Understanding Preferences: "Demand Types", and the Existence of Equilibrium with Indivisibilities

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Abstract

An Equivalence Theorem between geometric structures and utility functions allows new methods for understanding preferences. Our classification of valuations into "Demand Types" incorporates existing definitions (substitutes, complements, "strong substitutes", etc.) and permits new ones. Our Unimodularity Theorem generalises previous results about when competitive equilibrium exists for any set of agents whose valuations are all of a "demand type". Contrary to popular belief, equilibrium is guaranteed for more classes of purely-complements, than of purely-substitutes, preferences. Our Intersection Count Theorem checks equilibrium existence for combinations of agents with specific valuations by counting the intersection points of geometric objects. Applications include matching and coalition-formation, and the "Product-Mix Auction" introduced by the Bank of England in response to the financial crisis.

Keywords: consumer theory; equilibrium existence; general equilibrium; competitive equilibrium; duality; indivisible goods; geometry; tropical geometry; convex geometry; auction; product mix auction; product-mix auction; substitute; complement; demand type; matching

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‡This paper extends and supersedes much of the material in Baldwin and Klemperer (2012, 2014). However, we plan to develop large parts of Sections 2.5, 6.2, and 6.3.1 of the 2014 version in future work applying our "demand types" framework and tools to stable multi-agent matching, and to incorporate Sections 4.2, 4.3, and 5 of the 2014 version in future work on "demand types" and comparative statics with indivisibilities. This work was supported by ESRC grant ES/L003058/1. We are grateful for useful discussions with, and much helpful advice from Elie Alhajjar, Sushil Bikhchandani, Vincent Crawford, Vladimir Danilov, Tore Ellingsen, Johannes Horner, Ian Jewitt, Gleb Koshevoy, Elena Kreines, Sara Lamboglia, Christine Lang, Samuel Lings, Diane Maclagan, Margaret Meyer, Paul Milgrom, Kazuo Murota, Timothy O'Connor, Sam Payne, Kevin Roberts, Alexander Teytelboym, Ngoc Mai Tran, Rakesh Vohra, Jorgen Weibull, John Weymark, Zaifu Yang, Josephine Yu, Dylan Zwick, the referees and the coeditor (Dirk Bergemann) of the journal, and many other colleagues.

1 Introduction

This paper introduces a new way to think about preferences for indivisible goods, and obtains new results about the existence of competitive equilibrium.

An Equivalence Theorem. Our starting point is an Equivalence Theorem that a quasilinear utility function corresponds to a geometric structure that satisfies some simple conditions, and vice versa. So we analyse preferences by studying these geometric structures.

"Demand Types." We classify valuations into "demand types". A demand type is defined by a list of vectors that give the possible ways in which the individual or aggregate demand can change in response to a small generic price change. The vectors defining a demand type are thus analogous to the rows of a Slutsky matrix; they specify the possible comparative statics of any valuation of that "type".

For example, a purchaser of spectacles who values spare pairs might always buy lenses and frames in the ratio 2:1, so any increase or reduction in her demand, in response to any price change, would be in this ratio; her valuation is therefore of demand type $\pm\{(2,1)\}$. As another example, you might want to book a large hotel room, or a small room, or neither, but have no interest in both. So your response, if any, to a small change in prices would either be to substitute one room for the other, or to increase or decrease your demand for one of the rooms by 1 without altering your demand for the other. That is, your valuation is of demand type $\pm\{(1,-1),(0,1),(1,0)\}$.[1]

Our classification is parsimonious. For example, the demand type that comprises all possible substitutes preferences is defined by the set of all vectors with at most one positive integer entry, at most one negative integer entry, and all other entries zero; the demand type that is all complements preferences is defined by the set of all vectors in which all the non-zero entries (of which there may be any number) are integers of the same sign; the class of all "strong substitutes" preferences for n goods is a demand type with just $n(n+1)$ vectors.

Our classification clarifies the relationships between different classes of preferences. For example, the demand types descriptions above show clearly why the conditions for indivisible goods to all be (ordinary) substitutes are in general far more restrictive than the conditions for them to all be complements–although they are, of course, symmetric in the two-good case.

Our classification is general. It permits multiple units of each good; the agents can include sellers, buyers, and traders who can both buy and sell; and it can also be applied to matching models.

Importantly, we will see, our classification is also easy to work with.

Equilibrium-Existence Theorems. Our approach yields new theorems about the existence of competitive equilibrium with indivisibilities.

[1]In an auction in which goods' characteristics suggest natural rates of substitution, bidders might be asked to express valuations of the corresponding demand type; e.g., the Bank of England's Product-Mix Auction built one-for-one substitution into its design (see Klemperer, 2008, 2010).

Most previous results *either* (following Kelso and Crawford, 1982) show equilibrium is guaranteed whenever *every agent's individual valuation* has a certain property, *or* alternatively find conditions for equilibrium which must be checked against *every combination of agents' valuations* (as in Bikhchandani and Mamer, 1997, and Ma, 1998).

Our "Unimodularity Theorem" generalises results of the first kind: it states that competitive equilibrium always exists, whatever is the market supply, if and only if all agents' valuations are concave and drawn from a demand type that is defined by a unimodular set of vectors.[2] These assumptions are much weaker than those required by most earlier results, so our theorem immediately implies these earlier results, and extensions of them.

This theorem also easily identifies previously-unknown environments in which the existence of equilibrium is assured. Moreover, it is *not* the case that existence requires substitutes valuations (or a "basis change" thereof). Indeed every demand type for which equilibrium is guaranteed can be obtained as a basis change of a demand type involving only complements preferences (and for which equilibrium is guaranteed)–and the corresponding result is *not* true for substitute preferences.

Our "Intersection Count Theorem", by contrast, is a result of the second kind, although its conditions for equilibrium are very different from the conditions in the previous literature. Our theorem relates the number of price vectors at which more than one agent is indifferent between more than one bundle to whether equilibrium always exists, for given agents' valuations, whatever is the market supply. By contrast, equilibrium tests such as those of Bikhchandani and Mamer (1997) and Ma (1998) require examining every supply of interest separately. (A corollary of our theorem, our "Weak Intersection Count Theorem", can be particularly easy to apply.)

Both our Unimodularity Theorem and our Intersection Count Theorem follow from our most general, but more technical, "Subgroup Indices Theorem".

To illustrate our equilibrium-existence Theorems, recall the hotel-room example above. Like you, Elizabeth is interested in either room (the hotel only has two rooms), but not both. Both your and her valuations are therefore of demand type $\pm\{(1,-1),(0,1),(1,0)\}$, which is unimodular (because any matrix formed by two of these vectors has determinant 0 or ±1). So the Unimodularity Theorem tells us that whatever are your and Elizabeth's valuations (they will generally be different), there always exist competitive equilibrium prices, that is, prices such that demand exactly equals supply, if you and she are the only potential buyers.

Paul, however, requires two hotel rooms for his family; if they cannot have both, they will go elsewhere. So Paul's valuation is of demand type $\pm\{(1,1)\}$. The (smallest) demand type from which Elizabeth's and Paul's valuations are both drawn is therefore $\pm\{(1,1),(1,-1),(0,1),(1,0)\}$, which is not unimodular (because the determinant of $(1,1)$ and $(1,-1)$ is -2). So the Unimodularity Theorem tells us that, if Elizabeth and Paul are the potential buyers, then there are *some* valuation(s) of Elizabeth and Paul for which equilibrium does not exist–but this Theorem does not tell us what those valuation(s) are.

However, our Intersection Count Theorem does tell us whether a competitive equilibrium exists for any specific valuations: a competitive equilibrium exists for Elizabeth

[2] A unimodular set of vectors in n dimensions is one for which every subset of n of them has determinant 0 or ±1 (with an additional condition if they are not a spanning set).

and Paul if and only if *either* (i) there are exactly two price vectors at which both agents are indifferent between more than one bundle, *or* (ii) there exists a price vector at which one agent is indifferent between at least two bundles, and the other is indifferent between at least three (case (ii) is non-generic).

For example, imagine Elizabeth would pay up to £40 for the large room, or £30 for the small. Paul is indifferent between paying £50 for both, and going elsewhere. Then there is only one pair of prices, (£30, £20) for the large and small rooms respectively, such that both agents are indifferent between more than one option. (Paul is indifferent between taking both rooms, and taking neither, while Elizabeth is indifferent between the two rooms so, also, each agent is indifferent between only two bundles.) So the Intersection Count Theorem predicts–and it is not hard to check (and Section 5.1.1 will confirm[3])–that equilibrium will fail: at any prices at which Paul is prepared to take both rooms, Elizabeth will also demand a room.

If, however, Paul is willing to pay up to £100 for the rooms, then there are exactly two price vectors, (£40, £60) and (£70, £30), such that both Elizabeth and Paul are indifferent between more than one option (Paul between taking both rooms and neither, and Elizabeth between taking no room and the one room she considers good value). So the Intersection Count Theorem now predicts that equilibrium does exist. (In fact, any prices that exceed £40 for the large and £30 for the small, and add to not more than £100, clear the market.) Section 5.1.1 gives full details.

Outline of the paper. We work in a transferable-utility exchange economy, with linear and anonymous pricing of multiple indivisible units of each of multiple distinct goods. Our basic tools are convex and "tropical" geometry.[4] We begin, in Section 2, by studying two dual geometric objects:

The first, the "Locus of Indifference Prices" (LIP) comprises the price vectors at which the agent is indifferent between two or more bundles, that is, the price vectors at which the agent's demand changes. Quasilinearity implies that this is a piecewise-linear geometric object and, as we explain, its linear pieces fit together as a "polyhedral complex". Recent mathematical literature implies a Valuation-Complex Equivalence Theorem: any polyhedral complex that satisfies a simple condition corresponds to a quasilinear utility function, and vice versa. This equivalence theorem implies that we can develop our understanding of demand by simply working directly with the LIP and/or its dual.

The LIP's dual, the "demand complex", consists of "cells" defined by the sets of bundles (i.e., quantity vectors) among which the agent is indifferent at some price vector.

Section 3 then defines a "demand type", using a set of vectors. These vectors describe the ways in which the bundles demanded by the agent can change in response to a small generic change in prices. And these vectors relate very simply to our geometric objects: they are the normal vectors to generic points on the LIP; they are also the vectors of the edges of the cells of the demand complex. So we can easily check whether a demand

[3]That section extends agents' domains to include the possibilities of Paul buying just one room (for which he has value zero) and Elizabeth buying both rooms (which she values the same as buying the large room). The results are unaffected.

[4]Tropical geometry was recently developed by, among others, Mikhalkin (2004). We believe the first version of this paper, Baldwin and Klemperer (2012), was the first to apply it to economics. Matveenko (2014), Shiozawa (2015), Crowell and Tran (2016) and Weymark (2016) are other applications.

type is, for example, substitutes, or complements, or "strong substitutes", or "gross substitutes and complements", etc. It is straightforward that an aggregate valuation of multiple valuations of a demand type is of the same demand type.

Section 4 proves our Unimodularity Theorem. Danilov et al. (2001) provide a very similar sufficient condition for equilibrium, but our use of tropical ideas allows a simpler proof. We also show the necessity of the same condition, so that our theorem is a full characterisation of when equilibrium exists. Our concept of demand types also shows how this condition can be applied. For example, equilibrium existence results such as those in Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014), are easy special cases of the Unimodularity Theorem, but none of these papers present their results as specialisations of Danilov et al.'s earlier work, since the latter's relevance was unclear. (Analysing preferences in price space–as well as, like Danilov et al., in quantity space–also allows us to develop additional implications.)

One intuition for the Unimodularity Theorem is that, as described above, the individual demand response to a small generic price change must be in the direction of a vector defining the demand type. So the *aggregate* demand change, due to a small generic price change, must be an integer sum of these vectors. Thus, existence of equilibrium for any supply is guaranteed if it is possible to move between any available bundles using integer combinations of the vectors defining the demand type. It is a standard result that this requires that the vectors are a unimodular set.

An alternative intuition uses the demand complex for the aggregation of the agents. If a bundle is at a vertex of one of the cells of this complex, then it must be demanded on aggregate for some price vector. But a bundle in the "middle" of a cell is not necessarily demanded. That is, equilibrium fails if any of the aggregate-demand complex's cells is "too large". However, if a cell is a parallelepiped, its volume is just the determinant of the vectors of its edges, and we saw above that these vectors are vectors of the demand type. So we show unimodularity of the demand type's vectors ensures that all the cells' volumes are small enough to guarantee that equilibrium exists.

The latter intuition also explains Section 5's Intersection Count Theorem. If any of the aggregate-demand complex's cells are "too large" there must be "too few" of these cells, and therefore, by the duality we show, also "too few" cells in the intersection of the corresponding LIPs. Conversely, a sufficient number of LIP intersections guarantees equilibrium. Moreover, LIPs can be obtained as "tropicalising" transformations–which preserve intersection properties–of ordinary polynomial curves. And the number of LIP intersections that guarantees equilibrium then follows from a version of Bézout's classic theorem that the number of intersections of two polynomial curves, taking into account "multiplicities" such as tangencies, is equal to the product of their degrees.[5]

Section 6 discusses applications: Sections 6.1-6.2 show how to find new demand types for which equilibrium always exists, including purely-complements demand types unrelated to any substitutes preferences (even via any basis change). Section 6.3 uses our Intersection Count Theorem to understand equilibrium of families of combinations of valuations; Appendix A gives an algorithm for determining whether or not equilibrium exists for any given specific combination of valuations.

[5]For example, two lines intersect once (possibly at infinity). A quadratic curve and a line intersect at two points (possibly including points with complex coordinates and points at infinity, and double-counting tangencies). Two quadratic curves intersect four times (correctly counted), etc.

Sections 6.4 shows that our model encompasses classic models, and clarifies the relationships between them.

Sections 6.5-6.7 explain how our techniques apply to other contexts, including matching models, and auctions. The current paper was inspired by diagrams used for the Product-Mix Auctions introduced by the Bank of England during the financial crisis,[6] and our geometric methods have helped us extend these auctions further.

Section 7 concludes. The Appendix contains additional examples, and proofs of all results not proved in the text.

This paper has been written for economists, so we use standard mathematical economics (not "tropical") notation throughout. Some parts of our work (the connection to tropical geometry and the Unimodularity Theorem) have since been re-described for a mathematical audience by Tran and Yu (2017).

2 Representing Indivisible Demand Geometrically

2.1 Assumptions

An agent has a valuation $u : A \to \mathbb{R}$ on a finite set of bundles $\mathbf{x} \in A \subsetneq \mathbb{Z}^n$. That is, the bundles are formed of n distinct goods, which come in indivisible units. Each of these goods is available in multiple units, priced linearly. (We can handle independently-priced units of a good by treating them as different goods.)

Note that a bundle may be negative or mixed-sign: negative coordinates represent units of goods *sold*. So our model allows for sellers with non-trivial supply functions, and more general traders, as well as buyers. Note also that the *domain* A, of bundles that the agent considers possible, can be *any* finite set in $\mathbb{Z}^n$. So the agent can demand multiple units of each good. (Different units of the same good are, of course, indistinguishable to the agent.) Moreover, A need not contain every integer bundle in its convex hull. Nor need A include every bundle that is available in the economy. In particular, if a bundle is completely unacceptable to the agent, it is simply not in A. (This is equivalent to allowing the agent to value some bundles at "$-\infty$", and is more convenient for us.)

The agent has quasilinear utility, so maximises $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$, where $\mathbf{p} \in \mathbb{R}^n$ is the price vector. Valuations need not be positive or weakly increasing, and we allow negative prices, so our model covers "bads" as well as goods.

We will later (from Section 3.3) extend our model to a finite set of agents: agent j will have valuation u^j on integer bundles in a finite domain A^j. We will consider competitive equilibrium among these agents, given an exogenous supply. Thus our framework will encompass the case in which all traders (including all sellers) are explicitly modelled as agents, that is, exchange economies (for which the exogenous supply is $\mathbf{0}$).

The remainder of Section 2 interprets existing mathematics literature in the context of our basic (single-agent) model.

[6] Bidders in these auctions make sets of "either/or" bids for alternative objects. These bids can be represented geometrically as sets of points in multi-dimensional price space. The then-Governor of the Bank of England (Mervyn King) told *the Economist* that the Product-Mix Auction "is a marvellous application of theoretical economics to a practical problem of vital importance"; current-Governor Mark Carney announced plans for its greater use; and an updated version has been introduced–see Bank of England (2010, 2011), Milnes (2010), Fisher (2011), Frost et al (2015) and *the Economist* (2012).

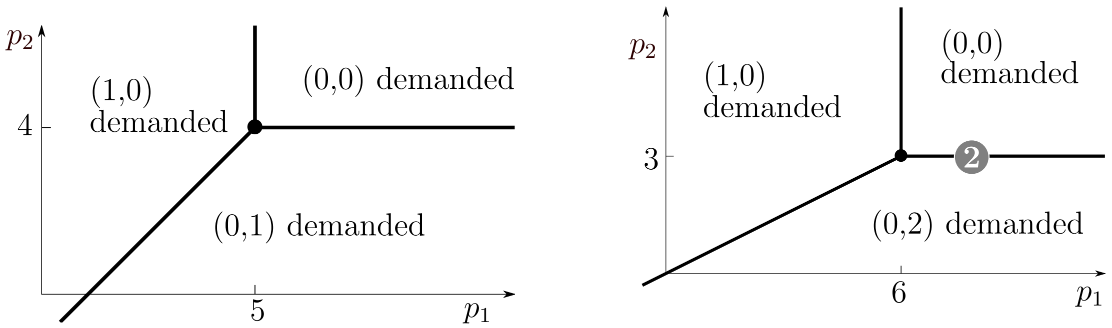
2.2 The Locus of Indifference Prices (LIP)

The prices at which the agent is *indifferent* among more than one bundle are those where her *demand set*, $D_u(\mathbf{p}) := \arg\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}\}$ contains multiple bundles:

Definition 2.1. The *Locus of Indifference Prices (LIP)* is $\mathcal{L}_u := \{\mathbf{p} \in \mathbb{R}^n : |D_u(\mathbf{p})| > 1\}$.

This set is known as a "tropical hypersurface" in the mathematics literature (see Mikhalkin, 2004, and others), but we are introducing new terminology to facilitate understanding among economists.

Because $u(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}$ is quasilinear (and therefore also continuous), the LIP comprises the only prices at which demand can *change* in response to a price change, and is the union of $(n-1)$-dimensional linear pieces, which we will call *facets*. The facets separate the *unique demand regions* (*UDR*s), in each of which some bundle is the unique demand.



(a) $u(0,0) = 0$, $u(1,0) = 5$, $u(0,1) = 4$. (b) $u(0,0) = 0$, $u(1,0) = 6$, $u(0,1) = 1$, $u(0,2) = 6$.

Figure 1: Examples of LIPs, $\mathcal{L}_u$, for two valuations, u. The facets are the line segments; the labelled facet in b has weight 2. The bundle demanded in each UDR is labelled.

Fig. 1a shows a simple example of a LIP. The agent uniquely demands one of the bundles $(0,0)$, $(0,1)$, and $(1,0)$ in the correspondingly-labelled 2-dimensional region, so these regions are the UDRs. The agent demands both the bundles $(0,0)$ and $(0,1)$ on the line segment $\{(p_1, 4) \in \mathbb{R}^2 : p_1 \geq 5\}$; this is a facet, as are the two other line segments shown. If, instead, bundles were formed from $n = 3$ distinct goods, then the facets would be formed of plane-segments, separating 3-dimensional UDRs, and so on in higher dimensions. So, formally:

Definition 2.2.

(1) A *unique demand region (UDR)* of a valuation u is the set of all prices at which a given bundle in A is uniquely demanded. That is, it has the form $\{\mathbf{p} \in \mathbb{R}^n : \{\mathbf{x}\} = D_u(\mathbf{p})\}$ for some $\mathbf{x} \in A$.

(2) A *facet* of $\mathcal{L}_u$ is a subset $F \subseteq \mathcal{L}_u$ such that there exist $\mathbf{x}^1, \mathbf{x}^2 \in A$, with $\mathbf{x}^1 \neq \mathbf{x}^2$, satisfying $F = \{\mathbf{p} \in \mathcal{L}_u : \mathbf{x}^1, \mathbf{x}^2 \in D_u(\mathbf{p})\}$ and such that $\dim F = n - 1$.[7]

[7] We always use the natural dimensions. Thus the dimension of a set $F \subseteq \mathbb{R}^n$ is the dimension of its affine span, i.e. the dimension of the smallest linear subspace $U \subseteq \mathbb{R}^n$ such that $F \subseteq \{\mathbf{c}\} + U$ for some fixed vector $\mathbf{c}$. Here, and throughout the text, we use Minkowski (set-wise) addition on sets.

So the UDRs comprise all the prices which are *not* in the LIP, and for each facet there is a pair of bundles which are demanded at all its prices.

The facets contain important economic information: at any price $\mathbf{p}$ in a given facet, F, the agent is indifferent between the bundles $\mathbf{x}$ and $\mathbf{x}'$ demanded in the UDRs on either side of F. That is, $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = u(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}', \forall \mathbf{p} \in F$. So $\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})$ is constant across all $\mathbf{p} \in F$. Therefore F is normal to the vector that gives the change in demand, $\mathbf{x}' - \mathbf{x}$, between the UDRs on either side of F. In Fig. 1a, again, the facet $\{(p_1, 4) \in \mathbb{R}^2 : p_1 \geq 5\}$ contains prices at which demand can change by $(0,0) - (0,1) = (0,-1)$, which vector is normal to this facet.

So the geometry of the LIP tells us the *directions* of demand changes between pairs of prices. To know *how much* demand changes in any direction that the LIP specifies, we need one more piece of information, the *facet weights*:

Definition 2.3. Let $\mathbf{x}, \mathbf{x}'$ be the bundles demanded in the UDRs on either side of facet F. The *weight* of F, $w_u(F)$, is the greatest common divisor of the entries of $\mathbf{x}' - \mathbf{x}$.

Now $\frac{1}{w_u(F)}(\mathbf{x}' - \mathbf{x})$ is a primitive integer vector, that is, the greatest common divisor of its entries is 1. So, since goods are indivisible, it is the smallest possible change in bundle in the direction $(\mathbf{x}' - \mathbf{x})$. The facet weight, $w_u(F)$, is therefore the number of times this smallest possible change in bundle is made as we cross the facet, F.

Fig. 1b shows a LIP with a facet of weight 2, namely $\{(p_1, 3) : p_1 \geq 6\}$, across which demand changes from $(0,0)$ to $(0,2)$, i.e., twice the smallest change in this direction.

The vector $\frac{1}{w_u(F)}(\mathbf{x}' - \mathbf{x})$ points from the UDR where $\mathbf{x}'$ is demanded, to the UDR where $\mathbf{x}$ is demanded, in the opposite direction to the demand change. But since F is $(n-1)$ dimensional, there is a unique primitive integer vector normal to F and pointing in this direction. So:

Proposition 2.4.

(1) If $\mathbf{x}$, $\mathbf{x}'$ are uniquely demanded on either side of facet F, then $\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})$ is constant for all $\mathbf{p} \in F$.

(2) The change in demand as price changes between the UDRs on either side of F, is $w_u(F)$ times the primitive integer vector that is normal to F, and points in the opposite direction to the change in price.

That is, the LIP and its vector, $\mathbf{w}_u$, of weights, taken together, embody *all* the information about how demand changes between UDRs.

2.2.1 The Price Complex

To develop our full Equivalence Theorem (Thm. 2.14) we need to understand how u defines a "*price complex*" of "*cells*"; these cells generalise facets.

Definition 2.5.

(1) A *price complex cell* of u is a non-empty set $C \subseteq \mathbb{R}^n$ such that there exist $\mathbf{x}^1, \ldots, \mathbf{x}^k \in A$, with $k \geq 1$, satisfying $C = \{\mathbf{p} \in \mathbb{R}^n : \mathbf{x}^1, \ldots, \mathbf{x}^k \in D_u(\mathbf{p})\}$.

(2) The *price complex* is the set of all price complex cells.

(3) The *cells of the LIP* are the price complex cells contained in the LIP.

By continuity, the closure of a UDR is a price complex cell: it comprises all the points at which a particular bundle is demanded. Any other price complex cell must be defined by the demand for two or more bundles, and so is a cell of the LIP.

In Fig. 1a, then, the price complex cells are the closures of the three UDRs; the three facets; and also the point $(5, 4)$, where the agent is indifferent between all the bundles $(0, 0), (1, 0)$, and $(0, 1)$. The 2-dimensional closures of the UDRs meet in the 1-dimensional facets, which meet in the 0-dimensional cell.

All this fits into a standard pattern from convex geometry, so we recall:

Definition 2.6.

(1) A *rational polyhedron* is the intersection of a finite collection of half-spaces $\{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{v} \le \alpha\}$ for some $\mathbf{v} \in \mathbb{Z}^n$ and $\alpha \in \mathbb{R}$ (we do not need to restrict α further).
(2) A *face* of a polyhedron C maximises $\mathbf{p} \cdot \mathbf{v}$ over $\mathbf{p} \in C$, for some fixed $\mathbf{v} \in \mathbb{R}^n$.
(3) The *interior* of polyhedron C is $C^\circ := \{\mathbf{p} \in C : \mathbf{p} \notin C' \text{ for any face } C' \subsetneq C\}$.
(4) A *rational polyhedral complex* Π is a finite collection of *cells* $C \subseteq \mathbb{R}^n$ such that:
   (i) if $C \in \Pi$ then C is a rational polyhedron and any face of C is also in Π;
   (ii) if $C, C' \in \Pi$ then either $C \cap C' = \emptyset$ or $C \cap C'$ is a face of both C and C'.
(5) A *k-cell* is a cell of dimension k. A *facet* is a cell of dimension $n - 1$.
(6) A polyhedral complex is *k-dimensional* if all its cells are contained in its k-cells.
(7) A *weighted* polyhedral complex is a pair $(\Pi, \mathbf{w})$ where Π is a polyhedral complex and $\mathbf{w}$ is a vector assigning a weight $w(F) \in \mathbb{Z}_{>0}$ to each facet $F \in \Pi$.

The price complex cells are defined by collections of linear equalities and weak inequalities. So it is straightforward (details in Appendix C.1) that:

Proposition 2.7.

(1) The price complex is an n-dimensional rational polyhedral complex.
(2) The LIP cells, paired with the facet weights, form an $(n-1)$-dimensional weighted rational polyhedral complex.

So if C is a price complex cell, then every face C' of C satisfying $C' \subsetneq C$ is also a price complex cell. So at prices in such C', the agent demands additional bundles to those that she demands in C. But her demand set is constant in the interior of the cell:

Lemma 2.8. *$D_u(\mathbf{p}^\circ)$ is constant across all $\mathbf{p}^\circ$ in the interior C° of a cell C. Moreover $D_u(\mathbf{p}^\circ)$ defines the cell: $C = \{\mathbf{p} \in \mathbb{R}^n : D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})\}$.*

Recall that the LIP $\mathcal{L}_u$ is the union of its facets. And conversely:

Lemma 2.9.

(1) UDRs are the connected components of the complement of $\mathcal{L}_u$, and so are convex, n-dimensional, open and dense in $\mathbb{R}^n$; and n-cells of the price complex are closures of UDRs.
(2) $C \subseteq \mathcal{L}_u$ is a cell iff it is the intersection of n-cells of the price complex.

Thus we can switch easily between the LIP $\mathcal{L}_u$ and its price complex, without reference to u or direct application of Definition 2.5(1). Fig. 1 illustrates all these points.

2.2.2 Concavity in valuations

We define concavity of the valuation u in the standard "concave-extensible" sense, but with an extra property since we allow the domain to be *any* finite subset of $\mathbb{Z}^n$:

Definition 2.10.
(1) A set $A \subseteq \mathbb{Z}^n$ is *discrete convex* if it contains every integer point within its convex hull, that is, $\operatorname{conv}(A) \cap \mathbb{Z}^n = A$.
(2) We write $\operatorname{conv}(u) : \operatorname{conv}(A) \to \mathbb{R}$ for the minimal weakly-concave function everywhere weakly greater than u (sometimes called the "concave majorant" of u).
(3) $u : A \to \mathbb{R}$ is *concave* if A is discrete-convex and $u(\mathbf{x}) = \operatorname{conv}(u)(\mathbf{x})$ for all $\mathbf{x} \in A$.

It is standard that concave valuations are those for which every possible bundle is demanded at some price, and for which the demand set at any price is discrete-convex, just as for divisible, weakly-concave valuations, and for essentially the same reasons:[8]

Lemma 2.11. *$u : A \to \mathbb{R}$ is concave*
iff *for all* $\mathbf{x} \in \operatorname{conv}(A) \cap \mathbb{Z}^n$ *there exists* $\mathbf{p}$ *such that* $\mathbf{x} \in D_u(\mathbf{p})$
iff $D_u(\mathbf{p})$ *is discrete-convex for all* $\mathbf{p}$.

The valuation of Fig. 1b illustrates failure of concavity: for it, no price $\mathbf{p}$ satisfies $(0,1) \in D_u(\mathbf{p})$ and, for example, $D_u(7,3) = \{(0,0),(0,2)\}$ is not discrete-convex.

If we weakly increase a valuation until it becomes concave, the only values we need change are those for bundles which were previously never demanded. And increasing any never-demanded bundle's value has no effect on the agent's behaviour until the bundle is just marginally demanded, when the value function becomes locally affine. The marginally defined bundle is then added to the demand set at some prices, but is never demanded uniquely. All other bundles are demanded exactly as they were previously, so the LIP is unchanged. For example, in Fig. 1b, increasing $u(0,1)$ to 3 yields a concave valuation but does not change the LIP. More generally:

Lemma 2.12. *Let $u : A \to \mathbb{R}$. Then:*
(1) for each $\mathbf{x} \in A$, $u(\mathbf{x}) = \operatorname{conv}(u)(\mathbf{x})$ *iff there exists* $\mathbf{p}$ *such that* $\mathbf{x} \in D_u(\mathbf{p})$;
(2) $\mathcal{L}_u = \mathcal{L}_{u'}$, *where u' is the restriction of* $\operatorname{conv}(u)$ *to* $\operatorname{conv}(A) \cap \mathbb{Z}^n$.

2.3 The Valuation-Complex Equivalence Theorem

We now state a mathematical result whose economic implications are important and, we believe, novel: the Valuation-Complex Equivalence Theorem shows that a set in $\mathbb{R}^n$ is the LIP of a valuation (that is, the locus of indifference points of a quasilinear utility function) *if and only if* it has some easily-checked geometric properties.

First recall from Prop. 2.4 that, once we know the demand in one particular UDR, and we know the weights of the LIP, we can infer the demand in every UDR, by stepping across a series of facets. But if we follow an agent along a price path that ends where it started, the demand at the end must be the same as that at the beginning. So the weights on the facets must satisfy the *balancing condition*:

[8] See Appendix C.1; these results are illustrated by the example in Section 2.4. For the divisible case see, e.g., Mas-Colell et al. (1995) pp. 135-138, especially Prop. 5.C.1(v), since a quasilinear utility function is equivalent to a standard profit function with a single-output technology.

Definition 2.13 (Mikhalkin, 2004, Defn. 3)**.** An $(n-1)$-dimensional weighted polyhedral complex Π is *balanced* if for every $(n-2)$-cell $G \in \Pi$, the weights $w(F_j)$ on the facets $F_1, \ldots, F_l$ that contain G, and primitive integer normal vectors $\mathbf{v}_{F_j}$ for these facets that are defined by a fixed rotational direction about G,[9] satisfy $\sum_{j=1}^{l} w(F_j)\mathbf{v}_{F_j} = 0$.

For example, Fig. 1b is balanced because $2 \times (0,1) + 1 \times (-1,0) + 1 \times (1,-2) = 0$. This balancing condition is, as we now see, the *only* condition that a weighted rational polyhedral complex has to satisfy, to consist of the cells of the LIP of some valuation.

However, that valuation is not unique. First, Lemma 2.12(2) showed us that every valuation gives rise to the same LIP as a concave valuation. Moreover, changing $u(\mathbf{x})$ by adding a constant, or by increasing the bundle demanded at any price by a fixed bundle, leaves the LIP unchanged. So, to pin down a unique concave valuation, we need to specify the demand set at some price, and the value of one bundle.

Theorem 2.14 (**The Valuation-Complex Equivalence Theorem**, Mikhalkin, 2004, Remark 2.3 and Prop. 2.4)**.** *Suppose that* $(\Pi, \mathbf{w})$ *is an* $(n-1)$*-dimensional weighted rational polyhedral complex in* $\mathbb{R}^n$*, that* $\mathcal{L}$ *is the union of the cells in* Π*, and* $\mathbf{p}$ *is any price not contained in* $\mathcal{L}$*.*

(1) There exists a finite set $A \subsetneq \mathbb{Z}^n$ *and a function* $u : A \to \mathbb{R}$ *such that* $\mathcal{L}_u = \mathcal{L}$ *and* $\mathbf{w}_u = \mathbf{w}$*, if and only if* $(\Pi, \mathbf{w})$ *is balanced.*

(2) If $(\Pi, \mathbf{w})$ *is balanced then there exists a finite set* $A \subsetneq \mathbb{Z}^n$ *and a unique* concave *valuation* $u : A \to \mathbb{R}$ *such that* $D_u(\mathbf{p}) = \{\mathbf{0}\}$*,* $u(\mathbf{0}) = 0$*,* $\mathcal{L}_u = \mathcal{L}$ *and* $\mathbf{w}_u = \mathbf{w}$*.*

Thm. 2.14 completes our demonstration of the equivalence between valuations u, LIPs $\mathcal{L}_u$, and suitable weighted polyhedral complexes $(\Pi, \mathbf{w})$. Our supplementary appendix (Baldwin and Klemperer, 2018) provides an example of its application.

The balancing condition is similar in flavour to integrability criteria such as that of Afriat's theorem (see e.g. Vohra, 2011, Thm. 7.2.1). But, while Afriat starts with a (finite) set of prices paired with demands, Thm. 2.14 uses only information about the geometrical divisions in price space, created by (unspecified) changes in demand. Thus, we can develop economic ideas, intuitions and (counter-)examples by reference *only* to such geometric objects–which can be considerably easier than working with explicit valuations. Subsequent sections will illustrate this.

2.4 The Demand Complex

Dual to our price complex is a "demand complex" (in quantity space).

Definition 2.15.

(1) A *demand complex cell*, σ, for u is a set $\sigma := \mathrm{conv}\,(D_u(\mathbf{p}))$ for some $\mathbf{p} \in \mathbb{R}^n$.
(2) The *demand complex* Σ_u is the set of all demand complex cells for u.
(3) The *vertices* of the demand complex are its 0-cells.
(4) The *edges* of the demand complex are its 1-cells.

[9] That is, take any sufficiently small circle, around a point in G, and embedded in a 2-dimensional plane perpendicular to G. The vectors $\mathbf{v}_F$ must all point in a consistent direction around this circle.

(5) The *length* of an edge is the number of primitive integer vectors, lying along it, of which it is formed (i.e., its Euclidean length divided by the Euclidean length of the parallel primitive integer vector).

It is easy to see that every cell in Σ_u is a rational polyhedron. Furthermore,

Proposition 2.16. *The demand complex is a rational polyhedral complex, with dimension equal to that of* $\mathrm{conv}(A)$.

We will understand this proposition via an alternative description of the demand complex, which aids intuition and also makes it easy to quickly develop examples.

First, note it is clear that:

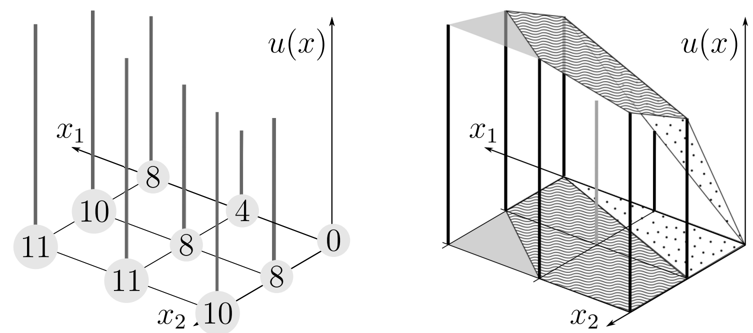
Lemma 2.17. $D_{\mathrm{conv}(u)}(\mathbf{p}) = \mathrm{conv}\left(D_u(\mathbf{p})\right)$ *for all* $\mathbf{p} \in \mathbb{R}^n$.

Now, $\mathrm{conv}(u)$ can be understood as a valuation function on divisible goods. So we can use the standard construction for a concave valuation: any price vector defines a hyperplane, tangent to the graph of the agent's valuation, which meets this graph at the agent's demand set for that price. But because $\mathrm{conv}(u)$ is only *weakly*-concave, some tangent hyperplanes meet the graph at more than one point, and some demand sets are multi-valued.

For example, Fig. 2a shows a valuation function, u, and Fig. 2b illustrates it, using bars to represent the valuations, $u(\mathbf{x})$, of bundles $\mathbf{x}$. We will always present the feasible bundles increasing to the *left*, and *down.* This will reveal the duality between the demand complex and the weighted price complex most clearly.

x_1: *2*	*1*	*0*	$u(\mathbf{x})$	
8	4	0	*0*	
10	8	8	*1*	x_2
11	11	10	*2*	

(a) Tabular representation of a valuation, $u(\mathbf{x})$.



(b) A representation of the valuation using bars.

(c) The "roof" of the valuation (the graph of $\mathrm{conv}(u)$).

Figure 2: A valuation and its roof.

We call the graph of $\mathrm{conv}(u)$ the "roof" of the valuation. Fig. 2c illustrates. At any price $\mathbf{p}$, the bundles, $\mathbf{x}$, demanded under the valuation $\mathrm{conv}(u)$, are those that maximise $\mathrm{conv}(u)(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = (-\mathbf{p}, 1) \cdot (\mathbf{x}, \mathrm{conv}(u)(\mathbf{x}))$. That is, $\mathbf{x}$ is demanded at $\mathbf{p}$ if the point $(\mathbf{x}, \mathrm{conv}(u)(\mathbf{x}))$ is farthest out from the origin in the direction of that price (i.e., in the direction $(-\mathbf{p}, 1)$). So an intersection between the roof and a supporting hyperplane is a set of the form $\hat{\sigma} = \{(\mathbf{x}, \mathrm{conv}(u)(\mathbf{x})) \in \mathbb{R}^{n+1} : \mathbf{x} \in D_{\mathrm{conv}(u)}(\mathbf{p})\}$, where $\mathbf{p}$ is such that $(-\mathbf{p}, 1)$ is normal to the hyperplane. We call these sets the *faces of the roof* (cf. Defn. 2.6(2)). And projecting such a face from $\mathbb{R}^{n+1}$ to its first n coordinates (in $\mathbb{R}^n$) just yields the set $D_{\mathrm{conv}(u)}(\mathbf{p}) = \mathrm{conv}\left(D_u(\mathbf{p})\right)$ for that $\mathbf{p}$. So:

Lemma 2.18. *$\hat{\sigma} \subsetneq \mathbb{R}^{n+1}$ is a face of the roof iff the projection of $\hat{\sigma}$ to its first n coordinates is a cell $\sigma \subsetneq \mathbb{R}^n$ of the demand complex.*

So projecting the faces of the roof into $\mathbb{R}^n$ yields the collection of all demand complex cells $\operatorname{conv}(D_u(\mathbf{p}))$. This is illustrated by the projection beneath the roof in Fig. 2c, and the demand complex in Fig. 3a.[10] Moreover, it is clear that the faces of the roof are faces of a polyhedron, namely, the convex hull of the points $(\mathbf{x}, u(\mathbf{x}))$. So these faces form a polyhedral complex. Prop. 2.16 follows from the fact that the projection of this complex to its first n coordinates is one-to-one. (Details are in Appendix C.2).

Fig. 3a shows the three 2-cells (areas), shaded to match the corresponding pieces of planes of the roof in Fig. 2c. The 2-cells are separated by nine *edges* (line-segments that are 1-cells), that themselves meet in the seven *vertices* (0-cells) of the demand complex.

Note that only the "white" circles represent vertices. The grey and black circles represent bundles that are *not* at vertices of the demand complex, since they are not uniquely demanded at any price. Indeed the demand complex cannot tell us whether non-vertex bundles such as these are ever demanded. However, it *does* tell us that if a non-vertex bundle is demanded at any price, then it is demanded at the price(s) corresponding to those cells in which it lies. This follows straightforwardly from Lemma 2.12(1)'s result that if a bundle, $\mathbf{x}$, is demanded at *any* price, then $u(\mathbf{x}) = \operatorname{conv}(u)(\mathbf{x})$, together with the observation that $D_u(\mathbf{p})=\{\mathbf{x} : u(\mathbf{x}) = \operatorname{conv}(u)(\mathbf{x})\} \cap D_{\operatorname{conv}(u)}(\mathbf{p})$, and Lemma 2.17. So we have proved:

Lemma 2.19. *If there is* any *price $\mathbf{p}$ at which $\mathbf{x}$ is demanded, and if $\mathbf{x} \in \operatorname{conv}(D_u(\mathbf{p}))$, then $\mathbf{x} \in D_u(\mathbf{p})$.*

2.5 Duality

We can now see an instructive (and beautiful) duality between the demand complex and the weighted price complex.[11]

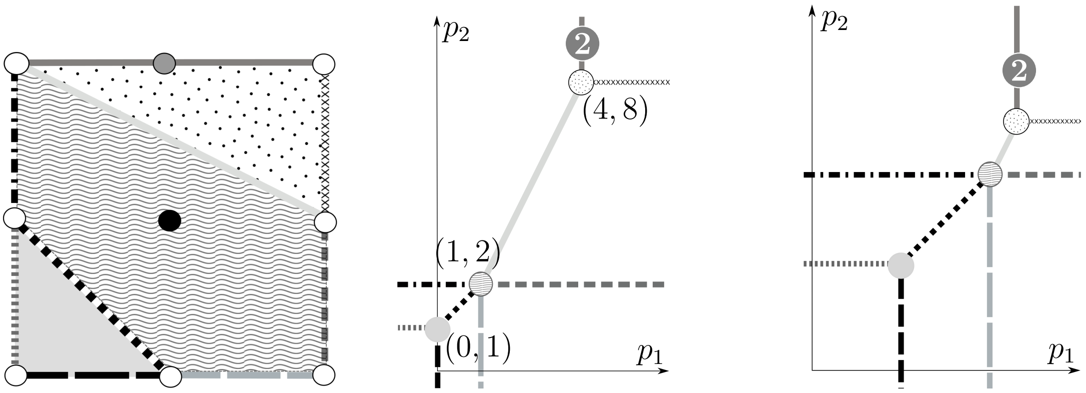
Since the vertices of the demand complex are at bundles which are uniquely demanded for some price, they correspond to UDRs. And an edge of the demand complex between vertices $\mathbf{x}$ and $\mathbf{x}'$ indicates the existence of prices, $\mathbf{p}$, for which the demand set contains both these bundles. Moreover, such $\mathbf{p}$ form an ($(n-1)$-dimensional) facet of the LIP, as they are defined by only one equality constraint $u(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x} = u(\mathbf{x}') - \mathbf{p}\cdot\mathbf{x}'$.[12] And as we saw in Prop. 2.4, $\mathbf{p}\cdot(\mathbf{x}' - \mathbf{x}) = \mathit{constant}$, for all these price vectors, $\mathbf{p}$. So each edge of the demand complex is normal to the facet that corresponds to it in the LIP. And more generally:

Proposition 2.20 (Duality). *There is a bijective correspondence between the demand complex and the price complex, associating: vertices of the demand complex with closures*

[10] We depict the demand complex by drawing its top-dimensional cells, on a grid of integer bundles. The remaining cells are easily identified as faces of the top-dimensional cells, while the grid allows us to identify the "lengths" of edges and the bundles in any cell. We omit axes, since replacing A with $A + \mathbf{x}$ for some $\mathbf{x} \in \mathbb{Z}^n$, and re-defining u correspondingly, yields a demand complex dual to the same weighted price complex.

[11] The construction uses Legendre-Fenchel duality; e.g. see Murota (2003). We use category-theoretic "duality", which allows an object to have multiple, equivalent, "duals".

[12] If there are additional points in A lying on the edge, they do not impose additional linearly independent constraints on such $\mathbf{p}$; see the discussion which follows, relating to the "dark-grey edge".



(a) Σ_u, with the grid of integer bundles in $\text{conv}(A)$.

(b) The weighted LIP $(\mathcal{L}_u, \mathbf{w}_u)$.

(c) Another weighted LIP, whose price complex is also dual to Σ_u.

Figure 3: a-b: The demand complex and weighted LIP of the valuation u given in Fig. 2a; dual geometric objects have the same style and shading (the black vertex in a has no dual object in b). The weighted LIP of a *different* valuation from u, *also* dual to the demand complex of a, is shown in c.

of UDRs; edges of the demand complex with weighted facets of the LIP; and k-cells σ of the demand complex with $(n-k)$-cells C_σ of the price complex, for $1 \leq k \leq \dim \text{conv}\, A$; such that:

(1) $\sigma = \text{conv}\,(D_u(\mathbf{p}))$ *iff* $\mathbf{p} \in C_\sigma^\circ$*;*
(2) $C_\sigma = \{\mathbf{p} \in \mathbb{R}^n : \sigma \subseteq \text{conv}\,(D_u(\mathbf{p}))\}$*;*
(3) inclusion relationships reverse: $\sigma \subsetneq \sigma' \Leftrightarrow C_{\sigma'} \subsetneq C_\sigma$*;*
(4) dual cells are orthogonal: $(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) = 0$ *for all* $\mathbf{p}, \mathbf{p}' \in C_\sigma$*,* $\mathbf{x}, \mathbf{x}' \in \sigma$*;*
(5) facets F_σ *correspond to edges* σ *of length* $w_u(F_\sigma)$*.*

The demand complex and weighted LIP of the valuation of Fig. 2a are pictured in Figs. 3a and 3b respectively; cells which are dual are depicted in the same style.

Thus the 0-cells of the LIP at the prices $(4, 8)$, $(1, 2)$, and $(0, 1)$ are dual to the dotted-, wavy-, and light-grey-, shaded 2-cells of the demand complex, respectively; the nine facets of the LIP are dual to the nine correspondingly-styled edges of the demand complex; and each of the seven UDRs, around the LIP, is dual to one of the seven bundles at the white circles that are the seven vertices of the demand complex.

Notice that the dark-grey horizontal edge at the top of the demand complex passes through a bundle, and has *length* 2 (in the sense of Defn. 2.15(5)). It is dual to the dark-grey vertical facet of the LIP, which correspondingly has *weight* 2, and is so labelled (see Prop. 2.4). All other edges of this demand complex have length 1; all other facets of the LIP correspondingly have weight 1.

As we noted in the previous subsection, neither the grey bundle, nor the black bundle, is at a vertex of the demand complex, since neither is ever *uniquely* demanded for any price, so nor do they correspond to any UDRs.

Furthermore, *neither* the LIP *nor* the demand complex can tell us whether a non-vertex bundle, such as one of these, is ever demanded. However we do know, from Lemma 2.19, that because the central wavy-shaded (five-sided) demand complex cell is the only demand complex cell that the black bundle lies in, the corresponding wavy-shaded 0-cell of the LIP in which that bundle is "hidden" indicates the only price, (1,2), at which that bundle *might* be demanded. Similarly, because the dark-grey horizontal edge at the top of the demand complex is the lowest-dimensional demand complex cell that the grey bundle lies in, the corresponding dark-grey vertical facet of the LIP in which that bundle is "hidden" indicates the only prices ($(4, p_2)$ for $p_2 \geq 8$–see Fig. 3b) at which that bundle might be demanded.

In fact, $(\mathbf{x}, u(\mathbf{x}))$ is in the roof for a non-vertex bundle, $\mathbf{x}$–and so the bundle is demanded–if and only if the valuation, u, is affine in the relevant range. The grey bundle is an example of this. It is at $(1, 0)$, and its valuation, 4, is the average of the valuations, 0 and 8, of the bundles $(0, 0)$ and $(2, 0)$, so it is demanded at the prices $\{(4, p_2) : p_2 \geq 8\}$.

However, if u is non-concave at a non-vertex bundle, the bundle's value lies strictly below the roof, so it is never demanded–it is "jumped over" as we cross between UDRs. The black bundle in the centre of demand complex illustrates this. Its value under u is strictly below its value under $\text{conv}(u)$, so it lies strictly under the "roof" (see Fig. 2c) and is never demanded at any price. (See Appendix C.2 for more discussion.)

Prop. 2.20, and the remark above Prop. 2.7, allow us to characterise the set of prices at which a bundle $\mathbf{x}$ is demanded, if it is demanded at any price:

Corollary 2.21. *Suppose that σ is the* minimal *cell of the demand complex such that $\mathbf{x} \in \sigma$, and that $\mathbf{x}$ is demanded for some price. Then $\mathbf{x} \in D_u(\mathbf{p})$ iff $\mathbf{p} \in C_\sigma$.*

Finally, note that, for any single demand complex, there are multiple weighted LIPs which satisfy the correspondences and orthogonality relationships of Prop. 2.20. For example, Figs. 3b and 3c give two different weighted LIPs–and therefore two different valuations–that are both dual to the demand complex of Fig. 3a. So it is natural to group together all valuations whose demand complexes are either the same, or differ only by a constant shift by some bundle, $\mathbf{x}$:

Definition 2.22. Two valuations u, u' have the same *combinatorial type* if they have the same demand complex, or if there exists $\mathbf{x} \in \mathbb{Z}^n$ such that $\sigma \in \Sigma_u$ iff $\{\mathbf{x}\} + \sigma \in \Sigma_{u'}$.

It is easy to list all the possible demand complexes, and examples of dual weighted LIPs which exhibit the combinatorial type (thus giving all "essentially-different" structures of demand) if the domain is not too large–see Figs. 10-11 of Ex. C.2 in Appendix C.2, where we also provide further discussion of Figs. 2-3. A three-dimensional example is given in our supplementary appendix (Baldwin and Klemperer, 2018).

2.6 Representation in Price Space *vs.* Quantity Space

Although the weighted LIP and demand complex are dual, there is an important distinction: the Valuation-Complex Equivalence Theorem applies *only* to price space. In quantity space, by contrast, it is *not* true that every way of subdividing $\text{conv}(A)$ into a rational polyhedral complex yields a demand complex. (See Maclagan and Sturmfels, 2015, Fig. 2.9 for a subdivision which corresponds to no LIP, and therefore to no valuation.) Nor does there seem to be any simple check of which polyhedral complexes in quantity space correspond to any valuation function. So while we can develop examples to, e.g., test conjectures, by working with geometric objects in price space, and be certain that the corresponding valuations will exist, it is hard to do this in quantity space.

Furthermore, LIPs show the *actual* prices at which bundles are demanded, whereas a demand complex shows only collections of bundles among which the agent is indifferent for *some* prices. Since it is also much easier to aggregate agents' valuations in price space (see Section 3.3), we mostly work in price space.

However, some information that is only implicit in the weighted LIP becomes obvious in the demand complex, in quantity space. For example, we will see in Sections 4.1 and 5 that a low-dimensional cell of the LIP sometimes "hides" important detail that is much more easily seen in the higher-dimensional dual object in the demand complex. Moreover, the easiest way to compute the LIP of a specific valuation is often by first finding the demand complex (e.g., it is easy to go from Fig. 2a to Fig. 3a and then, using duality, to Fig. 3a, see Appendix C.2); it is generally much harder to construct the LIP directly from the valuation.

The fact that the different representations are useful in different contexts makes the ability to move easily between them, using duality, especially valuable.

3 "Demand Types"

3.1 Definition of Demand Types, and Comparative Statics

We saw in the previous section that the LIP's facet normals describe how demand changes between UDRs (Prop. 2.4). They therefore give all the possible directions of change in demand (if any) that can generically result from a small change in prices. So it is natural to classify valuations into "demand types" according to these facet normals. A valuation's demand type then gives us comparative statics information, that is analogous to the information that the Slutsky matrix provides for a valuation on divisible goods at a single price point.

Definition 3.1. Let $\mathcal{D} \subsetneq \mathbb{Z}^n$ be a set of non-zero primitive integer vectors such that if $\mathbf{v} \in \mathcal{D}$ then $-\mathbf{v} \in \mathcal{D}$. The *demand type* defined by $\mathcal{D}$ comprises valuations u such that every facet of $\mathcal{L}_u$ has normal vector in $\mathcal{D}$.

For example, the valuation of Fig. 1a is of demand type $\pm\{(1,0),(0,1),(-1,1)\}$, as are many other valuations, such as all those shown in Figs. 8a-c. Note that a valuation is of *any* demand type which contains the facet normals of its LIP; we do *not* restrict to the minimal such set.[13] However:

Proposition 3.2. *Every demand type defined by a finite set of vectors, $\mathcal{D}$, is the minimal demand type for some valuation.*

Proof. For each pair of vectors $\pm\{\mathbf{v}\} \in \mathcal{D}$, choose (any) one hyperplane normal to them. The union of these hyperplanes is a rational polyhedral complex, and if we apply weight 1 to each facet then it is balanced. Now apply part 1 of the Valuation-Complex Equivalence Theorem (Thm. 2.14). □

By duality (Prop. 2.20), we could equivalently classify valuations according to the

[13]Our definition does not consider the weights on facets; see Baldwin and Klemperer (2014, note 42).

directions of their demand complexes' edges.[14] But our description makes clear that the demand type provides the generic comparative statics. As is usual, we say that a property holds for "generic" $\mathbf{p} \in \mathbb{R}^n$ if it holds for all $\mathbf{p}$ in a dense open subset of $\mathbb{R}^n$.

Proposition 3.3. *The following are equivalent for a valuation u:*

(1) u is of demand type $\mathcal{D}$.

(2) For valuation u, for any $\mathbf{t} \in \mathbb{R}^n$, and for generic $\mathbf{p} \in \mathbb{R}^n$, if $\exists\ \epsilon > 0$ such that $\mathbf{p}$ and $\mathbf{p} + \epsilon\mathbf{t}$ are in distinct UDRs, and such that $\nexists\ \epsilon' \in (0, \epsilon)$ such that $\mathbf{p} + \epsilon'\mathbf{t}$ is in a third distinct UDR, then the difference between bundles demanded at $\mathbf{p}$ and $\mathbf{p} + \epsilon\mathbf{t}$ is an integer multiple of some vector in $\mathcal{D}$.

Proof. It is generic for a price $\mathbf{p}$ to be a UDR price, and for a straight line from $\mathbf{p}$ in direction $\mathbf{t}$ to only intersect facets in their interiors. Under the conditions of (2), only one facet is crossed in this way, so the change in demand is given by one of the demand type's vectors. That (1)$\Leftrightarrow$(2) is now immediate from Prop. 2.4. $\square$

Furthermore, since the domain A is finite, the response to any specific price change can, generically, be broken down into a series of steps of this form. And importantly, as we will see, Prop. 3.3 reveals the close relationship between demand types and standard economic descriptions of comparative statics.

Baldwin and Klemperer (2014) discuss *non*-generic price changes (i.e., those which do not start in a UDR) and also give other equivalent characterisations of demand types, but Prop. 3.3 will suffice for our purposes.

3.2 Substitutes, Complements, and other "Demand Types"

It follows straightforwardly that demand types provide simple characterisations of familiar concepts such as ordinary substitutes, ordinary complements, and "strong substitutes". These characterisations are easier to generalise than standard ones based on imposing restrictions on u directly. Moreover, they more clearly reveal and explain features such as the lack of symmetry between substitutes and complements.

We begin by recalling standard definitions:

Definition 3.4 (Standard)**.**

(1) A valuation u is *ordinary substitutes* if, for any UDR prices $\mathbf{p}' \geq \mathbf{p}$ with $D_u(\mathbf{p}) = \{\mathbf{x}\}$ and $D_u(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \geq x_k$ for all k such that $p_k = p'_k$.[15]

[14] Danilov, Koshevoy and their co-authors' work (see Section 4.3) examine these vectors in quantity space. However, they do not use them to create a taxonomy of demand or, e.g., interpret them as giving comparative statics information. We, by contrast, develop a general framework to understand them in economic terms (see also Baldwin and Klemperer, 2012, 2014).

[15] We write, as is standard, $\mathbf{p}' \geq \mathbf{p}$ when the inequality holds component-wise.

We call "ordinary substitutes" what most others (e.g., Ausubel and Milgrom, 2002, Hatfield and Milgrom, 2005) simply call "substitutes". We do this for clarity, since some have defined "substitutes" in other ways. In particular, although Kelso and Crawford's (1982) definition is equivalent in their model, it is not generally equivalent if it is extended to multiple units of three or more goods (which yields Milgrom and Strulovici's, 2009, definition of "weak substitutes"); see Danilov et al., 2003, Ex. 6 and Thm. 1. Our definition (3.4(1)) seems the most natural one in the general case. It is also equivalent to several properties that seem to naturally characterise "substitutes", and to the indirect utility function

(2) A valuation u is *ordinary complements* if, for any UDR prices $\mathbf{p}' \geq \mathbf{p}$ with $D_u(\mathbf{p}) = \{\mathbf{x}\}$ and $D_u(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \leq x_k$ for all k such that $p_k = p'_k$.
(3) A valuation u is *strong substitutes* if, when we consider every unit of every good to be a separate good, it is a valuation for ordinary substitutes.[16]

It is easy to use Prop. 3.3 to provide alternative, equivalent, definitions of these concepts, as demand types. For substitutes:

Definition 3.5. The (n-dimensional) *ordinary substitutes vectors* are the set of non-zero primitive integer vectors $\mathbf{v} \in \mathbb{Z}^n$ with at most one positive coordinate entry, and at most one negative coordinate entry. They define the *ordinary substitutes demand type* (for n goods).

Proposition 3.6. *A valuation is an ordinary substitutes valuation* iff *it is of the ordinary substitutes demand type.*

Proof. Consider the change in demand from a UDR price $\mathbf{p}$ to a UDR price $\mathbf{p}' \geq \mathbf{p}$ such that $D_u(\mathbf{p}') \neq D_u(\mathbf{p})$. Write $\mathbf{t} := \mathbf{p}' - \mathbf{p}$ and $\{\mathbf{x}\} = D_u(\mathbf{p})$. By Prop. 3.3, we may choose $\widetilde{\mathbf{p}}$ arbitrarily close to $\mathbf{p}$ such that if $\widetilde{\mathbf{p}}'' = \widetilde{\mathbf{p}} + \epsilon\mathbf{t}$ is in the first UDR which is distinct from that of $\mathbf{x}$, on the line from $\widetilde{\mathbf{p}}$ in direction $\mathbf{t}$, and if $\mathbf{x}''$ is demanded at $\widetilde{\mathbf{p}}''$, then $\mathbf{x}'' - \mathbf{x}$ is an integer multiple of an ordinary substitutes vector. In particular (since UDRs are open) we can choose such $\widetilde{\mathbf{p}}$ so that it is in the same UDR as $\mathbf{p}$, and such that $\mathbf{p} + \epsilon\mathbf{t}$ is in the closure of the UDR containing $\widetilde{\mathbf{p}}''$, implying that $\mathbf{x}'' \in D_u(\mathbf{p} + \epsilon\mathbf{t})$.

By standard results $(\mathbf{x}'' - \mathbf{x}) \cdot (\widetilde{\mathbf{p}}'' - \widetilde{\mathbf{p}}) < 0$ (see e.g. Mas-Colell et al., 1995, Prop. 2.F.1). But $\widetilde{\mathbf{p}}'' - \widetilde{\mathbf{p}} = \epsilon\mathbf{t} = \epsilon(\mathbf{p}' - \mathbf{p}) \geq 0$. So, $\mathbf{x}'' - \mathbf{x}$ must have a strictly negative coordinate for some good whose price strictly increases from $\mathbf{p}$ to $\mathbf{p}'$. But $\mathbf{x}'' - \mathbf{x}$ is an integer multiple of an ordinary substitutes vector, and so has at most one negative coordinate, so *demand weakly increases for all goods whose price does not change.*

If we apply this process repeatedly, until we reach a final price in the same UDR as $\mathbf{p}'$, we make this same conclusion at every step. So in sum, Defn. 3.4(1) is satisfied. □

Figs. 1 and 3b-3c illustrate the substitutes property holding; Figs. 4b-4d and 12 show it failing. So a vector that is normal to a facet of a LIP for substitutes cannot have two non-zero entries of the same sign. To understand the necessity of this, consider a LIP with a facet whose normal vector's first and third coordinates have the same sign (Fig. 12 in Appendix C.2). Increasing the price on either good 1 or good 3 can take us across the facet–decreasing demand for *both* goods 1 and 3. So this facet generates complementarities at some prices, and so cannot be part of a LIP for substitutes.

For complements, a price change that reduces demand for a good can of course reduce (but not increase) demand for other goods:

($\max_{\mathbf{x} \in A}\{u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}\}$) being submodular–see Baldwin, Klemperer and Milgrom (in preparation). See also Baldwin and Klemperer (2014). Hatfield et al. (2013)–see our Section 6.4–and Danilov et al. (2003) use definitions equivalent to 3.4(1), and the latter authors make a similar observation to our Prop. 3.6 when they say "each cell of a valuation's parquet is a polymatroid".

[16] This is equivalent to Milgrom and Strulovici's (2009) definition–see Danilov et al. (2003, Cor. 5). There are many other equivalent definitions (see Shioura and Tamura, 2015), in particular $M^\natural$-concavity" of the valuation (Murota and Shioura, 1999). When there is only one unit of each good, it is also equivalent to Kelso and Crawford's (1982) 'gross substitutes', but we do not use that name as it does not distinguish between ordinary and strong substitutes when there are multiple units available.

Definition 3.7. The (n-dimensional) *ordinary complements vectors* are the set of non-zero primitive integer vectors $\mathbf{v} \in \mathbb{Z}^n$ whose non-zero coordinate entries are all of the same sign. They define the *ordinary complements demand type* (for n goods).

So, applying Prop. 3.3 in the same way as in the proof of Prop. 3.6:

Proposition 3.8. *A valuation is an ordinary complements valuation* iff *it is of the ordinary complements demand type.*

The lack of symmetry between substitutes and complements, and the reason for it, are now clear: ordinary complements vectors may have *any number* of non-zero entries (of the same sign), but any pair of non-zero entries in an ordinary substitutes vector must be of opposite signs, so ordinary substitutes vectors can have *at most two* non-zero entries (see Ex. C.3 for more discussion).

Our characterisation of strong substitutes as a demand type also gives an intuitive description of them:

Definition 3.9. The *strong substitute vectors* are those non-zero $\mathbf{v} \in \mathbb{Z}^n$ which have at most one $+1$ entry, at most one -1 entry, and no other non-zero entries. They define the *strong substitutes demand type.*

Proposition 3.10 (See Baldwin and Klemperer, 2014, Cor. 5.20; and Shioura and Tamura, 2015, Thm. 4.1(i))**.** *A valuation is strong substitutes iff it is concave and is of the strong substitutes demand type.*

So Figs. 1a, 4a, and 8a-c show examples of LIPs of strong substitutes valuations.[17]

We will see now, in Section 3.2, and later, in Section 6.1, that demand types also allow us to characterise significant new classes of valuations.

Re-packaging goods, so that any integer bundle can still be obtained by buying and selling an integer selection of the new packages, corresponds, of course, to a unimodular change of basis,[18] which distorts the LIP, but retains its nature as a "complex". Specifically, for a unimodular $n \times n$ matrix G, define (as is standard) the "pullback" of a valuation $u : A \to \mathbb{R}$ to be $G^*u : G^{-1}A \to \mathbb{R}$ via $G^*u(\mathbf{x}) := u(G\mathbf{x})$. Then:

Proposition 3.11 (cf. e.g. Gorman, 1976, p. 219-220)**.**
(1) $\mathbf{x} \in D_u(\mathbf{p}) \iff G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p})$.
(2) $\mathcal{L}_{G^*u} = G^T\mathcal{L}_u := \{G^T\mathbf{p} : \mathbf{p} \in \mathcal{L}_u\}$*;*
(3) $u(\cdot)$ *is of demand type* $\mathcal{D}$ *iff* $G^*u(\cdot)$ *is of demand type* $G^{-1}\mathcal{D} := \{G^{-1}\mathbf{v} : \mathbf{v} \in \mathcal{D}\}$.

Ex. C.4 gives an illustration.

Some economic properties of valuations are, of course, changed by such transformations from one demand type to another: in particular, the local trade-offs (so whether valuations are substitutes or complements, etc.) But many important properties are preserved–see Prop. 4.7 below on equilibrium, and Baldwin and Klemperer (2014, especially Sec 5). So it is useful to know, for example, that the following demand types are simply unimodular basis changes of strong substitutes:

[17] Our description is closely related to the "step-wise gross substitutes" of Danilov et al. (2003), which they link to the edge vectors of (what we call) the demand complex; these edge vectors are also linked to $M^\natural$-concavity by Murota and Tamura (2003), while Fujishige and Yang (2003) showed that $M^\natural$-concavity is equivalent to Kelso and Crawford's (1982) "gross substitutes".

[18] A unimodular matrix is an integer matrix with integer inverse (i.e., with determinant ± 1).

Specific cases of this observation have been made before (see, e.g., Sun and Yang 2006, and Hatfield et al., 2013); we lay out the general behaviour here.

"Consecutive Games" (see Greenberg and Weber, 1986, and also Danilov et al., 2013). Premultiplying the strong substitutes vectors, $\mathbf{e}^i$ and $(\mathbf{e}^i - \mathbf{e}^j)$, by the upper triangular matrix of 1s (of the appropriate dimension) yields the vectors $\sum_{k=1}^{i} \mathbf{e}^k$ and $\sum_{k=j+1}^{i} \mathbf{e}^k$ for $i > j$, respectively (and their negations). This is the demand type for goods which have a natural fixed order, and for which any contiguous collection of goods may be considered as complements by any agent. For example, valuations for bands of radio spectrum, or for "lots" of sea bed to be developed for offshore wind (see Ausubel and Cramton, 2011) may be of this form.

"Generalised Gross Substitutes and Complements Valuations". Premultiplying the strong substitutes vectors by a matrix formed of $\{\mathbf{e}^i : i \leq k\} \cup \{-\mathbf{e}^i : i > k\}$, for some k, yields the demand type in which goods can be separated into two groups, with goods within the same group being strong substitutes, and each good also may exhibit 1:1 complementarities with any good in the other group.[19]

3.3 Demand types and Aggregate Demand

An important feature of our demand types classification–that, in particular, greatly facilitates the study of equilibrium–is that the demand type when we aggregate valuations from multiple agents is just the union of the sets of vectors that form the individual agents' demand types.

So we now consider a finite set J of agents: agent $j \in J$ has valuation u^j for integer bundles in a finite set, A^j. Their *aggregate demand* is, of course, the (Minkowski) sum of the individual demands, but to apply our techniques to this, we want to treat it as the demand of a single "aggregate" agent.

Definition 3.12. An *aggregate valuation* of $\{u^j : j \in J\}$ is a valuation u^J with domain $A := \sum_{j \in J} A^j$ such that $D_{u^J}(\mathbf{p}) = \sum_{j \in J} D_{u^j}(\mathbf{p})$ $\forall \mathbf{p} \in \mathbb{R}^n$.

Note that aggregate valuations are not uniquely defined. However, this does not matter: since the aggregate demand sets are unambiguous, properties such as concavity of aggregate valuations are also unambiguous, and the aggregate weighted LIP is unique. (The fact that we can construct the aggregate LIP from the individual LIPs *without* knowing the form of u^J–so without using any cumbersome formula for u^J–is an important advantage of aggregation in price space.[20])

The rest of this subsection proves and discusses:

Lemma 3.13. *Given a finite set of valuations* $\{u^j : j \in J\}$*:*

(1) an aggregate valuation u^J *exists;*

[19] The demand type's vectors are $\{\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^i - \mathbf{e}^{i'}, \mathbf{e}^i + \mathbf{e}^j, \mathbf{e}^j - \mathbf{e}^{j'} : i, i' \in \{1, \ldots, k\}, j, j' \in \{k+1, \ldots, n\}\}$. This extends Sun and Yang (2006, 2009) by permitting multiple units of goods, and sellers as well as buyers. See also Shioura and Yang (2015).

[20] As is well known, with quasi-linear preferences, $\max\left\{\sum_{j \in J} u^j(\mathbf{x}^j) : \mathbf{x}^j \in A^j, \sum_{j \in J} \mathbf{x}^j = \mathbf{y}\right\}$ is an aggregate valuation. Mathematically, this aggregate valuation is the tropical product of the tropical polynomials that are the individual valuations. Economically, it says a bundle's aggregate value is the maximum sum of agents' values that can be obtained by apportioning the bundle among the agents.

Adding a constant to the valuation, and/or changing the value of never-demanded bundles to leave them never demanded, also provide aggregate valuations according to Definition 3.12.

(2) $\mathcal{L}_{u^J} = \bigcup_{j \in J} \mathcal{L}_{u^j}$*;*

(3) If F *is a facet of* $\mathcal{L}_{u^J}$*, then* $w_{u^J}(F) = \sum_{F^j \in \mathcal{F}} w_{u^j}(F^j)$*, in which* $\mathcal{F}$ *is the set of all facets of the individual* $\mathcal{L}_{u^j}$ *which contain* F*.*

Corollary 3.14. *A collection of individual valuations are all of demand type* $\mathcal{D}$ *iff every aggregate valuation of every finite subset of them is of demand type* $\mathcal{D}$*.*

For example, Figs. 4a-b show the LIPs of Elizabeth's and Paul's valuations for the hotel rooms of our introductory example, if we extend both valuations to the full domain $\{0,1\}^2$. Elizabeth regards the rooms as substitutes; her valuation is $u^s(x_1, x_2) = \max\{40x_1, 30x_2\}$ (Fig. 4a). Paul sees them as complements; his valuation is $u^c(x_1, x_2) = \min\{50x_1, 50x_2\}$ (Fig. 4b).

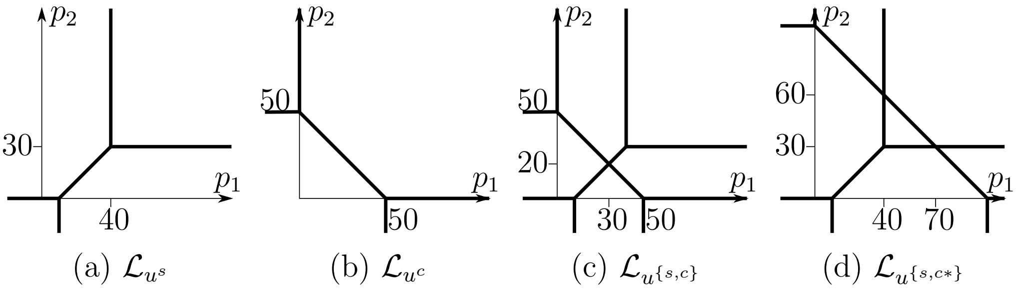


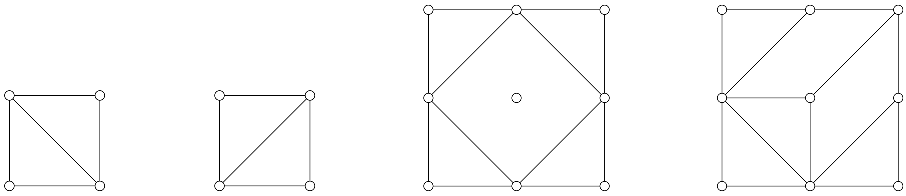
Figure 4: The LIPs of: (a) a simple substitutes valuation; and (b) a simple complements valuation. The aggregate LIPs of (c) the substitutes and complements valuations shown; and (d) the substitutes valuation shown and a complements valuation in which the bundle of both rooms together is valued at greater than 70.

It is easy to see that an aggregate demand set consists of a unique bundle iff all the individual demand sets do (and so to prove Lemma 3.13(2); see also Murota, 2003, Sec. 11.2). Thus Fig. 4c shows the aggregate LIP, $\mathcal{L}_{u^{\{s,c\}}}$, for the valuations u^s and u^c. It is obvious that a demand type contains the individual valuations iff it contains any aggregate valuation (Cor. 3.14).

From the aggregate LIP we can obtain a new price complex, in the usual way (Lemma 2.9). Its cells are intersections of cells from the individual price complexes. Thus the price $(30, 20)$ in Fig. 4c is a 0-cell, on the boundary of four distinct facets. Write Π for the subcomplex of cells of the LIP. The change in aggregate demand between any pair of prices is the sum of the changes of the individual demands. So the weight of any facet F of the aggregate LIP is the sum of the weights of all the facets F' of the individual LIPs for which $F \subseteq F'$ (which proves Lemma 3.13(3)). And since the weighted polyhedral complex $(\Pi, \mathbf{w})$ is derived from balanced complexes, it is itself balanced, and so (using Thm. 2.14(1)) it is the LIP of some valuation (so Lemma 3.13(1) holds).

Notice, however, that we *cannot* find the demand complex of an aggregate valuation using only the individual demand complexes: a demand complex does not correspond to a unique valuation, and different valuations may aggregate in different ways.

For example, the demand complexes corresponding to the LIPs of Figs. 4a-b are shown in Figs. 5a-b. The demand complex corresponding to their aggregate LIP (Fig. 4c) is shown in Fig. 5c; its domain is $\{0,1\}^2 + \{0,1\}^2 = \{0,1,2\}^2$. If Paul's valuation increases to $u^{c*}(x_1, x_2) = \min\{100x_1, 100x_2\}$, then his demand complex remains that of Fig. 5b. However, the LIP $\mathcal{L}_{u^{\{s,c*\}}}$ is shown in Fig. 4d, and its demand complex is that of



(a) Σ_{u^s} (b) $\Sigma_{u^c}, \Sigma_{u^{c*}}$ (c) $\Sigma_{u^{\{s,c\}}}$ (d) $\Sigma_{u^{\{s,c*\}}}$

Figure 5: The demand complexes dual to the LIPs in Figs. 4a-d, when every facet has weight 1. (Bundles in the domains of valuations are shown, without colour-coding.)

Fig. 5d. So there is no unique aggregate demand complex corresponding to the demand complexes of Fig. 5a and Fig. 5b.

4 The Unimodularity Theorem–when does Equilibrium *always* exist for a Demand Type?

We state the Unimodularity Theorem and some immediate corollaries in the next subsection; give intuition and the proof in Section 4.2; and explain the close connections with Danilov et al. (2001) in Section 4.3.

4.1 Statement of Results

We are interested in the standard notion of competitive equilibrium:

Definition 4.1. An *equilibrium* exists, for a market supply $\mathbf{x} \in \mathbb{Z}^n$ and a finite set of valuations, if $\mathbf{x}$ is in the valuations' aggregate demand set for some price.

It is standard (Lemma 2.11) that concavity of an aggregate valuation u^J is necessary and sufficient for equilibrium to exist for all integer bundles in the convex hull of the domain of u^J. We therefore refer to these bundles as the *relevant supply bundles*: equilibrium will clearly never exist for other bundles, as they are the wrong "size".

Concavity of individual valuations is therefore necessary even for all one-agent economies to have equilibrium, so our results will also restrict attention to concave valuations.

With *in*divisible goods (unlike with divisible goods), individual concavity is *not* sufficient to guarantee aggregate concavity (and so, for example, supporting hyperplanes do not necessarily exist). However, our geometric approach provides a simple additional condition that *is* sufficient to guarantee equilibrium. First we define:

Definition 4.2. A set of vectors in $\mathbb{Z}^n$ is *unimodular* if every linearly independent subset can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1.

By "the *determinant*" of n vectors we mean the absolute value of the determinant of the $n \times n$ matrix which has them as its columns.

Unimodularity is not too hard to check (see also Facts 4.9 and C.7, and Remark A.2). We refer to "unimodular demand types" in the obvious way. We can now state (see Section 4.2.4 for proof):

Theorem 4.3 (The Unimodularity Theorem). *An equilibrium exists for* every pair *of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles,* iff *$\mathcal{D}$ is unimodular.*

Corollary 4.4. *An equilibrium exists for* every finite set *of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles,* iff *$\mathcal{D}$ is unimodular.*

Proof. If the demand type is unimodular, then a valuation obtained by aggregating any two concave valuations is also concave, so we can apply Thm. 4.3 repeatedly. □

If the set $\mathcal{D}$ of vectors spans $\mathbb{R}^n$, then there exist sets of n of them that are linearly independent; then unimodularity follows if all n-element sets have determinant ± 1 or 0:

Corollary 4.5. *With n goods, if the vectors of $\mathcal{D}$ span $\mathbb{R}^n$, then an equilibrium exists for* every *finite set of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles,* iff *every subset of n vectors from $\mathcal{D}$ has determinant* 0 *or* ± 1.

Many standard results are immediate special cases. For example, it is well-known that "strong substitute" vectors form a unimodular set (Poincaré, 1900), and these valuations are (by definition) concave, so:

Proposition 4.6 (Danilov et al., 2001, 2003, and Milgrom and Strulovici, 2009). *An equilibrium exists for* every *finite set of strong substitutes valuations, for all relevant supply bundles.*

Other familiar equilibrium results, such as those of Kelso and Crawford (1982) and Gul and Stacchetti (1999), of course immediately follow. Straightforwardly, moreover:

Proposition 4.7. *An equilibrium exists for all relevant supply bundles and every pair of concave valuations of demand type $\mathcal{D}$, iff an equilibrium exists for all relevant supply bundles and every pair of concave valuations of demand type $G^{-1}\mathcal{D}$, where G is a unimodular $n \times n$ matrix.*

So, for example, equilibrium also always exists for "consecutive games" and for "generalised gross substitutes and complements valuations" of Sec. 3.2. See also Ex. C.5.

4.2 Intuition and Proof for the Unimodularity Theorem

4.2.1 The Role of Intersections

The first critical observation is that we can determine whether equilibrium exists by focusing only on intersections of individual LIPs: we know equilibrium always exists, that is, every relevant bundle is demanded at some price, *iff* any aggregate valuation is concave *iff* the aggregate demand set is discrete-convex at every price (Lemma 2.11). But if all but one of the agents have unique demand at some price, the aggregate demand set is simply the shift of the remaining agent's demand set by the other agents' (unique) demands. And this set must be discrete-convex, since we assumed that every individual valuation is concave. So we only need to check prices at which two or more agents have non-unique demand. This proves:

Lemma 4.8. *For concave valuations $u^1, \ldots, u^s$, an equilibrium exists for every relevant supply bundle* iff *the aggregate demand set is discrete-convex at every price in every* intersection $\mathcal{L}_{u^j} \cap \mathcal{L}_{u^{j'}}$ *for $j, j' = 1, \ldots, s$, $j \neq j'$.*

So, for example, for our hotel example in the introduction, which had simple two-goods substitutes and complements valuations $u^s(x_1, x_2) = \max\{40x_1, 30x_2\}$ (Fig. 4a) and $u^c(x_1, x_2) = \min\{50x_1, 50x_2\}$ (Fig. 4b) whose aggregate LIP is shown in Fig. 4c, the *only* price we need to analyse is the intersection $(30, 20)$.

The aggregate demand at this price is the sum of the individual demands: $D_{u^s}(30, 20) = \{(1, 0), (0, 1)\}$ and $D_{u^c}(30, 20) = \{(0, 0), (1, 1)\}$. So $D_{u^{\{s,c\}}}(30, 20)$ consists of four bundles, $\{(1, 0), (0, 1), (2, 1), (1, 2)\}$, the corners of the "diamond" square of Fig. 5c. However, this square also contains the non-vertex integer bundle $(1, 1)$. Since $(1, 1)$ is not demanded at $(30, 20)$, the demand set is not discrete-convex. Thus, although both u^s and u^c are concave, any aggregate valuation for them is *not* (Lemma 2.11). In particular, equilibrium fails if the supply is $(1, 1)$ (Lemma 2.19). Note we did *not* need to actually aggregate the valuations u^s and u^c to determine this. (However, Fig. 13 and Ex. C.6 show that $u^{\{s,c\}}$, calculated using the formula of Note 20, is indeed non-concave.)

4.2.2 The Role of Unimodularity

Our proof will use two equivalent conditions to unimodularity:

Fact 4.9. The following are equivalent for a set $\mathcal{D}$ of vectors in $\mathbb{Z}^n$:

(1) $\mathcal{D}$ is unimodular;

(2) for every linearly independent subset $\{\mathbf{v}^1, \ldots, \mathbf{v}^s\}$ of $\mathcal{D}$, there is no non-vertex integer point in the parallelepiped whose edges are these vectors, that is, the parallelepiped whose vertices are $\left\{\sum_{j=1}^{s} \lambda^j \mathbf{v}^j : \lambda^j \in [0, 1]\right\}$;

(3) for every linearly independent subset $\{\mathbf{v}^1, \ldots, \mathbf{v}^s\}$ of $\mathcal{D}$, if $\mathbf{x} \in \mathbb{Z}^n$ and $\mathbf{x} = \sum_{j=1}^{s} \alpha^j \mathbf{v}^j$ with $\alpha^j \in \mathbb{R}$, then $\alpha^j \in \mathbb{Z}$ for all j.

Fact 4.9(2) says that a parallelepiped (that is, an s-dimensional parallelogram), whose vertices are integer points, contains a non-vertex integer point iff the vectors along its edges do not form a unimodular set. Now recall two geometric facts for the case $s = n$: that the volume of a parallelepiped is given by the (absolute value of the) determinant of its edge vectors; and that a parallelepiped whose vertices are integer points contains a non-vertex integer point iff its volume strictly exceeds 1.

In our hotel example, the edges of the square demand complex cell in Fig. 5c are in directions $(1, 1)$ and $(-1, 1)$. Their determinant is 2, so the area of the cell is 2, and it therefore contains a non-vertex integer point at which equilibrium may fail. (Thm. 4.3 warned of this, since the determinant being 2 exhibits failure of unimodularity.)

So equivalent condition 4.9(2) will help to demonstrate the necessity of Thm. 4.3's condition for equilibrium (as we will see in Lemma 4.16).[21]

[21] When the set of vectors is *not* unimodular, the number of non-vertex bundles in such a parallelepiped is one less than the determinant (Fact 5.18). So we expect these determinants should give bounds on the extent to which supply constraints need to be relaxed to achieve equilibrium. (Moreover, the extension of our results to matching theory–see Section 6.6–might then yield results related to Nguyen and Vohra, 2014 and Nguyen et al., 2016.)

Fact 4.9(3) tells us that any vector in the space spanned by a given set of vectors can be created as an integer combination of the set iff the set is unimodular.

In our hotel example, the four corners of the square in Fig. 5c are vertices of the aggregate-demand complex cell, and are therefore dual to UDRs that each contain the price $(30, 20)$ in their boundary (Prop. 2.20). If we move between these UDRs, around the price $(30, 20)$, aggregate demand changes by the vector normal to any facet crossed (Prop. 2.4), that is, by the vector in the direction of one of the square's edges. Moreover, at $(30, 20)$, the only possible changes in individual demand, and hence the only possible changes in aggregate demand, are made up of these vectors. So the impossibility of demanding (1,1) on aggregate at this price vector, and so (by Lemma 2.19) anywhere, is equivalent to the impossibility of obtaining (1,1) by starting at any of the four bundles at the corners of the square ($(1, 0)$, $(0, 1)$, $(2, 1)$, and $(1, 2)$), and adding integer combinations of the edge vectors (1,1) and (-1,1). More generally if, by contrast, every demand complex cell's edge vectors were a unimodular set, this problem could not arise.

So equivalent condition 4.9(3) will be useful for demonstrating the sufficiency of Thm. 4.3's condition for equilibrium (as we will see in Sec. 4.2.4).

4.2.3 The Role of Transverse Intersections

Not all LIP intersections have the simple form of our hotel example. However, we will see that intersections are generically of a form, namely "transverse", for which we can apply the intuition for "sufficiency" developed above. Moreover, the genericity of transverse intersections implies that these cases are the only ones we need consider.

Definition 4.10 (see e.g. Maclagan and Sturmfels, 2015, Defn. 3.4.9)**.**

(1) The intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is *transverse at* $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, in which C^j is the minimal cell of $\mathcal{L}_{u^j}$ containing $\mathbf{p}$, for $j = 1, 2$.

(2) The intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is *transverse* if they intersect transversely at every point of their intersection.

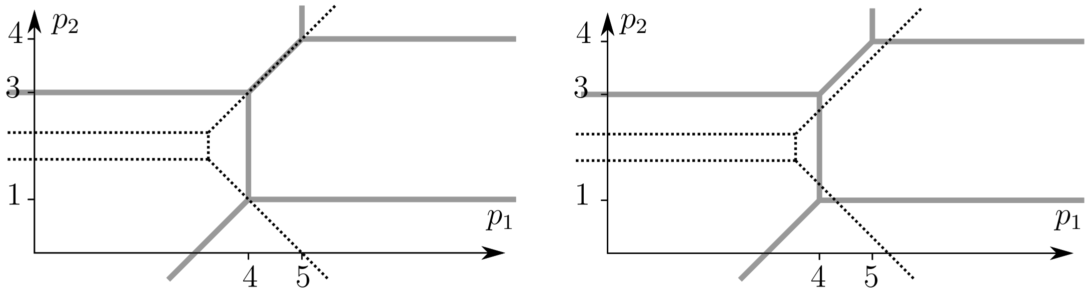
(3) The intersection of $\{\mathcal{L}_{u^j} : j \in \{1, \dots, k\}\}$ is *transverse at* $\mathbf{p}$ if the intersection of $\mathcal{L}_{u^{\{1,\dots,j\}}}$ and $\mathcal{L}_{u^{j+1}}$ is transverse at $\mathbf{p}$, for all $j = 1, \dots, k - 1$.[22]

For example, in two dimensions, two lines crossing at a single point are intersecting transversely. So the intersections in our hotel example (Figs. 4c and 4d) are transverse. However, two coincident lines do not intersect transversely, and nor does a line crossing through a 0-cell intersect transversely. So the grey LIP and the black dotted LIP of Fig. 6a, which intersect at $(4, 1)$ and along the line from $(4, 3)$ to $(5, 4)$, do not intersect transversely at *any* price. (For each of the three prices $(4, 1)$, $(4, 3)$ and $(5, 4)$, the minimal cell of the grey LIP containing the price is the 0-cell at the price itself.) In three dimensions, an intersection is transverse at all of the prices where a 1-cell meets a facet in a single point, or two facets meet along a line, or three facets meet in a single point.

Importantly, transversality of intersections of LIPs is generic:

Proposition 4.11 (Maclagan and Sturmfels, 2015, Prop. 3.6.12)**.** *For any $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, and generic $\mathbf{v} \in \mathbb{R}^n$, the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2} + \{\epsilon\mathbf{v}\}$ is transverse for all sufficiently small $\epsilon > 0$.*

[22] This definition is independent of the order in which the LIPs are taken; see e.g. Lemma C.8(2).



(a) The grey LIP and black dotted LIP do not intersect transversely at any price.

(b) After a small translation, the intersection is transverse, cf. Prop. 4.11.

Figure 6: Illustration of intersections that are (a) non-transverse; (b) transverse.

For example, a small translation of the black dotted LIP of Fig. 6a by $\{\epsilon(1,0)\}$ yields the transverse intersection, consisting of four isolated points, shown in Fig. 6b.

Now if $u^{[\epsilon]}(\mathbf{x}) := u(\mathbf{x}) + \epsilon\mathbf{v}\cdot\mathbf{x}$, then $\mathcal{L}_{u^{[\epsilon]}} = \mathcal{L}_u + \{\epsilon\mathbf{v}\}$, because $D_{u^{[\epsilon]}}(\mathbf{p}+\epsilon\mathbf{v}) = D_u(\mathbf{p})$. Moreover these translations of LIPs preserve *failure* of equilibrium, for small $\epsilon > 0$:

Proposition 4.12. *If equilibrium fails to exist for valuations u^1 and u^2, and some relevant supply $\mathbf{y}$, then for any $\mathbf{v} \in \mathbb{R}^n$, equilibrium also fails to exist for valuations u^1 and $u^{2[\epsilon]}$, in which $u^{2[\epsilon]}(\mathbf{x}) = u^2(\mathbf{x}) + \epsilon\mathbf{v}\cdot\mathbf{x}$, for supply $\mathbf{y}$ and all sufficiently small $\epsilon > 0$.*

Proof. First consider a specific allocation of $\mathbf{y}$ across two agents. If the agents' valuations are u^1 and u^2, then equilibrium fails. So the two sets of prices at which these agents demand their respective allocations must be disjoint. If either set is in fact empty, then the corresponding set is also empty when we consider the price sets for u^1 and $u^{2[\epsilon]}$ (u^1 is unchanged, and for $u^{2[\epsilon]}$ this follows because $D_{u^{2[\epsilon]}}(\mathbf{p}+\epsilon\mathbf{v}) = D_{u^2}(\mathbf{p})$ for all $\mathbf{p}$) so this allocation of $\mathbf{y}$ also cannot be an equilibrium for the valuations u^1 and $u^{2[\epsilon]}$. So assume neither set is empty. A non-empty set of prices at which an agent demands a specific bundle is a price complex cell, and so is a polyhedron (Prop. 2.7(1)). And if two polyhedra are disjoint, then there is a positive minimum Euclidean distance between any point in one and any point in the other (see, e.g., Gruber, 2007, p. 59).

Let δ be the minimum such distance, over all the pairs of polyhedra corresponding to the finite set of all possible allocations of $\mathbf{y}$ between the agents. Choose ϵ so that $\|\epsilon\mathbf{v}\| < \delta$. Since $D_{u^{2[\epsilon]}}(\mathbf{p}+\epsilon\mathbf{v}) = D_{u^2}(\mathbf{p})$, the polyhedron of prices in which an agent with valuation $u^{2[\epsilon]}$ demands a bundle is shifted by $\{\epsilon\mathbf{v}\}$, compared with the corresponding prices for u^2. But, for any allocation of $\mathbf{y}$ between agents with valuations u^1 and $u^{2[\epsilon]}$, the sets of prices at which these agents demand their allocations are still disjoint, by definition of δ and ϵ. So equilibrium will still fail. □

Now, suppose equilibrium fails for a pair of concave valuations and a relevant supply. By Props. 4.11 and 4.12 there exists a small translation, $\epsilon\mathbf{v}$, such that the intersection is transverse and equilibrium still fails. Demand types are unchanged by translations, so we need only prove sufficiency in Thm. 4.3 for transverse intersections:

Corollary 4.13. *If equilibrium exists for every pair of concave valuations of demand type $\mathcal{D}$ whose LIP intersection is transverse, and every relevant supply bundle, then equilibrium exists for every pair of concave valuations of demand type $\mathcal{D}$ and every relevant supply bundle.*

Another property of transverse intersections that is important for our proof of Thm. 4.3 is that the changes in the bundles considered by agents at any prices in these intersections are in fundamentally different directions, in the sense that:

Definition 4.14. The *linear span of changes in demand*, K_σ, associated to a demand complex cell σ, is the set of linear combinations of vectors in $\{\mathbf{y}-\mathbf{x} : \mathbf{x}, \mathbf{y} \in \sigma\}$.

Lemma 4.15. *Suppose $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ intersect at $\mathbf{p}$, and the two agents' individual demand complex cells at this price are σ^1 and σ^2, while the aggregate-demand complex cell is $\sigma^{\{1,2\}}$. Then the intersection is transverse at $\mathbf{p}$ iff every vector in $\mathrm{K}_{\sigma^{\{1,2\}}}$ can be written* uniquely *as a sum of a vector in K_{σ^1} and a vector in K_{σ^2}.*

In our example of Sections 4.2.1-4.2.2, at the transverse intersection at price $(30, 20)$, we have $\mathrm{K}_{\sigma^{\{1,2\}}} = \mathbb{R}^2$, while $\mathrm{K}_{\sigma^s} = \{\lambda(1,-1) : \lambda \in \mathbb{R}\}$ and $\mathrm{K}_{\sigma^c} = \{\lambda(1,1) : \lambda \in \mathbb{R}\}$. And any vector in $\mathbb{R}^2$ can be uniquely written as $\lambda_1(1,-1) + \lambda_2(1,1)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Lemma 4.15 allows us to apply Fact 4.9(3) to show, in the same way as we developed using our hotel example, that unimodularity is sufficient for equilibrium when intersections are transverse (see Section 4.2.4 below).

4.2.4 Proof of the Unimodularity Theorem

For Thm. 4.3, we need to show that unimodularity is necessary for equilibrium with only two agents. But it is easy to first generalise the situation of the simple example discussed in Sections 4.2.1-4.2.2 above to n agents.

Lemma 4.16. *Suppose $|J| \leq n$ and that $\{\mathbf{v}^j : j \in J\}$ are linearly independent vectors, and that, for $j \in J$, u^j satisfies $D_{u^j}(\mathbf{p}) = \{\mathbf{x}^j, \mathbf{x}^j + \mathbf{v}^j\}$. The set $\{\mathbf{v}^j : j \in J\}$ is unimodular iff every integer bundle in* $\mathrm{conv}(D_{u^J}(\mathbf{p}))$ *is demanded for some price.*

Proof. Observe $D_{u^J}(\mathbf{p}) = \left\{\sum_{j\in J}(\mathbf{x}^j + \delta_j\mathbf{v}^j) : \delta_j \in \{0,1\}; j \in J\right\}$. Since the $\mathbf{v}^j$ are linearly independent, this set is the vertices of a $|J|$-dimensional parallelepiped in $\mathbb{Z}^n$ with edges $\mathbf{v}^j$. By Fact 4.9(2) there exists a non-vertex integer bundle in this parallelepiped iff the set $\{\mathbf{v}^j : j \in J\}$ is not unimodular. The result follows from Lemma 2.19. $\square$

Proof of necessity of unimodularity in Thm. 4.3. Assume we have a non-unimodular linearly independent set of primitive integer vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s \in \mathcal{D}$. Find k such that the set $\mathbf{v}^1, \ldots, \mathbf{v}^k$ is unimodular, but the set $\mathbf{v}^1, \ldots, \mathbf{v}^{k+1}$ is not. ($1 \leq k \leq s-1$, since a single primitive integer vector is a unimodular set, by Fact 4.9(3).) Let $\mathbf{x}^j, \mathbf{x}^j + \mathbf{v}^j$ be the *only* bundles in the domain of agent j's valuation, $j = 1, \ldots, k+1$. Let $J = \{1, \ldots, k\}$. By Lemma 4.8, discrete convexity of any demand set could possibly fail for $\{u^j : j \in J\}$ only at the intersection of two or more of their LIPs. But by Lemma 4.16, this does not happen, since the set $\{\mathbf{v}^j : j \in J\}$ (and any subset thereof) *is* unimodular. So u^J is concave, by Lemma 2.11. But if u^J is the valuation of a new agent k^*, then $u^{\{k^*, k+1\}} = u^{J\cup\{k+1\}}$, for which equilibrium clearly fails (by Lemma 4.16). $\square$

Proof of sufficiency of unimodularity in Thm. 4.3. Cor. 4.13 shows that we can restrict attention to concave valuations whose intersection is transverse. So let u^1 and u^2 be such valuations of a unimodular demand type. By Lemma 4.8 it suffices to show that the demand set is discrete convex at any price $\mathbf{p}$ in their intersection. Write σ^1 and σ^2 respectively for the individual demand complex cells at $\mathbf{p}$, and let $\sigma^{\{1,2\}}$ be the aggregate-demand complex cell at $\mathbf{p}$. We want to show that any integer supply $\mathbf{y} \in \sigma^{\{1,2\}} = \operatorname{conv}(D_{u^{\{1,2\}}}(\mathbf{p}))$ is demanded, that is, also satisfies $\mathbf{y} \in D_{u^{\{1,2\}}}(\mathbf{p})$.

To do this, consider any vertex, $\mathbf{x}$, of $\sigma^{\{1,2\}}$. By Defn. 4.14, the change in demand, $\mathbf{y} - \mathbf{x}$, is in $\mathrm{K}_{\sigma^{\{1,2\}}}$. Fix a basis for $\mathrm{K}_{\sigma^{\{1,2\}}}$, composed of edge vectors of $\sigma^{\{1,2\}}$.[23] These vectors are, equivalently, the normals to the facets of $\mathcal{L}_{u^{\{1,2\}}}$ which contain $\mathbf{p}$ (see Prop. 2.20). So this set is unimodular and, by Fact 4.9(3), $\mathbf{y} - \mathbf{x}$ can therefore be written as an *integer* combination of these vectors. Furthermore, since $\mathcal{L}_{u^{\{1,2\}}} = \mathcal{L}_{u^1} \cup \mathcal{L}_{u^2}$, each of these vectors is normal to a facet of $\mathcal{L}_{u^1}$ or of $\mathcal{L}_{u^2}$. So we can separate the basis vectors into two sets, correspondingly. Then our presentation of $\mathbf{y} - \mathbf{x}$, in terms of this basis, splits as $\mathbf{y} - \mathbf{x} = \mathbf{z}^1 + \mathbf{z}^2$, in which $\mathbf{z}^j \in \mathrm{K}_{\sigma^j}$, for $j = 1, 2$, and the $\mathbf{z}^j$ are integer bundles.

On the other hand, since $\sigma^{\{1,2\}} = \sigma^1 + \sigma^2$ (see e.g. Cox et al. 2005, Section 7.4, Ex. 3) there exist $\mathbf{y}^j \in \sigma^j$, $j = 1, 2$, such that $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$. And since $\mathbf{x}$ is a vertex of $\sigma^{\{1,2\}}$, it is demanded in a UDR adjacent to $\mathbf{p}$, so $\mathbf{x} \in D_{u^{\{1,2\}}}(\mathbf{p})$, i.e. $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$ in which $\mathbf{x}^j \in D_{u^j}(\mathbf{p})$ (and so are integer bundles). So $\mathbf{y} - \mathbf{x} = (\mathbf{y}^1 - \mathbf{x}^1) + (\mathbf{y}^2 - \mathbf{x}^2)$, and we also have $\mathbf{y}^j - \mathbf{x}^j \in \mathrm{K}_{\sigma^j}$, $j = 1, 2$.

So, by transversality (Lemma 4.15), $\mathbf{y}^j - \mathbf{x}^j = \mathbf{z}^j$. And, since we already showed $\mathbf{x}^j$ and $\mathbf{z}^j$ are integer, it follows that $\mathbf{y}^j$ are also integer, for $j = 1, 2$.

So we have $\mathbf{y}^j \in \sigma^j = \operatorname{conv}(D_{u^j}(\mathbf{p}))$ and $\mathbf{y}^j \in \mathbb{Z}^n$. But since u^j is concave, its demand sets are all discrete convex (Lemma 2.11) and so $\mathbf{y}^j \in D_{u^j}(\mathbf{p})$, for $j = 1, 2$. Therefore, since $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$, we can conclude $\mathbf{y} \in D_{u^{\{1,2\}}}(\mathbf{p})$, as required.[24] □

4.3 Related Work

Danilov et al. (2001) have developed results that are very closely related to our Thm. 4.3. In particular, their Thms. 3 and 4 together provide a *sufficient* condition for equilibrium, which is analogous to our condition on demand types.[25] (Howard's (2007, Thm. 1) subsequent work is equivalent to Thm. 4 of Danilov et al.)

However, the economic interpretation or usefulness of this condition is not clear. By contrast, our Thm. 4.3 both demonstrates the applicability of the result, and clarifies the connections to existing economic results. We will see in Section 6 that our Thm. 4.3 generalises many results in well-known work *subsequent to* Danilov et al.'s, including results in Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014).[26]

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[23] That such a basis exists follows from, e.g., combining Gruber (2007) Thms. 14.2 and 15.8.

[24] Ex. C.9 in Appendix C.3 provides additional intuition for the sufficiency condition by illustrating the failure of this argument for the simple example discussed in Sections 4.2.1-4.2.2.

[25] Their Thm. 3 shows that equilibrium is guaranteed if the valuations are "$\mathscr{D}$-concave" for some "class of discrete convexity" $\mathscr{D}$. (This notation $\mathscr{D}$ is *not* connected with our use of $\mathcal{D}$ to represent demand types.) "$\mathscr{D}$-concave" valuations are concave valuations such that every demand set $D_u(\mathbf{p})$ belongs to a specified set "$\mathscr{D}$" of subsets of $\mathbb{Z}^n$. A collection of such sets is a "class of discrete convexity" if every set it contains, and every sum and every difference of these sets, is discrete convex.

Their Thm. 4 (which is proved by Danilov and Koshevoy, 2004, Thm. 2) is that $\mathscr{D}$ is a class of discrete convexity if the edges of the convex hulls of the sets in $\mathscr{D}$ form a unimodular set of vectors. But this is true if, in our language, all the valuations are of the demand type defined by this unimodular set. So the sufficiency part of our Thm. 4.3 follows from combining their two results.

[26] Danilov et. al.'s lack of our notion of demand types or of any economic interpretation of their "$\mathscr{D}$-

Danilov et al. also prove no *necessity* result. Because they have no concept of "demand types", and have not developed their definition as a taxonomy of demand, there is no natural result for them to give. But once our concept of demand types is introduced, a necessity result can easily be developed from their work (using our Lemma 2.19).

However, our methods seem simpler and more accessible to economists than Danilov et al.'s *extremely* advanced mathematics. So we prefer the proof we have given above (which we developed before we knew of their work). Tran and Yu (2017) provide another proof, via integer programming, in their recent exposition of our work.

Danilov et al. also state their results under different assumptions from ours. They assume the domain, A, of every agent's valuation is contained in $\mathbb{Z}^n_{\geq 0}$, which precludes, for example, the application to agents who both buy *and* sell, which our more general assumption permits. For example, our model, unlike theirs, applies to (and extends) Hatfield et al. (2013)–see Section 6.4.

Finally, our techniques lead to an additional set of results about when equilibrium exists for *specific* valuations; the next section turns to these.

5 The Intersection Count Theorem–when does Equilibrium exist for *Specific* Valuations?

The Unimodularity Theorem (Thm. 4.3) tells us exactly which demand types *always* have a competitive equilibrium. But even when equilibrium is not guaranteed to exist for a demand type, it of course exists for many *specific* combinations of valuations. Our Intersection Count Theorem (Thm. 5.1) concerns which combinations these are.

The key to the Unimodularity Theorem was to understand which demand types permitted "over-large" volumes to *potentially* arise in the demand complex; thus the key to the Intersection Count Theorem is to understand whether such large volumes *in fact* arise for agents' actual valuations.

As in Section 4.2's development of the Unimodularity Theorem, we need only analyse certain isolated points in the LIP intersection. Here we refine our earlier analysis of such points with our more general Subgroup Indices Theorem, which clarifies which points relate to cells which are "over-large" and for which points equilibrium is locally guaranteed. Furthermore, tropical intersection theory bounds the *number* of these points; a simple count of them often suffices to tell us whether there are any such "over-large" volumes, and hence demonstrate the existence or failure of equilibrium.

Section 5.1 provides a preview and explanation of the Intersection Count Theorem, focusing on the two-dimensional case. Sections 5.2-5.3 state the full theorem (Thm. 5.1), together with additional definitions needed for it. The theorem's proof is sketched in Section 5.4, alongside an explanation of "subgroup indices" and the Subgroup Indices Theorem (Thm. 5.19). Section 5.5 explains the limitations of the Intersection Count Theorem, but gives a small extension for transverse cases.

concavity", and the presentation of their work in relatively unfamiliar terms (namely the relationships between sets of primitive integer vectors which are parallel to edges of specific collections of integral pointed polyhedra and their "classes of discrete convexity") seem to have resulted in leading economists (as well as us) having been unaware of their work or of its implications.

5.1 Preview, and Explanation, of the Theorem

5.1.1 The simple "Hotel Rooms" example

Recall the introduction's hotel example. As discussed in Sections 4.2.1-4.2.2, equilibrium fails to exist when the supply is $(1,1)$ for the valuations u^s and u^c, illustrated in Figs. 4a-b. And this is shown by the fact that, if we look at the only price in the intersection of these LIPs, the dual demand complex cell has area 2, which exceeds 1 (see Fig. 5c).

But equilibrium exists for every supply for valuations u^s and u^{c*}, which have the same aggregate demand type. The two demand complex cells dual to the two prices at which their LIPs intersect both have area 1 (see Fig 5d). It is not a coincidence that the number of the points in the intersection, weighted by the corresponding areas, is constant: this follows from the tropical version of the Bernstein-Kouchnirenko-Khovanskii (BKK) Theorem (which we will state below, see Thm. 5.22).[27]

Specifically, in 2 dimensions, consider any two valuations with individual domains, A^1 and A^2, whose LIPs intersect transversely. The tropical BKK Theorem tells that *the number of price points at which the LIPs intersect, weighting each point by the* area *of its dual aggregate-demand complex cell, is a (known) constant.* We will call this constant $M^2(A^1, A^2)$.

Fig. 5 illustrates why the tropical BKK Theorem holds. We wish to calculate the total area of aggregate-demand complex cells that are dual to intersections of the individual LIPs. But the total area of any 2-dimensional aggregate-demand complex is the sum of the areas of its 2-cells. These 2-cells are dual, of course, to the 0-cells of the corresponding (aggregate) LIP. But each 0-cell of that LIP is either a 0-cell of one of the individual LIPs, or a point where the two individual LIPs intersect. (It cannot be both, because the intersection is transverse.) But the 0-cells of the individual LIPs are dual, in turn, to the 2-cells of their individual demand complexes. That is, the triangular 2-cells in each of Figs. 5c and 5d are exactly the collection of 2-cells in Figs. 5a and 5b (taken together). So the total area of the triangles in each of Figs. 5c and 5d is the sum of the areas of the individual demand complexes, that is the sum of the areas of $\mathrm{conv}(A^1)$ and $\mathrm{conv}(A^2)$. Meanwhile, the total area of each of the aggregate-demand complexes of Figs. 5c and 5d is the area of $\mathrm{conv}(A^1 + A^2)$. So in each of Figs. 5c and 5d, the total area of all the aggregate-demand complex cells that are dual to intersection prices, must equal *the area of* $\mathrm{conv}(A^1+A^2)$ *minus the sum of the areas of* $\mathrm{conv}(A^1)$ *and* $\mathrm{conv}(A^2)$, which is a constant; this is the constant we defined as $M^2(A^1, A^2)$ in the previous paragraph.

Moreover, in two dimensions, a price at which the intersection between two LIPs transverse is an intersection of two facets, and so the dual demand complex cell is a

[27] As noted in the introduction, Bézout (1779) showed that in two dimensions the number of intersection points of two (ordinary) geometric curves equals the product of their degrees (accounting for "multiplicities"–see Note 5). Bernstein (1975), Kouchnirenko (1976) and Khovanskii (1978) extended the theorem, including to higher dimensions. LIPs are particular limits of logarithmic transformations of algebraic hypersurfaces (in complex projective space) see, e.g., Maclagan and Sturmfels (2015), and analogous intersection theorems hold. Moreover, because a LIP is in real (not complex) space, we can "see" the intersection points.

A "Tropical Bézout-Bernstein Theorem" was first stated by Sturmfels (2002), and the full "tropical BKK theorem" could at that point be derived from earlier results (see, e.g., McMullen, 1993, Huber and Sturmfels, 1995) as we observe in Appendix B. However, a clear presentation of the full version was first provided by Bertrand and Bihan (2013), whose conventions we will follow.

parallelogram. It therefore has area at least 1. It follows that the number of points in the LIP intersection is bounded above by $M^2(A^1, A^2)$. Suppose the intersecting facets both have weight 1. It then follows from Sections 4.2.2-4.2.4 (by combining the discussion following Fact 4.9 with Lemma 4.16) that equilibrium exists at this intersection, if and only if the corresponding area exactly 1.

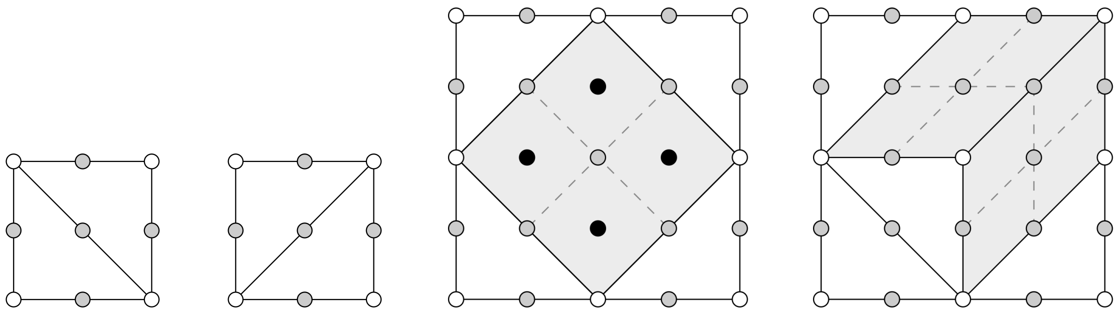
So, in two dimensions, if the valuations u^1 and u^2 are concave, all facets are weight 1, and the LIP intersection is transverse: (i) there are at most $M^2(A^1, A^2)$ points in the LIP intersection, and (ii) equilibrium exists for every possible supply if and only if there are exactly $M^2(A^1, A^2)$ points in this intersection.

In the hotel example, the domain of each individual valuation is $\{0,1\}^2$, so the aggregate domain is $\{0,1,2\}^2$, and $M^2(\{0,1\}^2, \{0,1\}^2) = 4 - 1 - 1 = 2$. Moreover, the intersection is transverse and the facets are weight 1. So the previous paragraph tells us that there are at most two points in the intersection, and equilibrium exists for every supply if and only if there are exactly two points in the intersection (which results are consistent with the cases analysed above).

Our full Intersection Count Theorem develops these ideas in several directions:

5.1.2 Facet Weights

Reconsider the hotel example, but with weight 2 on every facet: let concave valuations u^{2s}, u^{2c} and u^{2c*}, have the same LIPs as u^s, u^c and u^{c*}, respectively, but with weight 2 on every facet. (So, for example u^{2s} is equivalent to an aggregate valuation of two identical copies of Elizabeth.) The demand complexes of u^{2s}, u^{2c}, $u^{\{2s,2c\}}$ and $u^{\{2s,2c*\}}$ are pictured in Fig. 7. (The bundles are colour-coded as before, see figure caption.)



(a) $\Sigma_{u^{2s}}$ (b) $\Sigma_{u^{2c}}, \Sigma_{u^{2c*}}$ (c) $\Sigma_{u^{\{2s,2c\}}}$ (d) $\Sigma_{u^{\{2s,2c*\}}}$

Figure 7: Demand complexes dual to the LIPs in Figs. 4a-d, if every facet has weight 2. The cells dual to intersection prices of the LIPs are shaded. (The dashed lines show they are grids of copies of cells from Figs. 5c-d.) Bundles uniquely demanded for some price are white; those never demanded are black; the remainder, demanded non-uniquely because the valuation is locally linear, are grey. (As usual, we present the bundles increasing from top to bottom, and from right to left.)

The demand set $D_{u^{2s}}(30, 20)$ is $\{(2,0),(1,1),(0,2)\}$: the non-vertex bundle in the diagonal demand complex 1-cell in Fig. 7a *is* demanded at this price, because the valuation is concave–indeed it is locally linear. Similarly, $D_{u^{2c}}(30, 20) = \{(0,0),(1,1),(2,2)\}$. So aggregate demand at the intersection price of $\mathcal{L}_{u^{2s}}$ and $\mathcal{L}_{u^{2c}}$ is $D_{u^{\{2s,2c\}}}(30, 20) = \{(2,0),(1,1),(0,2)\} + \{(0,0),(1,1),(2,2)\}$. Thus the vertices of the cell highlighted in Fig. 7c, $(2,0),(0,2),(2,4),(4,2)$, the bundles on the mid-points of its edges, $(1,1)$, $(1,3)$,

$(3,1)$, $(3,3)$, and its central bundle, $(2,2)$, are all demanded at $(30,20)$, while the cell's remaining four bundles are never demanded.

Observe that this cell is therefore just a grid of $2 \times 2 = 4$ copies of the central cell of Fig. 5c, as shown by the dashed lines in Fig. 7. This corresponds to the fact that the relevant 1-cells of the individual demand complexes have "length" 2 (Defn. 2.15(5)), that is, each of the facets of the individual LIPs have weight 2. The intuition for which bundles of the central cell of Fig. 7c are demanded is exactly as for the central cell of Fig. 5c in Section 4.2.2–the issue is which bundles can be reached from a vertex by an integer combinations of vectors that are (primitive) edge vectors of the cell.

Likewise, at the two prices at which the LIPs $\mathcal{L}_{u^{2s}}$ and $\mathcal{L}_{u^{2c*}}$ meet, the dual cells of the aggregate-demand complex $\Sigma_{u^{\{2s,2c*\}}}$ (shaded in Fig. 7d) are each $2 \times 2 = 4$ copies of their corresponding cell in $\Sigma_{u^{\{s,c*\}}}$ (see Fig. 5d).

Ex. C.10 gives further details on both these cases.

This result is general: if the intersection is transverse, then multiplying any one facet weight multiplies the area of the demand complex cell by the same factor, without affecting the existence or otherwise of equilibrium for concave valuations. So, applying this to the discussion of the previous subsection, in the two-dimensional transverse intersection case: (i) the weighted count of points in the LIP intersection, where each point is weighted by *the product of the weights of the facets passing through it* is bounded above by $M^2(\cdot,\cdot)$; and (ii) equilibrium exists for every relevant supply for two concave valuations if and only if this bound holds with equality. Thus we will weight each intersection by its "naïve multiplicity", which is the product of its facet weights in the two-dimensional transverse case (Defn. 5.13 gives the general case).[28]

It is easy to check for the example of Fig. 7 that $M^2(A^{2s}, A^{2c}) = 8$, and that each intersection point has weight 4. So there are at most two points in the LIP intersection, and equilibrium is guaranteed if and only if there are *exactly* 2, just as before.

5.1.3 Non-Transverse Intersections

As in our development of the Unimodularity Theorem in Section 4, we can handle non-transverse intersections by considering the effects of small perturbations.

Return to the weight-one example of Figs. 4a-b, but modify the complements valuation to $u^{c\#}(x_1, x_2) = \min\{70x_1, 70x_2\}$. Then $\mathcal{L}_{u^{c\#}}$ intersects $\mathcal{L}_{u^s}$ non-transversely, exactly through its 0-cell at (40,30). This case is intermediate between those illustrated in Figs. 4c-d. So the aggregate-demand complex is like that of Fig. 5c, but without the edge that includes the bundles (1,0) and (0,1); equivalently, the aggregate-demand complex is like that of Fig. 5d, but without any of the three edges that include the bundle (1,1).

If we translate $\mathcal{L}_{u^{c\#}}$ by $\epsilon(1,1)$, for small $\epsilon > 0$, we return to the situation of Fig. 4d. So, by Prop. 4.12, equilibrium exists for all supplies: the bundle (1,1) is now "grey". So our count will give us the "right" result for this non-transverse case if we weight the intersection point by the sum of the weights that apply after *this* translation, that is, by 2: then the weighted count equals $M^2(\{0,1\}^2, \{0,1\}^2)$, as with the case of Fig. 4d.

But if we had translated in the other direction, and returned to the Fig. 4c case, the sum of the weights would have been only 1, that is, "too low". We therefore define the

[28]We call these multiplicities "naïve" because they can be distinct from the geometric "true multiplicities" (Defn. 5.21); indeed it is this distinction that allows us to identify equilibrium failure.

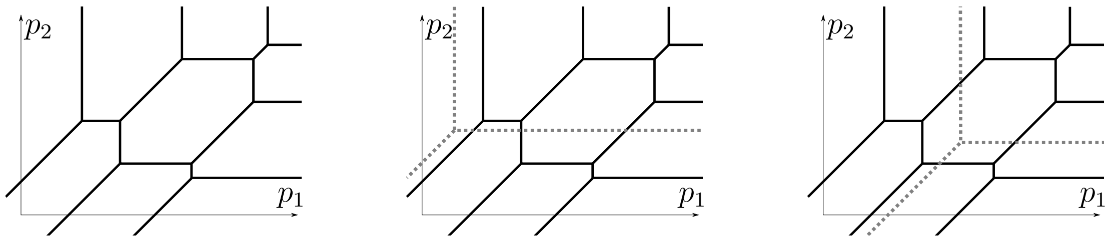
naïve multiplicity in this case by using the translation that yields the maximum possible sum of weights.

Weighting non-transverse intersection points in this way handles situations like this one. As in the transverse case, $M^2(\cdot,\cdot)$ is an upper bound on the possible count. That is, a weighted count equal to $M^2(\cdot,\cdot)$ remains sufficient, though, we will see, no longer necessary for equilibrium to exist for all supplies.

5.1.4 Illustration: the case of strong substitutes

An example of the power of our approach is that it provides an elegant illustration of Prop. 4.6 that strong substitutes valuations always have equilibrium:

Consider agents $j = 1, 2$ wanting up to d_j units, respectively, in total, of two goods. Figs. 8b and 8c show two different cases for the LIPs of generic "strong substitutes" valuations for $d_1 = 1, d_2 = 3$. Observe that the LIP intersection contains $1 \times 3 = 3$ points in both cases.[29] It is not hard to check that this will remain true after any generic translation of either LIP. Moreover, since all "strong substitutes" facet normals for two goods are in $\pm\{(1,0), (0,1), (-1,1)\}$ (Prop. 3.10), all generic "strong substitutes" LIPs have similar "honeycomb" structures, so it is a general result that any intersection has $d_1 d_2$ points. It is also straightforward that the area of conv(A^j) is $d_j^2/2$ and that of conv$(A^1 + A^2)$ is $(d_1 + d_2)^2/2$, so $M^2(A^1, A^2) = d_1 d_2$ (see also Fact 5.15(2)). So equilibrium exists for all supplies in the generic case.



(a) LIP of a strong substitutes valuation for up to 3 units.

(b) The LIP from (a), and a LIP of a valuation for up to 1 unit.

(c) As (b), but with a different valuation for up to 1 unit.

Figure 8: The LIPs of two generic strong substitutes valuations, one for up to 3 units, and one for a single unit, always intersect exactly $3 \times 1 = 3$ times.

For non-generic cases, it is obvious from Figs. 8b and 8c, that any translation of a non-transverse LIP intersection to create a transverse intersection yields the count $d_1 d_2$. Furthermore if, e.g., all the facets of the LIP in Fig. 8a had weight w, we would obtain the "correct" count $w d_1 d_2$. So the argument can be extended to confirm equilibrium always exists in non-generic cases too.

5.1.5 Higher Dimensions

Handling $n > 2$ dimensions is harder. First, intersections have dimension at least $n-2$, so do not consist of isolated points when $n > 2$. However, we will show it is sufficient to focus our analysis on certain isolated points in their intersection, namely "intersection

[29] The LIP of any valuation for up to d units in total is the tropical transformation of an "ordinary" polynomial of degree d. So Figs. 8a-c show tropical cubics and lines that intersect $3 \times 1 = 3$ times.

0-cells"(Defn. 5.3). Second, even if the intersection is transverse at such prices, the dual cell in the demand complex need not be a parallelepiped. Correspondingly, the individual LIP cells at an intersection point need not be facets, so to define "naïve multiplicities", we follow the mathematical literature in extending our definition of facet "weights" to cover all cells of the LIP.

As in two dimensions, our results can be thought of in terms of whether or not there are problematic bundles within appropriate parallelepipeds. But as in Section 4.2.4 (and the second intuition of Section 4.2.2), we use the alternative equivalent definition of unimodularity (Fact 4.9(3)): whether all integer bundles in a demand complex cell can be reached by combinations of appropriate vectors.

Our key tool will be "subgroup indices". These are used by mathematicians to generalise the 2-dimensional tropical BKK Theorem to higher dimensions. However, our Subgroup Indices Theorem (Thm. 5.19) shows that they are *also* intimately connected with the existence of equilibrium. Thus, the tropical BKK theorem tells us that, for all n, there is an upper bound, $M^n(A^1, A^2)$ (Defn. 5.14), for an appropriately weighted count of intersection 0-cells. We can then use our Subgroup Indices Theorem to see that the weighted count equalling this bound is a sufficient, and for $n \leq 3$ a necessary, condition for equilibrium to always exist–this gives us the general version of our Intersection Count Theorem (Thm. 5.1).

5.2 The Intersection Count Theorem

We delay the (rather involved) definitions for intersection 0-cells, naïve multiplicities and $M^n(A^1, A^2)$ to Section 5.3, stating the theorem first so that the end-point is in sight.

Theorem 5.1 (**The Intersection Count Theorem**). *For $j = 1, 2$, let u^j be concave valuations, on finite domains $A^j \subsetneq \mathbb{Z}^n$ such that* $\dim \operatorname{conv}(A^1 + A^2) = n$*. Then the number of intersection* 0*-cells for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, counted with naïve multiplicities, is bounded above by $M^n(A^1, A^2)$. If the number equals this bound, equilibrium exists for all relevant supplies.*

Suppose additionally that the intersection is transverse and that $n \leq 3$. The number of intersection 0*-cells for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, counted with naïve multiplicities, is equal to $M^n(A^1, A^2)$ iff equilibrium exists for all relevant supplies.*

We will see that naïve multiplicities are always at least 1 (see Section 5.3.3), and that there are never more 0-cells in $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$ than there are "intersection 0-cells" (see Section 5.3.1), so a weaker—but sometimes more immediately usable–result follows:

Corollary 5.2 (**The Weak Intersection Count Theorem**). *For $j = 1, 2$, let u^j be concave valuations, on finite domains $A^j \subsetneq \mathbb{Z}^n$ such that* $\dim \operatorname{conv}(A^1 + A^2) = n$*. If the number of* 0*-cells in $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$ equals $M^n(A^1, A^2)$, then equilibrium exists for all relevant supplies.*

The Weak Intersection Count Theorem suffices to demonstrate equilibrium existence in the hotel and the strong substitutes cases illustrated in Figs. 4-5, and Fig. 8, respectively (as discussed in Sections 5.1.1 and 5.1.4). Of course, we can also analyse these examples by examining each cell in the intersections and applying, locally, the same logic that we used to prove the unimodularity theorem.[30] But our results mean that we only need to *count*, instead of analyse, these cells. More important, a local analysis may not suffice to determine whether equilibrium exists for examples in which $n > 2$, while an intersection count may do so (see Ex. C.11).

Moreover, as discussed in Section 6.3, the Intersection Count Theorem (and Cor. 5.2) not only tell us about the existence of equilibrium for *every* relevant supply, for a given combination of valuations, but also illuminate how equilibrium varies within *families* of closely-related combinations of valuations; by contrast, the theorems of Bikhchandani and Mamer (1997) and Ma (1998) require testing *every* valuation of interest, for *every* supply of interest, *separately.*

5.3 Definitions for the Theorem

5.3.1 Intersection 0-cells

Recall that the existence of equilibrium depends on whether the aggregate demand set is discrete convex at all prices in the LIP intersection (Lemma 4.8). For more than two goods, and for some non-transverse intersections, these prices are not a set of isolated points. However, it will suffice to focus our analysis on a particular (finite) set of 0-cells in the intersection:

Definition 5.3. An *intersection* 0*-cell for LIPs* $\mathcal{L}_{u^1}$ *and* $\mathcal{L}_{u^2}$ is a 0-cell of their aggregate LIP, $\mathcal{L}_{u^{\{1,2\}}}$, contained in the intersection $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$.

Clearly, every 0-cell of $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$ is an intersection 0-cell. If the intersection is transverse, then all intersection 0-cells are of this kind, but if not, there may be additional intersection 0-cells. For example, if u^1 and u^2 are defined on $\{0, 1\}^2$ by $u^1(x_1, x_2) = x_1+x_2$, and $u^2(x_1, x_2) = x_1$, then $\mathcal{L}_{u^1}$ has two horizontal and two vertical facets, all meeting in one 0-cell (a vertical cross), while $\mathcal{L}_{u^2}$ just has one vertical facet, coinciding with the union of the two vertical facets of $\mathcal{L}_{u^1}$. So $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2} = \mathcal{L}_{u^2}$ which contains no 0-cells, but the 0-cell of $\mathcal{L}_{u^1}$ is an intersection 0-cell for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$.

Proposition 5.4. *If two individual valuations are concave, and their aggregate-demand complex has dimension* n*, then equilibrium exists for every relevant supply bundle iff the aggregate demand set is discrete-convex at every intersection* 0*-cell.*

If the aggregate-demand complex is n'-dimensional for $n' < n$, we can simply make a unimodular basis change so that its linear span is the span of the first n' coordinate directions. This transforms the problem to an equivalent one with n' new "goods", and the proposition can then be applied. (Otherwise we would have to analyse higher-dimensional cells.)

5.3.2 Parallel Lattices, and Cell Weights

As we saw in Section 5.1.2, the facets weights provide a measure of the "relative size" of the dual demand complex cell. To generalise this to lower-dimensional cells, we need a "lattice-volume" of a polytope analogous to the "length" of Defn. 2.15(5).

[30] The proof of sufficiency in Thm. 4.3 (Section 4.2.4) showed that aggregate demand was discrete convex at a price $\mathbf{p}$ by considering the normal vectors to "local" facets: those which contained $\mathbf{p}$.

Definition 5.5 (See, e.g. Cassels, 1971)**.**

(1) A *lattice* is a set $\Lambda \subseteq \mathbb{Z}^n$ such that $\mathbf{0} \in \Lambda$ and if $\mathbf{v}, \mathbf{v}' \in \Lambda$ then $\mathbf{v} - \mathbf{v}' \in \Lambda$.[31]
(2) Λ' is a *sublattice* of Λ if $\Lambda' \subseteq \Lambda$ and Λ' has the structure of a lattice.
(3) An *(integer) basis* for a lattice Λ is a set $\{\mathbf{v}^1, \ldots, \mathbf{v}^k\}$ such that any $\mathbf{v} \in \Lambda$ can be uniquely presented as $\mathbf{v} = \sum_j \alpha_j \mathbf{v}^j$ for $\alpha_j \in \mathbb{Z}$.[32]
(4) The *linear span* (in $\mathbb{R}^n$) of a lattice $\Lambda \subseteq \mathbb{Z}^n$ is the set of all linear combinations of finite subsets of Λ, and is written K_Λ.
(5) The *rank* of a lattice is the dimension of its linear span.

Fact 5.6. Every lattice has an integer basis, of size equal to the lattice rank.

The lattices important to us are (recalling–Defn. 4.14–that K_σ denotes the linear span of changes in demand associated with σ):

Definition 5.7 (See e.g. Maclagan and Sturmfels, 2015)**.** The *parallel lattice* to a demand complex cell σ is $\Lambda_\sigma := \mathrm{K}_\sigma \cap \mathbb{Z}^n$.

It is easy to see that if σ is a k-cell then Λ_σ has rank k, and that $K_{\Lambda_\sigma} = \mathrm{K}_\sigma$.

For our hotel example (Figs. 4 and 5), if we let σ^s, σ^c and $\sigma^{\{s,c\}}$ be the demand complex cells at $\mathbf{p} = (30, 20)$ of u^s, u^c and $u^{\{s,c\}}$ respectively, then $\Lambda_{\sigma^s} = \{m(-1, 1) : m \in \mathbb{Z}\}$, and $\Lambda_{\sigma^c} = \{m(1, 1) : m \in \mathbb{Z}\}$, while $\Lambda_{\sigma^{\{s,c\}}} = \mathbb{Z}^2$ (see Fig. 9 in Section 5.4.1). The parallel lattices are the same in the weight two version of this example (Section 5.1.2): at $\mathbf{p} = (30, 20)$ and cells σ^{2s}, σ^{2c} and $\sigma^{\{2s,2c\}}$ of $\Sigma_{u^{2s}}, \Sigma_{u^{2c}}$ and $\Sigma_{u^{\{2s,2c\}}}$ respectively, $\Lambda_{\sigma^{2s}} = \Lambda_{\sigma^s}$, $\Lambda_{\sigma^{2c}} = \Lambda_{\sigma^c}$ and $\Lambda_{\sigma^{\{2s,2c\}}} = \Lambda_{\sigma^{\{s,c\}}}$.

We now generalise the "weight" of Defn. 2.3. Given a rank-k lattice Λ, we can find a $k \times n$ matrix G_Λ such that $G_\Lambda \Lambda := \{G_\Lambda \mathbf{v} : \mathbf{v} \in \Lambda\} = \mathbb{Z}^k$.[33] We can use this identification to give volumes *relative* to the lattice Λ (as usual, the k-dimensional volume of $X \subsetneq \mathbb{R}^k$ is $\mathrm{vol}_k(X) := \int \cdots \int_X 1 \, dp_1 \ldots dp_k$):

Definition 5.8 (See e.g. Bertrand and Bihan, 2013)**.** If $X \subsetneq \mathbb{R}^n$ is a polytope with vertices in Λ, define the *lattice-volume* of X in Λ as $\mathrm{vol}_\Lambda(X) := \mathrm{vol}_k(G_\Lambda X)$, where G_Λ is a $k \times n$ matrix such that $G_\Lambda \Lambda = \mathbb{Z}^k$.

It is standard that this volume is independent of the choice of G_Λ: if also $\hat{G}_\Lambda \Lambda = \mathbb{Z}^k$ then the images of X under the two transformations are related by a change of basis matrix on $\mathbb{Z}^k$, which must be unimodular. So the volumes of the images are the same.

For example, given Λ_{σ^s} as above, we may set $G_{\Lambda_{\sigma^s}} = (-\frac{1}{2}, \frac{1}{2})$. Then $G_{\Lambda_{\sigma^s}}(-1, 1)' = 1$ and so, since $(-1, 1)$ is a basis for Λ_{σ^s}, it follows that $G_{\Lambda_{\sigma^s}} \Lambda_{\sigma^s} = \mathbb{Z}$. That is, operating with $G_{\Lambda_{\sigma^s}}$ on the lattice Λ_{σ^s} "rotates it" to "match it up" with $\mathbb{Z}$. So $G_{\Lambda_{\sigma^s}} \sigma^s$ is just the interval $[0, 1]$, which has length (1-dimensional volume) equal to 1, that is, its facet weight according to Defn. 2.3. In general, σ does not lie in K_σ, but for any $\mathbf{x} \in \sigma$, the shifted cell $\sigma' := \sigma + \{-\mathbf{x}\}$ lies in K_σ, and clearly has the same volume as σ.

[31] This group-theoretic meaning of "lattice" is completely different from the order-theoretic meaning of, e.g., Milgrom and Shannon (1994).

[32] We will refer to "*integer* bases" rather than just "bases" when there is ambiguity.

[33] Specifically, if H_Λ is an invertible $n \times n$ matrix whose first k columns give a basis $\{\mathbf{v}^1, \ldots, \mathbf{v}^k\}$ for Λ, then $H_\Lambda \mathbf{e}^i = \mathbf{v}^i$ and so $H_\Lambda^{-1} \mathbf{v}^i = \mathbf{e}^i$ for $i = 1, \ldots, k$: we set G_Λ to be the first k rows of H_Λ^{-1}.

Definition 5.9 (See e.g. Bertrand and Bihan, 2013)**.** Let C_σ be an $(n-k)$-cell of the LIP, dual to the demand complex k-cell σ. The *weight* of C_σ is $w_u(C_\sigma) := k!\text{vol}_{\Lambda_\sigma}(\sigma + \{-\mathbf{x}\})$, where $\mathbf{x} \in \sigma$.

So if $k = 1$ the weight of a cell is just its "length" in terms of the primitive integer vector in the direction of its demand complex cell. Thus this definition generalises Defn. 2.3. For example, $w_{u^c}(C_{\sigma^c}) = w_{u^s}(C_{\sigma^s}) = 1$ and $w_{u^{2c}}(C_{\sigma^{2c}}) = w_{u^{2s}}(C_{\sigma^{2s}}) = 2$.

The factor of $k!$ ensures that the cell weight is an integer (all cells are measured relative to a lattice simplex). So in our hotel example, the weights of the 0-cells of $\mathcal{L}_{u^s}$ are both 1 (see Figs. 4a and 5a). Since the lattice-volume of the central cell in the aggregate-demand complex, $\sigma^{\{s,c\}}$, is 2, this means that $w_{u^{\{s,c\}}}(C_{\sigma^{\{s,c\}}}) = 4$. Similarly, $w_{u^{\{2s,2c\}}}(C_{\sigma^{\{2s,2c\}}}) = 16$.

5.3.3 Naïve Multiplicities

Definition 5.10. If the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse at an intersection 0-cell C, define the *naïve multiplicity* $\hat{m}(C) := w_{u^1}(C^1) \cdot w_{u^2}(C^2)$, where C^j is the smallest cell of $\mathcal{L}_{u^j}$ containing C, for $j = 1, 2$.

To extend this to non-transverse cases, we first write:

Definition 5.11. If C is an intersection 0-cell for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, and if C' is an intersection 0-cell for $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\}+\mathcal{L}_{u^2}$ (for any $\epsilon > 0$ and $\mathbf{v} \in \mathbb{R}^n$), then we say C' *emerges from* C if there exist cells C^j of $\mathcal{L}_{u^j}$ for $j = 1, 2$, such that $C^1 \cap C^2 = C$ and $C^1 \cap (\{\epsilon\mathbf{v}\} + C^2) = C'$.

There can be several intersection 0-cells emerging from C under the same translation; for example, the intersection 0-cells at $(4, 1 + \epsilon)$ and $(4 + \epsilon, 1)$ in Fig. 6b emerge from the intersection 0-cell at $(4, 1)$ in Fig. 6a. There is always at least one:

Lemma 5.12. *For every $\mathbf{v} \in \mathbb{R}^n$ and sufficiently small $\epsilon > 0$, there exists an intersection 0-cell for $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ emerging from every intersection 0-cell for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$.*

Now if $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ intersect transversely, each of their intersection 0-cells has a naïve multiplicity. Moreover, for fixed $\mathbf{v}$ and small enough $\epsilon > 0$, the set of these multiplicities, and therefore also the sum of these multiplicities, is independent of ϵ. But as there are only finitely many cells in each LIP, there are only finitely many different sums that can be obtained in this way.

Take, for example, the intersection 0-cell at $(4, 1)$ in Fig. 6a. When $\mathbf{v} = (1, 0)$, as shown in Fig. 6b, we obtain intersection 0-cells whose naïve multiplicities sum to 2. But for $\mathbf{v} = (-1, 0)$, only one intersection 0-cell, with naïve multiplicity 1, would have emerged. We will always want the *maximum* of the sums, so:

Definition 5.13. The *naïve multiplicity* $\hat{m}(C)$ at an intersection 0-cell C for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is the maximum number that can be obtained by adding the naïve multiplicities of 0-cells emerging from C under a small translation of $\mathcal{L}_{u^2}$ which makes the intersection transverse at C.

Because the weight of a cell is always a positive integer, the same is true of naïve multiplicities.

5.3.4 General definition of $M^n(\cdot,\cdot)$

Definition 5.14. If $A^1, A^2 \subsetneq \mathbb{Z}^n$ are finite then, for $k = 1, \ldots, n-1$, define:

(1) $M^n_k(A^1, A^2) := \sum_{r=0}^{k} \sum_{s=0}^{n-k} (-1)^{n-r-s} \binom{k}{r} \binom{n-k}{s} \mathrm{vol}_n \, \mathrm{conv}(rA^1 + sA^2)$.

(2) $M^n(A^1, A^2) := \sum_{k=1}^{n-1} M^n_k(A^1, A^2)$

Thus $M^n(A^1, A^2)$ is a linear combination of ordinary volumes. In many of the cases that matter most in economics, it is easy to calculate:

Facts 5.15 (See e.g. Cox et al, 2005)**.** If $A^1, A^2 \subsetneq \mathbb{Z}^n$ are finite then, for $k = 0, \ldots, n$:

(1) $M^n_k(A, A) = n!\mathrm{vol}_n \, \mathrm{conv}(A)$.

(2) If $A^j = \{\mathbf{x} \in \mathbb{Z}^n_{\geq 0} : \sum_i x_i \leq d_j\}$ for $j = 1, 2$ then $M^n_k(A^1, A^2) = d_1^k d_2^{n-k}$

That is, the calculations are trivial if every agent j's valuation is over all bundles containing at most d_j goods (e.g., for $n = 2$, $M^2(A^1, A^2) = d_1 d_2$), or if every agent's valuation has the same domain (assuming this domain's volume is easy to compute). Moreover, $M^n(A^1, A^2) = 0$ whenever $\dim \mathrm{conv}(A^1 + A^2) < n$ (see Defn. 5.14(1)). More generally, $M^n(A^1, A^2)$ is always a non-negative integer–see Appendix B which provides an alternative characterisation of $M^n_k(A^1, A^2)$ as a "mixed volume" and also clarifies its role in the tropical BKK Theorem (Thm 5.22 below). See also Remark A.2.

5.4 Subgroup Indices and the Proof of the Intersection Count Theorem

This section explains the proof of Thm. 5.1. First, we show a fundamental connection between existence of competitive equilibrium, and an associated "subgroup index"; this Subgroup Indices Theorem expresses the underlying mathematics determining whether equilibrium exists. We then state the Tropical Bernstein-Kouchnirenko-Khovanskii Theorem, and connect it to our context. This section is necessarily more technical than the preceding sections, and is not necessary for understanding the remainder of the paper. Full details of proofs are in Appendix C.4.

5.4.1 Subgroup Indices

As usual, we will want to investigate whether, starting at a point at which we know equilibrium exists, a change in aggregate supply can be matched by a change in aggregate demand. Write σ^j and σ^J for the individual and aggregate-demand complex cells that correspond to bundles in the convex hull of aggregate demand at some price $\mathbf{p}$. Then changes in aggregate supply are in the directions given by the lattice Λ_{σ^J}, while changes in aggregate demand are in the directions in $\sum_{j \in J} \Lambda_{\sigma^j}$. So we will use a standard tool, the subgroup index, to compare these lattices:

Definition 5.16 (See e.g. Cassels, 1971)**.** Let Λ be a lattice, and $\Lambda' \subseteq \Lambda$ a sublattice of the same rank as Λ.

(1) A *fundamental parallelepiped* of Λ is a set $\Delta_\Lambda := \{\sum_j \lambda_j \mathbf{v}^j \in \mathbb{R}^n : 0 \leq \lambda_j < 1\}$, where $\{\mathbf{v}^1, \ldots, \mathbf{v}^k\}$ is a basis for Λ.

(2) The *subgroup index* $[\Lambda : \Lambda']$ is the lattice-volume in Λ of a fundamental parallelepiped of Λ', that is, $[\Lambda : \Lambda'] := \mathrm{vol}_\Lambda(\Delta_{\Lambda'})$.

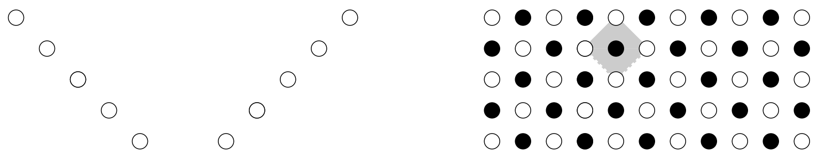
Different fundamental parallelepipeds are images of one another under unimodular basis changes. Thus the subgroup index does not depend on the choice of parallelepiped. (The standard group-theoretic definition is equivalent to ours; see Fact 5.18(5).)

We can use subgroup indices in our context because:

Lemma 5.17. *Suppose $\{u^j : j \in J\}$ are a finite set of valuations with individual and aggregate-demand complex cells σ^j and σ^J, respectively, at $\mathbf{p}$.*

(1) $\sum_{j\in J} \Lambda_{\sigma^j}$ is a sublattice of Λ_{σ^J} of the same rank as Λ_{σ^J}.
(2) If the intersection of $\{\mathcal{L}_{u^j} : j \in J\}$ is transverse at $\mathbf{p}$ then a basis for $\sum_{j\in J} \Lambda_{\sigma^j}$ is obtained by combining bases for each Λ_{σ^j}, for $j \in J$.

The parallel lattices for our hotel example (as given in Fig. 4), with $\mathbf{p} = (30, 20)$, are given in Fig. 9. The sublattice $\Lambda_{\sigma^s} + \Lambda_{\sigma^c} = \{m(1, -1) + m'(1, 1) : m, m' \in \mathbb{Z}\}$. (That is, this sublattice comprises the white bundles in Fig. 9c, while the lattice $\Lambda_{\sigma^{\{s,c\}}} = \mathbb{Z}^2$ comprises all the bundles in Fig. 9c.) A fundamental parallelepiped, $\Delta_{\Lambda_{\sigma^s}+\Lambda_{\sigma^c}}$, of this sublattice is shown; its lattice-volume in $\Lambda_{\sigma^{\{s,c\}}}$ is 2. So the subgroup index $[\Lambda_{\sigma^{\{s,c\}}} : \Lambda_{\sigma^s} + \Lambda_{\sigma^c}] = 2$.



(a) Λ_{σ^s} (b) Λ_{σ^c} (c) $\Lambda_{\sigma^{\{s,c\}}}$, with $\Lambda_{\sigma^s} + \Lambda_{\sigma^c}$ in white, and an example of $\Delta_{\Lambda_{\sigma^s}+\Lambda_{\sigma^c}}$ shaded.

Figure 9: The parallel lattices corresponding to demand complex cells σ^s, σ^c and $\sigma^{\{s,c\}}$ of, respectively, u^s, u^c and $u^{\{s,c\}}$ (as shown in Figs. 4a-c and Figs. 5a-c) at $\mathbf{p} = (30, 20)$.

Observe in Fig. 9c that the subgroup index also corresponds to the ratio of the total number of bundles to the number of white bundles. We can in fact understand subgroup indices generally in terms of such ratios (see Fact 5.18(5)). Subgroup indices generalise unimodularity, as is seen by comparing Fact 5.18(1), below, with Fact 4.9(2), and Fact 5.18(3) with Defn. 4.2:

Facts 5.18. Let Λ be a lattice, and $\Lambda' \subseteq \Lambda$ a sublattice of the same rank as Λ.

(1) $[\Lambda : \Lambda'] - 1$ is equal to the number of elements of Λ in $\Delta_{\Lambda'}$ which are not vertices.[34]
(2) $[\Lambda : \Lambda'] = 1$ iff $\Lambda = \Lambda'$.
(3) If $\Lambda = \mathbb{Z}^n$, and $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ is a basis for Λ', then $[\Lambda : \Lambda'] = |\det(\{\mathbf{v}^1, \ldots, \mathbf{v}^n\})|$.[35]
(4) Suppose $\Lambda = \mathrm{K} \cap \mathbb{Z}^n$ for some linear subspace K of $\mathbb{R}^n$. Then $[\Lambda : \Lambda'] = 1$ iff any basis for Λ' is unimodular.
(5) $[\Lambda : \Lambda']$ is the number of disjoint "cosets" $\{\mathbf{v}\} + \Lambda'$ where $\mathbf{v} \in \Lambda$.

[34]Since fundamental parallelepipeds are not *closed*, we do not count points on the "upper" boundary.
[35]When $\Lambda \neq \mathbb{Z}^n$, we can calculate $[\Lambda : \Lambda']$ by fixing a matrix G_Λ identifying Λ with $\mathbb{Z}^k$ for some k, as in Footnote 33, and then applying G_Λ to Λ'.

In our example, there is one element of $\Lambda_{\sigma^{\{s,c\}}}$ in $\Delta_{\Lambda_{\sigma^s}+\Lambda_{\sigma^c}}$ which is not at a vertex (see Fig. 9c). This illustrates Fact 5.18(1). As $\Lambda_{\sigma^{\{s,c\}}} = \mathbb{Z}^2$ we can use Fact 5.18(3) to calculate $[\Lambda_{\sigma^{\{s,c\}}} : \Lambda_{\sigma^s}+\Lambda_{\sigma^c}]$ as the determinant of the matrix with columns $(1,-1),(1,1)$ (cf. Section 4.2.2). This basis is not unimodular, and so Fact 5.18(4) verifies for us again that $\Lambda_{\sigma^s}+\Lambda_{\sigma^c} \neq \Lambda_{\sigma^{\{s,c\}}}$.

5.4.2 The Subgroup Indices Theorem

Using subgroup indices can establish that the aggregate demand set is discrete convex at a particular intersection price–and hence that equilibrium locally exists. The logic of Section 4.2.4 shows that it is sufficient that the facet normals at this price form a unimodular set. However, the following theorem gives a weaker sufficient condition, and is thus more general:

Theorem 5.19 (The Subgroup Indices Theorem). *Let u^j be concave for j in a finite set J, and suppose the intersection of those LIPs which contain* $\mathbf{p}$ *is transverse at* $\mathbf{p}$. *Write σ^j, σ^J for the demand complex cell at* $\mathbf{p}$ *of respectively u^j (where $j \in J$), u^J.*

(1) If $\left[\Lambda_{\sigma^J} : \sum_{j\in J}\Lambda_{\sigma^j}\right] = 1$ then $D_{u^J}(\mathbf{p})$ is discrete-convex.

(2) If $\left[\Lambda_{\sigma^J} : \sum_{j\in J}\Lambda_{\sigma^j}\right] > 1$ and if also $\exists j_0 \in J$ with $\dim\sigma^{j_0} \leq 2$, while $\dim\sigma^j \leq 1$ for $j \in J\setminus\{j_0\}$, then $D_{u^J}(\mathbf{p})$ is not discrete-convex.

To translate part (1) back to the familiar terms of Section 4.2, first suppose $|J| = 2$ and suppose both LIPs contain $\mathbf{p}$. Recall that $\Lambda_\sigma = \mathrm{K}_\sigma \cap \mathbb{Z}^n$ (Defn. 5.7). So, by Fact 5.18(4), $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1}+\Lambda_{\sigma^2}] = 1$ iff any basis for $\Lambda_{\sigma^1}+\Lambda_{\sigma^2}$ is unimodular. Moreover, the combination of a basis for Λ_{σ^1} with a basis for Λ_{σ^2} gives a basis for $\Lambda_{\sigma^1}+\Lambda_{\sigma^2}$ (Lemma 5.17(2)). The proof in Section 4.2.4 that $D_{u^{\{1,2\}}}(\mathbf{p})$ is discrete-convex depended on a unimodular basis for $\mathrm{K}_{\sigma^{\{1,2\}}}$, consisting of integer vectors in either K_{σ^1} or K_{σ^2}. The same arguments prove Thm. 5.19(1).

However, Thm. 5.19(1) shows that we need not restrict attention to facet normals, as we did in Section 4.2.4. Equilibrium requires that the combination of changes of demands from individual agents covers all possible supply bundles. But it does *not* require that each agent's demand differs from its demand in a UDR by an integer combination of facet normals. For example, if Figs. 4c and 5c correspond to a single agent, with a concave valuation, the bundle $(1,1)$ *is* then demanded (in contrast with our hotel example). The edges to this demand complex cell do not provide a basis for its parallel lattice, but the coordinate vectors do provide such a basis. By combining two agents of this kind in 4-dimensional space, Ex. C.14 shows that Thm. 5.19(1) demonstrates equilibrium in situations in which Thm. 4.3 does not.

Now we consider Thm. 5.19(2). If $\dim\sigma^j \leq 1$ for $j \in J$, then the corresponding minimal price complex cells are all either facets or closures of UDRs. In the latter case, the agent in question may be ignored. If additionally all the cells that are facets have weight 1, then the situation at the price $\mathbf{p}$ is exactly that described in Lemma 4.16. So, just as in Lemma 4.16, unimodularity is necessary (as well as sufficient) for equilibrium; now apply Fact 5.18(4). But, for any weights, σ^J is a grid of copies of a "small parallelepiped", as we saw in Section 5.1.2. This "small parallelepiped" is (the closure of) a fundamental parallelepiped of $\sum_{j\in J}\Lambda_{\sigma^j}$. So equilibrium fails for some supply if and only if such a fundamental parallelepiped contains a non-vertex integer point. Applying Fact 5.18(1) therefore yields Thm. 5.19(2) if $\dim \sigma^j \leq 1$ for all $j \in J$.

But if $\dim \sigma^j > 1$ for some j, then σ^J need not be a parallelepiped. In such cases, even if any fundamental parallelepiped of $\sum_{j \in J} \Lambda_{\sigma^j}$ contains a non-vertex integer point, it does not necessarily follow that a corresponding point lies in $\sum_{j \in J} \sigma^j = \sigma^J$ itself: it can fall outside the aggregate demand set at this price.

However, when we have one 2-cell, along with 1-cells, then we *can* capture this point. The reason is that a 2-cell will always contain a triangle, two of whose edges give a basis for its parallel lattice (see Appendix C.4). And this triangle is a copy of *half* of a fundamental parallelepiped. Meanwhile, each 1-cell contains an entire fundamental parallelepiped of its parallel lattice. The sum of these, σ^J, therefore contains at least *half* of a fundamental parallelepiped of $\sum_{j \in J} \Lambda_{\sigma^j}$. So by symmetry, if, in this case, $\left[\Lambda_{\sigma^J} : \sum_{j \in J} \Lambda_{\sigma^j}\right] > 1$, the aggregate-demand complex cell σ^J *does* contain a bundle not in $\sum_{j \in J} \Lambda_{\sigma^j}$, i.e., one that cannot be reached via changes in demand among the agents.

Exs. C.15-C.18 show that this is the furthest we can go:

Proposition 5.20. *For both the case in which* $\dim \sigma^1 = \dim \sigma^2 = 2$, *and the case in which* $\dim \sigma^1 = 3, \dim \sigma^2 = 1$, *there exist examples in which* $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] > 1$ *and in which* $D_{u^{\{1,2\}}}(\mathbf{p})$ is *discrete-convex, and other examples in which* $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] > 1$ *and in which* $D_{u^{\{1,2\}}}(\mathbf{p})$ *is* not *discrete-convex.*

5.4.3 The Tropical Bernstein-Kouchnirenko-Khovanskii (BKK) Theorem

We follow the conventions of Bertrand and Bihan (2013):

Definition 5.21 (Bertrand and Bihan, 2013, Defn. 5.2)**.** For valuations u^1, u^2 whose LIPs have an intersection 0-cell at $\mathbf{p}$, write $\sigma^1, \sigma^2, \sigma^{\{1,2\}}$ for their individual and aggregate-demand complex cells at $\mathbf{p}$; write $C_{\sigma^1}, C_{\sigma^2}, C_{\sigma^{\{1,2\}}}$ for the dual cells in $\mathcal{L}_{u^1}, \mathcal{L}_{u^2}, \mathcal{L}_{u^{\{1,2\}}}$.

(1) If the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse at $C_{\sigma^{\{1,2\}}}$, then the *true multiplicity* of $C_{\sigma^{\{1,2\}}}$ is $\mathrm{mult}(C_{\sigma^{\{1,2\}}}) := [\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] \cdot w_{u^1}(C_{\sigma^1}) \cdot w_{u^2}(C_{\sigma^2})$.

(2) In general, the *true multiplicity*, $\mathrm{mult}(C_{\sigma^{\{1,2\}}})$, of $C_{\sigma^{\{1,2\}}}$, is the sum of the multiplicities at all the intersection 0-cells emerging from $C_{\sigma^{\{1,2\}}}$ after any small translation of $\mathcal{L}_{u^2}$ which makes the intersection transverse.

So the distinction between "true" and "naïve" multiplicities is the use of subgroup indices. But these give us our equilibrium results (Thm. 5.19, the Subgroup Indices Theorem). For example, the intersection 0-cell at $(30, 20)$ in Fig. 4c has naïve multiplicity 1 but (see Section 5.4.1) true multiplicity 2 (and a failure of equilibrium). Note that both naïve and true multiplicities are positive integers (since cell weights are, and by Lemma 5.12 and Fact 5.18(1)).

Theorem 5.22 (Bertrand and Bihan, 2013, Thm. 6.1)**.**

(1) The number of intersection 0*-cells for* $\mathcal{L}_{u^1}$ *and* $\mathcal{L}_{u^2}$*, counting with their true multiplicities, is equal to* $M^n(A^1, A^2)$.

(2) For any fixed $k \in \{1, \ldots, n-1\}$*, if the intersection is transverse whenever a* k*-cell of* $\mathcal{L}_{u^1}$ *meets an* $(n-k)$*-cell of* $\mathcal{L}_{u^2}$*, then the number of such intersection* 0*-cells, counting with their true multiplicity, is equal to* $M^n_k(A^1, A^2)$.

This theorem also follows from Huber and Sturmfels (1995)'s analysis of the "mixed volumes" that define $M^n(A^1, A^2)$–see Appendix B.

5.4.4 Proving the Intersection Count Theorem

It is now easy to prove the sufficiency result for the transverse case. From Defns. 5.10 and 5.21, and Fact 5.18(1) (which implies that the subgroup index is always at least 1) we know that $\hat{m}(C) \leq \text{mult}(C)$ for any intersection 0-cell C. Equality holds iff the subgroup index at this price is 1. So, by Thm. 5.22(1), the number of intersection 0-cells, counted with naïve multiplicities, is bounded above by $M^n(A^1, A^2)$, and this bound holds with equality iff the subgroup index at every intersection 0-cell is 1. If this is true, then the aggregate demand is discrete-convex at every intersection 0-cell (Thm. 5.19(1)) and so equilibrium exists for every relevant supply (Prop. 5.4).

For necessity, we additionally assume $n \leq 3$. Every intersection 0-cell is the intersection of a 2-cell and a 1-cell of the two individual LIPs. So apply Thm. 5.19(2): if the number of intersection 0-cells, counted with naïve multiplicities, is strictly below $M^n(A^1, A^2)$ then the subgroup index is greater than 1 for some intersection 0-cell, and this implies a failure of equilibrium for some relevant supply.

When the intersection is not transverse, the logic is similar, but we need a little more care. We need to translate in the "right direction" to find the naïve multiplicity for each intersection 0-cell, but these directions need not be the same across all intersection 0-cells. However, we can combine Prop. 4.12 with Thms. 5.19(1) and 5.22(1) by working *locally* at each 0-cell. See Appendix C.4 for details.

5.5 Limitations, and an Extension, of the Theorem

5.5.1 Necessity

The condition of the Intersection Count Theorem is necessary for equilibrium only when the intersection is transverse and $n \leq 3$.

If the intersection is not transverse, it is possible for equilibrium to exist for every relevant supply, but to fail for some relevant supply after *any* small translation of the LIPs (see Ex. C.20). However, in such cases, Lemma 4.15 does not apply: a change in aggregate supply can be apportioned between individual agents in more than one way. So our tools cannot identify these "fragile" equilibria.

The failure of our condition to give necessity for $n \geq 4$ is immediate from Prop. 5.20. (Recall we needed Thm. 5.19(2) to prove necessity for $n \leq 3$.)

Note, however, that our upper bound $M^n(\cdot, \cdot)$ is tight (strong substitutes valuations illustrate this for both non-transverse cases and arbitrary n; see Prop. 4.6 and Section 5.1.4) so this bound cannot be improved on.

5.5.2 An Alternative Sufficient Condition

Thm. 5.22(2) concerns the weighted count of intersections between k-cells of the first LIP, and $(n-k)$-cells of the second, for some fixed k. So if the bound in Thm. 5.1 does not hold with equality, we can still restrict attention to such fixed k, to obtain a sufficiency result for equilibrium existing at particular intersection prices.

First note that an intersection is transverse at a 0-cell iff the minimal cells of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ that contain that 0-cell have dimensions k and $(n-k)$ for some k. Now, an argument analogous to that in Section 5.4.4 (for the transverse case of Thm. 5.1) shows:

Proposition 5.23. *Let $u^j : A^j \to \mathbb{R}$ be concave valuations for $j = 1, 2$, suppose $\dim \operatorname{conv}(A^1 + A^2) = n$, and fix $k \in \{1, \ldots, n-1\}$. Suppose the intersection is transverse at every intersection 0-cell C such that the minimal cell of $\mathcal{L}_{u^1}$ containing C has dimension k. Then the number of these intersection 0-cells, counted with naïve multiplicities, is bounded above by $M^n_k(A^1, A^2)$. If the number equals this bound, then equilibrium exists for all supplies in the convex hull of demand at each of these intersection 0-cells.*

6 Applications

6.1 New Demand Types which Guarantee Equilibrium

Our Unimodularity Theorem helps identify new demand types of economic interest, including ones which are *not* unimodular basis changes of ordinary substitutes, or any subset thereof, for which equilibrium is guaranteed.

For example, consider the demand type whose vectors are the columns of:

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This might, for example, model a firm's demand for "bundles of" four kinds of workers–three sorts of specialist (the first three goods) and a supervisor (the fourth good). The first three columns of D show that any one of the three kinds of specialist has value on his own; the middle three columns of D show that a supervisor increases the value of any specialist (that is, there are pairwise complementarities between any one of the first three "goods" together with the fourth); the last three columns of D show that there are also complementarities between any pair of different specialists if (but only if) a supervisor is also present; but a supervisor on her own is worthless.[36]

This demand type is pure complements, and is not a basis change of ordinary substitutes, or any subset thereof, as shown in the supplementary material to this paper (O'Connor, 2015; Baldwin and Klemperer, 2014, give a mathematical proof that it is not a basis change of strong substitutes). It is routine to check that it is unimodular. In fact, it is a basis change of a "cographic" unimodular set (see Seymour 1980[37]) but we think it has not previously been presented in this way as an example of purely complementary preferences. Cographic matrices are mathematically closely related to the "graphic" (also known as "network") matrices formed of the strong substitute vectors, but *not* via basis changes, so the arguments of Section 3.2 do *not* apply. Indeed, there appears to be no known way to translate between valuations of graphic and cographic demand types. Their economic properties are clearly very different.

Further demand types can be developed from D, and from the many other cographic unimodular sets. For *any* 4×9 matrix C, the columns of $D' = \left(\begin{smallmatrix} D & 0 \\ C & D \end{smallmatrix}\right)$ form a unimodular set that defines a demand type on 8 goods. Repeating the procedure, and using alternative C's, one can create many $4n$-dimensional unimodular demand types expressing

[36]The reason is that $\mathbf{e}^4 \notin \mathcal{D}$; perhaps each firm's owner is a supervisor, and an additional supervisor without any workers would merely "spoil the broth".

[37]Seymour (1980) characterised unimodular sets of vectors; Danilov and Grishukhin's (1999) extension fully describes, up to basis change, all maximal such sets (there are finitely many for any n).

alternative trade-offs (including purely complementary ones).

The vector $\mathbf{e}^4$ is not in these demand types; in general unimodular demand types need not contain all the coordinate vectors. Correspondingly, their entries need not be in $\{0, \pm 1\}^n$. (For example, $\pm\{(1, -3), (1, -4), (0, 1)\}$ is a unimodular demand type for substitutes.) But it is easy to find a basis change of any unimodular demand type that *does* include all the coordinate vectors (and so *must* have all its coordinates in $\{0, \pm 1\}^n$). Indeed, the inverse of any invertible matrix formed from vectors of this type provides such a basis change.

6.2 Relationships between Equilibrium and Complements; and between Equilibrium and Substitutes

Understanding the relationship between equilibrium and unimodularity shows that much conventional wisdom is mistaken in connecting the existence of equilibrium to substitutabilities: taking advantage of existing mathematical work on unimodular sets, demonstrates a link between equilibrium and *complements* that is, if anything, stronger.

First, we can easily extend the previous section's example to obtain:

Proposition 6.1. *With $n > 3$, there exist demand types which are* not *a unimodular basis change of ordinary substitutes, or a subset thereof, and for which an equilibrium exists for every finite set of concave valuations of the demand type, for all relevant supply bundles. For $n \leq 3$ there exist no such demand types.*

To see the result for $n > 3$, consider the demand type defined by the matrix D of Section 6.1, with its vectors extended by $n - 4$ zeros, and with the coordinate vectors $\mathbf{e}^i$, $i = 5, \ldots, n$ appended. Any basis change taking such a demand type to a substitutes demand type would restrict to a basis change taking D to substitutes, contradicting the result given in Section 6.1. The result for $n \leq 3$ follows from Seymour's (1980) characterisation.

We can also make use of mathematical results from Grishukhin et al. (2010) that imply that every unimodular set of vectors is a unimodular basis change of a set that contains only vectors in $\pm\{0, 1\}^n$. Since demand types containing these vectors contain only complements valuations, Prop. 4.7 tells us:

Proposition 6.2. *Every demand type for which an equilibrium is guaranteed (that is, exists for every finite set of concave valuations of the demand type, for all relevant supply bundles) is a unimodular basis change of a demand type which contains* only *complements valuations and for which equilibrium is guaranteed.*

Observe that Prop. 6.1 shows the corresponding statement *cannot* be made about substitutes.[38] It *is* true (and easy to show) that the strong substitutes vectors are *maximal* as a unimodular set of vectors. So, given any one valuation *not* for strong substitutes, we can find valuations which are strong substitutes such that equilibrium fails (see Section 4.2.4). Some have misinterpreted this result as a necessity of substitutes

[38] For two goods (but not more–see Section 3.2), substitutes are a basis change of complements via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see Section 3.2) so (only) for two goods, equilibrium fails "as often" for substitutes as for complements.

for equilibrium, but doing this overlooks the fact that a demand type need not contain all strong substitutes valuations.[39]

6.3 Equilibrium for *Specific* Combinations of Valuations, and for *Families* of Combinations of Valuations

We can use our Unimodularity and Intersection Count Theorems (Thms. 4.3 and 5.1) and related ideas from Section 5 to construct an algorithm (Algorithm A.1 in Appendix A) that determines whether equilibrium exists for any *specific* combination of valuations. Appropriate tools exist for each step of this algorithm: see Remark A.2.

The Intersection Count Theorem (Thm. 5.1) and its corollary (Cor. 5.2) help us understand equilibrium for *families* of combinations of valuations. Here, a "family" is any set, indexed by $k \in K$, of combinations of valuations of the form $\{u^{k,j} : j \in J\}$, where each combination can be indexed by the same set J, such that the domain of valuations $u^{k\,j}$ is independent of k. We are particularly interested in families in which for any given j, valuations $u^{k,j}$ have the same combinatorial type. For example, the combinations $\{u^s, u^c\}$ and $\{u^s, u^{c*}\}$ of Fig. 4 belong to such a family.

In that example, we saw that equilibrium is ensured iff there are two (naïvely counted) intersections; equivalently, iff the valuation for complements will accept the bundle $(1,1)$ at a price weakly higher than that at which the valuation for substitutes is indifferent between $(0,0)$, $(1,0)$ and $(0,1)$. Similarly, when $|J| = 2$, equilibrium holds for all combinations in such a family as long as variation in valuations leaves the naïvely weighted count of intersection 0-cells fixed at $M^n(A^1, A^2)$. Moreover, if $n \leq 3$, and this naïve count varies within the family, then the precise combinations for which equilibrium holds can be identified. See Ex. C.11 for an example of such a family: how existence varies within the family can be easily understood from the intersection count, but it is unclear how we would otherwise check existence without analysing each possible combination separately.

6.4 Interpreting Classic Models in a Unified Framework

Our framework developed in Sections 2 and 3 encompasses some classic studies as special cases. So it clarifies connections between them, and helps explain many of their results.

Kelso and Crawford's (1982) seminal analysis of n_1 firms, each of which is interested in hiring some of n_2 workers, can be understood as a model with $n_1 n_2$ distinct "goods", each of which is the "transfer of labour" by a specified worker (a "seller") to a specified firm (a "buyer"); the "price" of a good is the salary to be paid. So the full set of

[39] Gul and Stacchetti demonstrate this maximality result using n-dimensional variants of our simplest example, Fig. 1a. Such valuations are not, of course, in (e.g.) any purely-complements demand type, but Gul and Stacchetti's description of them as "unit demand" might suggest they should be in any set of interest (cf., the paper's remark (p. 96) "in a sense, the GS [gross substitutes] condition is necessary to ensure existence of a Walrasian equilibrium"). See also Kelso and Crawford (1982). And Azevedo et al.'s (2013, p. 286) remark "adding a continuum of consumers ... eliminates the existence problems created by complementarities" can also be misinterpreted. (However, Bikhchandani and Mamer, 1997, Prop. 3, previously noted the existence of non-substitute preferences which guarantee equilibrium if just two agents each value at most one unit of each good. And Sun and Yang (2014) show existence of competitive equilibrium with super-additive valuations and non-linear pricing.)

bundles is $\{-1,0,1\}^n$, in which $n = n_1 n_2$. However, each worker's valuation is defined only over a subset of this domain of the form $\{-1,0\}^{n_1}$ (that is, only over the n_1 goods that correspond to its own labour), and only over the subset of these vectors that have at most one non-zero entry (it can work for at most one firm). Obviously, its only possible demand complex edges are the strong substitute vectors (non-zero vectors with at most one $+1$ entry, at most one -1 entry, and no other non-zero entries, see Prop. 3.10). Similarly, each firm's valuation is defined only over a subset of the form $\{0,1\}^{n_2}$ (that is, it has preferences only over the n_2 goods that correspond to workers it can employ). Kelso and Crawford's assumptions imply that firms have ordinary substitutes preferences over workers (see Note 16). But, since there is only one unit of each good, the only substitute changes of demand are the strong substitutes vectors, so all valuations are of this type.

It may be less obvious that Hatfield et al.'s (2013) model of networks of trading agents, each of whom can both buy and sell, both fits into our framework, and is also closely related to Kelso and Crawford's model. To show this, we again treat each transfer of a product from a specified seller to a specified buyer as a distinct good, so each agent again has preferences over a subset of $\{-1,0,1\}^n$, where n is the number of distinct goods.

Since Hatfield et al. restrict each agent to be either a seller or a buyer (or neither) on any one good, an agent j which is the specified seller in n_1^j potential trades and is the specified buyer in n_2^j potential trades simply has preferences over a subset of the domain which, after an appropriate re-ordering of the goods for that agent, is of the form $\{-1,0\}^{n_1^j} \times \{0,1\}^{n_2^j}$. (As in Hatfield et al., we can restrict an agent's domain of preferences further so that, e.g., it cannot sell good 1 unless it also buys one of goods 2 or 3. They do this by using "$-\infty$" valuations, while we simply exclude bundles from the domain, but the effect is the same–see Section 2.1.) Furthermore, although Hatfield et al. describe goods to be sold as complements of goods to be bought, this is because they measure both buying and selling as non-negative quantities. So, since in our framework selling is just "negative buying", the "complementarities" disappear: the condition they impose is exactly ordinary substitutes (see Hatfield et al., 2015, Thm. B.1). Just as for Kelso and Crawford's model, the only demand complex edges of such a domain that are vectors of the ordinary substitutes demand type are also vectors of the strong substitutes demand type.

Trivially, any valuation over any subset of $\{-1,0\}^{n_1}$ or $\{0,1\}^{n_2}$ or $\{-1,0\}^{n_1^j} \times \{0,1\}^{n_2^j}$ is concave so, in both Kelso and Crawford's and Hatfield et al.'s models, the existence of equilibrium follows immediately from Section 4.1's discussion. And we showed above (see Sections 3.2 and 4.1) that "consecutive games" and "generalised gross substitutes and complements valuations" are also both easily related to the strong substitutes demand type, so also both always have equilibrium. As discussed at the end of Section 6.2, the maximality of the strong substitute vectors as a unimodular demand type implies "maximal domain" results for these cases, in the sense that if one valuation is not of the demand type, then there exist valuations which *are* of the type such that competitive equilibrium fails for the combination (e.g. Hatfield et al., 2013, Theorem 7).

Another model that is easy to analyse using our methods is a "circular ones" model (cf. Bartholdi et al., 1980) in which each of n kinds of agent is only interested in a single, specific, pair of goods, and these pairs form a cycle. The Unimodularity Theorem immediately tells us equilibrium is guaranteed to exist if and only if n is even. Furthermore, we can use Lemma 4.16 to find examples of equilibrium failure if n is odd; Ex. C.22 in Appendix C.4 gives details. More generally, Lemma 4.16 shows how to easily construct an explicit example of failure of equilibrium for *any* model that permits valuations from a non-unimodular demand type.

Reformulating models in our framework also shows clearly how we can generalise them. It is immediate, for example, that as long as we retain concavity and the strong substitutes demand type, we can remove Hatfield et al.'s restriction that an agent cannot be both a buyer and a seller on any one good (by simply extending their domain to be *any* subset of $\{-1, 0, 1\}^n$) and can also permit their agents to trade multiple units of the same products at linear prices (by enlarging the domain to any subset of $\mathbb{Z}^n$).

6.5 The Valuation-Complex Equivalence Theorem, and Understanding Individual Demand

Our Valuation-Complex Equivalence Theorem (Thm. 2.14) means we can explore quasi-linear preferences by considering simple geometric objects (as described in Sec. 2.2) *without* needing to construct explicit valuations. The latter is typically much harder. Our supplementary appendix (Baldwin and Klemperer, 2018) provides one illustration of this. Baldwin and Klemperer (2014, especially Sec 5) further explore the comparative statics of individual demand, including at non-UDR prices, and also generalise Gul and Stacchetti's (1999) "Single Improvement Property"–unimodularity plays a key role. Finally, Baldwin, Klemperer, and Milgrom (in prep) uses our framework to compare different notions of substitutability for indivisible goods.

6.6 Multi-Party Matching

Section 6.4 showed Kelso and Crawford's (1982) model of *bipartite* matching was a special case of our model. Our model also encompasses *multi*party matching. For example, in Section 6.1 we can interpret the columns of D as the coalitions of workers that create value: Baldwin and Klemperer (2014, Thm. 6.7) show that, assuming transferable utility, a stable match (that is, an allocation in the core of the game among the workers, so no subset can gain from re-matching) corresponds to an equilibrium allocation of workers in our model (in which every worker receives its competitive wage, and no gains from trade are possible). So, since the demand type is unimodular, it describes a class of problems for which a stable match always exists.

More generally, Baldwin and Klemperer (2014, Secs. 2.5, 6.2, 6.3.1) show that any model of coalition formation with transferable utility corresponds to a demand type, and a stable match exists if our Unimodularity Thm. or Intersection Count Thm. applies.[40] So also (recall Prop. 6.1) stable matchings are guaranteed for a broader class of preferences than often assumed.[41]

[40]Since our framework allows multiple agents of each kind, it also easily yields results along the lines of Chiappori et al. (2014).

[41]Consider, e.g., Hatfield and Milgrom's (2005, p.915) statement that "preferences that do not satisfy the substitutes condition cannot be guaranteed always to select a stable allocation", though the Proposition (p.921) that their introductory remark loosely summarises is, of course, correct in its context.

6.7 Auction Design

Practical auctions need to restrict the kinds of bids that can be made, thus limiting the preferences that bidders can express. Restricting to a demand type is natural, since the economic context often suggests appropriate trade-offs between goods. For example, the Bank of England expected bidders to have £1:£1 trade-offs, that is, a form of strong substitutes preferences, between the different "kinds" of money it loaned in the financial crisis.[42] So the Bank chose auction rules that made it easy for bidders to state such preferences, and was also unconcerned about ruling out the expression of other preferences.[43]

The diagrams used to analyse the Bank's crisis auction were precisely the figures of LIPs used in this paper–indeed it was diagrams from the Bank's auction that inspired the current work. Conversely, the current work helped us develop the Bank's auction further, inspired a related auction design for the government of Iceland (see Klemperer, 2018), and is now helping us develop additional versions of the Product-Mix Auction.[44] In all these auctions, participants' bids are aggregated in the same simple way that (weighted) LIPs are combined to find aggregate demand. This also makes the auctions "user-friendly", which is critical for getting them implemented in practice.

7 Conclusion

An agent's demand is completely described by its choices at all possible price vectors. So it can also be described by the divisions between the regions of price space in which the agent demands different bundles, and hence by the vectors that define these divisions. This suggests a natural way of classifying valuations into "demand types".

Using this classification, together with the duality between the geometric representations of valuations in price space and in quantity space, yields significant new insights into when competitive equilibrium exists.

A demand type's vectors also encode the possible comparative statics of demand, and we expect many other results can be understood more readily, and developed more efficiently, using our geometric perspective.

In future work,[45] we plan to use our framework and tools to obtain new results about the existence of stable matchings in multiple-agent matching models; to better understand individual demand; and to further develop the Product-Mix Auction implemented by the Bank of England in response to the 2007 Northern Rock bank run and the subsequent financial crisis.

[42] The different "goods" were long-term loans (repos) against different collaterals.

[43] Restricting to a demand type can allow quite complex bids, since there are easy software solutions to calculate the normal vectors of any LIP and so check the demand type (see Remark A.2). A simple modification of the Bank's Product-Mix auction (Klemperer 2008, 2010) allows any strong substitutes preferences to be expressed. The updated (2014) implementation allows some complements preferences.

[44] Our (open source) software implementing some of these can be found at http://pma.nuff.ox.ac.uk.

[45] Baldwin and Klemperer (2014) contains preliminary work on matching (in Sections 2.5, 6.2, and 6.3.1) and on comparative statics of individual demand (in Sections 4.2, 4.3, and 5).

A Algorithm to Check for Existence of Equilibrium

Algorithm A.1. Given concave valuations $u^j : A^j \to \mathbb{R}$ on finite $A^j \subsetneq \mathbb{Z}^n$ for $j = 1, \ldots, m$: apply steps (1)-(8), below, to $u^{k*} := u^{\{1,\ldots,k-1\}}$ and u^k, for each of $k = 2, \ldots, m$. If, for each k, equilibrium exists for u^k and u^{k*} for all relevant supplies, then equilibrium exists for $u^1, \ldots, u^m$, for all relevant supplies.

(1) Are u^k, u^{k*} of the same unimodular demand type?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies. End here.
  (ii) If no, continue.

(2) If $\dim \operatorname{conv}(A^k + A^{k*}) = n' < n$, make a unimodular change of basis so that $A^k, A^{k*} \subsetneq \mathbb{Z}^{n'}$, and use n' instead of n in the following steps.
(3) Calculate $M^n_k(A^k, A^{k*})$ for $k = 0, \ldots, n$ and $M^n(A^k, A^{k*})$. Find the intersection 0-cells.
(4) Does the *number* of intersection 0-cells equal $M^n(A^k, A^{k*})$?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies. End here.
  (ii) If no, continue.

(5) Find the naïve multiplicity of each intersection 0-cell C; note if the intersection is transverse at C, and the dimension of the minimal cell of $\mathcal{L}_{u^k}$ containing C.
(6) Does the naïvely-weighted count of all intersection 0-cells equal $M^n(A^k, A^{k*})$?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies. End here.
  (ii) If no, if $n \leq 3$, and if the intersection is transverse, then equilibrium fails for some relevant supply; end here. Otherwise continue.

(7) Identify all intersection 0-cells C such that the dimension k of the minimal cell of $\mathcal{L}_{u^k}$ containing C does not satisfy *both:*
  (i) the intersection of $\mathcal{L}_{u^k}$ and $\mathcal{L}_{u^{k*}}$ is transverse at every intersection 0-cell, C', for which the minimal cell of $\mathcal{L}_{u^k}$ containing C' has dimension k,
  and (ii) the number of all intersection 0-cells C' as in (7)(i), counted with naïve multiplicities, is equal to $M^n_k(A^k, A^{k*})$.

(8) Is $D_{u^{\{k,k*\}}}(\mathbf{p})$ discrete-convex for all $\mathbf{p}$ at intersection 0-cells identified in Step (7)?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies.
  (ii) If no, equilibrium fails for some relevant supply.

Step (1) summarises the Unimodularity Theorem (Thm. 4.3). Step (4) uses a special case of the Intersection Count Theorem (Thm. 5.1): if all naïve multiplicities equal 1, then the weighted count of intersection 0-cells is just the number of these cells. (This can save computations.) Step (6) is the Intersection Count Theorem itself. Finally, Steps (7)-(8) use Prop. 5.23 (with Prop. 5.4).

It is not surprising that we cannot rule out needing Step (8). The question of equilibrium with indivisible goods is closely related to a long-studied problem in mathematics: the question of when the integer points in the Minkowski sum of two polytopes, is equal to the Minkowski sum of the integer points in those polytopes (see, e.g., Haase et al., 2007). However, we know (Thm. 5.1 with Prop. 5.4) that we have to check at most $M^n(A^1, A^2)$ prices.

Note that, when $m > 2$, Algorithm A.1 only provides a *sufficient* condition for equilibrium. There exist cases in which equilibrium fails with two agents and some relevant supply, but exists for all relevant supplies once we add a suitable third agent (see e.g. Tran and Yu, 2017, Ex. 6).

Remark A.2. The LIP of a valuation is its associated "tropical hypersurface"; algorithms to calculate this are implemented by "gfan" (see Jensen, 2011), and "polymake" (see Gawrilow and Joswig, 2000). The primitive normal vectors are found in the process.

There exist polynomial time algorithms to test *total* unimodularity of a matrix (Schrijver, 2000, Ch. 20). A set of vectors is unimodular iff a basis change as described in Footnote 33 yields a totally unimodular matrix, so this allows us to easily check for unimodularity of a demand type.

Intersection 0-cells can be found using the "a-tint" extension of polymake (Hampe, 2014a). Hampe (2014b) shows these methods provide answers in reasonable time for $n \leq 10$ for harder problems than ours, if $|A|$ is relatively small.

The calculation of the "mixed volumes", $M_k^n(A^1, A^2)$, has high complexity in general (see Cox et al., 2005, Sec. 7.6). However, recent work develops ever-faster algorithms (see, e.g., Chen et al., 2014, and Jensen, 2016). And it is much simpler in the special cases of Facts 5.15.

Calculating cell weights requires finding the volumes of lattice polytopes (Defn. 5.9). This also has high complexity in general (Dyer and Frieze, 1988), but efficient approximate methods, such as those developed by Dyer et al. (1991), are likely to be applicable in our context. Finally, the "Quickhull" algorithm (Barber et al., 1996) helps check for discrete convexity in Step (8).

In conclusion, it seems hard to give clean results of computational complexity, or to compare the efficiency of using earlier stages of Algorithm A.1 with simply resorting to many "brute force" calculations, as in Step (8). And, because we may often need to resort to Step (8), it is also hard to compare with Bikhchandani and Mamer's (1997) methodology which checks for equilibrium by the "brute force" approach of comparing the solution of an integer programming problem with the corresponding linear programming problem for every relevant supply; Lenstra (1983), Kannan (1992) and Eisenbrand and Shmonin (2008) provide polynomial-time algorithms for this.

B Mixed volumes and $M_k^n(\cdot, \cdot)$

The quantity $M_k^n(\cdot, \cdot)$ arises from the "mixed volume" in algebraic geometry (see e.g. Sangwine-Yager, 1993 or Cox et al., 2005, Chapter 7). Recall in Figs. 7c and 7d, the value for $M^2(\{0,1,2\}^2, \{0,1,2\}^2)$ was equal to the sum of the areas shaded in grey. These grey areas are all the 2-cells of aggregate-demand complexes, with the property that one edge comes from the first individual demand complex and one edge comes from the second. We generalise:

Definition B.1 (See e.g. Cox et al. 2005, Defns. 7.6.4, 7.6.5, 7.6.6 and Thm. 7.6.7)**.** Suppose $Q = Q^1 + \cdots + Q^m \subsetneq \mathbb{R}^n$, where $Q^1, \ldots, Q^m$ are polytopes with vertices in $\mathbb{Z}^n$.

(1) A *subdivision* of Q is a collection of polytopes $R^1, \ldots, R^s$ such that $Q = R^1 \cup \cdots \cup R^s$ and such that, for $i \neq j$, the intersection $R^i \cap R^j$ is a face of both R^i and R^j.

(2) A subdivision $R^1, \dots, R^s$ of Q is a *mixed subdivision* if each R^i can be written as $R^i = F^1 + \cdots + F^m$, where F^j is a face of Q^j for each j, and where $n = \dim(F^1) + \cdots + \dim(F^m)$, and where if $R^j = F'^1 + \cdots + F'^m$, then $R^i \cap R^j = (F^1 \cap F'^1) + \cdots + (F^m \cap F'^m)$.
(3) A cell $R = F^1 + \cdots + F^m$ in a mixed subdivision is a *mixed cell* if $\dim(F^i) \leq 1$ for all i. In particular if $m = n$ then $\dim(F^i) = 1$ for all i.
(4) When $m = n$, define the *mixed volume* $MV_n(Q^1, \dots, Q^n) := \sum_R \mathrm{vol}_n(R)$, where the sum is over all mixed cells R of a mixed subdivision.

To understand these definitions, observe that the maximal cells of a demand complex form a subdivision of the convex hull of its domain. Similarly, the maximal cells of the aggregate-demand complex of m agents gives a subdivision of the convex hull of their aggregate domain. If the intersection between the individual LIPs is transverse, then this is a mixed subdivision. The mixed cells are dual to intersections of facets in their interiors; in Figs. 7c and 7d, these are the grey areas.

In both Figs. 7c and 7d, the sum of the areas of mixed cells is 2. Indeed, the sum of the volumes of mixed cells is always independent of the choice of mixed subdivision; this result is implicit in our definition of mixed volume above (see Huber and Sturmfels, 1995, Thm. 2.4; the standard definition is stated in their proof). So we can use very simple subdivisions to calculate mixed volumes: see Ex. B.4.

Recall that equilibrium fails for two LIPs, $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, iff it fails at an intersection 0-cell. Suppose cells C_{σ^1}, C_{σ^2} of the respective LIPs meet transversely at such a point. In the demand complexes, we correspondingly have cells σ^1, σ^2, of dimensions $k, n-k$, and such that $\sigma = \sigma^1 + \sigma^2$ is dual to the intersection 0-cell itself. As in Lemma 4.16, equilibrium will fail if the aggregate-demand complex cell $\sigma^1 + \sigma^2$ is "too big". So, as in Sections 5.1.1-5.1.2, we wish to add up the volumes of all aggregate-demand complex cells such as $\sigma^1 + \sigma^2$. And we can do this using mixed volumes.

To calculate a mixed volume we need n polytopes, with each mixed cell being a sum of pieces of dimension 1. But we have *two* polytopes: the convex hulls of the two domains. And we are interested in the sum of aggregate cells like $\sigma^1 + \sigma^2$, but $\dim \sigma^1 + \dim \sigma^2 = n$ (because the intersection is transverse). As Fact B.2 shows, the solution is to take $k := \dim \sigma^1$ copies of the first domain and $n-k$ copies of the second:

Fact B.2 (follows from Huber and Sturmfels, 1995, Thm. 2.4)**.** Suppose the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse. The total volume of aggregate-demand complex cells dual to intersection 0-cells at which an $(n-k)$-cell of $\mathcal{L}_{u^1}$ meets a k-cell of $\mathcal{L}_{u^2}$ is equal to $\frac{1}{k!(n-k)!} MV_n(\mathrm{conv}(A^1), \dots, \mathrm{conv}(A^1), \mathrm{conv}(A^2), \dots, \mathrm{conv}(A^2))$, in which we take k copies of $\mathrm{conv}(A^1)$ and $n-k$ copies of $\mathrm{conv}(A^2)$.

The additional factor of $\frac{1}{k!(n-k)!}$ perfectly cancels the factors we used in defining weights of cells–consistent with defining $M_k^n(\cdot, \cdot)$ as a mixed volume in this way.

Lemma B.3 (Cox et al., 2005, Thm 7.4.12.d)**.** *If $A^1, A^2 \subsetneq \mathbb{Z}^n$ are finite, then $M_k^n(A^1, A^2)$ is the mixed volume of k copies of* $\mathrm{conv}(A^1)$ *with $(n-k)$ copies of* $\mathrm{conv}(A^2)$*, for $k = 1, \dots, (n-1)$.*

Proof of Facts 5.15. (1) is Cox et al. (2005, Exercise 7.7.b). (2) is an elementary calculation. □

Example B.4. Let $n = 3$ and suppose that A^1 and A^2 are the discrete-convex sets with vertices $\{(0,0,0),(2,0,0),(0,2,0),(2,2,0)\}$ and $\{(0,0,0),(1,0,0),(0,0,2),(1,0,2)\}$ respectively: the domains of the demand complexes shown in Figs. 14a-b.

We calculate $M_1^3(A^1, A^2)$ and $M_2^3(A^1, A^2)$ as described in Section 5.3.4 by considering: agent $1'$, with valuation $u^{1'}(\mathbf{x}) = 0$ for all $\mathbf{x} \in A^1$; and agent $2'$, with valuation $u^{2'}(\mathbf{x}) = x_1 + x_3$ for all $\mathbf{x} \in A^2$. Then $\Sigma_{u^{1'}}$ has a single 2-cell of volume 4 (*not* the demand complexes pictured in Fig. 14a). The corresponding 1-cell of $\mathcal{L}_{u^{1'}}$ is in direction $\mathbf{e}^3$ and passes through $\mathbf{0}$. It therefore intersects a weight-2 facet of $\mathcal{L}_{u^{2'}}$ corresponding to the edge of $\operatorname{conv}(A^2)$ from $\mathbf{e}^1$ to $\mathbf{e}^1 + 2\mathbf{e}^3$, and so the demand complex cell corresponding to this intersection 0-cell has volume $4 \times 2 = 8$. So by Fact B.2 and Definition B.3 we know $M_2^3(A^1, A^2) = 2!1! \times 8 = 16$.

Similarly, $\Sigma_{u^{2'}}$ has a single 2-cell of volume 2, and the corresponding 1-cell is in direction $\mathbf{e}^2$ and passes through $(1, 0, 1)$. It therefore intersects a weight-2 facet of $\mathcal{L}_{u^{1'}}$ corresponding to the edge of $\operatorname{conv}(A^2)$ from $\mathbf{0}$ to $2\mathbf{e}^2$, and so the demand complex cell corresponding to this intersection 0-cell has volume $2 \times 2 = 4$. So by Fact B.2 and Defn. B.3 we know $M_1^3(A^1, A^2) = 2!1! \times 4 = 8$.

We conclude that $M^3(A^1, A^2) = 8 + 16 = 24$.

Proof of Thm. 5.22. See Bertand and Bihan, 2013, Thm. 6.1. Alternatively, see that if C_{σ^1} and C_{σ^2} intersect transversely, then it follows from our definitions of cell weights and subgroup indices that $\operatorname{mult}(C_{\sigma^{\{1,2\}}}) = k!(n-k)!\operatorname{vol}_n(\sigma^{\{1,2\}})$, where $k = \dim \sigma^1$. Thm 5.22 now follows from Fact B.2. □

C Proofs and Additional Examples

Results are given below in the order in which it is most convenient to prove them.

C.1 Proofs for Sections 2 and 3

Proof of Prop. 2.7. (1) There are finitely many cells in the price complex, as $A \subsetneq \mathbb{Z}^n$ is finite. Each is defined by a collection of equalities $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ and inequalities $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') \geq u(\mathbf{x}) - u(\mathbf{x}')$, so it follows (since $A \subsetneq \mathbb{Z}^n$) that each is a rational polyhedron. Conversely, a set defined by equalities and inequalities of this form is in the price complex. A face of a price complex cell C is the subset of C on which at least one of these inequalities holds with equality: it is therefore a cell of the LIP. To consider $C \cap C'$, where $C \neq C'$ are price complex cells, suppose X, X' are such that $C = \{\mathbf{p} \in \mathbb{R}^n : X \subseteq D_u(\mathbf{p})\}$ and $C' = \{\mathbf{p} \in \mathbb{R}^n : X' \subseteq D_u(\mathbf{p})\}$. Then $X \neq X'$ and $C \cap C' = \{\mathbf{p} \in \mathbb{R}^n : X \cup X' \subseteq D_u(\mathbf{p})\}$. If this is non-empty, it is itself a cell of the LIP and a face of both C and C'. Thus the price complex is a polyhedral complex.

(2) If more than one bundle is demanded then $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for some $\mathbf{x} \neq \mathbf{x}'$; the number of bundles is finite so it is generic for only one bundle to be demanded, and hence the UDRs are all n-dimensional. So the LIP is equal to the boundary of the n-cells of the price complex, and thus is contained in its $(n-1)$-dimensional components; and conversely any price complex k-cell, where $k < n$, is a cell of the LIP. Thus cells of the LIP are just the k-cells of the price complex, for all $k < n$, whence the result. □

Proof of Lemma 2.8. It is easier to prove these statements in the opposite order. The set $\{\mathbf{p} \in \mathbb{R}^n : D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})\}$ is non-empty (it contains $\mathbf{p}^\circ$ itself) and so defines a cell, C'. If $X \subseteq A$ is such that $C = \{\mathbf{p} \in \mathbb{R}^n : X \subseteq D_u(\mathbf{p})\}$ then $X \subseteq D_u(\mathbf{p}^\circ)$ and so any $\mathbf{p} \in C'$ satisfies $X \subseteq D_u(\mathbf{p})$, so $C' \subseteq C$. From Prop. 2.7, we know that $C \cap C'$ is a face of C and of C'. However $C \cap C'$ contains a point, $\mathbf{p}^\circ$, that is in the interior of C. This is only consistent if $C \cap C' = C$. As $C' \subseteq C$ we conclude $C' = C$.

So $D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})$ iff $\mathbf{p} \in C$. But since this holds for any $\mathbf{p}^\circ \in C^\circ$, we may reverse the roles of $\mathbf{p}$ and $\mathbf{p}^\circ$ to see that $D_u(\mathbf{p}^\circ) = D_u(\mathbf{p})$ if $\mathbf{p} \in C^\circ$. ☐

Proof of Lemma 2.9. (1) was proved in the proof of Prop. 2.7 above. For (2), first it follows from Prop. 2.7(1) that the intersection of two or more n-cells of the price complex is a cell of the LIP.

Conversely, given any cell C of the LIP and $\mathbf{p}^\circ \in C^\circ$, let X be the set of all bundles uniquely demanded at some price in a small neighbourhood of $\mathbf{p}^\circ$. Then $C' = \{\mathbf{p} \in \mathbb{R}^n : X \subseteq D_u(\mathbf{p})\}$ is a cell of the LIP. By continuity of the indirect utility in $\mathbf{p}$, that is, of $\mathbf{p} \mapsto u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$, every bundle in X is demanded at $\mathbf{p}^\circ$, so $\mathbf{p}^\circ \in C'$ and hence (since $\mathbf{p}^\circ$ is in the interior of C) $C \subseteq C'$. So C is a face of C': there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{p}' \cdot \mathbf{v} \le \mathbf{p} \cdot \mathbf{v}$ for all $\mathbf{p}' \in C'$, $\mathbf{p} \in C$. Suppose for a contradiction that $C \subsetneq C'$, so there exists $\mathbf{p}' \in C'$ such that $\mathbf{p}' \cdot \mathbf{v} < \mathbf{p}^\circ \cdot \mathbf{v}$. But because the UDRs are dense in $\mathbb{R}^n$, for any $\epsilon > 0$ there exists a UDR price $\mathbf{p} \in \mathbb{R}^n$ such that $0 < \mathbf{v} \cdot (\mathbf{p} - \mathbf{p}^\circ) < \epsilon$. Thus, if $\{\mathbf{x}\} = D_u(\mathbf{p})$, we must have $\mathbf{x} \in X$. However, again by continuity of indirect utility, $\mathbf{x}$ is only demanded in the closure of the UDR containing $\mathbf{p}$, and by construction, $\mathbf{p}'$ is not in this set. Thus, by contradiction, $C = C'$. But by construction, C' is the intersection of a set of n-cells of the price complex. ☐

Proof of Lemma 2.17. We prove this lemma early, for use in other proofs. It does not rely on intermediate results: Lemmas 2.11-2.12, Thm. 2.14 and Prop. 2.16.

Let G be the upper boundary, with respect to the final coordinate, of the set $\operatorname{conv}(\{(\mathbf{x}, u(\mathbf{x})) : \mathbf{x} \in A\})$. Then G is the graph of a weakly-concave function on $\operatorname{conv}(A)$, which clearly satisfies the definition of $\operatorname{conv}(u)$. By the supporting hyperplane theorem, for every $\mathbf{x} \in \operatorname{conv}(A)$ there exists a supporting hyperplane, H, to G, at $(\mathbf{x}, \operatorname{conv}(u)(\mathbf{x}))$. Both $\operatorname{conv}(D_u(\mathbf{p}))$ and $D_{\operatorname{conv}(u)}(\mathbf{p})$ are equal to the projection of $H \cap G$ to its first n coordinates, and thus are equal. ☐

Proof of Lemma 2.11. Suppose that A is discrete-convex. It is a standard application of the supporting hyperplane theorem that a function $u : A \to \mathbb{R}$ is concave iff, for all $\mathbf{x} \in A$, there exists $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} \in D_u(\mathbf{p})$. As we have defined concavity to require discrete-convexity of domain, this provides the first equivalence. Moreover, since the intersection between a supporting hyperplane and a convex set is always itself convex, discrete-convexity of $D_u(\mathbf{p})$ also follows.

Conversely, suppose $D_u(\mathbf{p})$ is discrete-convex for all $\mathbf{p}$. Let $u' : \operatorname{conv}(A) \cap \mathbb{Z}^n \to \mathbb{R}$ be the restriction of $\operatorname{conv}(u)$ to $\operatorname{conv}(A) \cap \mathbb{Z}^n$. Consider any $\mathbf{x} \in \operatorname{conv}(A) \cap \mathbb{Z}^n$. By the previous paragraph, there exists $\mathbf{p}$ such that $\mathbf{x} \in D_{u'}(\mathbf{p})$. As $\operatorname{conv}(u) = \operatorname{conv}(u')$, it follows by Lemma 2.17 that $\operatorname{conv}(D_u(\mathbf{p})) = \operatorname{conv}(D_{u'}(\mathbf{p}))$. But then, by assumption, $\mathbf{x} \in D_u(\mathbf{p})$. So the second property holds (and in particular A is discrete-convex). ☐

Proof of Lemma 2.12. Restrict u to $\operatorname{conv}(D_u(\mathbf{p})) \cap \mathbb{Z}^n$ to see that Part (1) follows from Lemma 2.11. For (2), recall that $\text{conv}(u) = \text{conv}(u')$ and so, by Lemma 2.17, $D_u(\mathbf{p}) = \{\mathbf{x}\}$ iff $D_{u'}(\mathbf{p}) = \{\mathbf{x}\}$, for any bundle $\mathbf{x}$. So $|D_u(\mathbf{p})| = 1$ iff $|D_{u'}(\mathbf{p})| = 1$. □

Proof of Prop. 2.16. The "roof" is $\{(\mathbf{x}, \text{conv}(u)(\mathbf{x})) \in \mathbb{R}^{n+1} : \mathbf{x} \in \text{conv}(A)\}$ (it is the graph of $\text{conv}(u)$). It is clearly the upper boundary, with respect to the final coordinate, of the convex hull of the points $\{(\mathbf{x}, u(\mathbf{x})) \in \mathbb{Z}^n \times \mathbb{R} : \mathbf{x} \in A\}$. The set of faces of the "roof" thus has the structure of a polyhedral complex (see e.g. Grünbaum and Shephard, 1969). Moreover, there is a clear bijection between $\text{conv}(A)$ and the "roof", which is linear on each of these faces. So the projections of these faces to their first n coordinates also form a polyhedral complex. Now the result follows from Lemma 2.18 (which is shown in the text to follow from Lemma 2.17, which we have already proved). □

Proof of Prop. 2.20. By Lemma 2.8, the demand set is constant in the interior of a price complex cell. So the correspondence in (1) is well-defined. Moreover, the affine span of C_σ is given by the set of prices $\mathbf{p}'$ such that $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = u(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}'$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, i.e. all prices such that $\mathbf{p}' \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all such $\mathbf{x}, \mathbf{x}'$. If $\sigma = \text{conv}\,(D_u(\mathbf{p}))$ is k-dimensional, there are k linearly independent vectors of the form $\mathbf{x} - \mathbf{x}'$, and so k linearly independent constraints $\mathbf{p}'$. So $\dim C_\sigma = n - k$.

For (2), recall from Lemma 2.8 that $C_\sigma = \{\mathbf{p} \in \mathbb{R}^n : D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p}\}$, where $\mathbf{p}^\circ \in C_\sigma^\circ$. But for such $\mathbf{p}^\circ$, we show $D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})$ iff $\sigma = \text{conv}\, D_u(\mathbf{p}^\circ) \subseteq \text{conv}\,(D_u(\mathbf{p}))$. Necessity is obvious, and sufficiency follows from Lemma 2.19: if we assume $\text{conv}\,(D_u(\mathbf{p}^\circ)) \subseteq \text{conv}\,(D_u(\mathbf{p}))$, then any $\mathbf{x} \in D_u(\mathbf{p}^\circ) \subsetneq \text{conv}\,(D_u(\mathbf{p}))$ must satisfy $\mathbf{x} \in D_u(\mathbf{p})$. So $\mathbf{p} \in C_\sigma$ iff $\sigma \subseteq \text{conv}\,(D_u(\mathbf{p}))$. Now (3) follows from the combination of (1) and (2).

For (4), we saw already that $\mathbf{p}' \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$ and all $\mathbf{p}' \in C_\sigma$ where $\sigma = \text{conv}(D_u(\mathbf{p}))$. Thus $(\mathbf{p}'' - \mathbf{p}') \cdot (\mathbf{x}' - \mathbf{x}) = 0$ for all $\mathbf{p}', \mathbf{p}'' \in C_\sigma$, also for any $\mathbf{x}, \mathbf{x}' \in \text{conv}\,(D_u(\mathbf{p})) = \sigma$. Now (5) is immediate from Defns. 2.3 and 2.15(5). □

Corollary C.1. *If the demand complex is n-dimensional then every k-cell C_σ of $\mathcal{L}_u$ has some 0-cell C_τ in its boundary, with $\sigma \subseteq \tau$. Moreover if $\mathbf{x} \in \sigma$ but $\mathbf{x} \notin D_u(\mathbf{p}_\sigma^\circ)$ for $\mathbf{p}_\sigma^\circ \in C_\sigma^\circ$, then also $\mathbf{x} \notin D_u(\mathbf{p}_\tau^\circ)$ for $\mathbf{p}_\tau^\circ \in C_\tau^\circ$.*

Proof. If σ is an $(n-k)$-cell of an n-dimensional demand complex, then σ is contained in an n-cell τ. So C_τ is a 0-cell of the LIP and $C_\tau \subseteq C_\sigma$ (Prop. 2.20(3)). By Prop. 2.20(1) we know $\sigma = \text{conv}(D_u(\mathbf{p}_\sigma^\circ))$, so by Lemma 2.19, $\mathbf{x} \notin D_u(\mathbf{p}_\tau^\circ)$. □

Proof of Prop. 3.11. (1): by definition, $\mathbf{x} \in D_u(\mathbf{p})$ if $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}' \in A$, with equality iff $\mathbf{x}' \in D_u(\mathbf{p})$ also. For any invertible matrix G, we may re-write $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') = \mathbf{p}^T G G^{-1}(\mathbf{x} - \mathbf{x}') = (G^T\mathbf{p})^T(G^{-1}\mathbf{x} - G^{-1}\mathbf{x}')$. If G is additionally unimodular, then $G^{-1}\mathbf{x}$ and $G^{-1}\mathbf{x}' \in \mathbb{Z}^n$. If we write $\mathbf{y} = G^{-1}\mathbf{x}$ and $\mathbf{y}' = G^{-1}\mathbf{x}'$ then $(G^T\mathbf{p})^T(\mathbf{y} - \mathbf{y}') \leq G^*u(\mathbf{y}) - G^*u(\mathbf{y}')$ holds iff $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$.

(2): since $\mathcal{L}_u = \{\mathbf{p} \in \mathbb{R}^n : |D_u(\mathbf{p})| > 1\}$, this follows from (1).

(3): if F is a facet of $\mathcal{L}_u$, then by (2), $G^T F = \{G^T\mathbf{p} : \mathbf{p} \in F\}$ is a facet of $\mathcal{L}_{G^*u}$. If $\mathbf{v}$ is normal to F, then $\mathbf{p}^T\mathbf{v}$ is constant for $\mathbf{p} \in F$, so $\mathbf{p}^T G G^{-1}\mathbf{v} = (G^T\mathbf{p})^T G^{-1}\mathbf{v}$ is constant for $G^T\mathbf{p} \in G^T F$. So $G^{-1}\mathbf{v}$ is a demand type vector for G^*u. As G has an integer inverse, the converse is also true. □

C.2 Examples for Sections 2 and 3

Additional Discussion of Figs. 2 and 3. In Figs. 3b-3c, the bundles demanded in the UDRs are (0,0), (0,1), (0,2), (1,2), (2,2), (2,1), and (2,0), clockwise from the top right of the LIP in Fig. 3a.

Observe that if the "black bundles"'s value was greater, so the corresponding bar in Figs. 2b and 2c just touched the roof, then it would still not be at a vertex of Fig. 3a but it would be demanded at the price corresponding to the wavy-shaded 0-cell (that is, $(1, 2)$) that is at a vertex of the LIP. And if it had an (even) higher valuation (so "poked through" the current roof), then the corresponding demand complex point would become a vertex, and the corresponding LIP 0-cell would "open up" to form a new UDR corresponding to the range of prices at which the bundle $(1, 1)$ would then be demanded.

To find the exact LIP of Fig. 2a's valuation using the demand complex of Fig. 3a, compare the values of bundles in adjacent UDRs: the valuations of bundles $(1, 0)$ and $(0, 1)$ show that the dotted 0-cell of the LIP is at $\mathbf{p} = (4, 8)$, since 4 and 8 are the prices below which the agent will first buy any of goods 1 and 2, respectively, when the other good's price is very high. And the wavy-shaded 0-cell must be at (1,2) since $10 - 8 = 2$ is the incremental value of a second unit of good 2, when the agent has none of good 1, and $11 - 10 = 1$ would be the incremental value from then adding a unit of good 1, etc.

So constructing the LIP via the demand complex separates the questions "in what directions are there line segments?" and "where in space are they?", and clarifies which bundle values have to be compared to fix the precise locations of the LIP's cells.

Example C.2. For $A = \{0, 1\}^2$ it is easy to draw every possible demand complex and so obtain every possible combinatorial type of weighted LIP–see Fig. 10. It is clear that

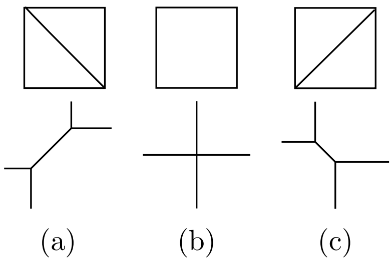


Figure 10: All possible demand complexes, and examples of their dual weighted LIPs, giving all the combinatorial types when $A = \{0, 1\}^2$.

Fig. 10a applies when $u(0, 0) + u(1, 1) < u(1, 0) + u(0, 1)$, so represents substitutes; Fig. 10b applies when $u(0, 0) + u(1, 1) = u(1, 0) + u(0, 1)$, so is additively separable demand; and Fig. 10c applies when $u(0, 0) + u(1, 1) > u(1, 0) + u(0, 1)$, so is complements. (See Section 3.2 for these distinctions). Importantly, it is also clear that these are the *only* possibilities.

Observe that Fig. 10b can be seen as a limit of Fig. 10a (or Fig. 10c). In the LIP, the two 0-cells become arbitrarily close and then coincide in the limit; in quantity space, the faces of the roof tilt until they are coplanar, so that a demand complex edge disappears.

More generally, any demand complex in which the subdivision is not maximal (that is, additional valid $(n - 1)$-faces could be added) can be recovered by deleting $(n - 1)$-faces from some demand complex whose subdivision *is* maximal. For example, Fig.

11 shows all the demand complexes with maximal subdivision, and examples of their dual weighted LIPs, for $A = \{0, 1\} \times \{0, 1, 2\}$; we can then easily recover the remaining combinatorial types of weighted LIP if desired.

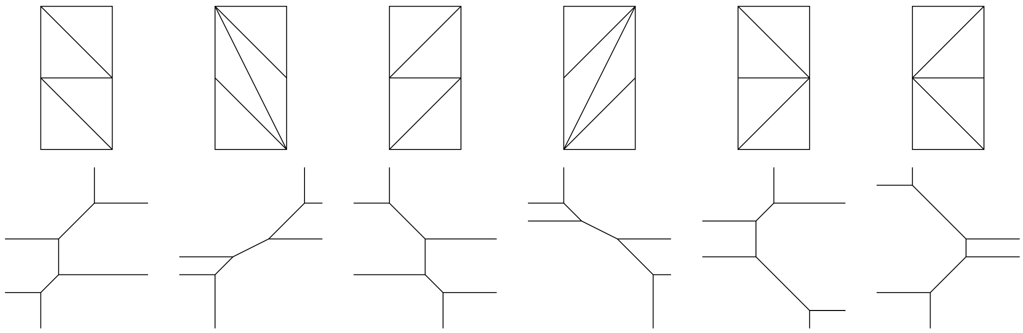


Figure 11: All possible demand complexes with maximal subdivision, and examples of their dual weighted LIPs, giving all such combinatorial types when $A = \{0, 1\} \times \{0, 1, 2\}$.

Example C.3. To illustrate why the condition for indivisible goods to be substitutes is so restrictive, consider three trips: trip A can be made only by bus or train; trip B only by car or train; and trip C only by car or bus. As divisible goods, the three modes of transport are all mutual substitutes. But if the price of *either* bus tickets or train tickets is raised, a consumer might buy a car and use less of *both* forms of public transport, which are therefore locally complements–the car takes the role of good 2 in Fig. 12.[46]

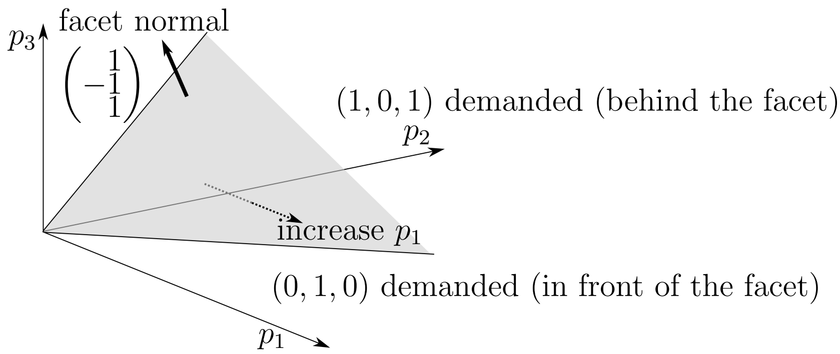


Figure 12: A facet (shaded) defined by $\{\mathbf{p} \in \mathbb{R}^3 : p_1 + p_3 = p_2;\ p_1, p_2, p_3 \geq 0\}$, with its normal (1,-1,1) (the bold arrow). Increasing either p_1 (dotted arrow), or p_3, demonstrates complementarities between goods 1 and 3, as the demand switches from (1,0,1) to (0,1,0).

Example C.4. To illustrate Prop. 3.11, we transform the valuations shown in Figs. 4a and 4b. Create a new good, 3, from two units of good 1 plus one unit of good 2, and consider the economy in which the goods traded are 1 and 3. Note that we can recreate

[46]Even if all goods are mutual substitutes, there can never be trade-offs between more than two of them across the whole of a facet. One mutual substitute might trade-off against two others at prices where more than one facet meet, if at least one of those facets has weight greater than 1. For example, an agent might switch 2 units of some good A for 1 each of two other goods, B and C, which it treats as indistinguishable, in the intersection of all three weight-2 facets where the agent switches between two of the three goods.

one unit of good 2 by buying one unit of good 3 and selling two units of good 1, and we can convert any bundle expressed in terms of goods 1 and 2 (as a column vector) to a bundle of goods 1 and 3 by pre-multiplying by $\left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right)$–this matrix plays the role of G^{-1}.

Observe that the "substitutes" agent of the original economy (who bought either $(1,0)$ or $(0,1)$ at price $(30,20)$) corresponds to an agent in the new economy who would "buy" either $(1,0)$ or $(-2,1)$. We can interpret this as an agent with an endowment of -2 units of good 1 (a contract requiring them to sell), and who buys either three units of good 1 or one unit of good 3. Thus this agent treats goods 1 and 3 as 3:1 substitutes. Similarly, the "complements" agent of the original economy (who bought neither or both of goods 1 and 2) corresponds to an agent in the new economy with an endowment of -1 unit of good 1, who buys one unit of either good 1 or good 3 (so is indifferent between bundles $(0,0)$ and $(-1,1)$) that is, an agent who treats goods 1 and 3 as 1:1 substitutes. So this is a pure substitutes economy.

C.3 Proofs and Additional Examples for Section 4

Proof of Prop. 4.7. Suppose there is always an equilibrium for every finite set of agents with concave valuations of type $G^{-1}\mathcal{D}$, and any relevant supply.

Let $\{u^j : j \in J\}$ be finitely many concave valuations of type $\mathcal{D}$ and let $\mathbf{x}$ be a relevant supply bundle. Then, by Prop. 3.11(3) the valuations $\{G^* u^j : j \in J\}$ are of demand type $G^{-1}\mathcal{D}$. It is obvious they are also all concave. By definition of pullback, $\mathbf{y} := G^{-1}\mathbf{x}$ is in the convex hull of the domain of their aggregate valuation, i.e. is a relevant supply. By assumption, there exists a price $\mathbf{p}$ at which the agent with valuation $G^* u^j$ demands $\mathbf{y}^j$ and $\sum_{j=1}^k \mathbf{y}^j = \mathbf{y}$. Define $\mathbf{x}^j := G\mathbf{y}^j \in D_{u^j}(G^{-T}\mathbf{p})$ (see Prop. 3.11(1)). Then $\mathbf{x} = \sum_{j=1}^k \mathbf{x}^j \in D_{u^J}(G^{-T}\mathbf{p})$. As G is invertible, the converse is shown similarly. $\square$

Example C.5. Recall Ex. C.4. Since the demand type of the original economy contained the columns of $\left(\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which have determinant -2, the demand type in the new economy contains the columns of $\left(\begin{smallmatrix} -3 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, that is, $\left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$which also have determinant -2. Thus equilibrium still fails after the change of basis, despite the fact that this is a pure substitutes economy.

Example C.6 (Failure of aggregate concavity in "hotel rooms" example). Figs. 13a-c show the valuations u^s, u^c, and an aggregate valuation for them, $u^{\{s,c\}}(\mathbf{y}) = \max\{u^s(\mathbf{x}^s) + u^c(\mathbf{x}^c) : \mathbf{x}^s, \mathbf{x}^c \in \{0,1\}^2,\ \mathbf{x}^s + \mathbf{x}^c = \mathbf{y}\}$, respectively. The failure of aggregate concavity at $(1,1)$ is clear from the fact that $\frac{1}{4}(u^{\{s,c\}}(1,0) + u^{\{s,c\}}(0,1) + u^{\{s,c\}}(2,1) + u^{\{s,c\}}(1,2)) = 60 > 50 = u^{\{s,c\}}(1,1)$. This is illustrated in Fig. 13d, which shows $u^{\{s,c\}}$ together with the cell of its roof that corresponds to the price vector $(30,20)$; the bundles $(1,0)$, $(0,1)$, $(2,1)$, and $(1,2)$ are all demanded at this price, but $(1,1)$ is not demanded at any price.

Proof of Fact 4.9. That (2) is equivalent to (1) follows from Cassels (1971) Lemma I.1 and Cor. I.3. For (2)$\Leftrightarrow$(3) consider a parallelepiped P whose vertices are $\mathbf{y} + \sum_{j=1}^s a_j \mathbf{v}^j$ for $a_j \in \{0,1\}$. If $\mathbf{z}$ is a non-vertex integer point in P, then $\mathbf{z} - \mathbf{y}$ exhibits the failure of (3). Conversely, if failure of (3) is exhibited by an integer vector $\sum_{j=1}^s b_j \mathbf{v}^j$ where b_j are not all integers, then $\mathbf{y} + \sum_{j=1}^s a_j \mathbf{v}^j$ exhibits failure of (2), where a_j is the non-integer part of b_j in each case. $\square$

x_1			
1	*0*	$u^s(\mathbf{x})$	
40	0	*0*	x_2
40	30	*1*	

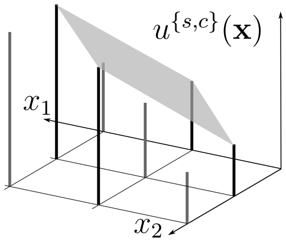
(a) $u^s(\mathbf{x})$.

x_1			
1	*0*	$u^c(\mathbf{x})$	
0	0	*0*	x_2
50	0	*1*	

(b) $u^c(\mathbf{x})$

x_1				
2	*1*	*0*	$u^{\{s,c\}}(\mathbf{x})$	
40	40	0	*0*	
90	50	30	*1*	x_2
90	80	30	*2*	

(c) $u^{\{s,c\}}(\mathbf{x})$.



(d) The $\mathbf{p} = (30, 20)$ cell of the "roof".

Figure 13: Illustration of an aggregate valuation $u^{\{s,c\}}$ which is not concave.

Fact C.7 (see, e.g. Cassels, 1971, Lemma I.2)**.** A set of $s \leq n$ linearly independent vectors in $\mathbb{Z}^n$ are unimodular iff, among the determinants of all the $s \times s$ matrices consisting of s rows of the $n \times s$ matrix whose columns are these s vectors, the greatest common factor is 1.

Lemma C.8. *For a finite set of valuations* $\{u^j : j \in J\}$*, and* $\mathbf{p} \in \mathbb{R}^n$*, write* $\sigma^j = \operatorname{conv}(D_{u^j}(\mathbf{p}))$ *and* $\sigma^J = \operatorname{conv}(D_{u^J}(\mathbf{p}))$*.*

(1) $\sigma^J = \sum_{j\in J} \sigma^j$ *and* $\mathrm{K}_{\sigma^J} = \sum_{j\in J} \mathrm{K}_{\sigma^j}$*.*

(2) If the LIPs intersect at $\mathbf{p}$*, they are transverse at* $\mathbf{p}$ *iff* $\mathrm{K}_{\sigma^J} = \bigoplus_{j\in J} \mathrm{K}_{\sigma^j}$*.*

Proof. (1) By definition $D_{u^J}(\mathbf{p}) = \sum_{j\in J} D_{u^j}(\mathbf{p})$, from which $\sigma^J = \sum_{j\in J} \sigma^j$ follows (see e.g. Cox et al. 2005, Section 7.4, Exercise 3). Thus, by Defn. 4.14, $\mathrm{K}_{\sigma^J} = \sum_{j\in J} \mathrm{K}_{\sigma^j}$.

(2) First suppose $r = 2$ and write $k^j = \dim \sigma^j = \dim K_{\sigma^j}$, and $k^J = \dim \sigma^J = \dim K_{\sigma^J}$. By Part (1) it follows that $k^J = k^1 + k^2 - \dim(\mathrm{K}_{\sigma^1} \cap \mathrm{K}_{\sigma^2})$. But also, $\dim(C_{\sigma^1} + C_{\sigma^2}) = \dim C_{\sigma^1} + \dim C_{\sigma^2} - \dim C_{\sigma^J} = (n - k^1) + (n - k^2) - (n - k^J)$. By Defn. 4.10, the intersection is transverse at $\mathbf{p}$ iff this is equal to n, that is, iff $k^1 + k^2 = k^J$. So we conclude that the intersection is transverse at $\mathbf{p}$ iff $\mathrm{K}_{\sigma^1} \cap \mathrm{K}_{\sigma^2} = \{\mathbf{0}\}$, which, together with (1), is the definition of $\mathrm{K}_{\sigma^J} = \mathrm{K}_{\sigma^1} \oplus \mathrm{K}_{\sigma^2}$. The $r \geq 3$ case now follows, as we check for transversality incrementally using the $r = 2$ case (Defn. 4.10). ☐

Proof of Lemma 4.15. This is a standard consequence of Lemma C.8(2). Every vector $\mathbf{x}$ in $\mathrm{K}_{\sigma^{\{1,2\}}}$ can be written as a sum of a vector $\mathbf{x}^1 \in \mathrm{K}_{\sigma^1}$ and a vector $\mathbf{x}^2 \in \mathrm{K}_{\sigma^2}$. If we also write $\mathbf{x} = \mathbf{x}^{1\prime} + \mathbf{x}^{2\prime}$, where $\mathbf{x}^{j\prime} \in \mathrm{K}_{\sigma^j}$, $j = 1, 2$, then $\mathbf{x}^1 - \mathbf{x}^{1\prime} = \mathbf{x}^2 - \mathbf{x}^{2\prime} \in \mathrm{K}_{\sigma^1} \cap \mathrm{K}_{\sigma^2} = \{\mathbf{0}\}$ and so $\mathbf{x}^1 - \mathbf{x}^{1\prime} = \mathbf{x}^2 - \mathbf{x}^{2\prime} = \mathbf{0}$. ☐

Example C.9 (**The "hotel rooms" example, and Section 4.2.4's argument**)**.** It helps intuition to see which parts of the argument of Section 4.2.4 are, and are not, valid for our simple two-goods substitutes, $u^s(x_1, x_2) = \max\{40x_1, 30x_2\}$ (Fig. 4a), and complements, $u^c(x_1, x_2) = \min\{50x_1, 50x_2\}$ (Fig. 4b), valuations, for which the failure of equilibrium was discussed in Sections 4.2.1-4.2.2.

Consider the (transverse) intersection price $\mathbf{p} = (30, 20)$, and the bundle $\mathbf{y} = (1, 1) \in \sigma^{\{s,c\}} = \operatorname{conv}(D_{u^{\{s,c\}}}(\mathbf{p}))$. So $\sigma^{\{s,c\}}$ is the central square 2-cell in Fig. 5c. In this example it is clear that the corresponding cells, $\sigma^s = \operatorname{conv}(D_{u^s}(\mathbf{p}))$ and $\sigma^c = \operatorname{conv}(D_{u^c}(\mathbf{p}))$, of Figs. 5a and 5b, respectively, are the two diagonal 1-cells. (The general procedure for identifying σ^s and σ^c is to examine the LIP intersection in Fig. 4c, hence to identify the relevant cells of the LIPs in Figs. 4a-b, and so the relevant cells in Figs. 5a-b.)

We have $\mathrm{K}_{\sigma^{\{s,c\}}} = \mathbb{R}^2$, and the edges of $\sigma^{\{s,c\}}$ are $\{(1,-1),(1,1)\}$ (see Fig. 5c), and in this case the whole set of edges is needed to form a basis of $\mathrm{K}_{\sigma^{\{s,c\}}}$. And (see Figs. 4a, 4b) K_{σ^s} and K_{σ^c} are the sets of scalar multiples of $(1,-1)$ and $(1,1)$, respectively (and these vectors give a basis for each).

Now say, for example, $\mathbf{x} = (1,0) \in \sigma^{\{s,c\}}$. We can write $\mathbf{y}-\mathbf{x} =(0,1)$ as $\lambda_1(1,-1) + \lambda_2(1,1)$ by choosing $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. So $\mathbf{y}-\mathbf{x} = \mathbf{z}^s + \mathbf{z}^c$ in which $\mathbf{z}^s = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\mathbf{z}^c = \left(\frac{1}{2}, \frac{1}{2}\right)$. (These are *not* integer bundles because unimodularity fails in this example.)

On the other hand, we use $\sigma^{\{s,c\}} = \sigma^s + \sigma^c$ to write $\mathbf{y} = \mathbf{y}^s + \mathbf{y}^c$ with $\mathbf{y}^s \in \sigma^s$ and $\mathbf{y}^c \in \sigma^c$. We can see from Figs. 5a, 5b that the unique way to do this is $\mathbf{y}^s = \mathbf{y}^c = \left(\frac{1}{2}, \frac{1}{2}\right)$. Also, since $\mathbf{x} \in D_{u^{\{s,c\}}}(\mathbf{p})$, we can write $\mathbf{x} = \mathbf{x}^s + \mathbf{x}^c$ in which $\mathbf{x}^s$ and $\mathbf{x}^c$ are integer bundles. In fact, $\mathbf{x}^s$=$(1,0) \in D_{u^s}(\mathbf{p})$ and $\mathbf{x}^c$=$(0,0) \in D_{u^c}(\mathbf{p})$, as can be seen by using Fig. 4c to see that the UDR of $\mathbf{x}$ is the region above the price $(30,20)$, hence the relevant UDRs in Figs. 4a and 4b are $(1,0)$ and $(0,0)$, respectively. So $\mathbf{y}^s - \mathbf{x}^s = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\mathbf{y}^c - \mathbf{x}^c = \left(\frac{1}{2}, \frac{1}{2}\right)$.

The intersection is transverse at $\mathbf{p}$ and so, as predicted, $\mathbf{y}^s - \mathbf{x}^s = \mathbf{z}^s$, and $\mathbf{y}^c - \mathbf{x}^c = \mathbf{z}^c$. However, in this example, the set of edges of $\sigma^{\{s,c\}}$, that is, $\{(1,-1),(1,1)\}$ is *not* unimodular, so $\mathbf{z}^s$ and $\mathbf{z}^c$, and hence also $\mathbf{y}^s$ and $\mathbf{y}^c$, are *not* integer bundles, so the method does *not* demonstrate equilibrium. (Indeed, in this case it demonstrates the failure of equilibrium.)

C.4 Proofs and Additional Examples for Sections 5 and 6

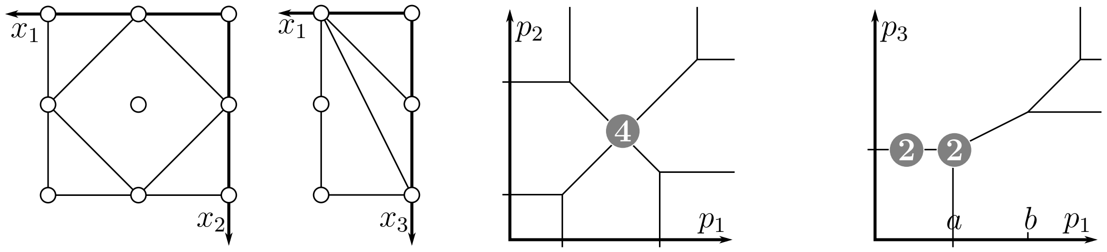
Example C.10 (The "hotel rooms" example with weight 2 facets–further discussion of Section 5.1.2). We can understand "mid-point bundles" such as (1,1) and (1,3) in Fig. 7c as being reached by starting from a vertex bundle, and then changing demand to move one "step" along an edge. These bundles correspond to demand changing part-way along a diagonal edge in Figs. 7a and 7b. Likewise the "central bundle" $(2,2)$ can be reached by a combination of these part-way diagonal steps, as shown by the dashed grey lines. Meanwhile the remaining four bundles (the black grid points in Fig. 7c) cannot be reached by any such moves, so are not demanded.

As in the discussion of Fig. 5c in Section 4.2.2, this illustrates the relevance of Fact 4.9(3). Unimodularity is the ability to create any vector in the space spanned by the vectors of the demand type as an integer combination of any spanning set of vectors of the demand type. $u^{\{2s,2c\}}$ is not of a unimodular demand type; there is therefore the possibility that some bundles cannot be reached using combinations of the edge vectors of the relevant demand complex cell; and indeed these four bundles are unreachable.

Fig. 7d can be understood similarly to Fig. 7c. Now all bundles can be reached via changes of demand that take the agents part-way along their individual demand complex cells (so are edge vectors of the relevant aggregate-demand complex cell; these are again denoted by dashed lines). So equilibrium always exists in this case.

Example C.11. Consider agents with concave valuations and demand complexes as given in Figs. 14a-14b. We do not specify precise valuations, and as discussed in Section 3.3, the aggregate-demand complex is therefore not uniquely specified. We wish to understand what combinations can arise and when equilibrium can fail for some relevant supplies. However, it is helpful to consider specific corresponding LIPs, and their projections to respectively (p_1, p_2) and (p_1, p_3) coordinate spaces are shown in Figs. 14c-14d, which also marks all cell weights greater than 1.

In Ex. B.4 we computed $M_1^3(A^1, A^2) = 16$ and $M_2^3(A^1, A^2) = 8$.



(a) Σ_{u^1}, with the grid of integer bundles in $\mathrm{conv}(A^1)$.

(b) Σ_{u^2}.

(c) The projection of $\mathcal{L}_{u^1}$ to (p_1, p_2)-space, with cell weights greater than 1.

(d) The projection of $\mathcal{L}_{u^2}$ to (p_1, p_3)-space, with cell weights greater than 1.

Figure 14: The demand complex (a-b) and weighted LIP (c-d) for Ex. C.11.

Because each of the five 2-cells of Σ_{u^1} lies in the space $x_3 = 0$, the LIP $\mathcal{L}_{u^1}$ has five 1-cells (pictured as 0-cells in Fig. 14c), each parallel to $\mathbf{e}^3$. The central 1-cell has weight 4, and the remainder have weight 1. Its eight facets all span this same coordinate direction. Similarly, the three 1-cells and seven facets of $\mathcal{L}_{u^2}$ are each seen as the extension, parallel to $\mathbf{e}^2$, of respectively the 0-cells and 1-cells in Fig. 14d.

So, if the intersection is transverse, each 1-cell of $\mathcal{L}_{u^2}$, running in direction $\mathbf{e}^2$, meets two facets of $\mathcal{L}_{u^1}$. As one of the 1-cells of $\mathcal{L}_{u^2}$ has weight 2, the total count of such intersections is 8: the mixed volume required. So all the corresponding intersection 0-cells are discrete convex. By Prop. 4.11, it follows that this is also true if the intersection is not transverse.

But, again starting in the transverse case, each 1-cell of $\mathcal{L}_{u^1}$, running in direction $\mathbf{e}^3$, could meet either one or two facets of $\mathcal{L}_{u^2}$. To be precise, write such a 1-cell as $\mathbf{p} + \{\lambda \mathbf{e}^3 : \lambda \in \mathbb{R}\}$. It meets two facets of $\mathcal{L}_{u^2}$ if $p_2 > b_2$, it meets one facet of weight 2 if $p_2 < a_2$, and it meets one facet of weight 1 if $a_2 < p_2 < b_2$.

There are five 1-cells of $\mathcal{L}_{u^1}$, one with weight 4. So the naïvely-weighted count of these intersections is equal to 16 *unless* $a_2 < p_2 < b_2$ holds for any of these 1-cells, in which case, it does not. This, then, is a necessary and sufficient condition for equilibrium. It is easy to translate this into conditions on the valuations themselves.

Finally, note that the existence of equilibrium when $p_2 < a_2$ or $p_2 > b_2$ for each of the 1-cells of $\mathcal{L}_{u^1}$, is not implied by any "local" application of the unimodularity theorem (as described at Footnote 30). The central 1-cell of $\mathcal{L}_{u^1}$ has adjacent facets with normals $\mathbf{e}^1 + \mathbf{e}^2$ and $\mathbf{e}^1 - \mathbf{e}^2$. No set containing these two vectors can be unimodular.

Proof of Prop. 5.4. Necessity of this condition follows from Lemma 2.11.

Conversely, suppose that equilibrium fails for some relevant supply. By Lemma 4.8, there exists a price $\mathbf{p}$ in $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$ such that $D_{u^{\{1,2\}}}(\mathbf{p})$ is not discrete-convex. So there exists $\mathbf{x} \in \mathbb{Z}^n$ such that $\mathbf{x} \in \sigma^{\{1,2\}} := \mathrm{conv}(D_{u^{\{1,2\}}}(\mathbf{p}))$, but $\mathbf{x} \notin D_{u^{\{1,2\}}}(\mathbf{p})$. By Lemma 2.8, $D_{u^{\{1,2\}}}(\mathbf{p}')$ is constant for any $\mathbf{p}' \in C^\circ_{\sigma^{\{1,2\}}}$, and thus both individual demand sets must be constant in $C^\circ_{\sigma^{\{1,2\}}}$. So $C_{\sigma^{\{1,2\}}} \subseteq \mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$.

Since the aggregate-demand complex is n-dimensional, we may apply Cor. C.1: there exists a 0-cell C_τ of $\mathcal{L}_{u^{\{1,2\}}}$ such that $\sigma \subseteq \tau$ and which also satisfies $\mathbf{x} \notin D_{u^{\{1,2\}}}(\mathbf{p}_\tau)$ for $\mathbf{p}_\tau \in C_\tau$. Moreover, $C_\tau \subseteq C_\sigma$ (by Prop. 2.20(3)) and so $C_\tau \subseteq \mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$. $\square$

Proof of Fact 5.6. Standard; see e.g. Cox et al. (2005, p. 334). $\square$

Lemma C.12. *Fix* $\mathbf{p} \in \mathbb{R}^n$ *and write* u^* *for the restriction of* u *to* $D_u(\mathbf{p})$. *Then there exists* $\delta > 0$ *such that* $D_u(\mathbf{p}') = D_{u^*}(\mathbf{p}')$ *for all* $\mathbf{p}' \in B_\delta(\mathbf{p}) = \{\mathbf{p}' \in \mathbb{R}^n : \|\mathbf{p}' - \mathbf{p}\| < \delta\}$. *In particular the LIPs coincide for all such* $\mathbf{p}'$.

Proof. There exists some $\epsilon > 0$ such that $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} > u(\mathbf{y}) - \mathbf{p} \cdot \mathbf{y} + \epsilon$ for every $\mathbf{x} \in D_u(\mathbf{p})$, $\mathbf{y} \notin D_u(\mathbf{p})$. So if we choose $\delta < \frac{\epsilon}{\|\mathbf{x}\|}$ for all $\mathbf{x} \in D_u(\mathbf{p})$ then it is easy to show that $u(\mathbf{y}) - \mathbf{p}' \cdot \mathbf{y} < u(\mathbf{x}) - \mathbf{p}' \cdot \mathbf{x}$ for all $\mathbf{p}' \in B_\delta(\mathbf{p})$, $\mathbf{x} \in D_u(\mathbf{p})$, $\mathbf{y} \notin D_u(\mathbf{p})$. Thus, at prices $\mathbf{p}' \in B_\delta(\mathbf{p})$, only bundles in $D_u(\mathbf{p})$ might be demanded; $D_u(\mathbf{p}') = D_{u^*}(\mathbf{p}')$. $\square$

Lemma C.13. *Suppose* $\mathcal{L}_{u^1}$ *and* $\mathcal{L}_{u^2}$ *have an intersection* 0*-cell* C *at* $\mathbf{p}$. *Suppose* $\mathbf{v} \in \mathbb{R}^n$ *is such that* $\mathcal{L}_{u^1}$ *and* $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ *intersect transversely for sufficiently small* $\epsilon > 0$, *and write* $\mathcal{C}$ *for the set of their intersection* 0*-cells emerging from* C. *Write* $\mathbf{x}^1$, $\mathbf{x}^2$ *for the bundles such that* $\{\mathbf{x}^1\} = D_{u^1}(\mathbf{p} + \epsilon\mathbf{v})$, $\{\mathbf{x}^2\} = D_{u^2}(\mathbf{p} - \epsilon\mathbf{v})$. *Then the demand complex cell* $\sigma_C \in \Sigma_{u^{\{1,2\}}}$ *dual to* C *is equal to* $(\{\mathbf{x}^2\} + \sigma^1) \cup (\{\mathbf{x}^1\} + \sigma^2) \cup \bigcup\{\sigma_{C'} : C' \in \mathcal{C}\}$, *where* $\sigma^j = \operatorname{conv}(D_{u^j}(\mathbf{p}))$ *and* $\sigma_{C'} \in \Sigma_{u^{\{1,2_\epsilon\}}}$ *is dual to* $C' \in \mathcal{C}$.

Proof. Use the notation of Lemma C.12; using that lemma, choose $\delta > 0$ sufficiently small so that, for all $\mathbf{p}' \in B_\delta(\mathbf{p})$, *both* $D_{u^1}(\mathbf{p}') = D_{u^{1*}}(\mathbf{p}')$ *and* $D_{u^2}(\mathbf{p}') = D_{u^{2*}}(\mathbf{p}')$.

We write $u^{2*[\epsilon]}$ for the valuation given by $u^{2*[\epsilon]}(\mathbf{x}) = u^{2*}(\mathbf{x}) + \epsilon\mathbf{v} \cdot \mathbf{x}$. Since $B_\delta(\mathbf{p})$ is topologically open, and since $\mathcal{L}_{u^{2[\epsilon]}} = \{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$, we can choose ϵ sufficiently small that $D_{u^{2[\epsilon]}}(\mathbf{p}') = D_{u^{2*[\epsilon]}}(\mathbf{p}')$ for all $\mathbf{p}' \in B_\delta(\mathbf{p})$. Moreover, by Defn. 5.11, emerging 0-cells get arbitrarily close to C for arbitrarily small ϵ. So we can choose ϵ sufficiently small that all intersection 0-cells for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^{2[\epsilon]}}$, emerging from C, are contained in $B_\delta(\mathbf{p})$. Now, it is sufficient to prove the result for $\mathcal{L}_{u^{1*}}$, $\mathcal{L}_{u^{2*}}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^{2*}} = \mathcal{L}_{u^{2*[\epsilon]}}$.

The cell σ_C is, by definition, the convex hull of the domain of $u^{\{1*,2*\}}$, and hence also convex hull of the domain of $u^{\{1*,2*_\epsilon\}}$. So σ_C is equal to the union of the top-dimensional cells in the demand complex $\Sigma_{u^{\{1*,2*_\epsilon\}}}$, i.e., the demand complex cells dual to 0-cells of $\mathcal{L}_{u^{\{1*,2*_\epsilon\}}}$. But the 0-cells of $\mathcal{L}_{u^{\{1*,2*_\epsilon\}}}$ are: those in $\mathcal{C}$; and, if they exist, 0-cells at $\mathbf{p}$ of $\mathcal{L}_{u^{1*}}$, $\mathcal{L}_{u^{2*}}$. Take duals, and in the latter cases account for the demand of the other agent. Since in every case $\{\mathbf{x}^1\} + \sigma^2 \subseteq \sigma_C$ and $\{\mathbf{x}^2\} + \sigma^1 \subseteq \sigma_C$, we obtain the result. $\square$

Proof of Lemma 5.12. Write σ^1, σ^2 and $\sigma^{\{1,2\}}$, respectively, for the demand complex cells at $\mathbf{p}$ of valuations u^1, u^2 and $u^{\{1,2\}}$. For any $\mathbf{v}$ and ϵ, if there exist no intersection 0-cells emerging from C then, by Lemma C.13, $\operatorname{vol}_n(\sigma^1) + \operatorname{vol}_n(\sigma^2) \geq \operatorname{vol}_n(\sigma^{\{1,2\}})$. But this is impossible since $\sigma^{\{1,2\}} = \sigma^1 + \sigma^2$, since both σ^1 and σ^2 contain more than one element, and since $\dim \sigma^{\{1,2\}} = n$, which implies $\operatorname{vol}_n(\sigma^{\{1,2\}}) \neq 0$. $\square$

Proof of Lemma 5.17. From Lemma C.8(1), $\sum_{j\in K} \mathrm{K}_{\sigma^j} = \mathrm{K}_{\sigma^J}$. It follows that $\mathrm{K}_{\sigma^j} \subseteq \mathrm{K}_{\sigma^J}$, and so $\Lambda_{\sigma^j} \subseteq \Lambda_{\sigma^J}$, for all $j \in J$. Thus, as Λ_{σ^J} is additively closed, $\sum_{j\in J} \Lambda_{\sigma^j} \subseteq \Lambda_{\sigma^J}$. It is straightforward that it is a sublattice.

The linear span of Λ_{σ^J} is K_{σ^J}. The linear span of $\sum_{j\in J} \Lambda_{\sigma^j}$ contains K_{σ^j} for all $j \in J$, and so it contains their sum; on the other hand $\sum_{j\in J} \Lambda_{\sigma^j} \subseteq \sum_{j\in J} \mathrm{K}_{\sigma^j}$ and the latter is linear; so $\sum_{j\in J} \mathrm{K}_{\sigma^j}$ is the linear span of $\sum_{j\in J} \Lambda_{\sigma^j}$. Since, again, $\sum_{j\in J} \mathrm{K}_{\sigma^j} = \mathrm{K}_{\sigma^J}$, the ranks are clearly the same, and (1) follows. Finally, if the intersection of $\{\mathcal{L}_j : j \in J\}$ is transverse at $\mathbf{p}$ then (2) follows immediately from Lemma C.8(2). $\square$

Proof of Facts 5.18. Cassels (1971) assumes that all lattices $\Lambda \subseteq \mathbb{Z}^n$ have rank n. To adapt his results to our conventions, first fix a $k \times n$ matrix G_Λ such that $G_\Lambda \Lambda = \mathbb{Z}^k$. Then

$G_\Lambda\Lambda'$ is a rank-k sublattice of $\mathbb{Z}^k$. From our definitions it is clear that $[G_\Lambda\Lambda : G_\Lambda\Lambda'] = [\Lambda : \Lambda']$. Now Cassels (1971) Lemma I.1 gives 5.18(5). And Fact 5.18(1) follows: if $\mathbf{v}, \mathbf{w} \in \Delta_{\Lambda'} \cap \Lambda$ but $\{\mathbf{v}\} + \Lambda' = \{\mathbf{w}\} + \Lambda'$ then $\mathbf{v} - \mathbf{w} \in \Lambda'$, so $\mathbf{v} - \mathbf{w} = \sum_i \alpha_i \mathbf{v}'^i$ where $\mathbf{v}'^i$ are our basis for Λ', and $\alpha_i \in \mathbb{Z}$; then $\mathbf{w} \in \Delta_{\Lambda'}$ and $\mathbf{v} = \mathbf{w} + (\mathbf{v} - \mathbf{w}) \in \Delta_{\Lambda'}$ are consistent only if $\mathbf{v} = \mathbf{w}$; the converse is argued similarly. Fact 5.18(2) follows immediately from Fact 5.18(5). Fact 5.18(3) is given by Cassels (1971, p. 69). Fact 5.18(4) follows from (2) and Fact 4.9(2). □

Proof of Thm. 5.19. Part (1) is very similar to the proof of "sufficiency" for Thm. 4.3, given in Section 4.2.4. Let $\mathbf{x} \in \sigma^J \cap \mathbb{Z}^n$. By Lemma C.8, $\sigma^J = \sum_{j \in J} \sigma^j$ and so $\mathbf{x} = \sum_{j \in J} \mathbf{x}^j$, where $\mathbf{x}^j \in \sigma^j$ for $j \in J$. We wish to show that each $\mathbf{x}^j \in \mathbb{Z}^n$; then $\mathbf{x}^j \in D_{u^j}(\mathbf{p})$ by concavity of u^j (Lemma 2.11) and so $\mathbf{x} \in D_{u^J}(\mathbf{p})$.

Fix $\mathbf{y} \in D_{u^J}(\mathbf{p})$, so $\mathbf{y} = \sum_{j \in J} \mathbf{y}^j$ where $\mathbf{y}^j \in D_{u^j}(\mathbf{p}) \subsetneq \mathbb{Z}^n$ for $j \in J$. Then $\mathbf{y} - \mathbf{x} \in \Lambda_{\sigma^J}$. But $\Lambda_{\sigma^J} = \sum_{j \in J} \Lambda_{\sigma^j}$ (by Fact 5.18(2)) and so $\mathbf{y} - \mathbf{x} = \sum_{j \in J} \mathbf{z}^j$, where $\mathbf{z}^j \in \Lambda_{\sigma^j} \subseteq \mathbb{Z}^n$. But now $\sum_{j \in J} \mathbf{z}^j = \sum_{j \in J} (\mathbf{y}^j - \mathbf{x}^j)$. By Lemma 4.15, $\mathbf{y}^j - \mathbf{x}^j = \mathbf{z}^j$ for $j \in J$. Thus $\mathbf{x}^j = \mathbf{y}^j - \mathbf{z}^j \in \mathbb{Z}^n$ for $j \in J$, as required.

We prove Part (2) in the case when $\dim \sigma^{j_0} = 2$. The case in which $\dim \sigma^{j_0} = 1$ may be seen from the following argument by ignoring the role of σ^{j_0}. So suppose $\dim \sigma^{j_0} = 2$.

Assume that $\mathbf{0} \in \sigma^j$ for $j \in J$ (otherwise the following arguments are simply augmented by a fixed shift). If $\dim \sigma^j = 0$ for any $j \in J$ then $\Lambda_{\sigma^j} = \{\mathbf{0}\}$ and inclusion of σ^j has no effect on σ^J. So we assume $\dim \sigma^j = 1$ for all $j \in J \setminus \{j_0\}$.

For $j \in J \setminus \{j_0\}$, fix a minimal integer non-zero vector $\mathbf{v}^j \in \sigma^j$. In each case this vector then forms a basis for the corresponding lattice Λ_{σ^j}.

We also need a basis for $\Lambda_{\sigma^{j_0}}$ consisting of vectors $\mathbf{v}^0, \mathbf{v}^1$ contained inside σ^{j_0}. Start by taking $\mathbf{w}^0$,$\mathbf{w}^1 \in \sigma^{j_0}$ which are linearly independent integer vectors. If these are a basis for $\Lambda_{\sigma^{j_0}}$, we are done. If not, they span a sublattice Λ'_1 of $\Lambda_{\sigma^{j_0}}$, such that $[\Lambda_{\sigma^{j_0}} : \Lambda'_1] > 1$, and so there must exist $\mathbf{w} \in \Lambda_{\sigma^{j_0}}$ which is a non-vertex point of $\Delta_{\Lambda'_1}$. Then $\mathbf{w} = \alpha^0\mathbf{w}^0 + \alpha^1\mathbf{w}^1$ with $\alpha^0, \alpha^1 \in [0, 1)$. If $\alpha^0 + \alpha^1 \leq 1$ then we fix $\mathbf{w}^2 := \mathbf{w}$; if $\alpha^0 + \alpha^1 > 1$ then let $\mathbf{w}^2 = \mathbf{w}^0 + \mathbf{w}^1 - \mathbf{w}$. In either case now $\mathbf{w}^2 = \beta^0\mathbf{w}^0 + \beta^1\mathbf{w}^1$ with $\beta^0 + \beta^1 \leq 1$. As σ^{j_0} is convex we conclude that $\mathbf{w}^2 \in \sigma^{j_0}$.

Recalling by definition that $\mathbf{w} \notin \Lambda'_1$, we know $\mathbf{w}^2$ is distinct from $\mathbf{w}^0, \mathbf{w}^1, \mathbf{0}$. So $\mathbf{w}^2$ is a non-vertex point of the convex hull Δ^0 of $\mathbf{0}, \mathbf{w}^0, \mathbf{w}^1$. Hence the convex hull Δ^1 of $\mathbf{0}, \mathbf{w}^1, \mathbf{w}^2$ has strictly smaller area than Δ^0. Moreover, the parallelepipeds spanned by $\mathbf{w}^0, \mathbf{w}^1$ and by $\mathbf{w}^1, \mathbf{w}^2$ have areas equal to twice the areas of Δ^0, Δ^1, respectively. So, if Λ'_2 is the sublattice of $\Lambda_{\sigma^{j_0}}$ spanned by $\mathbf{w}^1, \mathbf{w}^2$, then $[\Lambda_{\sigma^{j_0}} : \Lambda'_2] < [\Lambda_{\sigma^{j_0}} : \Lambda'_1]$.

As all subgroup indices are positive-integer-valued, after a finite number of repetitions of this process, the subgroup index is 1, and hence (by Fact 5.18(2)) we have obtained vectors in σ^{j_0} which are a basis of $\Lambda_{\sigma^{j_0}}$, as required. Label these vectors $\mathbf{v}^{j_0}$ and $\mathbf{v}^{j_1}$ where $j_1 \notin J$ and write $J' = J \cup \{j_1\}$.

Now, by Fact 5.18(2), there exists $\mathbf{x} \in \Lambda^J_\sigma$, $\mathbf{x} \notin \sum_{j \in J} \Lambda_{\sigma^j}$. And by Lemma 5.17(2), our identified vectors $\{\mathbf{v}^j : j \in J'\}$ are an integer basis for $\sum_{j \in J} \Lambda_{\sigma^j}$. By Fact 5.6 and Lemma 5.17(1), they are thus a vector space basis for K_{σ^J}, so we can write $\mathbf{x} = \sum_{j \in J'} \alpha^j \mathbf{v}^j$. Moreover, since subtracting integer multiples of the $\mathbf{v}^j$ from $\mathbf{x}$ yields a new element of Λ_{σ^J}, we can assume that $\alpha^j \in [0, 1)$ for $j \in J'$. Additionally, we can assume that $\alpha^{j_0} + \alpha^{j_1} \leq 1$: if $\alpha^{j_0} + \alpha^{j_1} > 1$ then replace $\mathbf{x}$ with $\sum_{j \in J'} \mathbf{v}^j - \mathbf{x} \in \Lambda_{\sigma^J}$. Now $\alpha^{j_0}\mathbf{v}^{j+0} + \alpha^{j_1}\mathbf{v}^{j_1} \in \sigma^{j_0}$ and, for all other $j \in J$, also $\alpha^j\mathbf{v}^j \in \sigma^j$. So, $\mathbf{x} \in \sum_{j \in J} \sigma^j = \sigma^J =$

conv $(D_{u^J}(\mathbf{p}))$. Moreover, $\mathbf{x} \in \Lambda_{\sigma^J} \subseteq \mathbb{Z}^n$. But, by assumption, $\mathbf{x} \notin \sum_{j\in J} \Lambda_{\sigma^j}$, and so $\mathbf{x} \notin \sum_{j\in J} D_{u^j}(\mathbf{p}) = D_{u^J}(\mathbf{p})$. □

Example C.14 (Equilibrium demonstrated by the Subgroup Indices Theorem but not the Unimodularity Theorem). Suppose $n = 4$ and that $D_{u^1}(\mathbf{p}) = \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^1 + \mathbf{e}^2, 2\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 + 2\mathbf{e}^2\}$ while $D_{u^2}(\mathbf{p}) = \{\mathbf{e}^3, \mathbf{e}^4, \mathbf{e}^3 + \mathbf{e}^4, 2\mathbf{e}^3 + \mathbf{e}^4, \mathbf{e}^3 + 2\mathbf{e}^4\}$ (Prop. 2.20 and Thm. 2.14(1) show how to construct u^1, u^2 with these properties). Write $\sigma^j = \mathrm{conv}(D_{u^j}(\mathbf{p}))$ for $j = 1, 2, \{1, 2\}$. Then $\Lambda_{\sigma^1} = \{\mathbf{v} \in \mathbb{Z}^4 : v_1 = v_2 = 0\}$ and $\Lambda_{\sigma^2} = \{\mathbf{v} \in \mathbb{Z}^4 : v_3 = v_4 = 0\}$, while $\Lambda_{\sigma^{\{1,2\}}} = \mathbb{Z}^4$, so $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] = 1$, and $D_{u^{\{1,2\}}}(\mathbf{p})$ is discrete-convex (as may also be checked directly).

However, the facets to Agent 1's demand at $\mathbf{p}$ have normal vectors $(1, 1, 0, 0)$ and $(1, -1, 0, 0)$, while the facets normals for Agent 2 are $(0, 0, 1, 1)$ and $(0, 0, 1, -1)$. This set is *not* unimodular, and so equilibrium is not assured by application of Thm. 4.3.

Example C.15. Let $n = 4$. Agent 1 has valuation $u^1(0, 0, 0, 0) = 0$, $u^1(1, 1, 0, 0) = 6$, $u^1(0, 0, 1, 1) = 6$. So for prices in the LIP 2-cell $\{\mathbf{p} \in \mathbb{R}^4 : p_1 + p_2 = 6,\ p_3 + p_4 = 6\}$, Agent 1 is indifferent between these three bundles; the dual demand complex cell is $\sigma^1 = \mathrm{conv}((0, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 1))$.

Agent 2 has valuation $u^2(0, 0, 0, 0) = 0$, $u^2(0, 1, 1, 0) = 9$, $u^2(4, 0, 0, 1) = 6$. So for prices in the LIP 2-cell $\{\mathbf{p} \in \mathbb{R}^4 : p_2+p_3 = 9,\ 4p_1+p_4 = 6\}$, Agent 2 is indifferent between these three bundles; the dual demand complex cell is $\sigma^2 = \mathrm{conv}((0, 0, 0, 0), (0, 1, 1, 0), (4, 0, 0, 1))$.

These two individual LIP 2-cells intersect at $\mathbf{p} = (1, 5, 4, 2)$. At this price, the individual demand complex cells are σ^1 and σ^2 as above. The aggregate-demand complex cell $\sigma^{\{1,2\}}$ is 4-dimensional and so $\Lambda_{\sigma^{\{1,2\}}} = \mathbb{Z}^4$. Meanwhile Λ_{σ^1} and Λ_{σ^2} are rank-2 lattices, and we check that the non-zero vectors we already know in each lattice do give a basis in each case, by checking that the sets $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$ and $\{(0, 1, 1, 0), (4, 0, 0, 1)\}$ are unimodular (use Fact C.7). Thus the union of these sets is a basis for $\Lambda_{\sigma^1} + \Lambda_{\sigma^2}$ (Lemma 5.17(2)). But this union is *not* unimodular, and calculating its determinant tells us (by Fact 5.18(3)) that $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] = 3$.

So, by Fact 5.18(1), we now know that there are exactly 2 non-vertex integer points to a fundamental parallelepiped of $\Lambda_{\sigma^1} + \Lambda_{\sigma^2}$. In terms of the basis vectors, these are:

$$\frac{2}{3}\big(1, 1, 0, 0\big) + \frac{2}{3}\big(0, 0, 1, 1\big) + \frac{1}{3}\big(0, 1, 1, 0\big) + \frac{1}{3}\big(4, 0, 0, 1\big) = \big(2, 1, 1, 1\big) \tag{1}$$

$$\frac{1}{3}\big(1, 1, 0, 0\big) + \frac{1}{3}\big(0, 0, 1, 1\big) + \frac{2}{3}\big(0, 1, 1, 0\big) + \frac{2}{3}\big(4, 0, 0, 1\big) = \big(3, 1, 1, 1\big) \tag{2}$$

These expressions show clearly that $(2, 1, 1, 1)$ can be decomposed to give a part in σ^2 and a part *not* in σ^1, whereas $(3, 1, 1, 1)$ can be decomposed to give a part in σ^1 and a part *not* in σ^2. Moreover, by linear independence of this set of four vectors, these are the *only* possible decompositions into sums of bundles in the affine spans of σ^1, σ^2. So neither is in $\sigma^1 + \sigma^2 = \sigma^{\{1,2\}} = \mathrm{conv}\, D_{u^{\{1,2\}}}(1, 5, 4, 2)$. So the only integer vectors in $\mathrm{conv}\, D_{u^{\{1,2\}}}(1, 5, 4, 2)$ are in fact in $D_{u^{\{1,2\}}}(1, 5, 4, 2)$ itself: it is discrete-convex.

Examples C.16–C.18 (Modifications of Ex. C.15). In each of these examples we specify a demand complex containing a single maximal cell σ^j, for each of our two agents $j = 1, 2$. In each case a dual LIP is easy to find, and so (Thm. 2.14(2)) there exists a concave valuation with these properties.[47] The analysis then proceeds as in Ex. C.15.

Example C.16. Swap a pair of the bundles: $\sigma^1 = \mathrm{conv}((0,0,0,0),(1,1,0,0),(0,1,1,0))$ and let $\sigma^2 = \mathrm{conv}((0,0,0,0),(0,0,1,1),(4,0,0,1))$. In this case the decompositions (1) and (2) of Ex. C.15 show that both $(2,1,1,1)$ and $(3,1,1,1)$ *are* in $\sigma^1+\sigma^2$. Thus they are in $\mathrm{conv}\left(D_{u^{\{1,2\}}}(\mathbf{p})\right)$ but not in $D_{u^{\{1,2\}}}(\mathbf{p})$: this set is in this case *not* discrete-convex.

Example C.17. Let $\sigma^1 = \mathrm{conv}((0,0,0,0),(1,1,0,0),(0,1,1,0),(0,0,1,1))$ while $\sigma^2 = ((0,0,0,0),(4,0,0,1))$. This time $\dim\sigma^1 = 3$ and $\dim\sigma^2 = 1$, and expressions (1) and (2) show that in neither case do these vectors decompose to give a part in σ^1, and so cannot be in $\sigma^1+\sigma^2 = \sigma^{\{1,2\}}$. So in this case, $D_{u^{\{1,2\}}}(\mathbf{p})$ *is* discrete-convex.

Example C.18. Let σ^1 be as in Ex. C.17, but let $\sigma^2 = \mathrm{conv}((0,0,0,0),(4,1,1,1))$. By the same techniques as in Ex. C.15, see that $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1}+\Lambda_{\sigma^2}] = 3$. A non-vertex integer point in the fundamental parallelepiped of $\Lambda_{\sigma^1}+\Lambda_{\sigma^2}$ is given by

$$\frac{1}{3}\big(1,1,0,0\big)+\frac{1}{3}\big(0,0,1,1\big)+0\cdot\big(0,1,1,0\big)+\frac{2}{3}\big(4,1,1,1\big)=\big(3,1,1,1\big)$$

This point *does* lie in $\sigma^1+\sigma^2$, and hence does demonstrate failure of discrete-convexity.

Lemma C.19. *$\hat{m}(C) \le \mathrm{mult}(C)$ for an intersection 0-cell C, with equality holding iff there exists a small translation such that the subgroup indices are 1 at all emerging intersection 0-cells.*

Proof. If the intersection is transverse at C then $\mathrm{mult}(C)$ is the product of $\hat{m}(C)$ with a subgroup index (Defns. 5.10 and 5.21). Moreover, $\hat{m}(C) > 0$ by definition, and any subgroup index is at least 1 (by Fact 5.18(1)), so the result follows. The non-transverse case is similar; here $\hat{m}(C) > 0$ because emerging 0-cells always exist (Lemma 5.12). □

Proof of Thm. 5.1. It is clear from Thm. 5.22(1) and Lemma C.19 that $M^n(A^1,A^2)$ is an upper bound for intersection 0-cells counted with naïve multiplicities, and that the count equals this bound iff $\hat{m}(C) = \mathrm{mult}(C)$ for every intersection 0-cell C.

Suppose that equilibrium does not exist for all relevant supplies. By Prop. 5.4 there exists an intersection 0-cell C at price $\mathbf{p}$, and a bundle $\mathbf{y}\in\mathbb{Z}^n$, such that $\mathbf{y}\notin D_{u^{\{1,2\}}}(\mathbf{p})$, but $\mathbf{y}\in\sigma^{\{1,2\}} := \mathrm{conv}(D_{u^{\{1,2\}}}(\mathbf{p}))$. Let $\mathbf{v}\in\mathbb{R}^n$ and small $\epsilon>0$ be such that $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\}+\mathcal{L}_{u^2}$ intersect transversely and the sum of the naïve multiplicities of intersection 0-cells emerging from C is equal to $\hat{m}(C)$. By Lemma C.13, there exists an intersection 0-cell C' at $\mathbf{p}'$ for $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\}+\mathcal{L}_{u^2}$ emerging from C, such that $\mathbf{y}\in\mathrm{conv}(D_{u^{\{1,2[\epsilon]\}}}(\mathbf{p}'))$ (where $u^{2[\epsilon]}(\mathbf{x}) = u^2(\mathbf{x})+\epsilon\mathbf{v}\cdot\mathbf{x}$); the fact $\mathbf{y}\notin D_{u^{\{1,2\}}}(\mathbf{p})$ rules out the other cases. By Prop. 4.12 we know $\mathbf{y}\notin D_{u^{\{1,2[\epsilon]\}}}(\mathbf{p}')$. By Theorem 5.19(1) we conclude that the subgroup index for $u^1, u^{2[\epsilon]}$ and $u^{\{1,2[\epsilon]\}}$ at $\mathbf{p}'$ is greater than 1. By Lemma C.19, $\hat{m}(C) < \mathrm{mult}(C)$, so as argued above, the count with naïve multiplicities is strictly below $M^n(A^1,A^2)$.

"Necessity" with transverse intersections and $n\le 3$ is presented in the text. □

Example C.20 (A "fragile" equilibrium). Consider two identical agents whose demand sets at price $(2,2)$ are the bundles $(0,0),(1,2),(2,1)$ and $(1,1)$. (For example,

[47] In these examples it is not hard to find u^1 and u^2 once $\mathbf{p}$ has been chosen. For example, for Ex. C.16, let $\mathbf{p} = (1,5,4,2)$. Then $u^1(0,0,0,0)=0$, $u^1(1,1,0,0)=6$, $u^1(0,1,1,0)=9$ and $u^2(0,0,0,0)=0$, $u^2(0,0,1,1)=6$, $u^2(4,0,0,1)=6$ have the required properties.

$u : \{0,1,2\}^2 \to \mathbb{R}$ and $u(x_1, 0) = x_1; u(0, x_2) = x_2; u(1,1) = 4; u(1,2) = u(2,1) = 6; u(2,2) = 7$). Then the bundle $(2,2)$ is in the aggregate demand set at this price: we assign bundle $(1,1)$ to both agents. But observe that bundle $(1,1)$ is an interior point of each agent's demand complex cell with the vertices $(0,0), (1,2)$, and $(2,1)$. So if we make any small translation to either agent's valuation, then equilibrium fails: for any prices close to $(2,2)$ the perturbed agent's demand must be some subset of these vertices, and it is easy to see that then $(2,2)$ cannot be an aggregate demand.

Example C.21. In both Exs. C.15 and C.16, we can calculate $M^4(A^1, A^2)$ to be 3 (indeed $M^4_2(A^1, A^2) = 3$; and $M^4_k(A^1, A^2) = 0$ for $k \neq 2$). The weights of the individual's 2-cells that meet at $(1,5,4,2)$ are both 1. This illustrates again that the condition of Thm. 5.1 is sufficient, but *not* necessary, for existence of equilibrium when $n \geq 4$.

Example C.22 (Circular Ones Model, cf. Bartholdi et al, 1980). Consider "complements" consumers, each of whom is only interested in a single, specific, pair of goods, such that these pairs form a cycle. Thus there are n kinds of consumers and n goods, and we can number both goods and consumers $1, \ldots, n$, such that every consumer of kind $i < n$ demands goods i and $i+1$, which it sees as perfect complements, while consumers of kind n demand goods n and 1. It is easy to check that:

$$\det \begin{pmatrix} 1 & 0 & 0 & & & 0 & 1 \\ 1 & 1 & 0 & & & 0 & 0 \\ 0 & 1 & 1 & & & 0 & 0 \\ \cdot & 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & 1 & 0 \\ 0 & 0 & 0 & & & 1 & 1 \end{pmatrix} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

So if n is odd, our Unimodularity Theorem tells us equilibrium does *not* always exist. Furthermore, Lemma 4.16 constructs an explicit example of equilibrium failure for any non-unimodular demand type. Here we simply select a single agent of each kind, each of which values its desired pair at v, so that they are all indifferent between purchase and no purchase (and hence their facets all intersect) if every good's price is $v/2$.[48]

If n is even, the columns of this matrix are not linearly independent. However, if we exclude the i^{th} column, for any i, the remaining $n-1$ columns are then linearly independent, and can trivially be extended to n linearly independent vectors with determinant 1 by adding the column $\mathbf{e}^i$. So using Thm. 4.3, equilibrium *always* exists if n is even, since the valuations are, trivially, concave.[49,50]

[48]So equilibrium fails if aggregate supply is exactly 1 unit of each good (the "middle of the parallelepiped") since the minimum and maximum aggregate demands are zero, and 2 units of each good, respectively, at this price. (It is easy to check failure of equilibrium for $x_i = 1$, for all i, by contradiction. At least one good, w.l.o.g. good 1, would not be part of a pair being allocated together. So good 1 has value 0 to whoever receives it, hence $p_1 \leq 0$. Therefore $p_2 \geq v$, since otherwise consumer 1 would demand both goods 1 and 2. Therefore $p_3 \leq 0$, since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore $p_4 \geq v$, etc., so $p_j \leq 0$ if j is odd. But consumer n then wishes to buy goods n and 1, which is a contradiction.)

[49]For example, the aggregate demand of 1 unit of each good is supported by price $v/2$ for every good, when there is exactly one consumer of each kind, each of which values its preferred pair at v.

[50]Sun and Yang (2011) and Teytelboym (2014) have independently used alternative methods to show these results for a version of this model; the even n case is a special case of the "generalised gross substitutes and complements" demand type that we discuss in Section 3.2; one can also use the relationship with matching (see Section 6.6), together with Pycia (2008), to obtain the $n = 3$ case.

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