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THE SYMPLECTIC SEMIGROUP AND RICCATI DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study close connections that exist between the Riccati operator (differential) equation that arises in linear control systems and the symplectic group and its subsemigroup of symplectic Hamiltonian operators. A canonical triple factorization is derived for the symplectic Hamiltonian operators, and their closure under multiplication is deduced from this property. This semigroup of Hamiltonian operators, which we call the symplectic semigroup, is studied from the viewpoint of Lie semigroup theory, and resulting consequences for the theory of the Riccati equation are delineated. Among other things, these developments provide an elementary proof for the existence of a solution of the Riccati equation for all $t \geq 0$ under rather general hypotheses.

1. Introduction

The main purpose of this paper is to demonstrate how the Lie theory of subsemigroups of a matrix group or, more generally, a Lie group can be applied to problems in geometric control theory. We have chosen to do this in the form of a case study of a basic tool of control theory: the familiar Riccati equation that arises in the context of linear control systems with quadratic costs. The bulk of our theory carries through in the infinite dimensional setting with little additional effort, and we develop our theory in this context. This generalization is perhaps of some interest since both classical control theory and the Lie theory of semigroups have typically been developed in the finite-dimensional setting.

We recall the primary connection of Lie semigroup theory with geometric control theory. Suppose that the states of a control problem are points of a Lie group and the controls are right invariant vector fields, or that the control problem can be reinterpreted so that this is the case. If the control functions are closed under concatenation, then the attainable set

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from the identity forms an infinitesimally generated subsemigroup of the Lie group, and all attainable sets are translates of this semigroup. If this attainability semigroup is closed in the group, then it is an example of a Lie semigroup (i.e., a closed infinitesimally generated semigroup). In addition to the techniques of geometric control, one has available the vast machinery and structure theory of Lie groups and Lie algebras to study control problems on Lie groups and the attainability semigroup. The attainability semigroup has been used primarily to study questions of controllability (see, for example, the survey [13]), but in this paper we seek to make a case that a detailed understanding of the attainability semigroup can be useful for attacking a variety of control questions. We refer the reader to [4, 5] for the theory of Lie semigroups; a good background article on semigroups and control is the survey article [8].

Connections between linear control theory, the Riccati equation, and the symplectic group are well known (see, e.g., [3, 6, 14, 15] and the references therein). In this paper, we focus on connections with the symplectic subsemigroup, which consists of those symplectic transformations that are sometimes called Hamiltonian. This semigroup has largely been overlooked in the control context; see, however, Bougerol [1], which was an important inspiration for our investigations. We exploit properties of the symplectic group and symplectic semigroup both to rederive some familiar results concerning the Riccati equation from this vantage point, hopefully with some new insights along the way, and to further extend and generalize the theory. We employ (and thus illustrate) a variety of basic tools from Lie group and Lie semigroup theory such as pushing forward control systems from groups to homogeneous spaces (Sec. 5) and the subtangential set of a semigroup, called its Lie wedge (Sec. 8). But the primary structure theorem for the symplectic semigroup, which is crucial to many of our applications, is its triple decomposition as given in Sec. 6. The triple decomposition often allows us to break up problems into much simpler subcases.

There exists an important order on the symmetric operators called the Loewner order. In the last two sections we consider this order and its connections with the symplectic semigroup and the Riccati equation.

Many of the results of this paper are not new, but rather are new derivations of known results from the perspective of Lie group and Lie semigroup theory. Part of the purpose, as was already mentioned, is to give an accessible case study of Lie semigroup theory and its connections with the control theory. However, this paper is also foundational for more advanced and original applications of the symplectic semigroup to the study of Riccati equations that we plan to publish in a subsequent paper or papers.

2. Symplectic spaces and the symplectic group

In this section, we recall basic results concerning symplectic spaces and the symplectic group. These results are well known, in particular, in the finite-dimensional setting, but it will be convenient to have them at hand for the general setting of this paper. Crucial for later purposes are the familiar results Proposition 2.5 through Proposition 2.7 at the end of the section, and the reader may choose simply to glance at them and move on.

Let V be a vector space over $\mathbb{R}$, the real numbers. A *symplectic form* on V is a nondegenerate, skew-symmetric bilinear form $Q : V \times V \to \mathbb{R}$.

Definition 2.1. We give a standard construction for symplectic forms. Let E be a Hilbert space over $\mathbb{R}$ with inner product $\beta(\cdot,\cdot) : E \times E \to \mathbb{R}$. We seet $V_E := E \oplus E$; we denote elements of V_E by column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in E$. We define the symplectic form $Q := Q_E$ on V_E by

$$Q_E\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) := \beta(x_1, y_2) - \beta(y_1, x_2).$$

The pair (V_E, Q_E) is called a *standard symplectic space* over $\mathbb{R}$.

Example 2.2. Let $E = \mathbb{R}$ be a one-dimensional Hilbert space over $\mathbb{R}$. Then $V_E = \mathbb{R} \oplus \mathbb{R}$ and

$$Q_E\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \det \begin{bmatrix} w_1 & z_1 \\ w_2 & z_2 \end{bmatrix}, \quad w_1, w_2, z_1, z_2 \in \mathbb{R}.$$

Example 2.3. Let $E = \mathbb{R}^n$ with its usual Hilbert space structure. Then $V_E = \mathbb{R}^{2n}$. The symplectic form restricted to the standard basis is given by

$$Q(e_i, e_j) = \begin{cases} 1 & \text{if } j = i + n, \\ -1 & \text{if } i = j + n \\ 0 & \text{otherwise.} \end{cases}$$

The symplectic form Q can be expressed as the matrix product

$$Q(x, y) = x^* \cdot J \cdot y,$$

where x^* denotes the transpose and J denotes the $(2n \times 2n)$-matrix given in the block form as

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Remark 2.4. Given any symplectic form Q on a finite-dimensional vector space V, there exists a basis $\{\epsilon_1, \ldots, \epsilon_{2n}\}$ such that Q restricted to this basis is given by the formulas in Example 2.3. Thus, the symplectic space (V, Q) is isomorphic as a symplectic space to the standard one of Example 2.3 under the isomorphism that carries ϵ_i to e_i.

Let $(V_E, Q_E = Q)$ be a standard symplectic space, where E is a Hilbert space. Then any bounded linear operator $A : V_E \to V_E$ has a block matrix representation of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where} \quad A_{ij} := \pi_i \circ A \circ \iota_j : E \to E,$$

where $\iota_j : E \to V_E$ is the natural embedding into the jth coordinate and $\pi_i : V_E \to E$ is the projection into the ith coordinate, for $i, j = 1, 2$. We denote by $\mathrm{End}(V_E)$ (respectively, $\mathrm{End}(E)$) the set of bounded linear operators on V_E (respectively, E) and by $\mathrm{GL}(V_E)$ (respectively, $\mathrm{GL}(E)$) the set of invertible linear operators. We will always assume that the topology is generated by the operator norm. Note that the operator norm topology on $\mathrm{End}(V_E)$ is the product topology of the operator norm topology for the four block matrix operators in $\mathrm{End}(E)$.

We define an operator $J \in \mathrm{End}(V_E)$ by

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the identity operator on E. Note that

$$J^2 = -I_{V_E}, \quad J^4 = I_{V_E};$$

in particular, J is invertible.

There is a Hilbert space inner product $\beta(\cdot, \cdot)$ defined on V_E by

$$\beta\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) := \beta(u, x) + \beta(v, y).$$

In terms of β, the symplectic form Q is given by $Q(\mathbf{a}, \mathbf{b}) = \beta(\mathbf{a}, J\mathbf{b})$.

For a bounded linear transformation A on E or V_E, let A^* denote the unique linear operator such that $\beta(Ax, y) = \beta(x, A^*y)$ for all x and y in E or V_E, respectively. The operator A^* is called the *adjoint* operator of A. Observe that for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathrm{End}(V_E)$, we have by a straightforward computation:

$$\beta\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \beta\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Thus,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}.$$

It follows that

$$J^* = -J, \quad JJ^* = I.$$

The operator A is said to be *symmetric* if $A^* = A$.

We denote by $M^\sharp$ for $M \in \mathrm{End}(V_E)$ the unique linear operator such that $Q(Mx, y) = Q(x, M^\sharp y)$ for all $x, y \in V_E$. Since

$$\begin{aligned} Q(x, M^\sharp y) &= Q(Mx, y) = \beta(Mx, Jy) = \beta(x, M^* Jy) \\ &= \beta(x, JJ^* M^* Jy) = Q(x, J^* M^* Jy), \end{aligned}$$

we conclude that $M^\sharp = J^* M^* J = -JM^* J$. The operator $M^\sharp$ is called the *symplectic conjugate* of M.

For a standard symplectic space (V_E, Q), we set

$$\mathrm{Sp}(V_E) := \Big\{ M \in \mathrm{GL}(V_E) : \forall x, y \in V_E,\ Q(Mx, My) = Q(x, y) \Big\}.$$

Suppose that $M \in \mathrm{Sp}(V_E)$. Then for all $x, y \in V_E$,

$$\beta(x, Jy) = Q(x, y) = Q(Mx, My) = \beta(Mx, JMy) = \beta(x, M^* JMy).$$

Thus, $J = M^* JM$, and the argument reverses to yield that $M \in \mathrm{Sp}(V_E)$ if M is invertible and $M^* JM = J$. We conclude that for $M \in \mathrm{GL}(V_E)$,

$$M^* JM = J \iff M \in \mathrm{Sp}(V_E).$$

Proposition 2.5. *Let $M \in \mathrm{GL}(V_E)$. The following conditions are equivalent*:

(1) $M \in \mathrm{Sp}(V_E)$, *i.e., M preserves $Q(\cdot, \cdot)$*;
(2) $M^* JM = J$;
(3) *if* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *then*
  (a) $A^* C$ *and* $B^* D$ *are symmetric*;
  (b) $A^* D - C^* B = I$;
(4) $M^{-1} = M^\sharp$.

Thus, the set $\mathrm{Sp}(V_E)$ is a group.

Proof. The equivalence of (1) and (2) was established in the remarks preceding the proposition. The equivalence of (2) and (3) is obtained by a straightforward computation. The implication (2) $\Rightarrow$ (4) follows from multiplying both sides of (2) on the left by $-J$, and left multiplying of $M^\sharp M = I$ by J gives the reverse implication.

It follows from the definition that $\mathrm{Sp}(V_E)$ is closed under composition and from (4) that it is closed under inversion (since $M^{\sharp\sharp} = M$); hence it is a group. □

Corollary 2.6. *The set $\mathrm{Sp}(V_E)$ is closed under taking adjoint. Hence* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E)$ *if and only if*

(1) AB^* *and* CD^* *are symmetric*;
(2) $AD^* - BC^* = I$.

Proof. Let $M \in \mathrm{Sp}(V_E)$. By item (4) of the preceding proposition, we have

$$(M^*)^{-1} = (M^{-1})^* = (M^\sharp)^* = (J^* M^* J)^* = J^* M^{**} J = (M^*)^\sharp,$$

and, therefore, $M^* \in \mathrm{Sp}(V_E)$. The remaining assertion follows from applying the preceding proposition to M^*. $\square$

Recall that the Lie algebra $\mathfrak{sp}(V_E)$ consists of all $X \in \mathrm{End}(V_E)$ such that $\exp(tX) \in \mathrm{Sp}(V_E)$ for all $t \in \mathbb{R}$.

Proposition 2.7. *Let $X \in \mathrm{End}(V_E)$. The following assertions are equivalent*:

(1) $X \in \mathfrak{sp}(V_E)$;
(2) $X^* J + JX = 0$;
(3) *if* $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *then*
   (a) B *and* C *are symmetric*;
   (b) $D = -A^*$.

Proof. (1) $\iff$ (2) Assume that $X \in \mathfrak{sp}(V_E)$. Then by item (2) of Proposition 2.5, $e^{tX^*} J e^{tX} = J$ for all $t \in \mathbb{R}$. Differentiating with respect to t and evaluating at $t = 0$ yields the desired result. Conversely, if (2) holds, then the function $t \mapsto e^{tX^*} J e^{tX}$ has the derivative (with respect to t) $e^{tX^*}(X^* J + JX)e^{tX} = 0$, and hence is a constant function. Evaluating at $t = 0$ establishes that the constant is J.

(2) $\iff$ (3) This is a straightforward computation using the block operator representation. $\square$

Remark 2.8. Results similar to the last three can be deduced for the complex symplectic group obtained by starting from a complex Hilbert space E and carrying out a similar construction. In this case, however, the role of the adjoint in the preceding material is now played by the *complex conjugate of the adjoint*, i.e., the transpose. With this modification, the results in the rest of the paper can also be applied to the complex setting.

3. The Riccati equation

In this section, we continue with known results, but present them in a manner that will be convenient for us. The results are a special case of a theorem of J. Levin [11, Theorem 1] (the earliest reference that we are aware of), except that we have merely observed that they be extended to the infinite-dimensional case as well.

Let E be a Hilbert space and let $V_E = E \oplus E$ be as in the previous section. We consider the control system given by the *basic group control equation* (BGCE) on $\mathrm{Sp}(V_E)$:

$$\dot{g}(t) = u(t) g(t), \qquad \text{(BGCE)}$$

where $u : \mathbb{I} \to \mathfrak{sp}(V_E)$, $\mathbb{I}$ is a (finite or infinite) subinterval of $\mathbb{R}$, is called a *steering* or *control* function. In the case where E is finite-dimensional, we assume that $u(\cdot)$ belongs to the class of measurable functions from $\mathbb{I}$ into $\mathfrak{sp}(V_E)$ which are locally bounded, i.e., bounded on every finite subinterval, and in the case of general E we assume that $u(\cdot)$ is a regulated function, i.e., a function that on each finite subinterval of its domain is a uniform limit of a piecewise-constant functions. A solution of (BGCE), called a *trajectory*, is an absolutely continuous function $x(\cdot)$ from $\mathbb{I}$ into G such that Eq. (BGCE) holds a.e., where a.e. means "on the complement of a set of measure 0" in the finite-dimensional setting and "on the complement of a countable set" otherwise. Control systems such as the one just described are called *right invariant*, since right translates of solutions of (BGCE) are again solutions. Using the homogeneity of $\mathrm{Sp}(V_E)$, one readily obtains that global solutions on all of $\mathbb{I}$ exist whenever local solutions exist. Thus, global solutions always exist in the settings we are considering (see [8, Sec. 3] and [4, Sec. IV.5]). The solution for the initial condition $g(0) = \mathfrak{id}_{V(E)}$ is called the *fundamental solution* of the basic group control equation and is denoted by $\Phi(t)$. By the right invariance, the general solution of (BGCE) with initial condition $g(t_0) = g_0$ is then given by $g(t) = \Phi(t)(\Phi(t_0))^{-1}g_0$.

We turn now to Ricatti equations.

Definition 3.1. An (*operator*) *Riccati equation* is a differential equation on the space $\mathrm{Sym}(E)$ of bounded symmetric operators on E of the form

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \quad K(t_0) = K_0, \tag{R}$$

where $R(t)$, $S(t)$, and K_0 are all in $\mathrm{Sym}(E)$.

There is a close connection between the basic group control equation and the Riccati equation.

Lemma 3.2. *Suppose that $g(\cdot)$ is a solution of the following* (BGCE) *on an interval $\mathbb{I}$:*

$$\dot{g}(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A^*(t) \end{bmatrix} \begin{bmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{bmatrix}, \quad R(t), S(t) \in \mathrm{Sym}(E).$$

If g_{22} is invertible for all $t \in \mathbb{I}$, then $K(t) := g_{12}(t)(g_{22}(t))^{-1}$ satisfies

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t)$$

on $\mathbb{I}$. Furthermore, if

$$g(t_0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}$$

for some $t_0 \in \mathbb{I}$, then $K(t_0) = K_0$.

Proof. Using the product rule and the power rule for inverses and the equality of the second columns in the basic group control equation, we obtain

$$\begin{aligned}\dot{K} &= \dot{g}_{12}(g_{22})^{-1} - g_{12}g_{22}^{-1}\dot{g}_{22}g_{22}^{-1} \\ &= (Ag_{12} + Rg_{22})g_{22}^{-1} - K(Sg_{12} - A^*g_{22})g_{22}^{-1} \\ &= AK + R - KSK + KA^*.\end{aligned}$$

The last assertion is immediate. □

Corollary 3.3. *Local solutions exist for the Riccati equation* (R)*.*

Proof. Global solutions exist for the basic group control equation (BGCE) with the initial condition

$$g(t_0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}$$

and the $g_{22}(t)$-entry will be invertible in some neighborhood of t_0. Now apply the previous theorem. □

4. The Spaces Λ and $\mathcal{M}$

We fix the Hilbert space E and define

$$\Lambda := \left\{ \begin{bmatrix} B \\ D \end{bmatrix} : \exists A, C \in \mathrm{End}(E) \text{ such that } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) \right\}.$$

We also consider the lower block triangular subgroup $\mathbf{P}$ of $\mathrm{Sp}(V_E)$ given by

$$\mathbf{P} := \left\{ \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : A, C, D \in \mathrm{End}(E) \right\}.$$

We note from Proposition 2.5 that such a lower triangular block matrix is in $\mathrm{Sp}(V_E)$ if and only if $A^* = D^{-1}$ and $A^*C = D^{-1}C$ is symmetric.

Proposition 4.1. *Let* $\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} \in \Lambda$*. The following assertions are equivalent*:

(1) *there exist* $M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$, $M_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \in \mathrm{Sp}(V_E)$ *such that* $M_1\mathbf{P} = M_2\mathbf{P}$;

(2) *for all* $M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$, $M_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \in \mathrm{Sp}(V_E)$*, we have* $M_1\mathbf{P} = M_2\mathbf{P}$;

(3) *there exists* $Q \in \mathrm{GL}(E)$ *such that* $B_1Q = B_2$ *and* $D_1Q = D_2$.

Proof. (1) $\Rightarrow$ (3) This implication follows directly from the fact $M_2 \in M_1\mathbf{P}$; the matrix Q is the lower right entry of the matrix $P \in \mathbf{P}$ such that $M_2 = M_1P$.

(3) $\Rightarrow$ (1) Assume that (3) holds. Take $\begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} \in \mathrm{Sp}(V_E)$. Then

$$\begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} \begin{bmatrix} (Q^*)^{-1} & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} A(Q^*)^{-1} & B_1 Q = B_2 \\ C(Q^*)^{-1} & D_1 Q = D_2 \end{bmatrix}.$$

Note that the right-hand side is in $\mathrm{Sp}(V_E)$, since the left-hand factors are. Also the second factor on the left-hand side is in $\mathbf{P}$, and (1) follows.

(1) $\iff$ (2) The implication from the right to the left is trivial. Assume that (1) holds. It suffices to show that if there is another matrix $M = \begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} \in \mathrm{Sp}(V_E)$, then $M_1\mathbf{P} = M\mathbf{P}$. Since M_1 and M have the same second column, when left multiplied by ${M_1}^{-1}$, the second columns must remain equal, i.e., $\hat{M} := {M_1}^{-1}M$ must be of the form $\begin{bmatrix} * & 0 \\ * & I \end{bmatrix}$. Since $\hat{M} \in \mathrm{Sp}(V_E)$, it follows that $\hat{M} \in \mathbf{P}$, since it is block lower triangular. Thus, $M_1\mathbf{P} = M\mathbf{P}$. □

Definition 4.2. We set $\begin{bmatrix} B_1 \\ D_1 \end{bmatrix} \sim \begin{bmatrix} B_2 \\ D_2 \end{bmatrix}$ if the equivalent conditions of Proposition 4.1 hold. The relation $\sim$ is an equivalence relation (from part (2) or (3)) and the quotient space $\Lambda/\sim$ is denoted by $\mathcal{M}$. We denote the equivalence class of $\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}$ by $\begin{pmatrix} B_1 \\ D_1 \end{pmatrix}$.

There exists a natural projection $\pi : \mathrm{Sp}(V_E) \to \Lambda$ which sends a matrix to its second column. Let $\rho : \Lambda \to \mathcal{M}$ be the natural projection from Λ to $\mathcal{M}$ which sends a column to its $\sim$-equivalence class. We endow $\mathcal{M}$ with the quotient topology from $\rho \circ \pi$.

Corollary 4.3. *Consider $\psi := \rho \circ \pi : \mathrm{Sp}(V_E) \to \mathcal{M}$. Then $\psi(M_1) = \psi(M_2)$ if and only if $M_1\mathbf{P} = M_2\mathbf{P}$. Thus the left transformation group $(\mathrm{Sp}(V_E), \mathcal{M})$, where the action is given by the left block matrix multiplication by any representative of a $\sim$-equivalence class, is topologically conjugate to the coset transformation group $(\mathrm{Sp}(V_E), \mathrm{Sp}(V_E)/\mathbf{P})$ and the mapping ψ is open.*

Proof. The first assertion follows readily from Proposition 4.1, and the second assertion follows readily from the first. It is standard that the quotient mapping onto a homogeneous space is an open mapping. □

This corollary allows us to identify $\mathcal{M}$ and the homogeneous space $\mathrm{Sp}(V_E)/\mathbf{P}$. In the finite dimensional setting $\mathbf{P}$ is a parabolic subgroup and the homogeneous space is a flag manifold of $\mathrm{Sp}(V_E)$.

We say that a point $\begin{pmatrix} B \\ D \end{pmatrix} \in \mathcal{M}$ is *finite* if D is invertible (note from item (3) of Proposition 4.1 that this invertibility is independent of which representative is chosen). In this case, we may rewrite the point as $\begin{pmatrix} BD^{-1} \\ I \end{pmatrix}$. Since for appropriate A and C

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D^* & 0 \\ 0 & D^{-1} \end{bmatrix} = \begin{bmatrix} * & BD^{-1} \\ * & I \end{bmatrix} \in \mathrm{Sp}(V_E),$$

we conclude that $(BD^{-1})^* I$ is symmetric, and hence BD^{-1} is symmetric. Conversely, if $E \in \mathrm{End}(E)$ is symmetric, then $\begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \in \mathrm{Sp}(V_E)$ and, therefore, $\begin{pmatrix} E \\ I \end{pmatrix} \in \mathcal{M}$.

Proposition 4.4. *The correspondence* $A \leftrightarrow \begin{pmatrix} A \\ I \end{pmatrix}$ *is a homeomorphism between the set* $\mathrm{Sym}(E)$ *of symmetric operators in* $\mathrm{End}(E)$ *and the open set* $\mathcal{M}_0$ *of finite points in* $\mathcal{M}$.

Proof. Since by item (3) of Proposition 4.1 we can represent each member of $\mathcal{M}$ in at most one way with bottom entry I, we have from the preceding discussion that the correspondence is a bijection.

Since the operator norm topology for the block operator matrices agrees with the product topology from the operator norm topologies in each block, we conclude that the mapping

$$\beta : \mathrm{Sym}(E) \to \mathcal{M} \ \text{ defined by } \ A \mapsto \begin{bmatrix} A \\ I \end{bmatrix} \mapsto \begin{pmatrix} A \\ I \end{pmatrix}$$

is continuous.

Conversely, consider the open subset $U \subseteq \mathrm{Sp}(V_E)$ of elements such that the $(2,2)$-block entry D is invertible. The open set U is the inverse image of the set of finite points of $\mathcal{M}$. Thus, the set $\mathcal{M}_0$ of finite points is open in $\mathcal{M}$, since the quotient mapping is open. The mapping

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto BD^{-1} : U \to \mathrm{Sym}(E)$$

is continuous and induces $\begin{pmatrix} B \\ D \end{pmatrix} \mapsto BD^{-1}$ on the quotient space $\mathcal{M}$ and, therefore, the latter mapping is also continuous (note that the mappings do go into $\mathrm{Sym}(E)$ by the paragraph preceding the proposition). Thus, the correspondence is a homeomorphism. $\square$

Remark 4.5. Our presentation of the manifold $\mathcal{M}$ is nonstandard. Typically, at least in the finite-dimensional setting, one considers maximal isotropic subspaces of V_E, sometimes called Lagrangian subspaces or polarizations. The "horizontal" subspace $E_H = E \oplus \{0\}$ and the "vertical" subspace $E_V = \{0\} \oplus E$ are examples of them. If in our context we define a polarization of V_E to be an image of E_V under a member of $\mathrm{Sp}(V_E)$, then we can identify the members of $\mathcal{M}$ with the polarizations of V_E via the correspondence

$$\begin{pmatrix} B \\ D \end{pmatrix} \leftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} E_V, \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E).$$

Another way of saying this is that we associate with $\begin{pmatrix} B \\ D \end{pmatrix}$ the column space of $\begin{bmatrix} B \\ D \end{bmatrix}$.

5. Extended solutions of Riccati equations

The results of the previous section allow us to extend the solution of a Ricatti equation by considering it to be a differential equation on a larger $\mathcal{M}$ with $\mathrm{Sym}(E)$ embedded as the set of finite points as outlined in the previous section.

Consider on E the Riccati equation

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \quad K(t_0) = K_0, \quad \text{(R)}$$

where t varies over some interval $\mathbb{I}$ containing t_0. (We recall our standard assumption that coefficient functions are locally bounded and measurable in the finite dimensional case and regulated otherwise.) As we have seen in Sec. 3, we can obtain a solution of the Riccati equation from the solution of the basic group control equation

$$\dot{g}(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A^*(t) \end{bmatrix} \begin{bmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{bmatrix}, \quad g(t_0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}$$

by setting $K(t) = g_{12}(t)(g_{22}(t))^{-1}$ on any interval containing t_0, where $g_{22}(t)$ is invertible.

Note that, by the uniqueness of solutions we have $g(t) = k(t)$, where we define

$$k(t) := \Phi(t)(\Phi(t_0))^{-1} \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix},$$

since both satisfy (BGCE) and have the same initial point at t_0.

Suppose that on the interval $\mathbb{I}$ we define a function $\tilde{K}(\cdot)$ on $\mathcal{M}$ by

$$\begin{aligned} \tilde{K}(t) = g(t) \begin{pmatrix} 0 \\ I \end{pmatrix} &= k(t) \begin{pmatrix} 0 \\ I \end{pmatrix} \\ &= \Phi(t)(\Phi(t_0))^{-1} \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = \Phi(t)(\Phi(t_0))^{-1} \begin{pmatrix} K_0 \\ I \end{pmatrix}. \quad \text{(ES)} \end{aligned}$$

We observe that on any interval where $g_{22}(t)$ is invertible

$$\tilde{K}(t) = \begin{pmatrix} g_{12}(t) \\ g_{22}(t) \end{pmatrix} = \begin{pmatrix} g_{12}(t)(g_{22}(t))^{-1} \\ I \end{pmatrix} = \begin{pmatrix} K(t) \\ I \end{pmatrix},$$

where the last equality follows from Lemma 3.2. The last expression also agrees with the embedded image in $\mathcal{M}$ of the solution $K(t)$. The function $\tilde{K}(\cdot)$ on $\mathbb{I}$ is called the *extended solution of the Riccati equation*.

Consider the maximal interval around t_0 for which $g_{22}(t)$ is invertible. This interval is open since g_{22} is continuous and $\mathcal{M}_0$ is open in $\mathcal{M}$. By the uniqueness of solutions, any solution $K_1(\cdot)$ of the Riccati equation (R) and $\tilde{K}(\cdot)$ must agree on this interval (we consider $\mathrm{Sym}(E)$ as embedded in $\mathcal{M}$). If $K_1(\cdot)$ admits a solution at the endpoint t_1, then by the continuity $K_1(t_1) = \tilde{K}(t_1)$, which is impossible since one is a finite point and the other is not. Thus, we have established the following result.

Proposition 5.1. *The Riccati equation* (R) *admits an extended solution throughout the interval* $\mathbb{I}$ *on which it is defined. The maximal interval on which* (R) *admits a solution is the largest interval containing* t_0 *such that the extended solution is finite.*

The basic group control equation (BGCE) pushes forward to a control system on the manifold $\mathcal{M}$ so that the restriction to $\mathrm{Sym}(E)$, the space of finite points, agrees with the Riccati equation. We briefly describe this "push-forward" construction. Let M be a smooth (of class C^∞) manifold. Assume that $\Psi : G \times M \to M$ is a smooth action of a Lie group G on M. In our case $G = \mathrm{Sp}(V_E)$ and $M = \mathcal{M}$ endowed with the appropriate smooth structure to make the action of $\mathrm{Sp}(V_E)$ on $\mathcal{M}$ smooth. (This smooth structure arises by taking the inverse of the embedding of $\mathrm{Sym}(E)$ into $\mathcal{M}$ and its translates by members of $\mathrm{Sp}(V_E)$ acting on $\mathcal{M}$ as an atlas of charts.) We typically denote $\Psi(g, x)$ by gx or $g \cdot x$. Let $V^\infty(M)$ denote the Lie algebra of smooth vector fields on M. For $x \in M$, the smooth mapping $\Psi_x : G \to M$ given by $\Psi_x(g) = g \cdot x$ has derivative at e, $d\Psi_x : T_eG \to T_xM$; alternatively, $d\Psi_x(v)$, $v \in T_eG$, is given by $v \mapsto \dot{\alpha}(0)$, where $\alpha : \mathbb{R} \to M$ is defined by $\alpha(t) = \exp(tX) \cdot x$, where $\exp : \mathfrak{g} \to G$ is the exponential mapping and $X(e) = v$. The mappings $d\Psi_x$ give rise to a Lie algebra homomorphism $d\Psi : \mathfrak{g} \to V^\infty(M)$ given by $d\Psi(X)(x) = d\Psi_x(X(e))$ (note that the appropriate match-up to obtain a Lie algebra homomorphism is right invariant vector fields with left actions). We denote the vector field $d\Psi(X)$ by $\vec{X}$. We consider the *basic manifold control equation* on M given by the control differential equation

$$\dot{x}(t) = \vec{u}(t)(x(t)), \tag{BMCE}$$

where $u(\cdot) : \mathbb{I} \to \mathfrak{g}$ is locally bounded and $\vec{u}(t) = d\Psi(u(t))$.

Proposition 5.2. *The solution of* (BMCE)

$$\dot{x}(t) = \vec{u}(t)(x(t)), \quad x(t_0) = x_0$$

on M is given by $x(t) = \Phi(t)(\Phi(t_0))^{-1} \cdot x_0$. The basic control differential equation on M has a global solution for any initial value.

Proof. The first assertion follows from

$$\begin{aligned}\dot{x}(t) &= d\Psi_{x_0}\dot{\Phi}(t)(\Phi(t_0))^{-1} = d\Psi_{x_0}\left(u(t)\Phi(t)(\Phi(t_0))^{-1}\right)\\ &= d\Psi_{x_0} \circ d\rho_{\Phi(t)(\Phi(t_0))^{-1}}(u(t)e)\\ &= d\Psi_{\Phi(t)(\Phi(t_0))^{-1}.x_0}(u(t)e) = \vec{u}(x(t)),\end{aligned}$$

where $\rho_g(h) = hg$ is a right translation in G. The existence of global solutions now follows from the corresponding assertion for (BGCE). The last assertion follows readily from the first. ☐

Remark 5.3. In the case $G = \mathrm{Sp}(V_E)$ and $M = \mathcal{M}$ under consideration, we note from equation (ES) above that

$$\tilde{K}(t) = \Phi(t)(\Phi(t_0))^{-1}\begin{pmatrix} K_0 \\ I \end{pmatrix},$$

which is the solution of (BMCE) for the initial condition $\begin{pmatrix} K_0 \\ I \end{pmatrix}$ at the time t_0. Hence the extended solution of the Riccati equation is the solution of (BGCE) pushed forward to (BMCE).

6. The symplectic semigroup

Let (V_E, Q) be a standard symplectic space constructed from a real Hilbert space H. A bounded symmetric operator A on E is *positive semidefinite* if $\langle x, Ax\rangle \geq 0$ for all $x \in E \setminus \{0\}$. We denote by $\mathcal{P}$ (respectively, $\mathcal{P}_0$) all positive semidefinite (respectively, positive semidefinite invertible) bounded operators on E. We use the standard fact from the operator theory that a positive semidefinite operator has a unique positive semidefinite square root.

Lemma 6.1. *If $P, Q \in \mathcal{P}$, then $I + PQ$ is invertible. If $P \in \mathcal{P}_0$ and $Q \in \mathcal{P}$, then $P + Q \in \mathcal{P}_0$.*

Proof. We first show that $I + PQ$ is injective. If $(I + PQ)(x) = 0$, then

$$0 = \langle Q(x), (I + PQ)(x)\rangle = \langle Q(x), x\rangle + \langle Q(x), PQ(x)\rangle.$$

Since both latter terms are nonnegative by the hypothesis, we have that

$$0 = \langle Qx, x\rangle = \langle Q^{1/2}x, Q^{1/2}x\rangle,$$

and, therefore, $Q^{1/2}(x) = 0$. It follows that

$$0 = (I + PQ)(x) = x + PQ^{1/2}(Q^{1/2}x) = x,$$

and, therefore, $I + PQ$ is injective.

The same argument can be applied to the adjoint $I + QP$ to conclude that it is also injective, and hence its adjoint $I + PQ$ has a dense image.

Suppose that $(I + PQ)(x_n) \to 0$. We claim that $x_n \to 0$. If not, then we obtain a subsequence, again denoted by x_n, such that x_n is bounded away from 0, i.e., there exists $\beta > 0$ such that $\beta < \|x_n\|$ for all n. Then $u_n := x_n/\|x_n\| = (\epsilon_n/\beta)x_n$ for some $0 < \epsilon_n < 1$. Since $(1/\beta)(I + PQ)(x_n) \to 0$, it follows that $(I + PQ)(u_n) \to 0$. Since $\|Q(u_n)\| \le \|Q\|$ for all n, we have

$$\langle Q(u_n), u_n\rangle + \langle Q(u_n), PQ(u_n)\rangle = \langle Q(u_n), (I + PQ)(u_n)\rangle \to 0.$$

Since both terms on the left-hand side are nonnegative, we have

$$\langle Q^{1/2}(u_n), Q^{1/2}(u_n)\rangle = \langle Q(u_n), u_n\rangle \to 0,$$

and, therefore, $Q^{1/2}(u_n) \to 0$. By the continuity of $PQ^{1/2}$, it follows that $PQ(u_n) \to 0$ and hence $\|(I + PQ)(u_n)\| \to 1$, a contradiction with $(I + PQ)(u_n) \to 0$.

We conclude by showing that $I+PQ$ is surjective. Let $y \in E$. Then there exists $x_n \in E$ such that $y_n := (I + PQ)(x_n) \to y$, since $I + PQ$ has a dense image. Then the double indexed sequence $y_n - y_m = (I + PQ)(x_n - x_m)$ tends to 0 as $m, n \to \infty$. Then it follows from the previous paragraph that the double indexed sequence $x_n - x_m$ tends to 0 as $m, n \to \infty$, i.e., the sequence $\{x_n\}$ is a Cauchy sequence. Let x be its limit. By the continuity, $(I + PQ)(x) = y$. Thus, $I + PQ$ is surjective. By the Banach open-mapping theorem, it is open and, therefore, the inverse is a bounded linear operator. The last assertion follows from the relation $P + Q = P(I + P^{-1}Q)$. □

Remark 6.2. Note that the proof is considerably simpler in the finite-dimensional case. Indeed, it follows from the injectivity of $I + PQ$ that it is invertible.

We define the following four subsets:

$$\mathcal{S} = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible, } B^*D \in \mathcal{P},\ CD^* \in \mathcal{P} \right\},$$
$$\mathcal{S}_1 = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible, } B^*D \in \mathcal{P}_0,\ CD^* \in \mathcal{P} \right\},$$
$$\mathcal{S}_2 = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : D \text{ is invertible, } B^*D \in \mathcal{P},\ CD^* \in \mathcal{P}_0 \right\},$$
$$\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2.$$

Remark 6.3. Note that $\mathcal{S}_2$ is the adjoint dual of $\mathcal{S}_1$ and that $\mathcal{S}$ is self-dual, i.e., $\mathcal{S}$ is closed under adjoints.

Members of $\mathcal{S}$ are sometimes called *Hamiltonian* operators of $\mathrm{Sp}(V_E)$.

We define

$$\Gamma^U = \left\{ \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} : B \in \mathcal{P} \right\}, \quad \Gamma_0^U = \left\{ \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} : B \in \mathcal{P}_0 \right\},$$
$$\Gamma^L = \left\{ \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} : C \in \mathcal{P} \right\}, \quad \Gamma_0^L = \left\{ \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} : C \in \mathcal{P}_0 \right\}.$$

Further, we define a group H of block-diagonal matrices by

$$H = \left\{ \begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix} : A \in \mathrm{GL}(E) \right\}.$$

The following lemma is straightforward.

Lemma 6.4. *All four sets Γ^U, Γ^L, Γ_0^U, and Γ_0^L are semigroups under composition, the first two are closed, and Γ_0^U (respectively, Γ_0^L) is a semigroup ideal in Γ^U (respectively, Γ^L). The semigroup Γ^U (respectively, Γ_0^U) consists of all unipotent block upper triangular operators contained in $\mathcal{S}$ (respectively, $\mathcal{S}_1$). Similarly, the semigroup Γ^L (respectively, Γ_0^L) consists of all unipotent block lower triangular operators contained in $\mathcal{S}$ (respectively, $\mathcal{S}_2$). The group H is closed in $\mathrm{GL}(V_E)$ and consists of all block diagonal matrices in $\mathrm{Sp}(V_E)$. Furthermore, each of the four semigroups Γ^U, Γ^L, Γ_0^U, and Γ_0^L is invariant under conjugation by members of H.*

Lemma 6.5. *We have that $\mathcal{S} = \Gamma^U H \Gamma^L$, $\mathcal{S}_1 = \Gamma_0^U H \Gamma^L$, $\mathcal{S}_2 = \Gamma^U H \Gamma_0^L$, and $\mathcal{S}_0 = \Gamma_0^U H \Gamma_0^L$. Furthermore, these "triple decompositions" are unique. The multiplication mapping from $\Gamma^U \times H \times \Gamma^L$ to $\mathcal{S}$ is a homeomorphism.*

Proof. Each member of $\mathcal{S}$ admits a triple decomposition of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (D^{-1})^* & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}. \tag{6.1}$$

The triple decomposition is verified by a direct multiplication (applying the equations $A^*D - C^*D = I$ and $B^*D = D^*B$ to see that the $(1,1)$-entry is A). Further, we note that if $B^*D = D^*B \in \mathcal{P}$ (respectively, $\mathcal{P}_0$), then $BD^{-1} = (D^{-1})^* D^* B D^{-1} \in \mathcal{P}$ (respectively, $\mathcal{P}_0$), and hence the first factor in the triple decomposition belongs to Γ^U (respectively, Γ_0^U). Similar reasoning applies to the third factor after noting $D^{-1}C = D^{-1}CD^*(D^{-1})^*$.

Conversely, consider a product

$$\begin{bmatrix} D^{-1} + BD^*C & BD^* \\ D^*C & D^* \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} D^{-1} & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \in \Gamma^U H \Gamma^L.$$

Then the $(2,2)$-entry in the product is precisely D^* and the middle block diagonal matrix in the factorization is determined. Multiplying the $(1,2)$-entry of the product on the right by $(D^*)^{-1}$ gives B and the $(2,1)$-entry on the left by $(D^*)^{-1}$ gives C. Hence the triple factorization is uniquely defined. Finally, note that $(BD^*)^*D^* = DB^*D^*$ is positive semidefinite since B is positive semidefinite (since the first block matrix belongs to Γ^U).

Also $(D^*C)(D^*)^* = D^*CD$, which is positive semidefinite since C is positive semidefinite. Thus, the product block matrix belongs to $\mathcal{S}$. Further, note that DB^*D^* (respectively, D^*CD) is invertible if B (respectively, C) is invertible and, therefore, the decomposition holds also in $\mathcal{S}_i$, $i = 0, 1, 2$.

In regard to the last statement, we have seen that the mapping is a bijection, it is continuous since multiplication (i.e., composition) is bijective, and from (6.1) we see that the inverse factorization is also continuous on $\mathcal{S}$. $\square$

Related triple decompositions in the finite dimensional setting have been obtained by Wojtkowski [16] for the real symplectic group, by Koufany [7] in the setting of euclidean Jordan algebras, and by the authors in the setting of Lie algebras of Cayley type [9].

Remark 6.6. Similar triple decompositions occur for a larger set of symplectic block matrices for which the $(2, 2)$-block D is invertible. For this set of matrices, one has unique triple decompositions in the set product $N^U HN^L$, where N^U (respectively, N^L) denotes the group of upper (respectively, lower) block unipotent matrix operators. The preceding proof adapts directly to this case.

The following semigroup property appears in the finite-dimensional setting in [1, 16, 17].

Theorem 6.7. *We have that $\mathcal{S}$ is a semigroup. Furthermore, $\mathcal{S}\mathcal{S}_i\mathcal{S} \subseteq \mathcal{S}_i$ for $i = 0, 1, 2$, i.e., each $\mathcal{S}_i$ is a semigroup ideal.*

Proof. Let $s_1 = u_1h_1l_1$ and $s_2 = u_2h_2l_2$ be the triple decompositions for $s_1, s_2 \in \mathcal{S}$. Suppose that $l_1u_2 = u_3h_3l_3 \in \Gamma^U H\Gamma^L$. Then Lemma 6.4 implies that

$$\begin{aligned} s_1s_2 &= u_1h_1l_1u_2h_2l_2 = u_1h_1u_3h_3l_3h_2l_2 \\ &= [u_1(h_1u_3h_1^{-1})](h_1h_3h_2)[(h_2^{-1}l_3h_2)l_2] \end{aligned}$$

belongs to $\Gamma^U H\Gamma^L$. We observe that, indeed,

$$l_1u_2 = \begin{bmatrix} I & 0 \\ C_1 & I \end{bmatrix} \begin{bmatrix} I & B_2 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & B_2 \\ C_1 & I + C_1B_2 \end{bmatrix},$$

and that the $(2, 2)$-entry is invertible by Lemma 6.1. Further, we have that

$$B_2^*(I + C_1B_2) = B_2^* + B_2^*C_1B_2$$

is positive semidefinite (and is in $\mathcal{P}_0$ if $B_2 \in \mathcal{P}_0$ by Lemma 6.1) and

$$C_1(I + C_1B_2)^* = C_1 + C_1B_2C_1^*$$

is positive semidefinite since C_1 and B_2 are positive semidefinite (and is in $\mathcal{P}_0$ if C_1 is in $\mathcal{P}_0$). Thus, l_1u_2 has the desired triple decomposition $u_3h_3l_3$ and $\mathcal{S}$ is a semigroup by Lemma 6.5. The assertion that the $\mathcal{S}_i$ are ideals now follows readily from Lemmas 6.4 and 6.5 in a similar fashion. $\square$

Definition 6.8. The semigroup $\mathcal{S}$ from Theorem 6.7 is called the *symplectic semigroup.*

Corollary 6.9. *The symplectic semigroup can be alternatively characterized as*

$$\mathcal{S} = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E) : A \text{ is invertible, } C^*A \in \mathcal{P},\ BA^* \in \mathcal{P} \right\}.$$

Proof. Let $\mathcal{S}'$ denote the set defined on the right-hand side of the equation in the statement of the corollary. We observe that

$$\Delta \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Delta = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \quad \text{for} \quad \Delta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

The inner automorphism $M \mapsto \Delta M \Delta : \mathrm{GL}(V_E) \to \mathrm{GL}(V_E)$ carries $\mathrm{Sp}(V_E)$ onto itself (verify, e.g., that it preserves condition (3) of Proposition 2.5), interchanges the semigroups Γ^U and Γ^L, carries the group H to itself, and interchanges the semigroup $\mathcal{S}$ and the set $\mathcal{S}'$. Therefore, $\mathcal{S}'$ is a semigroup and

$$\mathcal{S}' = \Gamma^L H \Gamma^U \subseteq \mathcal{S}\mathcal{S}\mathcal{S} = \mathcal{S}.$$

Dually, $\mathcal{S} \subseteq \mathcal{S}'$. □

7. Fractional transformations

If $M \in \mathrm{Sp}(V_E)$ and $x, Mx \in \mathcal{M}_0$, the set of finite points, then

$$Mx = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} X \\ I \end{pmatrix} = \begin{pmatrix} AX + B \\ CX + D \end{pmatrix} = \begin{pmatrix} (AX+B)(CX+D)^{-1} \\ I \end{pmatrix}.$$

Identifying $X \in \mathrm{Sym}(E)$ with $\begin{pmatrix} X \\ I \end{pmatrix}$, we have that

$$MX = (AX+B)(CX+D)^{-1},$$

as long as MX is finite.

Proposition 7.1. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(V_E)$. *Then identifying finite points of $\mathcal{M}$ with symmetric operators and restricting M to the set of finite points whose image under M is again finite, we have that M acts on this set as the fractional transformation*

$$Z \mapsto (AZ+B)(CZ+D)^{-1}.$$

Members of $\mathcal{S}$ carry the set $\mathcal{P}$ (respectively, $\mathcal{P}_0$) into $\mathcal{P}$ (respectively, $\mathcal{P}_0$) via such fractional transformations.

Proof. We have already observed the first statement. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathcal{S}$ and Z in $\mathcal{P}$ (respectively, $\mathcal{P}_0$), we have the product

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} * & AZ+B \\ * & CZ+D \end{bmatrix}.$$

The product on the right-hand side belongs to $\mathcal{S}$ (respectively, $\mathcal{S}_1$) by Theorem 6.7. Therefore,

$$(AZ+B)^*(CZ+D) = (CZ+D)^*(AZ+B)$$

is in $\mathcal{P}$ (respectively, $\mathcal{P}_0$) by the definition of $\mathcal{S}$ and $\mathcal{S}_1$. Since the matrix product on the right-hand side belongs to $\mathcal{S}$, we have that $CZ+D$ is invertible. Hence

$$(AZ+B)(CZ+D)^{-1} = ((CZ+D)^{-1})^*[(CZ+D)^*(AZ+B)](CZ+D)^{-1}$$

belongs to $\mathcal{P}$ (respectively, $\mathcal{P}_0$). □

Proposition 7.2. *The symplectic semigroup $\mathcal{S}$ is closed in* $\mathrm{Sp}(V_E)$.

Proof. Let $\overline{\mathcal{S}}$ denote the closure of $\mathcal{S}$ in $\mathrm{Sp}(V_E)$. By the continuity of the multiplication, $\overline{\mathcal{S}}$ is again a subsemigroup. For $M \in \overline{\mathcal{S}}$, let

$$M_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \to M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $M_n \in \mathcal{S}$ for all n. Since $B_n^* D_n \to B^*D$ and the set $\mathcal{P}$ of positive semidefinite operators is closed in $\mathrm{End}(E)$, we conclude that $B^*D \geq 0$. Similarly, $CD^* \geq 0$ and the dual conditions $C^*A \geq 0$ and $BA^* \geq 0$ hold.

It is standard that $\mathcal{P}_0$ is open in $\mathrm{Sym}(E) = \mathcal{M}_0$ (we prove this later in Lemma 9.2) and hence in $\mathcal{M}$, since $\mathcal{M}_0$ is open in $\mathcal{M}$. By the continuity of the action of $\mathrm{Sp}(V_E)$ on $\mathcal{M}$, we conclude for $M \in \overline{\mathcal{S}}$ that $M(\mathcal{P}_0) \subseteq \overline{\mathcal{P}}_0$, the closure being taken in $\mathcal{M}$. Since $M(\mathcal{P}_0)$ is open, we conclude that there exists $P > 0$ such that

$$M\begin{pmatrix} P \\ I \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{pmatrix} P \\ I \end{pmatrix} = \begin{pmatrix} AP+B \\ CP+D \end{pmatrix} \in \mathcal{P}_0.$$

It follows that $CP+D$ is invertible. We have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AP+B \\ C & CP+D \end{bmatrix}.$$

The final product is in $\overline{\mathcal{S}}$ since it is a semigroup. Since the positivity conditions hold for any member of $\overline{\mathcal{S}}$ and since the $(2,2)$-entry is invertible, we conclude that the product is actually in $\mathcal{S}$. But then, by the dual conditions of Corollary 6.9, we conclude that the $(1,1)$-entry A is invertible. If we now apply Corollary 6.9 to M, we conclude that $M \in \mathcal{S}$. □

8. Global Riccati solutions via semigroup theory

Lie's fundamental theorems, which relate the Lie groups and Lie algebras, have been extended to the Lie semigroups and their tangent objects. For a closed subsemigroup S of a Lie group G, we set

$$\mathfrak{L}(S) := \Big\{X \in \mathfrak{g} : \exp(tX) \in S \text{ for all } t \geq 0\Big\}.$$

It follows directly from the Trotter product formula that $\mathfrak{L}(S)$ is a closed convex cone (see [4,5]); it is usually referred to as the *Lie wedge* of S, since it is typically not a pointed cone. The semigroup S is said to be *infinitesimally generated* if it is the closure of the semigroup generated by $\exp(\mathfrak{L}(S))$.

Proposition 8.1. *The symplectic semigroup $\mathcal{S}$ has the Lie wedge*

$$\mathfrak{L}(\mathcal{S}) = \left\{\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : B, C \geq 0\right\}.$$

Proof. We initially set $\mathcal{W}$ equal to the right-hand side of the equation in the statement of the proposition and establish that $\mathcal{W} = \mathfrak{L}(\mathcal{S})$. First, note that any member X of $\mathcal{W}$ can be uniquely written as a sum

$$X = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} = U + D + L$$

of a strictly upper block triangular, a block diagonal, and a strictly lower block triangular matrix. Since $\exp(tU) = \begin{bmatrix} I & tB \\ 0 & I \end{bmatrix} \in \Gamma^U \subseteq \mathcal{S}$ for all $t \geq 0$, we conclude that $U \in \mathfrak{L}(\mathcal{S})$, and, similarly, $L \in \mathfrak{L}(\mathcal{S})$. Clearly, $\exp(tD) \in H \subseteq \mathcal{S}$ for all t and, therefore, $D \in \mathfrak{L}(\mathcal{S})$ also. Since $\mathfrak{L}(\mathcal{S})$ is a cone, it is closed under addition, and we have $X \in \mathfrak{L}(\mathcal{S})$. Therefore, $\mathcal{W} \subseteq \mathfrak{L}(\mathcal{S})$.

Conversely, assume that $\exp(tX) \in \mathcal{S}$ for all $t \geq 0$. Using the triple decompositions of Lemma 6.5, we can write

$$\exp(tX) = U(t)D(t)L(t) \quad \text{for each} \quad t \geq 0.$$

Differentiating both sides with respect to t and evaluating at 0 yields

$$X = \dot{U}(0) + \dot{D}(0) + \dot{L}(0).$$

Then

$$X_{12} = \dot{U}(0)_{12} = \lim_{t \to 0^+} \frac{U(t)_{12}}{t} \geq 0,$$

since by Eq. (6.1) in the proof of Lemma 6.5 and the following sentences $U(t)$ has its $(1,2)$-entry greater than or equal to 0 for $t \geq 0$. Similarly, $X_{21} \geq 0$. □

Members of $\mathfrak{L}(S)$ are often called *Hamiltonian* operators of the symplectic Lie algebra. They are typically the Hamiltonian operators that one considers in the context of continuous systems and differential equations, while the Hamiltonian operators that make up the symplectic semigroup are the Hamiltonian operators that appear in discrete systems. Lie semigroup theory clearly shows the relationship between Hamiltonian operators at the symplectic group level and Hamiltonian operators at the symplectic Lie algebra level.

Assume that we consider the basic group control equation for a general Lie group modified so that the controls come from some nonempty subset $\Omega \subseteq \mathfrak{g}$ with initial condition $g(0) = e$, the identity of the group. The *attainable set* $A(\Omega)$ is the set of points that appear on trajectories of this system for some $t \geq 0$.

Proposition 8.2. *The attainable set $A = A(\Omega)$ is a subsemigroup of G. If $g(\cdot)$ is a trajectory of the system with $[t_1, t_2]$, $t_1 < t_2$ in its domain, then $g(t_2) = sg(t_1)$ for some $s \in A$.*

Proof. Let $u_i(\cdot) : [0, T_i] \to \Omega$ be steering functions for $i = 1, 2$, and let $g_i(\cdot)$, $i = 1, 2$, be the corresponding trajectories. It is elementary to observe that the concatenation steering function $u = u_1 * u_2 : [0, T_1 + T_2] \to \Omega$ has a trajectory given by $g(t) = g_1(t)$ for $0 \leq t \leq T_1$ and $g(t) = g_2(t - T_1)g_1(T_1)$ for $T_1 \leq t \leq T_2$. In particular, $g(T_1 + T_2) = g_2(T_2)g_1(T_1)$ and, therefore, the attainable set is a semigroup.

Let $u(\cdot)$ be a steering function with domain containing $[t_1, t_2]$ and the corresponding trajectory $g(\cdot)$. Define $\gamma(t) = \Phi(t + t_1)(\Phi(t_1))^{-1}$. Define $\tilde{u}(\cdot)$ on $[0, t_2 - t_1]$ by $\tilde{u}(t) = u(t + t_1)$. Then

$$\dot{\gamma}(t) = u(t + t_1)\Phi(t + t_1)(\Phi(t_1))^{-1} = \tilde{u}(t)\gamma(t), \quad \gamma(0) = e.$$

It follows that $\gamma(t_2 - t_1) \in A$. But $\gamma(t_2 - t_1) = \Phi(t_2)(\Phi(t_1))^{-1}$, and hence $\Phi(t_2) = s\Phi(t_1)$ for $s = \gamma(t_2 - t_1)$. □

Restricting this argument to the set of steering functions consisting of piecewise constant mappings, one observes that the reachable set is the semigroup consisting of finite products of members of $\exp(\Omega)$ (see, e.g., [8]). For the case where S is a closed subsemigroup and $\Omega = \mathfrak{L}(S)$, we conclude from the definition of the latter that the attainable set for the class of piecewise constant functions is contained in S. The density of the set of piecewise constant controls and the continuous dependence of solutions on controls then yield that the attainable set is contained in the closed set S.

From these observations and Proposition 7.2 we have the first assertion of the following proposition.

Proposition 8.3. *Each solution $\Phi(t)$ for $t \geq 0$ of the basic group control equation on* $\mathrm{Sp}(V_E)$

$$\dot{g}(t) = u(t)g(t), \quad g(0) = \mathfrak{id}_{V(E)}, \quad u(t) \in \mathfrak{L}(\mathcal{S}),$$

is contained in the semigroup $\mathcal{S}$, i.e., the attainable set is contained in $\mathcal{S}$. If $\Phi(s) \in \mathcal{S}_i$ for some s and some $i = 0, 1$, or 2, then $\Phi(t) \in \mathcal{S}_i$ for all $t > s$.

Proof. It remains to prove only the last assertion. But this follows from the second assertion of Proposition 8.2 and the fact that each $\mathcal{S}_i$ is an ideal of $\mathcal{S}$. □

Remark 8.4. If one considers the basic group control equation

$$\dot{g}(t) = u(t)g(t), \quad g(0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}, \quad u(t) \in \mathfrak{L}(\mathcal{S}), \quad K_0 \in \mathcal{P},$$

then the solution $\Phi(t)g(0)$ evolves in $\mathcal{S}$ for $t \geq 0$ by Proposition 8.3 and the semigroup property of $\mathcal{S}$. Thus, one can form the triple factorization of $\Phi(t)g(0)$ as in Lemma 6.5. We note from the proof of Lemma 6.5 that the $(1,2)$-entry of the upper block-triangular factor is given by $g_{12}(t)(g_{22}(t))^{-1}$, which has initial value K_0 and by Lemma 3.2 satisfies the Riccati equation defined by the steering function $u(\cdot)$. Thus, the function that sends t to the $(1,2)$-block of the upper triangular factor of the triple decomposition yields the solution of the corresponding Riccati equation. Similar remarks apply over any interval, where $g_{22}(t)$ is invertible.

Our results on the symplectic semigroup lead to a semigroup-theoretic proof of the following global existence result concerning the Riccati equation.

Theorem 8.5. *The Riccati equation*

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \quad K(t_0) = K_0,$$

has a solution in $\mathcal{P}$ for all $t \geq t_0$ if $R(t), S(t) \geq 0$ for all $t \geq t_0$ and $K_0 \geq 0$. If, in addition, $K(t_1) \in \mathcal{P}_0$ for some $t_1 \geq t_0$, then $K(t) \in \mathcal{P}_0$ for all $t > t_1$.

Proof. The Riccati equation has an extended solution on $\mathcal{M}$ given by

$$\tilde{K}(t) = \Phi(t)(\Phi(t_0))^{-1} \begin{pmatrix} K_0 \\ I \end{pmatrix}$$

(see Eq. (ES) of Sec. 5). By Proposition 8.2, $\Phi(t)(\Phi(t_0))^{-1} \in \mathcal{S}$ for all $t \geq t_0$. Then it follows from Proposition 7.1 that $\tilde{K}(t) \in \mathcal{P}$ for $t \geq t_0$ and thus is equal to $K(t)$ (see Proposition 5.1).

If $K(t_1) \in \mathcal{P}_0$ for some $t_1 \geq t_0$, then for $t > t_1$,

$$\begin{aligned} K(t) = \tilde{K}(t) &= \Phi(t)(\Phi(t_0))^{-1} \begin{pmatrix} K_0 \\ I \end{pmatrix} \\ &= s\Phi(t_1)(\Phi(t_0))^{-1} \begin{pmatrix} K_0 \\ I \end{pmatrix} = sK(t_1) \end{aligned}$$

for some $s \in \mathcal{S}$ by Propositions 8.3 and 8.2. The conclusion follows from Proposition 7.1. □

The next corollary gives a more general setting in which solutions of the Riccati solution are considered (see, e.g., [15, Chap. 8.2]).

Corollary 8.6. *The Riccati equation*

$$\dot{K}(t) = -R(t) - A(t)K(t) - K(t)A^*(t) + K(t)S(t)K(t), \quad K(t_0) = K_0,$$

has a solution in $\mathcal{P}$ for all $t \leq t_0$ if $R(t), S(t) \geq 0$ for all $t \leq t_0$ and $K_0 \geq 0$. If, in addition, $K(t_1) \in \mathcal{P}_0$ for some $t_1 \leq t_0$, then $K(t) \in \mathcal{P}_0$ for all $t < t_1$.

Proof. The equation

$$\dot{L}(t) = R(t_0 - t) + A(t_0 - t)L(t) + L(t)A(t_0 - t)^* - L(t)S(t_0 - t)L(t)$$

with the initial condition $L(0) = K_0$ has a solution for all $t \geq 0$ by the previous theorem and, therefore, $K(t) := L(t_0 - t)$ is the desired solution of the differential equation given in the corollary. ☐

9. The Loewner order

For $X, Y \in \mathrm{Sym}(E)$, we define

$$X < Y \iff Y - X \in \mathcal{P}_0,$$
$$X \leq Y \iff Y - X \in \mathcal{P}.$$

The order $\leq$ is sometimes called the *Loewner order*. For $X \leq Y$ (respectively, $X < Y$) we define the order intervals

$$[X, Y] = \{Z \in \mathrm{Sym}(E) : X \leq Z \leq Y\},$$
$$(X, Y) = \{Z \in \mathrm{Sym}(E) : X < Z < Y\},$$

respectively.

Lemma 9.1. *If $A \in \mathrm{Sym}(E)$ satisfies $\|A\| < 1$, then $I + A \in \mathcal{P}_0$. Hence $\{I + A : \|A\| < 1\}$ is an open set containing I in $\mathcal{P}_0$.*

Proof. Choose $r \in \mathbb{R}$ so that $\|A\| < r < 1$. Then by the Cauchy–Schwarz inequality

$$-\langle x, Ax\rangle \leq |\langle x, Ax\rangle| \leq \|x\| \, \|Ax\| \leq \|A\| \, \|x\|^2 \leq r\|x\|^2 = \langle x, rI(x)\rangle,$$

and hence $rI + A \geq 0$. Then $I + A = (rI + A) + (1 - r)I \geq 0$ belongs to $\mathcal{P}_0$ by Lemma 6.1. The last assertion now follows immediately. ☐

Lemma 9.2. *The set $\mathcal{P}_0$ is open in $\mathrm{Sym}(E)$, and hence open in $\mathcal{M}$, if we identify the symmetric operators with the finite points of $\mathcal{M}$.*

Proof. For $P \in \mathcal{P}_0$ the matrix

$$\begin{bmatrix} P^{1/2} & 0 \\ 0 & P^{-1/2} \end{bmatrix}$$

belongs to the symplectic semigroup $\mathcal{S}$ and carries I to P and $\mathcal{P}_0$ into $\mathcal{P}_0$ by Proposition 7.1. Thus, it carries the open set around I contained in $\mathcal{P}_0$ (Lemma 9.1) onto an open set around P that is contained in $\mathcal{P}_0$. Since $\mathrm{Sym}(E)$ is identified with the open set $\mathcal{M}_0$ of finite points in $\mathcal{M}$ (Proposition 4.4), the last assertion follows. ☐

Proposition 9.3. *For any $A, B \in \mathrm{Sym}(E)$ with $B < A$,*

(i) *the sets $(-\infty, A) = \{Y \in \mathrm{Sym}(E) : Y < A\}$ and $(B, +\infty) = \{Z \in \mathrm{Sym}(E) : B < Z\}$ are open*;

(ii) *the interval (B, A) is open.*

Proof. We have $(-\infty, A) = A - \mathcal{P}_0$ and $(B, \infty) = B + \mathcal{P}_0$, which are open since $\mathcal{P}_0$ is open. The intersection of these two sets is (B, A). □

Proposition 9.4. *For any $A \in \mathcal{P}_0$, the sets $\{(-(1/n)A, (1/n)A) : n \in \mathbb{N}\}$ form a basis of open sets at 0 in $\mathrm{Sym}(E)$.*

Proof. The sets $(-(1/n)A, (1/n)A)$ are open by Proposition 9.3. Suppose that $-(1/n)A < X < (1/n)A$ belongs to $\mathrm{Sym}(E)$. Then there exist $P, Q \in \mathcal{P}_0$ such that $X = P + (-(1/n)A)$ and $X + Q = (1/n)A$. Eliminating X, we obtain $P + Q = (2/n)A$ and, therefore, $P, Q < (2/n)A$. We have

$$\|P^{1/2}x\|^2 = \langle x, Px\rangle \le \langle x, (2/n)Ax\rangle \le (2/n)\|A\|\,\|x\|^2.$$

It follows that

$$\|P^{1/2}\| \le (\sqrt{2}/\sqrt{n})\|A\|^{1/2}$$

and, therefore,

$$\|P\| \le \|P^{1/2}\|^2 \le (2/n)\|A\|.$$

We conclude that

$$\|X\| \le \|P\| + (1/n)\|A\| \le (3/n)\|A\|.$$

Thus, the set $(-(1/n)A, (1/n)A)$ is contained in the open ball around 0 of radius $(3/n)\|A\|$. □

Proposition 9.5. *The closure of $\mathcal{P}_0$ in $\mathrm{Sym}(E)$ is $\mathcal{P}$.*

Proof. The fact that $\mathcal{P}$ is closed in $\mathrm{Sym}(E)$ follows immediately from its definition. Since for $A \in \mathcal{P}$, $A = \lim_{n\to\infty} A + (1/n)I$ and the members of the sequence are in $\mathcal{P}_0$ by Lemma 6.1, the proposition follows. □

A partial order $\le$ on a topological space X is *closed* if $\le = \{(x, y) : x \le y\}$ is closed in $X \times X$.

Proposition 9.6. *The Loewner order $\le$ is closed on $\mathrm{Sym}(E)$. Each order interval $[A, B] = \{X \in \mathrm{Sym}(E) : A \le X \le B\}$ for $A \le B$ is closed in $\mathcal{M}$ (where, as usual, we identify $\mathrm{Sym}(E)$ with the finite points $\mathcal{M}_0$ of $\mathcal{M}$). The interior of $[A, B]$ is equal to (A, B).*

Proof. We observe that

$$\{(X, Y) : X \le Y\} = \{(X, Y) : Y - X \in \mathcal{P}\},$$

and the latter set is closed since $\mathcal{P}$ is closed (by the previous proposition).

Consider the open set $(-I, I)$ around 0. Since $\mathcal{M}$ is regular (the fact that coset spaces are regular is a standard and elementary result in the theory of topological groups), there exists an open set U containing 0 such that $\overline{U} \subseteq (-I, I)$, where the closure is taken in $\mathcal{M}$. For any $A \in \mathcal{P}$, choose $n > 0$ so that $(-(1/n)A, (1/n)A) \subseteq U$; this is possible by Proposition 9.4. Then $[0, (1/2n)A] = \mathcal{P} \cap ((1/2n)A - \mathcal{P})$ is closed in $\mathrm{Sym}(E)$, is contained in $(-(1/n)A, (1/n)A)$ and hence in U and, therefore, is closed in $\overline{U}$ and in $\mathcal{M}$. The diagonal operator in $\mathrm{Sp}(V_E)$ with entries $\sqrt{2n}I$ and $(1/\sqrt{2n})I$ carries the closed interval $[0, (1/2n)A]$ onto $[0, A]$ and, therefore, the latter is also closed in $\mathcal{M}$. Since any closed interval $[B, A]$ is the image of $[0, A - B]$ under the operator with block matrix entries $\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$, we conclude they all are closed.

Consider a closed interval $[A, B]$ for $A \leq B$. Since (A, B) is open (Proposition 9.3), it is contained in the interior of $[A, B]$. Conversely, if X is in the interior of $[A, B]$, then there exists some open set U containing 0 such that $X + U \subseteq [A, B]$. By Proposition 9.4, there exists $n \in \mathbb{N}$ such that $(-(1/n)I, (1/n)I) \subseteq U$. Then $A \leq X - (1/n)I < X < X + (1/n)I \leq B$ and, therefore, $A < X < B$. This concludes the proof. □

Proposition 9.7. *For an element $A \in \mathrm{Sym}(E)$, the following assertions are equivalent*:

(1) $A \in \mathcal{P}$;
(2) $A + X$ *is invertible for all* $X \in \mathcal{P}_0$;
(3) $A + rI$ *is invertible for all* $r > 0$.

Proof. Item (2) follows from item (1) by Lemma 6.1 and item (3) is a trivial consequence of item (2). Assume (3) and suppose that $A \notin \mathcal{P}$. Consider the segment $\{tA + (1-t)I : 0 \leq t \leq 1\}$. This connected segment cannot lie entirely in the union of two disjoint open sets $\mathcal{P}_0$ and $\mathrm{Sym}(E) \setminus \mathcal{P}$ and, therefore, must intersect the set $\mathcal{P} \setminus \mathcal{P}_0$, which, by definition, consists of noninvertible elements. Hence $tA + (1-t)I$ is not invertible for some $0 < t < 1$. We conclude that the scalar multiple $A + ((1-t)/t)I$ is not invertible, a contradiction. □

Proposition 9.8. *The inversion on $\mathcal{P}_0$ is order reversing.*

Proof. If $A \in \mathcal{P}_0$ and $I \leq A$, then

$$\begin{aligned} \langle x, A^{-1}x \rangle &= \langle A^{-1/2}x, A^{-1/2}x \rangle \leq \langle A^{-1/2}x, A(A^{-1/2}x) \rangle \\ &= \langle A^{-1/2}x, A^{1/2}x \rangle = \langle x, x \rangle, \end{aligned}$$

and, therefore, $A^{-1} \leq I$. Now

$$X \leq Y \Rightarrow I = X^{-1/2} X X^{-1/2} \leq X^{-1/2} Y X^{-1/2}.$$

Thus inverting, we see that

$$I \geq X^{1/2} Y^{-1} X^{1/2} \Rightarrow X^{-1} = X^{-1/2} I X^{-1/2} \geq Y^{-1}.$$

Hence the inversion on $\mathcal{P}_0$ is order-reversing. $\square$

Proposition 9.9. *The closure $\overline{\mathcal{P}}$ of $\mathcal{P}$ in $\mathcal{M}$ has the interior $\mathcal{P}_0$.*

Proof. Since $\mathcal{P}_0$ is open in $\mathcal{M}$ (Lemma 9.2), it is contained in the interior of $\overline{\mathcal{P}}$. For the converse, we consider the symplectic mappings on $\mathcal{M}$ given by

$$t_I := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}, \quad t_{-I} := \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Note that

$$Jt_I \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} D \\ -(B+I) \end{pmatrix};$$

$$Jt_I \begin{pmatrix} X \\ I \end{pmatrix} = \begin{pmatrix} I \\ -(X+I) \end{pmatrix} = \begin{pmatrix} -(I+X)^{-1} \\ I \end{pmatrix}, \quad X \in \mathcal{P}.$$

Since inversion is order-reversing on $\mathcal{P}_0$ and $X \to -X$ is order-reversing on $\mathrm{Sym}(E)$, we conclude that Jt_I is order preserving on $\mathcal{P}$.

We observe that

$$Jt_I \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} -I \\ I \end{pmatrix}; \quad Jt_I \begin{pmatrix} (n-1)I \\ I \end{pmatrix} = \begin{pmatrix} (-1/n)I \\ I \end{pmatrix}.$$

We conclude that

$$Jt_I(\mathcal{P}) \subseteq \bigcup_n \left\{ X \in \mathrm{Sym}(E) : -I < X < \frac{-1}{n} I \right\},$$

since $\mathcal{P} = \bigcup [0, nI)$. (If $A \in \mathcal{P}_0$, then $(1/n)A \in (-I, I)$ for some n by Proposition 9.4 and, therefore, $A < nI$.) Thus, $\overline{Jt_I(\mathcal{P})} \subseteq \overline{[-I, 0)} \subseteq [-I, 0]$.

Suppose that $\begin{pmatrix} B \\ D \end{pmatrix}$ is in the interior of $\overline{\mathcal{P}}$, the closure is taken in $\mathcal{M}$. Then

$$Jt_I \begin{pmatrix} B \\ D \end{pmatrix} \in \operatorname{int} \overline{Jt_I(\mathcal{P})} \subseteq \operatorname{int}[-I, 0] = (-I, 0),$$

the last equality comes from Proposition 9.6. Hence

$$Jt_I \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} -P \\ I \end{pmatrix}$$

for some $P \in \mathcal{P}$, $0 < P < I$. Since the inverse of Jt_I is given by $t_{-I}(-J)$, we have

$$\begin{pmatrix} B \\ D \end{pmatrix} = t_{-I}(-J) \begin{pmatrix} -P \\ I \end{pmatrix} = t_{-I} \begin{pmatrix} -I \\ -P \end{pmatrix} = t_{-I} \begin{pmatrix} P^{-1} \\ I \end{pmatrix} = \begin{pmatrix} P^{-1} - I \\ I \end{pmatrix}.$$

Since $P < I$ implies $I < P^{-1}$, we conclude that $P^{-1} - I > 0$ and, therefore, $\begin{pmatrix} B \\ D \end{pmatrix} \in \mathcal{P}_0$. $\square$

The next proposition gives another important property of the symplectic semigroups.

Proposition 9.10. *Members of the symplectic semigroup $\mathcal{S}$ satisfy the following monotonicity properties*:

(i) *for $g \in \mathcal{S}$ and $X, Y \in \mathcal{P}_0$, $X \leq Y$ if and only if $g(X) \leq g(Y)$;*
(ii) *for $g \in \mathcal{S}$ and $X, Y \in \mathcal{P}$, $X \leq Y$ implies $g(X) \leq g(Y)$.*

Proof. (i) We verify this for each of the factors in the triple decomposition of Lemma 6.5. This is straightforward for the upper triangular and diagonal factors. Suppose that $X, Y \in \mathcal{P}_0$ and $X \leq Y$. Then for $C \in \mathcal{P}$

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{pmatrix} X \\ I \end{pmatrix} = \begin{pmatrix} X \\ CX + I \end{pmatrix} = \begin{pmatrix} X(CX+I)^{-1} \\ I \end{pmatrix},$$

and, similarly, the image of Y is $Y(CY+I)^{-1} = (C+Y^{-1})^{-1}$. Since, by the previous proposition, inversion is order-reversing on $\mathcal{P}_0$, we conclude that $C + X^{-1} \geq C + Y^{-1}$ and, therefore, $(C + X^{-1})^{-1} \leq (C + Y^{-1})^{-1}$. These steps are reversible. Hence lower triangular matrices in $\mathcal{S}$ also preserve the order on $\mathcal{P}_0$.

(ii) For $X \leq Y$ in $\mathcal{P}$, we have $X + (1/n)I \leq Y + (1/n)I$ for each $n > 0$. By the previous paragraph and Proposition 9.7, $g(X + (1/n)I) \leq g(X + (1/n)Y$ for each n. Since the order relation $\leq$ is closed (Proposition 9.6), we have by taking the limit as $n \to \infty$ that $g(X) \leq g(Y)$. □

10. Order and the Riccati equation

In this section, we briefly consider relationships that exist between the Riccati equation and the Loewner order.

If the Riccati equation

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \quad K(t_0) = K_0,$$

has a solution on the interval $[t_0, t_1]$, then we denote $K(t_1)$ by $\Gamma(t_0, K_0, t_1)$.

Proposition 10.1. *Assume that in the Riccati equation $R(t), S(t) \geq 0$ for all $t \in \mathbb{R}$ and $K_0 \geq 0$. Then for all $t_0 \leq t_1$, $\Gamma(t_0, K_0, t_1)$ exists and is in $\mathcal{P}$. Furthermore, for $K_0 = 0$ we have*

$$\Gamma(t_1, 0, t_2) \leq \Gamma(t_0, 0, t_2) \quad \textit{for all} \quad t_0 < t_1 < t_2.$$

Thus, the mapping $t \mapsto \Gamma(t, 0, t_1) : (-\infty, t_1] \to \mathcal{P}$ is a continuous order-reversing mapping.

Proof. The first assertion follows from Theorem 8.5. By Proposition 5.2 we have that

$$\Gamma(t_0, 0, t_2) = \Phi(t_2)(\Phi(t_0))^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \Gamma(t_1, 0, t_2) = \Phi(t_2)(\Phi(t_1))^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

By Proposition 8.2, there exist $s, s' \in \mathcal{S}$ such that $\Phi(t_2) = s\Phi(t_1)$ and $\Phi(t_1) = s'\Phi(t_0)$. Then

$$\begin{aligned}\Gamma(t_1, 0, t_2) = \Phi(t_2)(\Phi(t_1))^{-1}\begin{pmatrix}0\\I\end{pmatrix} &= s\begin{pmatrix}0\\I\end{pmatrix}\\ &\leq ss'\begin{pmatrix}0\\I\end{pmatrix} = \Phi(t_2)(\Phi(t_0))^{-1}\begin{pmatrix}0\\I\end{pmatrix} = \Gamma(t_0, 0, t_2),\end{aligned}$$

where the inequality follows from the facts that s is order-preserving on $\mathcal{P}$ (Proposition 9.10), $s'(0) \in \mathcal{P}$ (Proposition 7.1) and 0 is the least element in $\mathcal{P}$. Since

$$\Gamma(t, 0, t_1) = \Phi(t_1)(\Phi(t))^{-1}\begin{pmatrix}0\\I\end{pmatrix},$$

we conclude that the mapping $t \mapsto \Gamma(t, 0, t_1)$ is continuous on $(-\infty, t_1)$. ☐

We recall another important connection of the Loewner order with the Riccati equation. The elegant, quick proof is taken from [2], although the theorem appeared earlier in [12].

Proposition 10.2. *Consider the Hamiltonian matrices in the symmetric Lie algebra* $\mathfrak{sp}(V_E)$

$$H(t) = \begin{bmatrix}A(t) & R(t)\\ S(t) & -A^*(t)\end{bmatrix}, \quad \tilde{H}(t) = \begin{bmatrix}\tilde{A}(t) & \tilde{R}(t)\\ \tilde{S}(t) & -\tilde{A}^*(t)\end{bmatrix}$$

and the corresponding Riccati equations

$$\begin{aligned}\dot{K}(t) &= R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t), \quad K(0) = K_0,\\ \dot{\tilde{K}}(t) &= \tilde{R}(t) + \tilde{A}(t)\tilde{K}(t) + \tilde{K}(t)\tilde{A}^*(t) - \tilde{K}(t)\tilde{S}(t)\tilde{K}(t), \quad \tilde{K}(0) = \tilde{K}_0.\end{aligned}$$

Assume that

$$\tilde{H}J \leq HJ,$$

i.e.,

$$\begin{bmatrix}\tilde{R} - R & A - \tilde{A}\\ A^* - \tilde{A}^* & S - \tilde{S}\end{bmatrix} \geq 0$$

and $0 \leq K_0 \leq \tilde{K}_0$. *Then for every* $t \geq 0$, *we have* $K(t) \leq \tilde{K}(t)$.

Proof. Global solutions $K(\cdot)$ and $\tilde{K}(\cdot)$ exist for all $t \geq 0$ by Theorem 8.5. The symmetric operator function $U(t) := \tilde{K}(t) - K(t)$ satisfies the Riccati differential equation

$$\dot{U} = (\tilde{A} - K\tilde{S})U + U(\tilde{A} - K\tilde{S})^* - U\tilde{B}U + \begin{bmatrix}I & -X\end{bmatrix}(HJ - \tilde{H}J)\begin{bmatrix}I\\ -X\end{bmatrix}$$

with a positive-semidefinite initial condition. The result now follows from Theorem 8.5. ☐

11. Applications and future directions

In addition to developing and demonstrating important connections between control theory and Lie semigroup theory, the preceding material is intended as a foundation for future research and development. The authors have already completed a draft of a manuscript [10] in which the contraction property of symplectic Hamiltonian operators acting on the convex cone $\mathcal{P}_0$ of positive definite operators equipped with the natural Finsler metric called Thompson's metric is established. Indeed, a Birkhoff formula for computing the constant of contraction of a Hamiltonian operator is derived in this context. Such contraction formulas have a variety of applications in analysis and in control theory (see, e.g., [1] for applications to Kalman filtering theory). In [10] we exploit the contraction property of sympletic Hamiltonians with explicitly given contraction coefficients to study the convergence of iterative schemes to find the solution of discrete algebraic Riccati equations and rates of convergence of the solution.

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