{-1/2}||, ||X^{1/2}Y^{-1}X^{1/2}||\}.$$

Identifying the tangent bundle $T\mathcal{P}_0$ of $\mathcal{P}_0$ with $\mathcal{P}_0 \times \mathrm{Sym}(E)$, we define a Finsler structure on $\mathcal{P}_0$ by

$$|X|_A := ||A^{-1/2}XA^{-1/2}||$$

for $A \in \mathcal{P}_0$, $X \in \mathrm{Sym}(E)$. Then it is easy to see that $|\cdot|_A$ is a norm on the tangent space $\mathrm{Sym}(E)$ at A.

THEOREM 5.5 (see Theorem 1.1 of [21]). *Let $A, B \in \mathcal{P}_0$. Then*

$$p(A, B) = \inf\left\{\int_0^1 |\psi'(t)|_{\psi(t)} dt\right\},$$

where the infimum is taken over all piecewise C^1 maps ψ from $A = \psi(0)$ to $B = \psi(1)$. In particular,

$$p(A, B) = \int_0^1 |\gamma'(t)|_{\gamma(t)} dt,$$

where $\gamma(t) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$.

For notational convenience, we denote for $A \in \mathcal{P}$ and $D \in \mathrm{GL}(E)$,

$$\begin{aligned} t_A &:= \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \in \Gamma^U, \\ \tilde{t}_A &:= \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \in \Gamma^L, \\ h_D &:= \begin{bmatrix} D^* & 0 \\ 0 & D^{-1} \end{bmatrix} \in H. \end{aligned}$$

Then under the action of fractional transformation (3.1),

$$t_A(B) = A + B, \quad h_D(B) = D^*BD, \quad \tilde{t}_A(B) = (A + B^{-1})^{-1} = (jt_Aj)(B)$$

for $B \in \mathcal{P}_0$, where $j(A) = A^{-1}$, the inversion operator on $\mathcal{P}_0$.

PROPOSITION 5.6. *Let $X, Y \in \mathcal{P}_0$ and let $D \in \mathrm{GL}(E)$. Then*

$$t_X \circ h_D \circ \tilde{t}_Y = h_{Y^{-1/2}D} \circ t_{Y^{1/2}(D^{-1})^*XD^{-1}Y^{1/2}} \circ \tilde{t}_I \circ h_{Y^{1/2}}.$$

Proof. The proof is straightforward. □

Set $\infty := \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]\mathbf{P} \in \mathcal{M}$. It is easy to see that for $g \in \mathrm{Sp}(V_E)$, $g \cdot \infty = \infty$ if and only if g is an upper triangular block matrix.

LEMMA 5.7. *Let $0 < A \leq B$. If $X, Y \in [A, B]$, then $p(X, Y) \leq p(A, B)$.*

Proof. Suppose that $p(X, Y) = \log M(X/Y)$. Since $A \leq X \leq M(X/A)A \leq M(X/A)Y$, we have $M(X/Y) \leq M(X/A)$. The fact $A \leq X$ implies that $m(X/A) \geq 1$ and hence $M(A/X) = m(X/A)^{-1} \leq 1$. Thus $M(X/A) \geq 1$. Therefore

$$p(X, Y) = \log M(X/Y) \leq \log M(X/A) = p(A, X). \tag{5.1}$$

Now, $X \leq B \leq M(B/A)A$ implies that $M(X/A) \leq M(B/A)$ and hence by (5.1)

$$p(X, A) = \log M(X/A) \leq \log M(B/A) \leq p(B, A).$$

Therefore $p(X, Y) \leq p(A, B)$. Similarly, we have that $p(X, Y) \leq p(A, B)$ when $p(X, Y) = \log M(X/Y)$. □

THEOREM 5.8. *Let $g \in \mathcal{S}_0$. Then $g(\mathcal{P}_0) = (g(0), g(\infty))$, $g(\overline{\mathcal{P}}) = [g(0), g(\infty)] \subseteq \mathcal{P}_0$, and the diameter $\Delta(g)$ of $g(\mathcal{P}_0)$ for the metric p is the distance $p(g(0), g(\infty))$. If $g \in \mathcal{S} \setminus \mathcal{S}_0$, then $\Delta(g) = \infty$.*

Proof. Let $g = t_X \circ h_D \circ \tilde{t}_Y \in \mathcal{S}_0$. By Theorem 2.3, $X, Y \in \mathcal{P}_0$. Then $g(0) = X$ and $g(\infty) = X + D^*Y^{-1}D$. Suppose that $Z \in (X, X + D^*Y^{-1}D)$. Then $Z = X + A = X + D^*Y^{-1}D - B$ for some $A, B \in \mathcal{P}_0$. Note that $A = D^*Y^{-1}D - B$, so $A < D^*Y^{-1}D$. Since the inversion j is order-reversing on $\mathcal{P}_0$ (cf. Proposition 9.8 of [15]), $W := (DA^{-1}D^* - Y)^{-1} \in \mathcal{P}_0$. This implies that $Z = X + A = g(W) \in g(\mathcal{P}_0)$. Conversely, suppose that $Z \in g(\mathcal{P}_0)$. Then $Z = g(W) = X + D^*(Y + W^{-1})^{-1}D$ for some $W \in \mathcal{P}_0$. It is obvious that $X < Z$. Since $W^{-1} \in \mathcal{P}_0$, we have that $Y^{-1} > (Y + W^{-1})^{-1}$. Thus $D^*Y^{-1}D > D^*(Y + W^{-1})^{-1}D$. This implies that

$$g(\infty) - Z = (X + D^*Y^{-1}D) - (X + D^*(Y + W^{-1})^{-1}D) > 0.$$

Therefore $Z \in (g(0), g(\infty))$. So, $g(\mathcal{P}_0) = (g(0), g(\infty))$. The second assertion follows from this, Proposition 3.1, the fact that g acts as a homeomorphism on $\mathcal{M}$, and our computation of $g(0)$ and $g(\infty)$.

That the diameter of $g(\mathcal{P}_0) = (g(0), g(\infty))$ is the Thompson distance $p(g(0), g(\infty))$ follows from the preceding lemma.

Suppose that $g = t_A \circ h_D \circ \tilde{t}_B \in \mathcal{S} \setminus \mathcal{S}_0$. Then by Theorem 2.3 either A or B lies in $\mathcal{P} \setminus \mathcal{P}_0$. Suppose that $A \in \mathcal{P} \setminus \mathcal{P}_0$. Pick $C \in g(\mathcal{P}_0)$ with $C > 0$. Let $Y_n = g(\frac{1}{n}I) \in \mathcal{P}_0$. Then $Y_n \to g(0) = A$. Since $g(0) = A \in \mathcal{P} \setminus \mathcal{P}_0$, for each $k > 0$, there exists $n_k > 0$ such that $Y_{n_k} \notin [\frac{1}{k}C, kC]$, that is, $Y_{n_k} \nleq kC$ or $\frac{1}{k}C \nleq Y_{n_k}$. By definition, $M(C/Y_{n_k}) \geq k$ or $M(Y_{n_k}/C) \geq k$. Therefore $\log k \leq p(C, Y_{n_k}) \to \infty$, and hence $\Delta(g) = \infty$. Similarly, if $g(\infty) = B \in \mathcal{P} \setminus \mathcal{P}_0$, then $\Delta(g) = \infty$. □

LEMMA 5.9. *Let $A, X \in \mathcal{P}_0$. Then*

$$|(I + A)^{-1}U(I + A)^{-1}|_{X + (I + A^{-1})^{-1}} \leq \left(\sqrt{m(X/I)} + \sqrt{1 + m(X/I)}\right)^{-2} |U|_A$$

for all $U \in \mathrm{Sym}(E)$.

Proof. First, we show that

$$(I + A)X(I + A) + A^2 + A \geq \left(\sqrt{m(X/I)} + \sqrt{1 + m(X/I)}\right)^2 A.$$

It immediately follows from $m(X/I)I \leq X$ that

$$m(X/I)(I+A)^2 = m(X/I)(I+A)I(I+A) \leq (I+A)X(I+A).$$

We then have

$$\begin{aligned}
&(I+A)X(I+A) + A^2 + A \\
&\geq m(X/I)(I+A)^2 + A^2 + A \\
&= (m(X/I)+1)A^2 + m(X/I)I + (2m(X/I)+1)A \\
&\geq 2\sqrt{m(X/I)(m(X/I)+1)}A + (2m(X/I)+1)A \\
&= (\sqrt{m(X/I)} + \sqrt{1+m(X/I)})^2 A,
\end{aligned}$$

where the second inequality follows from the fact that the square of $\sqrt{m(X/I)+1}A - \sqrt{m(X/I)}I$ is positive semidefinite.

Set $k := (\sqrt{m(X/I)} + \sqrt{1+m(X/I)})^{-2}$. Then $-tI \leq kA^{-1/2}UA^{-1/2} \leq tI$ for some $t > 0$, or equivalently $(-t/k)A \leq U \leq (t/k)A$. From the first paragraph, we obtain that

$$-t\big((I+A)X(I+A) + A^2 + A\big) \leq \frac{-t}{k}A \leq U \leq \frac{t}{k}A \leq t((I+A)X(I+A) + A^2 + A).$$

Since $(I+A)^{-1}(A^2+A)(I+A)^{-1} = (I+A^{-1})^{-1}$, this implies that

$$-t(X + (I+A^{-1})^{-1}) \leq (I+A)^{-1}U(I+A)^{-1} \leq t(X + (I+A^{-1})^{-1})$$

and hence

$$-tI \leq (X + (I+A^{-1})^{-1})^{-1/2}(I+A)^{-1}U(I+A)^{-1}(X + (I+A^{-1})^{-1})^{-1/2} \leq tI.$$

Therefore from the definition of the order unit norm,

$$||(X+(I+A^{-1})^{-1})^{-1/2}(I+A)^{-1}U(I+A)^{-1}(X+(I+A^{-1})^{-1})^{-1/2}|| \leq k||A^{-1/2}UA^{-1/2}||,$$

and the lemma follows immediately. ◻

LEMMA 5.10. *Let $X \in \mathcal{P}_0$ and let $0 < \alpha < \beta$. Then*

$$M(X+\beta I/X+\alpha I) \geq \frac{m(X/I)+\beta}{m(X/I)+\alpha} \geq 1.$$

In particular,

$$M\left(X + \frac{\beta}{\beta+1}I/X + \frac{\alpha}{\alpha+1}\right) \geq \frac{m(X/I) + \frac{\beta}{\beta+1}}{m(X/I) + \frac{\alpha}{\alpha+1}} \geq 1.$$

Proof. If $X + \beta I \leq t(X+\alpha I)$ for $t > 0$, then

$$m(X/I) + \beta = m(X+\beta I/I) \leq m(t(X+\alpha I)/I) = t(m(X/I)+\alpha). \qquad ◻$$

Let us introduce the Lipschitz constant (the least coefficient of contraction) of $g \in \mathcal{S}$,

$$N(g) = \sup_{\substack{A,B\in\mathcal{P}_0 \\ A\neq B}} \frac{p(g(A),g(B))}{p(A,B)}.$$

Note that $N(g_1 g_2) \le N(g_1)N(g_2)$.

THEOREM 5.11. *Let $g \in \mathcal{S}$. Then*

$$N(g) = \tanh\left(\frac{\Delta(g)}{4}\right).$$

Proof. Let $g \in \mathcal{S}_0$. Note that $N(g) = N(h \circ g \circ h')$ for any $h, h' \in H$ by the H-invariance of the metric. By Proposition 5.6, we may assume that $g = t_X \circ \tilde{t}_I$ for some $X \in \mathcal{P}_0$. Then $g(0) = X, g(\infty) = X + I$, and hence $\Delta(g) = p(X, X+I) = \log M(X+I/X) = \log(1+M(I/X)) = \log(1+\frac{1}{m(X/I)})$. A straightforward calculation yields

$$\begin{aligned}
\tanh\left(\frac{\Delta(g)}{4}\right) &= \tanh\left(\frac{1}{4}\log\left(1+\frac{1}{m(X/I)}\right)\right) \\
&= \frac{(1+\frac{1}{m(X/I)})^{\frac{1}{4}} - (1+\frac{1}{m(X/I)})^{-\frac{1}{4}}}{(1+\frac{1}{m(X/I)})^{\frac{1}{4}} + (1+\frac{1}{m(X/I)})^{-\frac{1}{4}}} \\
&= \left(\sqrt{m(X/I)} + \sqrt{1+m(X/I)}\right)^{-2}. \qquad (5.2)
\end{aligned}$$

Furthermore, for the differential of the mapping $g(Y) = X + (I + Y^{-1})^{-1}$, we have

$$\mathrm{d}g(A)(U) = (I + A^{-1})^{-1}(A^{-1}UA^{-1})(I + A^{-1})^{-1} = (I + A)^{-1}U(I + A)^{-1}$$

for $A \in \mathcal{P}_0, U \in \mathrm{Sym}(E)$.

Let $A, B \in \mathcal{P}_0$, and let $\gamma(t) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ be the minimal geodesic curve passing from A to B. Then by Lemma 5.9,

$$\begin{aligned}
p(g(A), g(B)) &\le \int_0^1 |(g\circ\gamma)'(t)|_{g(\gamma(t))} dt \\
&= \int_0^1 |\mathrm{d}g(\gamma(t))(\gamma'(t))|_{g(\gamma(t))} dt \\
&= \int_0^1 |(I+\gamma(t))^{-1}\gamma'(t)(I+\gamma(t))^{-1}|_{X+(I+\gamma(t)^{-1})^{-1}} dt \\
&\le (\sqrt{m(X/I)} + \sqrt{1+m(X/I)})^{-2} \int_0^1 |\gamma'(t)|_{\gamma(t)} dt \\
&= (\sqrt{m(X/I)} + \sqrt{1+m(X/I)})^{-2} p(A, B),
\end{aligned}$$

where in the last equality we have used the fact that the distance $p(A, B)$ is equal to the Finsler length of the geodesic curve $\gamma(t)$. Therefore

$$N(g) \le \left(\sqrt{m(X/I)} + \sqrt{1+m(X/I)}\right)^{-2}.$$

To show that equality holds, it is enough to show that

$$\left(\sqrt{m(X/I)} + \sqrt{1+m(X/I)}\right)^{-2} \le \sup_{\substack{\alpha,\beta\in\mathbb{R}^+ \\ \alpha<\beta}} \frac{p(g(\alpha I), g(\beta I))}{p(\alpha I, \beta I)}.$$

By Lemma 5.10, we obtain that

$$\begin{aligned}
\sup_{\substack{\alpha,\beta\in\mathbb{R}^+\\ \alpha<\beta}} \frac{p(g(\alpha I), g(\beta I))}{p(\alpha I,\beta I)} &= \sup_{\substack{\alpha,\beta\in\mathbb{R}^+\\ \alpha<\beta}} \frac{p(X+(I+\alpha^{-1}I)^{-1}), X+(I+\beta^{-1}I)^{-1})}{p(\alpha I,\beta I)} \\
&= \sup_{\substack{\alpha,\beta\in\mathbb{R}^+\\ \alpha<\beta}} \frac{\log M\left(X+\frac{\beta}{\beta+1}I/X+\frac{\alpha}{\alpha+1}I\right)}{\log\frac{\beta}{\alpha}} \\
&\geq \sup_{\substack{\alpha,\beta\in\mathbb{R}^+\\ \alpha<\beta}} \frac{\log\frac{m(X/I)+\frac{\beta}{\beta+1}}{m(X/I)+\frac{\alpha}{\alpha+1}}}{\log\frac{\beta}{\alpha}} = \sup_{\substack{\alpha,\beta\in\mathbb{R}^+\\ \alpha<\beta}} \frac{\log\frac{\underline{g}(\beta)}{\underline{g}(\alpha)}}{\log\frac{\beta}{\alpha}},
\end{aligned}$$

where $\underline{g} = \left[\begin{smallmatrix} 1+m(X/I) & m(X/I) \\ 1 & 1\end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{R})$ is the usual Möbius transformation on $\mathbb{R}$. By the Birkhoff formula on the positive reals [4],

$$\begin{aligned}
\sup_{\substack{\alpha,\beta\in\mathbb{R}^+\\ \alpha<\beta}} \frac{\log\frac{\underline{g}(\beta)}{\underline{g}(\alpha)}}{\log\frac{\beta}{\alpha}} &= \tanh\left(\frac{\Delta(\underline{g})}{4}\right) \\
&= \tanh\left(\frac{1}{4}\log\left(1+\frac{1}{m(X/I)}\right)\right) \\
&\overset{5.2}{=} \left(\sqrt{m(X/I)}+\sqrt{1+m(X/I)}\right)^{-2}.
\end{aligned}$$

This shows that the Birkhoff formula holds for $\mathcal{S}_0$.

It follows from previous result that every member of $\mathcal{S}_0$ is a strict contraction. By definition we have that the operator $g_n := \left[\begin{smallmatrix} I & 0 \\ (1/n)I & I\end{smallmatrix}\right]$ is in $\mathcal{S}_2$, the operator $h_n := \left[\begin{smallmatrix} I & (1/n)I \\ 0 & I\end{smallmatrix}\right]$ is in $\mathcal{S}_1$, and $g_n, h_n \to e$, the identity element of $\mathrm{Sp}(V_E)$. Then for any $g \in \mathcal{S}$, $g_n h_n g \to g$ and $g_n h_n g \in \mathcal{S}_0$ since $\mathcal{S}_1$ and $\mathcal{S}_2$ are ideals by Theorem 2.3 and $\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2$ by definition. It follows from standard continuity arguments and the density of $\mathcal{S}_0$ in $\mathcal{S}$ that all members of $\mathcal{S}$ are contractions.

Define $\sigma : \mathcal{S} \to \mathbb{R}^+ = [0,\infty)$ by

$$\sigma(t_A h_D \tilde{t}_B) = \left(\sqrt{m(Q/I)}+\sqrt{1+m(Q/I)}\right)^{-2}, \quad \text{where } Q = B^{1/2}(D^{-1})^* A D^{-1} B^{1/2}.$$

Then σ is well defined from the unique triple factorization of $\mathcal{S}$ (Theorem 2.3) and is continuous by Lemma 5.4. By Proposition 5.6 and the calculation above, $\sigma(g) = N(g) = \tanh\left(\frac{\Delta(g)}{4}\right)$ for any $g \in \mathcal{S}_0$. Let $g = t_A h_D \tilde{t}_B \in \mathcal{S}\setminus\mathcal{S}_0$. Then either A or B is not invertible; thus $m(B^{1/2}(D^{-1})^* A D^{-1} B^{1/2}/I) = 0$, and hence $\sigma(g) = 1$. By Theorem 5.8, $\tanh(\Delta(g)/4) = 1$. For small positive ϵ, pick $g_\epsilon \in \mathcal{S}_0$ sufficiently close to the identity such that $\sigma(g) - \epsilon \leq \sigma(g_\epsilon g)$. Then

$$\sigma(g) - \epsilon \leq \sigma(g_\epsilon g) = N(g_\epsilon g) \leq N(g_\epsilon) N(g) \leq N(g) \leq 1,$$

which shows that $N(g) = 1 = \sigma(g) = \tanh(\Delta(g)/4)$. Thus the Birkhoff formula holds for $\mathcal{S}\setminus\mathcal{S}_0$, which completes the proof. $\square$

We refer the reader to the references [6] and [12] for applications of contraction results to Riccati transformations and control. There it is shown that the Riccati transformation of linear filtering/control theory is a contraction on the space of positive definite matrices. The metric used there is the standard Riemannian metric on the symmetric space of positive definite matrices. Since we extend these results to the infinite dimensional case as well, it has been necessary to substitute the Thompson metric for the Riemannian metric. We have sharpened the results in another sense by calculating the constant of contraction, the one given by the Birkhoff formula. These formulas have been derived in the finite dimensional case in [18].

6. The Birkhoff formula on the Lie wedge. For the symplectic semigroup $\mathcal{S}$, which is a closed subsemigroup of the symplectic Lie group $\mathrm{Sp}(V_E)$, the Lie wedge of $\mathcal{S}$,

$$\mathfrak{L}(\mathcal{S}) := \{X \in \mathfrak{g} : \exp(tX) \in S \ \forall\ t \geq 0\},$$

which is the tangent object of $\mathcal{S}$ in the Lie algebra, is explicitly described as follows.

PROPOSITION 6.1 (see Proposition 8.1 of [15]). *The symplectic semigroup $\mathcal{S}$ has Lie wedge*

$$\mathfrak{L}(\mathcal{S}) = \left\{ \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : B, C \geq 0 \right\}.$$

Setting

$$\mathfrak{h} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} : A \in \mathrm{End}(V_E) \right\},$$
$$W = \left\{ \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} : R, S \geq 0 \right\},$$

we have $\mathfrak{L}(\mathcal{S}) = \mathfrak{h} \oplus W$. In particular, $\mathfrak{h}$ is the Lie subalgebra of the subgroup H of block diagonal matrices.

We recall the Birkhoff constant map

$$N : \mathcal{S} \to [0,1], \quad N(g) = \tanh\left(\frac{\triangle(g)}{4}\right) = \sup_{\substack{X,Y>0 \\ X \neq Y}} \frac{p(g(X), g(Y))}{p(X,Y)},$$

and define

$$f : \mathfrak{L}(\mathcal{S}) \to [0,\infty), \quad \begin{bmatrix} A & R \\ S & -A^* \end{bmatrix} \mapsto \sqrt{m(S^{1/2}RS^{1/2}/I)}.$$

Then f is a continuous, homogeneous, Ad_H-invariant function and is an extension of the map $X \mapsto m(X/I)$ on $\mathcal{P}$.

THEOREM 6.2. *We have* $\log \circ N \circ \exp \leq -2f$ *on* $\mathfrak{L}(\mathcal{S})$.

Proof. Let $\left[\begin{smallmatrix} 0 & R \\ S & 0 \end{smallmatrix}\right] \in W^\circ$, the interior of W, i.e., $R, S > 0$. Then

$$\exp \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} = \begin{bmatrix} S^{-1/2} & 0 \\ 0 & S^{1/2} \end{bmatrix} \cdot \exp \begin{bmatrix} 0 & S^{1/2}RS^{1/2} \\ I & 0 \end{bmatrix} \cdot \begin{bmatrix} S^{1/2} & 0 \\ 0 & S^{-1/2} \end{bmatrix},$$

and it follows by homogeneity of the Thompson metric that

$$N\left(\exp \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right) = N\left(\exp \begin{bmatrix} 0 & S^{1/2}RS^{1/2} \\ I & 0 \end{bmatrix}\right).$$

Setting $X = S^{1/2}RS^{1/2}$ and $g = \exp\left[\begin{smallmatrix} 0 & X \\ I & 0 \end{smallmatrix}\right]$, we have

$$g = \begin{bmatrix} \cosh X^{1/2} & X^{1/2}\sinh X^{1/2} \\ X^{-1/2}\sinh X^{1/2} & \cosh X^{1/2} \end{bmatrix}$$

and therefore

$$g(0) = X^{1/2}\sinh X^{1/2}(\cosh X^{1/2})^{-1} = X^{1/2}\tanh X^{1/2}, \qquad g(\infty) = X^{1/2}\coth X^{1/2}.$$

Then

$$\begin{aligned} \triangle(g) &= p(g(0), g(\infty)) \\ &= p(X^{1/2}\tanh X^{1/2}, X^{1/2}\coth X^{1/2}) \\ &= p(I, \coth^2 X^{1/2}) \\ &= \log M(\coth X^{1/2}/I)^2 \\ &\leq \log\coth^2 m(X^{1/2}/I), \end{aligned}$$

where the third equality follows from the homogeneity of the metric, the fourth from $\coth X^{1/2} \geq I$, and the last inequality from the fact that for $t > 0$, $tI \leq X$ implies $t^n I \leq X^n$ and hence $(\exp t)I \leq \exp X$ and consequently $(\coth t)I \geq \coth X$. By direct computation we have

$$N(g) = \tanh\left(\frac{\triangle(g)}{4}\right) \leq \tanh\left(\frac{1}{2}\log\coth m(X^{1/2}/I)\right) = e^{-2m(X^{1/2}/I)} = e^{-2\sqrt{m(X/I)}}.$$

By continuity of $N(\cdot)$ and $m(\cdot/I)$, the asserted inequality holds for arbitrary members of W.

Finally, the assertion of the inequality on all of $\mathfrak{L}(S)$ follows from the preceding, from the fact that both sides of the inequality reduce to 0 on $\mathfrak{h}$, from the Lie–Trotter product formula, and from the multiplicative property of the Birkhoff constant function: $N(gh) \leq N(g)N(h)$. ∎

Remark. In the finite dimensional case, the inequality in Theorem 6.2 becomes an equality on W: $\log\circ N\circ\exp = -2f$. This follows from the fact that

$$||\coth X|| = \coth||X^{-1}||^{-1}, \quad X > 0.$$

Set $R = \left\{\left[\begin{smallmatrix} 0 & X \\ I & 0 \end{smallmatrix}\right] : X > 0\right\} \subseteq W^\circ$, the interior of W.

THEOREM 6.3. *We have*

$$\mathcal{S}_0 = \Gamma_0^U H \Gamma_0^L = H(\exp R)H = H(\exp W^\circ).$$

Proof. The first equality follows by Theorem 2.3. We have observed in the proof of Theorem 6.2 that $\exp W^\circ \subseteq H(\exp R)H$ and hence $H\exp W^\circ \subseteq H(\exp R)H$. Since W° is Ad_H-invariant, $(\exp R)H \subseteq H\exp W^\circ$, and therefore $H(\exp R)H \subseteq H\exp W^\circ$, the third equality is proved.

Let $X > 0$. Then

$$\begin{aligned} \exp\begin{bmatrix} 0 & X \\ I & 0 \end{bmatrix} &= \begin{bmatrix} \cosh X^{1/2} & X^{1/2}\sinh X^{1/2} \\ X^{-1/2}\sinh X^{1/2} & \cosh X^{1/2} \end{bmatrix} \\ = \begin{bmatrix} I & X^{1/2}\tanh X^{1/2} \\ 0 & I \end{bmatrix} & \begin{bmatrix} (\cosh X^{1/2})^{-1} & 0 \\ 0 & \cosh X^{1/2} \end{bmatrix} \begin{bmatrix} I & 0 \\ X^{-1/2}\tanh X^{1/2} & I \end{bmatrix} \\ &\in \mathcal{S}_0 = \Gamma_0^U H \Gamma_0^L \end{aligned}$$

because $X^{1/2}\tanh X^{1/2} > 0$ and $X^{-1/2}\tanh X^{1/2} > 0$. The ideal property of $\mathcal{S}_0$ implies that $H(\exp R)H \subseteq \mathcal{S}_0$. However, the explicit triple decomposition and Proposition 5.6 imply that

$$\exp \begin{bmatrix} 0 & X \\ I & 0 \end{bmatrix} \in H \cdot \begin{bmatrix} 0 & (\sinh X^{1/2})^2 \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \cdot H,$$

and thus each element in the right-hand side belongs to $H(\exp R)H$. Suppose that $g \in \mathcal{S}_0 = \Gamma_0^U H \Gamma_0^L$. Then by Proposition 5.6, $g = h_{D_1}\left[\begin{smallmatrix} I & A \\ 0 & I \end{smallmatrix}\right]\left[\begin{smallmatrix} I & 0 \\ I & I \end{smallmatrix}\right] h_{D_2}$ for some $A > 0$ and $D_i \in \mathrm{GL}(E)$. Set $X = [\log(A^{1/2} + (A+I)^{1/2})]^2$. Then $X > 0$, and by direct computation $\sinh X^{1/2} = A^{1/2}$, so that $g \in H(\exp R)H$. ∎

7. Discrete algebraic Riccati equations. The discrete algebraic Riccati equation (DARE) arises in the context of minimizing a quadratic cost for discrete-time linear time-invariant systems (see, for example, [23, Chapter 8.4]). We consider (DARE) on a Hilbert space E:

$$X = A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA + H, \tag{7.1}$$

where R and H are symmetric and positive definite [10].

It will be convenient to work with a simpler form of (DARE). We begin with the following result.

LEMMA 7.1.
1. *If $A + B$ is invertible, then $A(A+B)^{-1}B = A - A(A+B)^{-1}A$.*
2. $B(I + B^*XB)^{-1} = (I + BB^*X)^{-1}B$.

Proof. For the first assertion move the longer term from the right-hand side to the left, factor, and simplify. For the second eliminate the inverses by moving the expressions to the other side of the equation. ∎

Lemma 7.1 can be used to show that (DARE) is equivalent to

$$X = A^*X(I + GX)^{-1}A + H, \qquad G = BR^{-1}B^*. \tag{7.2}$$

Indeed,

$$\begin{aligned} & X - XB(R + B^*XB)^{-1}B^*X \\ = \;& X - XBR^{-1/2}(I + R^{-1/2}B^*XBR^{-1/2})^{-1}R^{-1/2}B^*X \\ \overset{C=BR^{-1/2}}{=} \;& X - XC(I + C^*XC)^{-1}C^*X \\ \overset{Lemma\ 7.1(2)}{=} \;& X - X(I + CC^*X)^{-1}CC^*X \\ \overset{G=CC^*}{=} \;& X - X(I + GX)^{-1}GX \\ \overset{Lemma\ 7.1(1)}{=} \;& X - X\Big(I - (I + GX)^{-1}\Big) \\ = \;& X(I + GX)^{-1}. \end{aligned}$$

THEOREM 7.2. *If A is invertible and $G = BR^{-1}B^*$ is positive definite, then* (DARE) *has a unique positive definite solution X_∞ and the iteration*

$$X_{n+1} = H + A^*X_n(I + GX_n)^{-1}A$$

starting at any point $X_0 \in \mathcal{P}_0$ converges to X_∞ with

$$p(X_\infty, X_n) \le \frac{L^n}{1 - L} p(X_1, X_0),$$

where $\Lambda = H^{-1/2}A^*G^{-1/2}$*,* $L = \tanh\big((1/4)\log\|I+\Lambda\Lambda^*\|\big)$.

Proof. We note that positive definite solutions of (DARE) correspond to positive definite fixed points of the map

$$X \mapsto A^*X(I+GX)^{-1}A + H \tag{7.3}$$

on $\mathcal{P}_0$. Under the fractional transformation, the mapping (7.3) becomes

$$X \mapsto H + A^*X(I+GX)^{-1}A = \begin{bmatrix} I & H \\ 0 & I \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.X.$$

The operator of the right-hand side,

$$\begin{bmatrix} I & H \\ 0 & I \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ G & I \end{bmatrix}, \tag{7.4}$$

belongs to $\mathcal{S}_0$ and hence is a strict contraction for Thompson's metric p by Theorems 5.11 and 5.8. By completeness of the metric, it has a unique fixed point on the positive definite cone $\mathcal{P}_0$, and therefore (DARE) has a unique positive definite solution. Obviously the solution X_∞ is represented as a limit of iteration $X_{n+1} = H + A^*X_n(I+GX_n)^{-1}A$ with initial point in $\mathcal{P}_0$. Set $X_\infty = \lim_{n\to\infty} X_n, X_0 > 0$. The p-diameter of the map $X \mapsto H + A^*X(I+GX)^{-1}A$ is computed from Theorem 5.8:

$$\Delta = p(H, H + A^*G^{-1}A) = p(I, I+\Lambda\Lambda^*) = \log ||I+\Lambda\Lambda^*||,$$

where $\Lambda =: \Lambda(H,R,A,B) = H^{-1/2}A^*G^{-1/2}$. Then its contraction constant is

$$L := \tanh\left(\frac{\Delta}{4}\right) = \tanh\left(\frac{\log ||I+\Lambda\Lambda^*||}{4}\right),$$

and the error bound may be estimated by $p(X_\infty, X_n) \le \frac{L^n}{1-L}\, p(X_1, X_0)$. $\square$

Remark. We observe that the unique positive definite solution $S(H,R,A,B)$ in the above theorem depends on the parameters H,R,A,B, where H,R vary over the positive definite operators and A,B over invertible operators on E. This defines the solution map of DARE

$$S : \mathcal{P}_0 \times \mathcal{P}_0 \times \mathrm{GL}(E) \times \mathrm{GL}(E) \to \mathcal{P}_0, \quad (H,R,A,B) \to S(H,R,A,B).$$

In the finite dimensional case it is shown in [2] that the solution map is continuous and extends to the set of singular A. Under the additional condition that A is stable (leaves the unit ball invariant) but without the invertibility condition of $G = BR^{-1}B^*$, (DARE) has a unique positive definite solution (Corollary 5.7 of [11]).

Remark. In [9] (DARE) is called the *standard symplectic form* (SSF) when both H and G are invertible, which is the case of Theorem 7.2; that is, the associated symplectic Hamiltonian (7.4) is in $\mathcal{S}_0$. An efficient numerical method is developed, the so-called *structure-preserving doubling* (SDP) algorithm, which requires the fact that the (positive) powers of the associated symplectic Hamiltonian Z remain in SSF (Theorem 2.1 of [9]). We have already obtained the semigroup property (even ideal property) of $\mathcal{S}_0$ in Theorem 2.3. The SDP algorithm produces the sequence $Z^{2^k}, k = 1,2,\ldots,$ and the rate of convergence is estimated in terms of eigenvalues of the associated symplectic pencil (Theorem 3.1 of [9]; see also [10]).

Remark. The unique positive definite solution of (DARE) or (7.3) lies in the open order interval

$$(H, H + A^*G^{-1}A).$$

Thus it is more effective to begin the iteration method starting at a point in this interval. There are three positive definite operators lying in the interval. The harmonic-geometric-arithmetic inequalities of the positive definite operators $A, B > 0$ (cf. [1]),

$$2(A^{-1} + B^{-1})^{-1} \leq A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \leq \frac{A+B}{2},$$

imply that the open order interval $(H, H+A^*G^{-1}A)$ contains the harmonic, geometric, and arithmetic means of H and $H + A^*G^{-1}A$:

$$2(H^{-1} + (H + A^*G^{-1}A)^{-1})^{-1}, \quad H\#(H + A^*G^{-1}A), \quad H + \frac{A^*G^{-1}A}{2}.$$

One can also show that $\frac{1}{2}(H + H\#(H + 4A^*G^{-1}A))$ lies in the interval $(H, H + A^*G^{-1}A)$.

8. Stability of Riccati differential equations. We consider the control system given by the *basic group control equation* (BGCE) on $\mathrm{Sp}(V_E)$:

$$\text{(BGCE)} \qquad \dot{g}(t) = u(t)g(t),$$

where $u : \mathbb{I} \to \mathfrak{sp}(V_E)$, $\mathbb{I}$ a (finite or infinite) subinterval of $\mathbb{R}$, is called a *steering* or *control* function. In the case that E is finite dimensional, we assume that $u(\cdot)$ belongs to the class of measurable functions from $\mathbb{I}$ into $\mathfrak{sp}(V_E)$, which are locally bounded, that is, bounded on every finite subinterval, and in the case of general E we assume that $u(\cdot)$ is a regulated function, that is, a function that on each finite subinterval of its domain is a uniform limit of piecewise constant functions. A solution of (BGCE), called a *trajectory*, is an absolutely continuous function $x(\cdot)$ from $\mathbb{I}$ into G such that the equation (BGCE) holds a.e., where a.e. means on the complement of a set of measure 0 in the finite dimensional setting and the complement of a countable set otherwise. The solution for initial condition $g(0) = \mathrm{id}_{V(E)}$ is called the *fundamental solution* of the basic group control equation and denoted $\Phi(t)$. By right invariance the general solution to (BGCE) with initial condition $g(t_0) = g_0$ is then given by $g(t) = \Phi(t)(\Phi(t_0))^{-1}g_0$.

PROPOSITION 8.1 (see [15, Proposition 8.3]). *Each solution $\Phi(t)$ for $t \geq 0$ of the basic group control equation on $Sp(V_E)$,*

$$\dot{g}(t) = u(t)g(t), \quad g(0) = id_{V(E)}, \quad u(t) \in \mathfrak{L}(\mathcal{S}),$$

is contained in the semigroup $\mathcal{S}$; i.e., the attainable set is contained in $\mathcal{S}$. If $\Phi(s) \in \mathcal{S}_i$ for some s and some $i = 0, 1,$ or 2, then $\Phi(t) \in \mathcal{S}_i$ for all $t > s$.

We return to the material on the Lie wedge of the symplectic semigroup at the beginning of section 6. Note that $\mathfrak{L}(S) = \mathfrak{h} \oplus W$ has interior $\mathfrak{h} \oplus W^\circ$ in $\mathfrak{sp}(V_E)$, where

$$W^\circ = \left\{ \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix} : R, S > 0 \right\}.$$

Since the exponential function is locally a homeomorphism in a neighborhood $N(0)$ of the 0-matrix, we conclude that members of $(\mathfrak{h} + W^\circ) \cap N(0)$ are carried by the exponential map into the interior of $\mathcal{S}$. Since for any $Y \in \mathfrak{sp}(V_E)$, $\exp(Y) = (\exp(1/n)Y)^n$

and $\mathcal{S}$ is a subsemigroup, we conclude that $\exp(\mathfrak{h}+W^\circ)$ is carried into the interior of $\mathcal{S}$. It follows readily from the homeomorphic triple decomposition of Theorem 2.3 that the interior of $\mathcal{S}$ is contained in $\mathcal{S}_0$ (indeed they are equal), so $\exp(\mathfrak{h}+W^\circ) \subseteq \mathcal{S}_0$.

We need the following elementary lemma.

Lemma 8.2. *Let $\Phi : \mathbb{R}^+ \times X \to X$ be a continuous semiflow of the nonnegative reals on a Hausdorff space X. Set $\phi_t(x) = \Phi(t, x)$. If ϕ_t has exactly one fixed point for each $t = 1/2^n$, $n \in \mathbb{N}$, then the fixed point is a common one for all ϕ_t, $t \in \mathbb{R}^+$.*

Consider on the Hilbert space E the Riccati differential equation

$$\text{(RDE)} \qquad \dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \quad K(t_0) = K_0,$$

where the coefficient functions are locally bounded and measurable in the finite dimensional case and regulated otherwise. It was shown in [15, section 5] that the solution of equation (RDE) for the case that $R(t), S(t), K_0 \geq 0$ arises through the fundamental solution of basic control equation (BGCE) acting by fractional transformations on $\mathcal{P} \subseteq \mathcal{M}$:

$$K(t) = \Phi(t)(\Phi(t_0))^{-1}(K_0), \qquad \text{where } u(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A(t)^* \end{bmatrix}.$$

In the case of constant coefficients with $R, S > 0$, then for $t > 0$, $\Phi(t) = \exp(tM)$ lies in $\mathcal{S}_0$ (as we have seen), where $M = \left[\begin{smallmatrix} A & R \\ S & -A^* \end{smallmatrix}\right]$. It follows that for each $t > 0$, $\exp(tM)$ is a strict contraction on $\mathcal{P}_0$ by Theorems 5.8 and 5.11 and hence has a unique fixed point. Hence by the preceding Lemma 8.2 we conclude that there is a common fixed point P^* for all ϕ_t, $t \geq 0$. Hence the vector field, given by (RDE), must have a 0-vector at P^*, i.e., the algebraic Riccati equation (ARE)

$$R + AK + KA^* - KSK = 0, \qquad R, S > 0,$$

must have a unique positive definite solution. (Note that another solution would yield another fixed point for the ϕ_t.) We have thus rederived from our machinery the following familiar result.

Proposition 8.3. *The ARE*

$$R + AK + KA^* - KSK = 0, \qquad R, S > 0,$$

has a unique positive definite solution.

Recall the homogeneous function defined on the Lie wedge $\mathfrak{L}(\mathcal{S})$,

$$f : \mathfrak{L}(\mathcal{S}) \to [0, \infty), \qquad \begin{bmatrix} A & R \\ S & -A^* \end{bmatrix} \mapsto \sqrt{m(S^{1/2}RS^{1/2}/I)}.$$

Corollary 8.4. *Let $\Phi(t)$ be the fundamental solution of the basic control equation*

$$\dot{g}(t) = u(t)g(t), \quad g(0) = \mathrm{id}_{V(E)}, \quad u(t) \in \mathfrak{L}(\mathcal{S}).$$

If there exists $\mu > 0$ such that $u(t) \in f^{-1}([\mu, \infty))$ for all $t \geq 0$, then $N(\Phi(t)) \leq e^{-2t\mu}$ for each $t \geq 0$.

Proof. The density of the set of piecewise constant controls yields that $\Phi(t)$ is a limit of finite products of elements of the form

$$\exp(\alpha_1 X_1)\exp(\alpha_2 X_2)\cdots\exp(\alpha_n X_n),$$

where $\sum_{i=1}^{n}\alpha_i = t$ and $\alpha_i \geq 0, X_i \in f^{-1}([\mu,\infty))$ for $i = 1,2,\ldots,n$. Theorem 6.2 ensures that

$$N(\exp \alpha_i X_i) \leq e^{-2f(\alpha_i X_i)} = e^{-2\alpha_i f(X_i)} \leq e^{-2\alpha_i \mu}$$

and hence

$$N(\exp(\alpha_1 X_1)\exp(\alpha_2 X_2)\cdots\exp(\alpha_n X_n)) \leq e^{-2\alpha_1\mu}e^{-2\alpha_2\mu}\cdots e^{-2\alpha_n\mu} = e^{-2t\mu}.$$

By continuity (see the last part of the proof of Theorem 5.11), $N(\Phi(t)) \leq e^{-2t\mu}$. ◻

Example. Let $u(t) = \left[\begin{smallmatrix} A(t) & R(t) \\ S(t) & -A(t)^* \end{smallmatrix}\right] \in \mathfrak{L}(\mathcal{S})$. Then $u(t) \in f^{-1}([\mu,\infty))$ for all $t \geq 0$ if and only if $m(S^{1/2}(t)R(t)S^{1/2}(t)/I) \geq \mu^2$ for all $t \geq 0$, and this includes the case when $S(t)$ is invertible and $R(t) \geq \mu^2 S^{-1}(t)$ for all $t \geq 0$.

The next theorem shows that under general conditions two solutions of the Riccati differential equation (RDE) exponentially converge toward each other.

THEOREM 8.5. *Let $K_1(t), K_2(t)$ be two solutions with initial values $K_1(t_0) = K_1 > 0$ and $K_2(t_0) = K_2 > 0$ of the Riccati differential equation*

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \ \text{where } R(t), S(t) \geq 0.$$

If there exists $\mu > 0$ and $t_1 \geq t_0$ such that $m(S(t)^{1/2}R(t)S(t)^{1/2}/I) \geq \mu^2$ for all $t \geq t_1$, then

$$p(K_1(t), K_2(t)) \leq e^{-2(t-t_1)\mu}p(K_1, K_2)$$

for $t \geq t_1$.

Proof. Let $u(t) = \left[\begin{smallmatrix} A(t) & R(t) \\ S(t) & -A(t)^* \end{smallmatrix}\right]$. Since $R(t), S(t) \geq 0$, then $u(t) \in \mathfrak{L}(\mathcal{S}), t \geq t_0$. Let $\Phi(t)$ be the fundamental solution of the basic group control equation

$$\dot{g}(t) = u(t)g(t), g(0) = \mathrm{id}_{V_E}.$$

Then

$$K_1(t) = \Phi(t)\Phi(t_0)^{-1}(K_1) = \Phi(t)\Phi(t_1)^{-1}\Phi(t_1)\Phi(t_0)^{-1}(K_1)$$

and $K_2(t) = \Phi(t)\Phi(t_1)^{-1}\Phi(t_1)\Phi(t_0)^{-1}(K_2)$. Note that $\Psi(t) := \Phi(t+t_1)\Phi(t_1)^{-1}$ is the fundamental solution of the basic group control equation

$$\dot{g}(t) = u(t+t_1)g(t), \qquad g(0) = \mathrm{id}_{V(E)}.$$

By assumption,

$$u(t+t_1) \in f^{-1}([\mu,\infty)) = \left\{ \begin{bmatrix} A & R \\ S & -A^* \end{bmatrix} \in \mathcal{L}(S) : m(S^{1/2}RS^{1/2}) \geq \mu^2 \right\},$$

and by the previous corollary $N(\Psi(t)) \leq e^{-2t\mu}$ for all $t \geq 0$. Similarly, $\Phi(t_1)\Phi(t_0)^{-1} \in \mathcal{S}$, a contraction. Therefore for $t \geq t_1$

$$\begin{aligned} p(K_1(t), K_2(t)) &= p(\Psi(t-t_1)\Phi(t_1)\Phi(t_0)^{-1}(K_1), \Psi(t-t_1)\Phi(t_1)\Phi(t_0)^{-1}(K_2)) \\ &\leq e^{-2(t-t_1)\mu}p(\Phi(t_1)\Phi(t_0)^{-1}(K_1), \Phi(t_1)\Phi(t_0)^{-1}(K_2)) \\ &\leq e^{-2(t-t_1)\mu}p(K_1, K_2). \qquad ◻ \end{aligned}$$

There has been extensive study of conditions for the existence of a (positive definite) solution K to the ARE

$$R + AK + KA^* - KSK = 0, \quad R, S > 0;$$

see, for example [23, Chapter 8.4]. The preceding allows one to draw results in the converse direction.

COROLLARY 8.6. *If K^* is the unique positive definite solution for the constant coefficient ARE, then all solutions of the corresponding Riccati differential equation $\dot{K} = R + AK + KA^* - KSK$, $R, S > 0$, that enter the space of positive definite operators converge exponentially toward K^*.*

Proof. We consider a trajectory of the given Riccati differential equation that takes on a value $K_0 > 0$ at some time t_0. Then the trajectory satisfies (RDE) with initial condition K_0 at time t_0. Since K^* satisfies ARE, the value of the Riccati differential equation at K^* is 0, and thus the solution through K^* is constant. Since the coefficients R, S are constant and positive definite, the appropriate boundedness condition of the previous theorem for $S^{1/2}RS^{1/2}$ is satisfied. Hence the first trajectory converges exponentially toward the second trajectory with constant value K^*. ∎

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