METRIC ESTIMATES AND MEMBERSHIP COMPLEXITY FOR ARCHIMEDEAN AMOEBAE AND TROPICAL HYPERSURFACES

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Abstract. Given any complex Laurent polynomial f, Amoeba(f) is the image of its complex zero set under the coordinate-wise log absolute value map. We give an efficiently constructible polyhedral approximation, ArchTrop(f), of Amoeba(f), and derive explicit upper and lower bounds, solely as a function of the number of monomial terms of f, for the Hausdorff distance between these two sets. We thus obtain an Archimedean analogue of Kapranov's Non-Archimedean Amoeba Theorem and a higher-dimensional extension of earlier estimates of Mikhalkin and Ostrowski. We also show that deciding whether a given point lies in ArchTrop(f) is doable in polynomial-time, for any fixed dimension, unlike the corresponding problem for Amoeba(f), which is **NP**-hard already in one variable.

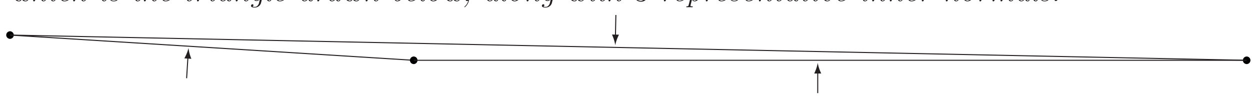
In memory of Mikael Passare.

1. Introduction

One of the happiest coincidences in algebraic geometry is that the norms of roots of polynomials can be estimated through polyhedral geometry. Perhaps the earliest incarnation of this fact was Isaac Newton's use of a polygon to determine series expansions for algebraic functions. This was detailed in a letter, dated October 24, 1676 [New76], that Newton wrote to Henry Oldenburg. In modern terminology, Newton counted the (s*-adic*) norms of roots of univariate polynomials over the *Puiseux* series field $\mathbb{C}\langle\langle s\rangle\rangle$, i.e., the union of formal Laurent series fields $\bigcup_{d\in\mathbb{N}}\mathbb{C}\left(\left(s^{1/d}\right)\right)$.

Definition 1.1. *Let* $[N]:=\{1,\ldots,N\}$. *We define the* s-adic valuation *of any element* $\zeta=\sum_{j=k}^{\infty}c_js^{j/d}$ *of* $\mathbb{C}\langle\langle s\rangle\rangle$ *to be* $\mathrm{ord}_s\zeta:=\min_{c_j\neq 0} j/d$. *(We set* $\mathrm{ord}_s 0:=\infty$ *and thus* $\mathrm{ord}_s:\mathbb{C}\langle\langle s\rangle\rangle\longrightarrow\mathbb{Q}\cup\{\infty\}$.*) We also let* $\mathrm{Conv}(U)$ *denote the* convex hull *of (i.e., smallest convex set containing) a set* $U\subseteq\mathbb{R}^n$. *For any* $f\in\mathbb{C}\langle\langle s\rangle\rangle[x_1]$ *written* $f(x_1)=\sum_{i=1}^{t}c_ix_1^{a_i}$, *with* $t\geq 2$, $a_1<\cdots<a_t$, *and* $c_i\neq 0$ *for all* i, *we define its* s-adic Newton polygon *to be* $\mathrm{Newt}_s(f)$ $:=\mathrm{Conv}(\{(a_i,\mathrm{ord}_s c_i)\mid i\in[t]\})$. *Finally, we define the* (s-adic) tropical variety of f *to be* $\mathrm{Trop}_s(f):=\{v\in\mathbb{R}\mid (v,1)$ *is an inner normal of an edge of* $\mathrm{Newt}_s(f)\}$. $\diamond$

Example 1.2. *The trinomial* $f(x_1):=s-x_1^{16}+x_1^{49}$ *has exactly* 49 *roots in* $\mathbb{C}\langle\langle s\rangle\rangle$*:* 16 *of the form* $e^{2\pi\sqrt{-1}j/16}s^{1/16}+\sum_{i=2}^{\infty}\alpha_{i,j}s^{i/16}$ *(for* $j\in[16]$*) and* 33 *of the form* $e^{2\pi\sqrt{-1}j/33}+\sum_{i=1}^{\infty}\beta_{i,j}s^{i}$ *(for* $j\in[33]$*), where* $\alpha_{i,j}\in\mathbb{Q}\left[e^{2\pi\sqrt{-1}/16}\right]$ *and* $\beta_{i,j}\in\mathbb{Q}\left[e^{2\pi\sqrt{-1}/33}\right]$. *Newton's technique from* [New76], *in more recent language, gives us the initial exponents* $1/16$ *and* 0 **exactly as the points of** $\mathbf{Trop}_s(f)$. *In particular,* $\mathrm{Newt}_s(f)$ *here is the convex hull of* $\{(0,1/16),(16,0),(49,0)\}$, *which is the triangle drawn below, along with* 3 *representative inner normals:*



There are just two upward-pointing inner normals, and thus just two inner normals of the form $(v,1)$*:* $(1/16,1)$ *and* $(0,1)$. *So* $\mathrm{Trop}_s(f)=\{1/16,0\}$, *and the horizontal lengths (*16 *and* 33*) of the two lower edges count the number of roots with corresponding valuation.* $\diamond$

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Newton's result has since been extended to other fields, such as algebraic extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ (see, e.g., [Dum06, Wei63]). Tropical geometry (see, e.g., [EKL06, LS09, IMS09, BR10, ABF13, MS15]) continues to deepen the links between algebraic, arithmetic, and polyhedral geometry. However, finding an analogous approach for roots in $\mathbb{C}$ presents a metric complication: Unlike $\mathbb{C}$, each field $\mathbb{C}\langle\langle s\rangle\rangle$, $\mathbb{Q}_p$, and $\mathbb{F}_p((t))$ is endowed with a (non-trivial) *non-Archimedean* norm, i.e., a norm which is bounded on the embedded copy of $\mathbb{Z}$ in the respective underlying field. For instance, one can set $|\zeta|_s := e^{-\text{ord}_s \zeta}$ for any $\zeta \in \mathbb{C}\langle\langle s\rangle\rangle$ and easily prove that $|\cdot|_s$ satisfies the Triangle Inequality, as well as the stronger *Ultrametric Inequality* $|x+y|_s \leq \max\{|x|_s, |y|_s\}$. In particular, $|\mathbb{C}|_s = \{0,1\}$, but this unfortunately renders $|\zeta|_s$ useless for estimating the usual *Archimedean* norm $|\zeta|$ of a nonzero root $\zeta \in \mathbb{C}$.

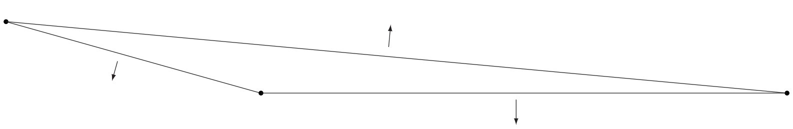
However, with some care, we can still study Archimedean norms of roots of polynomials in a polyhedral/tropical way: Jacques Hadamard was possibly the first to define an analogue of Newt_s for the usual norm on $\mathbb{C}$ [Had93] (see also [Ost40a] and [Val54, Ch. IX, pp. 193–202]). Here, we formulate a version applicable in arbitrary dimension. (See also [Mik04, PR04, PRS11, TdW13] for important precursors.)

Definition 1.3. *We call any* $f \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ *of the form* $f(x) = \sum_{i=1}^t c_i x_1^{a_i}$, *with* $c_i \neq 0$ *for all* i *and* $\{a_1, \ldots, a_t\}$ *of cardinality* t, *an* n-variate t-nomial. *(The notation* $x = (x_1, \ldots, x_n)$ *and* $x^{a_i} = x_1^{a_{1,i}} \cdots x_n^{a_{n,i}}$ *is understood.) We then define the* (ordinary) Newton polytope *of* f *to be* $\text{Newt}(f) := \text{Conv}\left(\{a_i\}_{i \in [t]}\right)$, *and the* Archimedean *Newton polytope of* f *to be* $\text{ArchNewt}(f) := \text{Conv}\left(\{(a_i, -\log|c_i|)\}_{i \in [t]}\right)$. *We also define the* Archimedean *tropical variety of* f *(provided* $t \geq 2$*) to be*
$\text{ArchTrop}(f) := \{w \in \mathbb{R}^n \mid (v, -1) \text{ is an outer normal of a positive-dimensional face of ArchNewt}(f)\}$. $\diamond$

Example 1.4. *It is easily checked that for any univariate binomial* f, $\text{ArchTrop}(f)$ *is a single point in* $\mathbb{R}$ *and all the complex roots of* f *lie on a circle of radius* $e^{\text{ArchTrop}(f)}$ *centered at the origin. More generally, for any* n*-variate binomial,* $\text{ArchTrop}(f)$ *is an affine hyperplane in* $\mathbb{R}^n$ *which is exactly the image of the complex roots of* f *under the coordinate-wise log absolute value map.* $\diamond$

While the norms of complex roots are not always described exactly by $\text{ArchTrop}(f)$, $\text{ArchTrop}(f)$ nevertheless provides an approximation within an explicit tolerance.

Example 1.5. *For a root* $\zeta \in \mathbb{C}$ *of* $f(x_1) := \frac{1}{89} - x_1^{16} + x_1^{49}$ *there are exactly* 26 *possible values for* $|\zeta|$. *However, these norms cluster tightly about just* **2** *values: Exactly* 16 *roots have norm near* $89^{-1/16} \approx 0.7553...$ *(to at least* 4 *decimal places) and exactly* 33 *roots have norm near* 1 *(to* 3 *decimal places). Here,* $\text{ArchNewt}(f)$ *is the convex hull of* $\left\{\left(0, -\log\frac{1}{89}\right), (16,0), (49,0)\right\}$, *which is the triangle with outer normals as shown below:*



There are just two downward-pointing outer normals, and thus just two outer normals of the form $(v,-1)$*:* $\left(\frac{1}{16}\log\frac{1}{89}, -1\right)$ *and* $(0,-1)$. *So* $\text{ArchTrop}(f) = \left\{\log\left(89^{-1/16}\right), \log 1\right\}$, *and the horizontal lengths (*16 *and* 33*) of the two lower edges count the number of roots with norm in the corresponding cluster.* $\diamond$

Theorem 1.6. *For any univariate* t*-nomial* f *with root* $\zeta \in \mathbb{C}\setminus\{0\}$ *and* $t \geq 3$, *we have that* $\#\text{ArchTrop}(f) \leq t-1$ *and* $\log|\zeta|$ *lies in the union of open intervals* $\bigcup\limits_{v \in \text{ArchTrop}(f)} (v - \log 3, v + \log 3)$. $\blacksquare$

Theorem 1.6 follows from the stronger Theorem 1.7 below. Theorem 1.6 already improves an earlier bound of Ostrowski [Ost40a, Bound (25, 3), pg. 145] which, letting d denote the degree of f, implies that $\log|\zeta|$ lies in the union $\bigcup_{v\in \mathrm{ArchTrop}(f)} (v-\log(d+1), v+\log(d+1))$. (Note that $3\leq t\leq d+1$ in Theorem 1.6.)

It is also the case that, for any $v\in\mathrm{ArchTrop}(f)$, there actually exists a root $\zeta\in\mathbb{C}$ of f with $\log|\zeta|$ close to v. In particular, the clustering of $\mathrm{ArchTrop}(f)$ determines certain annuli guaranteed to contain a positive number of roots of f. In what follows, for any line segment $L\subset\mathbb{R}^2$ with vertices (a,b) and (c,d), we define its *horizontal length* to be $\lambda(L):=|c-a|$.

Theorem 1.7. *Given any univariate t-nomial f with $t\geq 3$, let Γ be any connected component of the union of open intervals*

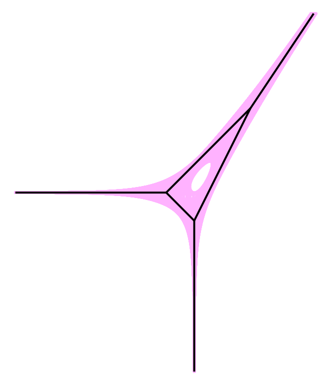
$$U_f := (\min \mathrm{ArchTrop}(f) - \log 2, \max \mathrm{ArchTrop}(f)+\log 2) \ \cap \bigcup_{v\in\mathrm{ArchTrop}(f)} (v-\log 3, v+\log 3)$$

and let Λ_Γ be the sum of $\lambda(L)$ over all edges L of $\mathrm{ArchNewt}(f)$ with outer normal $(v,-1)$ and $v\in\Gamma$. Then the number of roots $\zeta\in\mathbb{C}$ of f with $\log|\zeta|\in\Gamma$, counting multiplicity, is exactly Λ_Γ. In particular, $\Lambda_\Gamma\geq 1$ and every *root $\zeta\in\mathbb{C}$ of f satisfies $\log|\zeta|\in U_f$.*

Theorem 1.7 is proved in Section 2.1, where a slight sharpening is also provided for $t=3$.

We can in fact polyhedrally approximate norms of complex roots in arbitrary dimension.

Definition 1.8. *Let us set* $\mathrm{Log}|\zeta| := (\log|\zeta_1|,\ldots,\log|\zeta_n|)$ *and, for any $f\in\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$, define* $\mathrm{Amoeba}(f)$ *to be* $\{\mathrm{Log}|\zeta| \mid f(\zeta)=0\ ,\ \zeta\in(\mathbb{C}\setminus\{0\})^n\}$. $\diamond$



Example 1.9. *Taking $f(x)=1+x_1^3+x_2^2-10x_1x_2$, it is easily checked that* $\mathrm{Newt}(f)$ *is a triangle, while* $\mathrm{ArchNewt}(f)$ *is a pyramid. In particular,* $\mathrm{ArchTrop}(f)$ *is a* polyhedral complex *consisting of* 3 *vertices and* 6 *edges (*3 *of which are unbounded rays). An illustration of* $\mathrm{Amoeba}(f)\cap[-7,7]^2$ *and* $\mathrm{ArchTrop}(f)\cap[-7,7]^2$ *appears to the right.* $\mathrm{Amoeba}(f)$ *is lightly shaded and contains* $\mathrm{ArchTrop}(f)$ *(drawn darker).* $\diamond$

Our main result is that every point of $\mathrm{Amoeba}(f)$ is within an explicit distance of some point of $\mathrm{ArchTrop}(f)$, and vice-versa, independent of the degree or number of variables of f. We use $|\cdot|$ for the standard ℓ_2-norm on $\mathbb{C}^n$.

Definition 1.10. *For any $\varepsilon>0$ and $X\subseteq\mathbb{R}^n$ we define the* open ε-neighborhood of X *to be $X_\varepsilon:=\{x\in\mathbb{R}^n \mid |x-x'|<\varepsilon$ for some $x'\in X\}$, and let $\overline{X}_\varepsilon$ denote its Euclidean closure.* $\diamond$

Theorem 1.11. *For any $f\in\mathbb{C}\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$ with exactly $t\geq 2$ monomial terms and* $\mathrm{Newt}(f)$ *of dimension k, we have $1\leq k\leq \min\{n,t-1\}$ and:*

(0) *For $k=1$ we have that* $\mathrm{ArchTrop}(f)$ *is a non-empty disjoint union of at most $t-1$ parallel affine hyperplanes in $\mathbb{R}^n$, while for $k\geq 2$ we have that* $\mathrm{ArchTrop}(f)$ *is a path-connected $(n-1)$-dimensional polyhedral complex with at most $t(t-1)/2$ faces of dimension $n-1$.*

(1) *For $t=k+1$ we have* $\mathrm{ArchTrop}(f)\subseteq\mathrm{Amoeba}(f)$ *and both* $\mathrm{Amoeba}(f)$ *and* $\mathrm{ArchTrop}(f)$ *are contractible. In particular,* $t=2 \Longrightarrow \mathrm{Amoeba}(f)=\mathrm{ArchTrop}(f)$.

(2) *For all $t\geq k+1$ we have (a)* $\mathrm{Amoeba}(f)\subset\overline{\mathrm{ArchTrop}(f)_{\log(t-1)}}$ *and (b)* $\mathrm{ArchTrop}(f)\subset \mathrm{Amoeba}(f)_{\varepsilon_{k,t}}$, *where* $\varepsilon_{1,t}:=(\log 9)t-\log\frac{81}{2}<2.2t-3.7$, $\varepsilon_{2,t}:=\sqrt{2}(t-2)\left((\log 9)t-\log\frac{81}{2}\right)$ $<(t-2)(3.11t-5.23)$, *and* $\varepsilon_{k,t}:=\sqrt{k}\left\lceil\frac{1}{4}t(t-1)\right\rceil\left((\log 9)t-\log\frac{81}{2}\right)$ *for* $k\geq 3$.

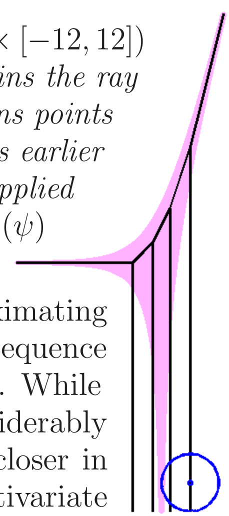
(3) *Let $\varphi(x):=1+x_1+\cdots+x_{t-1}$ and $\psi(x):=(x_1+1)^{t-k}+x_2+\cdots+x_k$. Then (a)* $\mathrm{Amoeba}(\varphi)$ *contains a point at distance* $\log(t-1)$ *from* $\mathrm{ArchTrop}(\varphi)$ *and (b)* $\mathrm{ArchTrop}(\psi)$ *contains points approaching distance* $\log(t-k)$ *from* $\mathrm{Amoeba}(\psi)$.

We prove Theorem 1.11 in Section 3. Our main contribution is Assertion (2): For multivariate polynomials, our bounds appear to be the first allowing dependence on just the number of terms t. In particular, Assertion (2a) sharpens, and extends to arbitrary dimension, an earlier bound of Mikhalkin for the case $n=2$: Letting L denote the number of lattice points in the Newton polygon of f, [Mik05, Lemma 8.5, pg. 360] asserts that $\mathrm{Amoeba}(f)$ is contained in the possibly larger neighborhood $\overline{\mathrm{ArchTrop}(f)_{\log(L-1)}}$. Assertion (3a) of Theorem 1.11 shows that the size of the neighborhood from Assertion (2a) is in fact optimal. (Note also that when $t\geq 4$, Theorem 1.6 refines the special case $n=1$ of Assertion (2a) above: When $n=1$, $\mathrm{ArchTrop}(f)_\varepsilon$ is simply a finite union of open intervals of width 2ε.)

We have included Assertions (0) and (1) for completeness, since they are implicit in earlier topological results on amoebae (see, e.g., [For98, Prop. 3.1.8] or [Rul03, Thms. 8 & 12]). For the convenience of the reader, we provide elementary proofs for Assertions (0) and (1) in Sections 3.1 and 3.2.

Finding the tightest neighborhood of $\mathrm{Amoeba}(f)$ containing $\mathrm{ArchTrop}(f)$ appears to be an open problem: We are unaware of any earlier multivariate version of Assertion (2b). The only other earlier distance bound between an amoeba and a polyhedral approximation we know of is a result of Viro [Vir01, Sec. 1.5] on the distance between the graph of a univariate polynomial (drawn on log paper) and a piece-wise linear curve that is ultimately a piece of the $n=2$ case of $\mathrm{ArchTrop}(f)$ here.

Example 1.12. *Setting* $\psi(x)=(x_1+1)^4+x_2$ *we see* $\mathrm{Amoeba}(\psi)\cap([-7,7]\times[-12,12])$ *and* $\mathrm{ArchTrop}(\psi)\cap([-7,7]\times[-12,12])$ *on the right.* $\mathrm{ArchTrop}(\psi)$ *contains the ray* $(\log 4, 4\log 4)+\mathbb{R}_+(0,-1)$ *and this rightmost downward-pointing ray contains points with distance from* $\mathrm{Amoeba}(\psi)$ *approaching* $\log 4$. *We also observe that Viro's earlier polygonal approximation of graphs of univariate polynomials on log paper, applied here, would result in the polygonal curve that is the subcomplex of* $\mathrm{ArchTrop}(\psi)$ *obtained by deleting all* 4 *downward-pointing rays.* $\diamond$



It is worth comparing Theorem 1.11 to two other methods for approximating complex amoebae: Purbhoo, in [Pur08], describes a uniformly convergent sequence of outer polyhedral approximations to any amoeba, using cyclic resultants. While $\mathrm{ArchTrop}(f)$ lacks this refinability, the computation of $\mathrm{ArchTrop}(f)$ is considerably simpler: see Section 1.2 below and [AGGR15]. $\mathrm{ArchTrop}(f)$ is actually closer in spirit to the *spine* of $\mathrm{Amoeba}(f)$. The latter construction, based on a multivariate version of Jensen's Formula from complex analysis, is due to Passare and Rullgård [PR04, Sec. 3] and results in a polyhedral complex that is always contained in, and is homotopy equivalent to, $\mathrm{Amoeba}(f)$. Unfortunately, the computational complexity of the spine is not as straightforward as that of $\mathrm{ArchTrop}(f)$. Further background on the computational complexity of amoebae can be found in [The02, SdW13, TdW15].

Our final main results concern the complexity of deciding whether a given point lies in a given amoeba or Archimedean tropical variety. However, let first us observe a consequence of our metric estimates for systems of polynomials.

1.1. **Coarse, but Fast, Isolation of Roots of Polynomial Systems.** An immediate consequence of Assertion (2a) of Theorem 1.11 is an estimate for isolating the possible norm vectors of complex roots of arbitrary systems of multivariate polynomial equations.

Corollary 1.13. *Suppose* $f_1,\ldots,f_m\in\mathbb{C}\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$ *where* f_i *has exactly* t_i *monomial terms for all* i. *Then any root* $\zeta\in(\mathbb{C}^*)^n$ *of* $F=(f_1,\ldots,f_m)$ *satisfies*

$$\mathrm{Log}|\zeta| \in \overline{\mathrm{ArchTrop}(f_1)_{\varepsilon_1}} \cap \cdots \cap \overline{\mathrm{ArchTrop}(f_m)_{\varepsilon_m}},$$

where $\varepsilon_i := \log(t_i - 1)$ *for all* i. ■

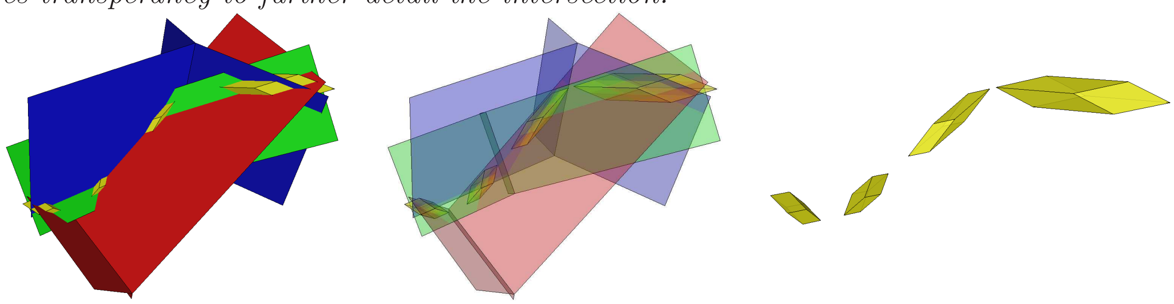
Example 1.14. *We can isolate the log-norm vectors of the complex roots of the* 3×3 *system*

$$F := (f_1, f_2, f_3) := (x_1x_2 - x_1^2 - 1/16^6, x_2x_3 - 1 - x_1^2/16^6, x_3 - 1 - x_1^2/16^{18})$$

via Corollary 1.13 as follows: Find the points of $X := \mathrm{ArchTrop}(f_1) \cap \mathrm{ArchTrop}(f_2) \cap \mathrm{ArchTrop}(f_3)$ *by searching through suitable triplets of edges of the* $\mathrm{ArchNewt}(f_i)$*, and then create isolating parallelepipeds about the points of* X*. More precisely, observe that*
$\mathrm{Conv}(\{(1,1,0,0),(2,0,0,0)\})$, $\mathrm{Conv}(\{(0,1,1,0),(0,0,0,0)\})$, $\mathrm{Conv}(\{(0,0,1,0),(0,0,0,0)\})$ *are respective edges of* $\mathrm{ArchNewt}(f_1)$*,* $\mathrm{ArchNewt}(f_2)$*, and* $\mathrm{ArchNewt}(f_3)$*, and the vector* $(0,0,0,-1)$ *is an outer normal to each of these edges. So* $(0,0,0)$ *is a point of* X*. Running through the remaining triplets we then obtain that* X *in fact consists of exactly* 4 *points:*

$$\mathrm{Log}\left|\left(\tfrac{1}{16^6},1,1\right)\right| \ , \ \mathrm{Log}|(1,1,1)| \ , \ \mathrm{Log}\left|(16^6,16^6,1)\right| \ , \ \textit{and} \ \mathrm{Log}\left|(16^{12},16^{12},16^6)\right|.$$

So Corollary 1.13 tells us that the points of $Y := \mathrm{Amoeba}(f_1) \cap \mathrm{Amoeba}(f_2) \cap \mathrm{Amoeba}(f_3)$ *lie in the union of the* 4 *parallelepipeds drawn below to the right: Truncations of* $\mathrm{ArchTrop}(f_1)$*,* $\mathrm{ArchTrop}(f_2)$*, and* $\mathrm{ArchTrop}(f_3)$ *are drawn below on the left, and the middle illustration uses transperancy to further detail the intersection.*



Suitably ordered, each point of X *is actually within distance* 0.11×10^{-6} $(< 0.693\ldots = \log 2)$ *of some point of* Y *(and vice-versa), well in accordance with Corollary 1.13.* ⋄

See [PR13] for the relevance of the preceding system to fewnomial theory over general local fields.

1.2. **On the Computational Complexity of ArchTrop(*f*) and Amoeba(*f*).** The complexity classes **P**, **NP**, **PSPACE**, and **EXPTIME** — from the classical Turing model of computation — can be identified with families of *decision* problems, i.e., problems with a yes or no answer. Larger complexity classes correspond to problems with larger worst-case complexity. We refer the reader to [Sip92, Pap95, AB09, Sip12] for further background. Aside from the basic definitions of input size and **NP**-hardness, it will suffice here to simply recall that $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXPTIME}$, and that the properness of each inclusion (aside from $\mathbf{P} \subsetneq \mathbf{EXPTIME}$, which is well-known) is a famous open problem. All algorithmic complexity results below count bit operations, and do so as a function of some underlying notion of input size.

Deciding membership in an amoeba can easily be rephrased as a problem within the Existential Theory of the Reals. The latter setting has been studied extensively in the $20^{\underline{\text{th}}}$ century (see, e.g., [Tar51, BKR86]) and the current state of the art implies that amoeba membership can be solved efficiently by a parallel algorithm. More precisely, we define the *input size* of a polynomial $f \in \mathbb{Z}[x_1,\ldots,x_n]$, written $f(x) = \sum_{i=1}^t c_i x^{a_i}$, to be $\mathrm{size}(f) := \sum_{i=1}^t \log_2\left((2+|c_i|)\prod_{j=1}^n (2+|a_{i,j}|)\right)$, where $a_i = (a_{i,1},\ldots,a_{i,n})$ for all i. (Put another way, up to a constant additive error, $\mathrm{size}(f)$ is just the sum of the bit-sizes of all the coefficients and exponents.) Similarly, we define $\mathrm{size}(v)$, for any $v=(v_1,\ldots,v_n)\in\mathbb{Q}^n$, to be the sum of the sizes of the numerators and denominators of the v_i (written in lowest terms). We similarly extend the notion of input size to polynomials in $\mathbb{Q}[x_1,\ldots,x_n]$. Considering real and imaginary parts, we can extend further still to polynomials in $\mathbb{Q}\left[\sqrt{-1}\right][x_1,\ldots,x_n]$.

Theorem 1.15. *There is a* **PSPACE** *algorithm to decide, for any input pair* $(z,f)\in\bigcup_{n\in\mathbb{N}}\left(\mathbb{Q}^n\times\mathbb{Q}\left[\sqrt{-1}\right][x_1,\ldots,x_n]\right)$*, whether* $\mathrm{Log}|z|\in\mathrm{Amoeba}(f)$*. Furthermore, the special case where* $z=1$ *and* $f\in\mathbb{Z}[x_1]$ *in the preceding membership problem is already* **NP***-hard.*

Theorem 1.15 is implicit in earlier work of Ben-Or, Kozen, and Reif [BKR86] and Plaisted [Pla84], so for the convenience of the reader, we provide a short proof in Section 2.2.

Remark 1.16. *For our notion of input size, polynomial-time for sufficiently sparse polynomials implies polynomiality in the* logarithm *of the degree of the polynomial. This is in contrast to a looser notion of input size implicit in* [The02, Cor. 2.7] *where "polynomial-time" point membership detection for amoebae in fixed dimension is stated: The methods there yield complexity polynomial in the degree when* n *is fixed, thus yielding* exponential *worst-case complexity relative to the input size we use here. The* **NP***-hardness lower bound from Theorem 1.15 tells us that speeding up point membership for amoebae to polynomial-time (relative to our notion of input size here) would imply* $\mathbf{P}=\mathbf{NP}$*.* $\diamond$

Since we now know that $\mathrm{ArchTrop}(f)$ is provably close to $\mathrm{Amoeba}(f)$, $\mathrm{ArchTrop}(f)$ would be of great practical value if $\mathrm{ArchTrop}(f)$ were easier to work with than $\mathrm{Amoeba}(f)$. This indeed appears to be the case. For example, when the dimension n is fixed and all the coefficient absolute values of f have rational logarithms, standard high-dimensional convex hull algorithms (see, e.g., [Ede87]) enable us to describe every face of $\mathrm{ArchTrop}(f)$, as an explicit intersection of half-spaces, in polynomial-time (see, e.g., [AGGR15]).

The case of rational coefficients presents some subtleties because the underlying computations, done naively, involve arithmetic on rational numbers with exponentially large bit-size. Nevertheless, point membership for $\mathrm{ArchTrop}(f)$ can be decided in polynomial-time when n is fixed.

Theorem 1.17. *Suppose* $z=(z_1,\ldots,z_n)\in\mathbb{Q}^n$*, and* $f\in\mathbb{Q}\left[\sqrt{-1}\right][x_1,\ldots,x_n]$ *(written* $f(x)=\sum_{i=1}^t c_i x^{a_i}$*) has exactly* t *monomial terms, degree at most* d *with respect to any variable, and the bit-sizes of the* z_i *and* c_i *are at most* σ*. Then there is an* $nt(\log d)^{1+o(1)}(25.2\sigma)^{2n+2+o(1)}$ *algorithm to decide, for any such input pair* (z,f)*, whether* $\mathrm{Log}|z|\in\mathrm{ArchTrop}(f)$*.*

Furthermore, if we instead assume that both $\log|z_i|,\log|c_i|\in\mathbb{Q}$ *have bit size* $\leq\sigma$ *for all* i*, then there is an* $O\left(nt(\sigma+\log d)\log^2(\sigma d)\right)$ *algorithm to decide whether* $\mathrm{Log}|z|\in\mathrm{ArchTrop}(f)$*.*

We prove Theorem 1.17 in Section 4. The complexity of finding the distance to $\mathrm{ArchTrop}(f)$ from a given query point v, and the relevance of such distance computations to polynomial system solving, is explored further in [AGGR15].

1.3. **Ideas Behind the Proofs and Simplified Maslov Dequantization.**

A key idea behind our metric results is the following fact: Knowing where a polynomial f has at least two monomials of largest norm is enough to recover useful information about the complex roots of f. In particular, it is easy to show that $\mathrm{ArchTrop}(f)$ admits the following alternative characterization.

Proposition 1.18. *For any* n*-variate* t*-nomial* $f(x)=\sum_{i=1}^t c_i x^{a_i}$ *we have*

$$\mathrm{ArchTrop}(f)=\left\{v\in\mathbb{R}^n \;\middle|\; \max_i |c_i e^{a_i\cdot v}| \textit{ is attained for at least two distinct values of } i\right\}. \quad\blacksquare$$

Remark 1.19. *We adopt the natural conventions* $\mathrm{Trop}_s(0)=\mathrm{ArchTrop}(0)=\mathbb{R}^n$ *and* $\mathrm{Trop}_s(cx^a)=\mathrm{ArchTrop}(cx^a)=\emptyset$ *for any* $c\neq 0$ *and* $a\in\mathbb{Z}^n$. $\diamond$

It is also conceptually important to recall a non-Archimedean precursor to our main result: Letting $\mathrm{ord}_s\zeta:=(\mathrm{ord}_s\zeta_1,\ldots,\mathrm{ord}_s\zeta_n)$ for any $\zeta=(\zeta_1,\ldots,\zeta_n)\in\mathbb{C}\langle\langle s\rangle\rangle^n$, and making the natural extension $\mathrm{Trop}_s(f):=\{v\in\mathbb{R}^n \mid (v,1)$ is an inner normal of a face of $\mathrm{Newt}_s(f)$ of positive dimension$\}$, the statement is as follows:

Kapranov's Non-Archimedean Amoeba Theorem. *(Special case)* [EKL06] *For any* $f\in\mathbb{C}\langle\langle s\rangle\rangle\left[x_1^{\pm1},\ldots,x_n^{\pm1}\right]$, *we have* $\{\mathrm{ord}_s\zeta \mid f(\zeta)=0$ *and* $\zeta\in(\mathbb{C}\langle\langle s\rangle\rangle\setminus\{0\})^n\}=\mathrm{Trop}_s(f)\cap\mathbb{Q}^n$. $\blacksquare$

Kapranov's result above (derived no later than 2000) is to our Theorem 1.11 as Newton's Puisueux series characterization [New76] is to our Theorem 1.7.

In Kapranov's Theorem, the containment of s-adic root valuation vectors in $\mathrm{Trop}_s(f)\cap\mathbb{Q}^n$ follows easily from the Ultrametric Inequality. Here, proving that $\mathrm{Amoeba}(f)$ is contained in a suitable neighborhood of $\mathrm{ArchTrop}(f)$ requires a more delicate application of the Triangle Inequality. Proving that $\mathrm{ArchTrop}(f)$ is contained in a suitable neighborhood of $\mathrm{Amoeba}(f)$ involves specializing to a curve (similar to a trick in the non-Archimedean setting) to reduce to the univariate case, and then applying Rouché's Theorem. An estimate on lattice points visible from the origin (Theorem 3.3 in Section 3.4 below) helps improve one of our bounds in the bivariate case.

We close with some topological observations. First observe that $\mathrm{ArchTrop}(f)$ need *not* be contained in $\mathrm{Amoeba}(f)$, nor even have the same homotopy type as $\mathrm{Amoeba}(f)$, already for $n=1$: The example $f(x_1)=(x_1+1)^2$ yields $\mathrm{ArchTrop}(f)=\{\pm\log 2\}$ but $\mathrm{Amoeba}(f)=\{0\}$. However, one can in fact always recover $\mathrm{ArchTrop}(f)$ as the limit of a *sequence of suitably scaled* amoebae. To clarify this, first recall that the *Hausdorff distance* between any two subsets $X,Y\subseteq\mathbb{R}^n$ is

$$\Delta(X,Y):=\max\left\{\sup_{x\in X}\inf_{y\in Y}|x-y|,\sup_{y\in Y}\inf_{x\in X}|x-y|\right\}.$$

Also, the *support* of a Laurent polynomial $f(x)=\sum_{i=1}^t c_ix^{a_i}$ is $\mathrm{Supp}(f):=\{a_i \mid c_i\neq 0\}$.

Corollary 1.20. *Let* $f\in\mathbb{C}\left[x_1^{\pm1},\ldots,x_n^{\pm1}\right]$ *be a* t*-nomial with* $t\geq 2$ *and* $k:=\dim\mathrm{Newt}(f)$. *Then:*

(1) $\Delta(\mathrm{ArchTrop}(f),\mathrm{Amoeba}(f))\leq\sqrt{k}\left\lceil\frac{1}{4}t(t-1)\right\rceil\left((\log 9)t-\log\frac{81}{2}\right)=O\left(t^{7/2}\right)$.

(2) *There exists a family of Laurent polynomials* $(f_\mu)_{\mu\geq1}$ *with* $\mathrm{Supp}(f_\mu)=\mathrm{Supp}(f)$ *for all* $\mu\geq 1$ *and* $\Delta\left(\frac{1}{\mu}\mathrm{Amoeba}(f_\mu),\mathrm{ArchTrop}(f)\right)\longrightarrow 0$ *as* $\mu\longrightarrow\infty$.

One of the consequences of *Maslov dequantization* (see, e.g., [LMS01, Vir01] and [Mik04, Cor. 6.4]) is a way to obtain a *non*-Archimedean tropical variety as a limit of a family of scaled Archimedean amoebae. Assertion (2) thus shows how $\mathrm{ArchTrop}(f)$ provides a fully Archimedean version of this limit. Another precursor to Assertion (2), involving the piece-wise linear structure approached by the intersection of $\mathrm{Amoeba}(f)$ with a large sphere, appears in [Ber71] and [GKZ94, Prop. 1.9, pg. 197]. Thanks to Assertion (1), we can prove Assertion (2) in just three lines.

Proof of Corollary 1.20: Assertion (1) of Corollary 1.20 follows immediately from Assertion (2) of Theorem 1.11, and the fact that $k\leq t-1$. Let us write $f(x)=\sum_{i=1}^t c_ix^{a_i}$, define $f_\mu(x):=\sum_{i=1}^t c_i^\mu x^{a_i}$, and observe that $f_1=f$.

Since $|c_ie^{a_i\cdot v}|\geq|c_je^{a_j\cdot v}| \iff |c_ie^{a_i\cdot v}|^\mu\geq|c_je^{a_j\cdot v}|^\mu$, we immediately obtain that $\mathrm{ArchTrop}(f_\mu) = \mu\mathrm{ArchTrop}(f)$. So then $\Delta(\mathrm{Amoeba}(f_\mu),\mathrm{Trop}(f_\mu)) = \mu\Delta\left(\frac{1}{\mu}\mathrm{Amoeba}(f_\mu),\mathrm{Trop}(f)\right)$ and Assertion (1) thus implies $\Delta\left(\frac{1}{\mu}\mathrm{Amoeba}(f_\mu),\mathrm{Trop}(f)\right)=\frac{O(t^{7/2})}{\mu}$ for all $\mu\geq 1$. ■

2. Background on Univariate Bounds and the Complexity of Amoeba Membership

To prepare for the proofs of our main metric results we will first review some classical root norm bounds in the univariate case, in order to recast them in terms of $\mathrm{ArchTrop}(f)$. We then prove a refinement of Theorem 1.6 (Corollary 2.3), Theorem 1.7 (along with a refinement for $t=3$), and conclude this section with a sketch of the proof of Theorem 1.15 (on the hardness of deciding point membership for amoebae).

We begin with a pair of bounds dating back to 1923, if not earlier.

Theorem 2.1.
(1) Suppose $f(x_1)=\sum_{i=0}^d c_ix_1^i\in\mathbb{C}[x_1]$ has a root $\zeta\in\mathbb{C}$ and $c_0c_d\neq 0$. Then

$$\frac{1}{2}\min_{i\in\{1,\ldots,d\}}\left|\frac{c_0}{c_i}\right|^{1/i} < |\zeta| < 2\max_{i\in\{0,\ldots,d-1\}}\left|\frac{c_i}{c_d}\right|^{1/(d-i)}.$$

(2) Suppose $f(x_1):=c_0+\cdots+c_px_1^p+\gamma_1x_1^{n_1}+\cdots+\gamma_qx_1^{n_q}\in\mathbb{C}[x_1]$, $c_p\neq 0$, and $1\leq p<n_1<\cdots<n_q$. Then f has a root with absolute value $\leq\left|\frac{c_0}{c_p}\right|^{1/p}\binom{p+q}{q}^{1/p}$. ■

Bound (1) dates back to early 20th-century work of Fujiwara [Fuj16] (see also [RS02, pp. 243–249], particularly Bound 8.1.11 on pg. 247). Bound (2) was proved by Montel [Mon23] (see also [RS02, Thm. 9.5.1, pg. 304]). If one makes an elementary observation on the definition of $\mathrm{ArchTrop}(f)$ then one immediately obtains a refinement of Theorem 1.6 for the roots of f of largest and smallest (nonzero) norm. In what follows, a *lower edge* of a polygon in $\mathbb{R}^2$ is simply an edge possessing an outer normal of the form $(v,-1)$.

Proposition 2.2. *For any univariate t-nomial with $t\geq 1$ we have that* $\mathrm{ArchTrop}(f)$ *is the set of slopes of the lower edges of* $\mathrm{ArchNewt}(f)$. ■

Corollary 2.3. *Suppose $f\in\mathbb{C}[x_1]$ is a univariate t-nomial with $t\geq 2$, degree d, and nonzero roots $\zeta_1,\ldots,\zeta_d$ (counting multiplicity) ordered so that $|\zeta_1|\leq\cdots\leq|\zeta_d|$. Then*

(a) $-\log 2 < \log|\zeta_1| - \min\mathrm{ArchTrop}(f) \leq \log(t-1)$,

(b) $-\log(t-1) \leq \log|\zeta_d| - \max\mathrm{ArchTrop}(f) < \log 2$,

(c) The $\log 2$ (resp. $\log(t-1)$) terms above can not be replaced by any smaller constant (resp. function of t solely).

Proof: The lower bound from Part (a) and the upper bound from Part (b) follow immediately from Proposition 2.2, upon taking the log absolute value of both sides of Bound (1) from Theorem 2.1. In particular, we see that the lower and upper bounds from Bound (1) are exactly $\frac{1}{2}e^{\min\mathrm{ArchTrop}(f)}$ and $2e^{\max\mathrm{ArchTrop}(f)}$.

The upper bound from Part (a) follows similarly, but employing Bound (2) from Theorem 2.1 instead of Bound (1). In particular, one must apply Bound (2) in the following way: Take p so that the $(p,-\log|c_p|)$ is the right-hand vertex of the left-most lower edge of $\mathrm{ArchNewt}(f)$. By construction, this edge has slope $\frac{\log|c_0|-\log|c_p|}{p}$. Observing that $\binom{p+q}{q}^{1/p}=$

$\left(\frac{(q+p)\cdots(q+1)}{p!}\right)^{1/p} = \left(\left(\frac{q}{p}+1\right)\cdots\left(\frac{q}{1}+1\right)\right)^{1/p} \leq ((q+1)^p)^{1/p} = q+1$, and that the number of terms is $t = p+q+1$ with $p \geq 1$, we are done.

The lower bound from Part (b) follows by applying the preceding paragraph to the polynomial $x_1^d f(1/x_1^d)$: This has the effect of reflecting $\mathrm{ArchNewt}(f)$ across the vertical line $\frac{d}{2}\times\mathbb{R}$, and thus $\mathrm{ArchTrop}(f)$ is replaced $-\mathrm{ArchTrop}(f)$. So we ultimately prove an upper bound of $\log(t-1)$ on $-\log|\zeta_d| - (-\max \mathrm{ArchTrop}(f))$ and we are done.

The optimality of the $\log 2$ terms is evinced by the polynomials $f_1(x_1) := x_1^{t-1} - x_1^{t-2} - \cdots - 1$ and $f_2(x_1) := -1 + x_1 + \cdots + x_1^{t-1}$: One need only show that f_1 (resp. f_2) has a unique positive root increasing toward a limit of 2 (resp. decreasing toward a limit of $\frac{1}{2}$) as $t \longrightarrow \infty$. Uniqueness follows from Descartes' Rule, while the limiting behavior of the positive root is easily obtained by Rolle's Theorem and geometric series.

The optimality of the $\log(t-1)$ terms is easily seen via the polynomial $g(x_1) := (x_1+1)^{t-1}$: The left-most (resp. right-most) lower edge of $\mathrm{ArchNewt}(g)$ has slope $-\log(t-1)$ (resp. $\log(t-1)$), by the log-concavity of the binomial coefficients. So by Proposition 2.2, $\min \mathrm{ArchTrop}(g) = -\log(t-1)$ and $\max \mathrm{ArchTrop}(g) = \log(t-1)$. Since $\mathrm{Amoeba}(g) = \{0\}$, we are done. ■

We now recall a seminal collection of bounds due to Ostrowski:

Theorem 2.4. [Ost40a, Cor. IX, pg. 143][1] *Following the notation of Corollary 2.3, let d be the degree of f, and let v_i denote the slope of the unique lower edge of the polygon* $\mathrm{ArchNewt}(f) \cap ([i-1,i]\times\mathbb{R})$. *Then*

(1) $-\log 2 < \log|\zeta_1| - v_1 \leq \log d,$

(2) $-\log d \leq \log|\zeta_d| - v_d < \log 2,$

(3) $\log\left(1 - \dfrac{1}{2^{1/i}}\right) < \log|\zeta_i| - v_i < -\log\left(1 - \dfrac{1}{2^{1/(d-i+1)}}\right)$ *for all* $i \in \{2,\ldots,d-1\}$.

In particular,

$$-0.5348 \leq \log\left(1 - \frac{1}{2^{1/i}}\right) - (-\log i) < -0.3665 \text{ and}$$

$$0.3665 < -\log\left(1 - \frac{1}{2^{1/(d-i+1)}}\right) - \log(d-i+1) \leq 0.5348. \quad ■$$

Remark 2.5. *Thanks to Proposition 2.2 we have* $\mathrm{ArchTrop}(f) = \{v_1,\ldots,v_d\}$. *In particular, note that our Theorem 1.6 implies that any given* $\log|\zeta_i|$ *lies within distance* $\log 3$ *of some* v_j, possibly with $j \neq i$. *In this sense, the final assertion of Theorem 2.4 tells us that Theorem 1.6 isolates each* $\log|\zeta_i|$ *strictly better than Ostrowski's bounds, except possibly in the cases* $i \in \{2, d-1\}$ *or* $t = d+1 = 3$. *Corollary 2.9 in Section 2.1 below matches Ostrowski's bounds when* $t = d+1 = 3$. ⋄

We have so far concentrated on showing that each $\log|\zeta_i|$ is close to some v_j, with near-optimal distance bounds. Showing that each v_j is close to some $\log|\zeta_i|$ requires more preparation, which we now detail.

2.1. **Proving Theorem 1.7.** We will need three technical results, on bounding the norms of summands of sparse polynomials, and counting roots of polynomials in annuli, before proving Theorem 1.7.

[1] There was a typo in Ostrowski's original statement of the upper bound from Assertion (3), later corrected in an addendum by Ostrowski [Ost40b].

Proposition 2.6. *Suppose $f(x_1):=\sum_{j=1}^t c_j x_1^{a_j}\in\mathbb{C}[x_1^{\pm 1}]$ satisfies $t\geq 3$, $a_1<\cdots<a_t$, and $c_j\neq 0$ for all j. Suppose further that $v\in\mathrm{ArchTrop}(f)$ and ℓ is the unique index such that $(a_\ell,-\log|c_\ell|)$ is the right-hand vertex of the lower edge of $\mathrm{ArchNewt}(f)$ of slope v (so $2\leq\ell$). Then for any $N\in\mathbb{N}$ and x_1 with $|x_1|\geq(N+1)e^v$ we have $\left|\sum\limits_{j=1}^{\ell-1} c_j x_1^{a_j}\right|<\frac{1}{N}\left|c_\ell x_1^{a_\ell}\right|$.*

Proof: First note that $2\leq\ell\leq t$ by construction. Letting $r:=\log|x_1|$ and $\beta_j:=\log|c_j|$ we obtain $\left|\sum_{j=1}^{\ell-1} c_j x_1^{a_j}\right|\leq\sum_{j=1}^{\ell-1}\left|c_j x_1^{a_j}\right|=\sum_{j=1}^{\ell-1} e^{a_j r+\beta_j}=\sum_{j=1}^{\ell-1} e^{a_j(r-v)+a_j v+\beta_j}$. Clearly, $a_j\leq a_\ell-(\ell-j)$, so for $r\geq v$ we have

$$\left|\sum_{j=1}^{\ell-1} c_j e^{a_j r}\right|\leq\sum_{j=1}^{\ell-1} e^{(a_\ell-(\ell-j))(r-v)+a_j v+\beta_j}\leq\sum_{j=1}^{\ell-1} e^{(a_\ell-(\ell-j))(r-v)+a_\ell v+\beta_\ell},$$

where the last inequality follows from Proposition 2.2 and the definition of $\mathrm{ArchTrop}(f)$. So then

$$\begin{aligned}\left|\sum_{j=1}^{\ell-1} c_j x_1^{a_j}\right| &\leq e^{(a_\ell-(\ell-1))(r-v)+a_\ell v+\beta_\ell}\sum_{j=1}^{\ell-1} e^{(j-1)(r-v)}\\ &= e^{(a_\ell-(\ell-1))(r-v)+a_\ell v+\beta_\ell}\left(\frac{e^{(\ell-1)(r-v)}-1}{e^{(r-v)}-1}\right)\\ &< e^{(a_\ell-(\ell-1))(r-v)+a_\ell v+\beta_\ell}\left(\frac{e^{(\ell-1)(r-v)}}{e^{r-v}-1}\right)=\frac{e^{a_\ell r+\beta_\ell}}{e^{r-v}-1}\end{aligned}$$

So to prove our desired inequality it clearly suffices to enforce $e^{r-v}-1\geq N$. The last inequality clearly holds for all $r\geq v+\log(N+1)$, so we are done. ■

A simple consequence of our preceding term domination trick is that we can give explicit annuli in $\mathbb{C}$ free of roots of f.

Corollary 2.7. *Suppose $f(x_1):=\sum_{j=1}^t c_j x_1^{a_j}\in\mathbb{C}\left[x_1^{\pm 1}\right]$ satisfies $a_1<\cdots<a_t$, $c_j\neq 0$ for all j, and that v_1 and v_2 are* consecutive *points of $\mathrm{ArchTrop}(f)$ satisfying $v_2\geq v_1+\log 9$. Let ℓ be the unique index such that $(a_\ell,-\log|c_\ell|)$ is the unique vertex of $\mathrm{ArchNewt}(f)$ incident to lower edges of slopes v_1 and v_2 (so $2\leq\ell\leq t-1$). Then f has* no *root $\zeta\in\mathbb{C}$ satisfying $3e^{v_1}\leq|\zeta|\leq\frac{1}{3}e^{v_2}$.*

Proof: Let A denote the stated annulus. By Proposition 2.6, we have $\left|\sum_{j=1}^{\ell-1} c_j\zeta^{a_j}\right|<\frac{1}{2}\left|c_\ell\zeta^{a_\ell}\right|$ when $|\zeta|\geq 3e^{v_1}$. Employing the substitution $x_1\mapsto\frac{1}{x_1}$ (which has the effect of replacing $\mathrm{ArchTrop}(f)$ by $-\mathrm{ArchTrop}(f)$) we also obtain $\left|\sum_{j=\ell+1}^{t} c_j\zeta^{a_j}\right|<\frac{1}{2}\left|c_\ell\zeta^{a_\ell}\right|$ when $\frac{1}{\zeta}\geq 3e^{-v_2}$. So we obtain $\left|\sum_{j\neq\ell} c_j\zeta^{a_j}\right|<\left|c_\ell\zeta^{a_\ell}\right|$ in A, and this inequality clearly contradicts the existence of a root of f in A. ■

Our next key step will be to relate clusters of points in $\mathrm{ArchTrop}(f)$ to certain sub-summands of f. To relate the roots of high (or low) order summands of f to an explicit portion of the roots of f, let us first recall the following classical result.

Rouché's Theorem. *(See, e.g.,* [Con78, pp. 125–126]*.) Suppose $U\subset\mathbb{C}$ is a simply connected open set with compact closure $\bar{U}$. Let $g_1,g_2:\bar{U}\longrightarrow\mathbb{C}$ be meromorphic, with only finitely many zeroes and poles in $\bar{U}$, no removable singularities in $\bar{U}$, and no zeroes or poles on the boundary ∂U. Assume also that $|g_1|<|g_2|$ on ∂U. Then g_1+g_2 and g_2 have the same number of roots, counting multiplicities, in U. (We count poles as zeroes with negative multiplicity.)* ■

Lemma 2.8. *Let $f(x_1) := \sum_{j=1}^{t} c_j x_1^{a_j}$ with $a_1 < \cdots < a_t$ and $c_j \neq 0$ for all j, and set $v_{\min} := \min \mathrm{ArchTrop}(f)$ and $v_{\max} := \max \mathrm{ArchTrop}(f)$. Also let v_1 and v_2 be* consecutive *points of* ArchTrop(f) *satisfying $v_2 \geq v_1 + \log 9$, and let ℓ be the unique index such that $(a_\ell, -\log|c_\ell|)$ is the unique vertex of* ArchNewt(f) *incident to lower edges of slopes v_1 and v_2 (so $2 \leq \ell \leq t-1$). Then, counting multiplicities, f has exactly $a_\ell - a_1$ (resp. $a_t - a_\ell$) roots $\zeta \in \mathbb{C}$ satisfying $\frac{1}{2}e^{v_{\min}} < |\zeta| < 3e^{v_1}$ (resp. $\frac{1}{3}e^{v_2} < |\zeta| < 2e^{v_{\max}}$).*

In what follows, we let $D(r)$ (resp. $\bar{D}(r)$) denote the open (resp. closed) disk of radius r, centered at the origin, in $\mathbb{C}$.

Proof of Lemma 2.8: By symmetry (with respect to replacing x_1 by $\frac{1}{x_1}$) it clearly suffices to prove the first root count. Proposition 2.6 then tells us that $\frac{1}{2}|c_\ell \zeta^{a_\ell}| > \left|\sum_{j=1}^{\ell-1} c_j \zeta^{a_j}\right|$ and, by another application of Proposition 2.6 to $f(1/x_1)$ (remembering that $v_2 - v_1 \geq \log 9$), we also obtain $\frac{1}{2}|c_\ell \zeta^{a_\ell}| > \left|\sum_{j=\ell+1}^{t} c_j \zeta^{a_j}\right|$ when $|\zeta| = 3e^{v_1}$. So $|c_\ell \zeta^{a_\ell}| > \left|\sum_{j \neq \ell}^{t} c_j \zeta^{a_j}\right|$ when $|\zeta| = 3e^{v_1}$. By Rouché's Theorem we must then have that the monomial $c_\ell x_1^{a_\ell}$ and f have the same total number of roots and poles, counting multiplicities (poles counted with negative multiplicity), in the open disk $D(3e^{v_1})$. Since f has no roots in the closed disk $\bar{D}\left(\frac{1}{2}e^{v_{\min}}\right)$, other than a root/pole of multiplicity a_1 at the origin, we obtain that f has exactly $a_\ell - a_1$ roots $\zeta \in \mathbb{C}$ (and *no* poles), counting multiplicity, satisfying $\frac{1}{2}e^{v_{\min}} < |\zeta| < 3e^{v_1}$. ■

We use $\#S$ to denote the cardinality of a set S.

Proof of Theorem 1.7: First note that $\#\mathrm{ArchTrop}(f) \leq t-1$, thanks to Proposition 2.2: $\#\mathrm{ArchTrop}(f)$ is the number of lower edges of ArchNewt(f) and ArchNewt(f) has at most t vertices. Note also that, by definition, Γ must contain at least 1 point of ArchTrop(f) and thus Λ_Γ is a positive integer.

Suppose $\mathrm{ArchTrop}(f)_{\log 3}$ is connected. Then $\Lambda_\Gamma = a_t - a_1$, Corollary 2.3 tells us that $\mathrm{Amoeba}(f) \subset \Gamma$, and we are done.

So assume $\mathrm{ArchTrop}(f)_{\log 3}$ has at least two distinct connected components. Lemma 2.8 then immediately yields the conclusion of Theorem 1.7 when Γ is either the left-most or right-most connected component of $\mathrm{ArchTrop}(f)_{\log 3}$: We simply take w_1 to be the right-most point of $\Gamma \cap \mathrm{ArchTrop}(f)$ and w_2 the left-most point of ArchTrop(f) in the connected component of $\mathrm{ArchTrop}(f)_{\log 3}$ immediately to the right of Γ, or we take w_2 to be the left-most point of $\Gamma \cap \mathrm{ArchTrop}(f)$ and w_1 the right-most point of ArchTrop(f) in the connected component of $\mathrm{ArchTrop}(f)_{\log 3}$ immediately to the left of Γ.

Noting that

$$\sum_{\substack{\Gamma \text{ a connected component} \\ \text{of } \mathrm{ArchTrop}(f)_{\log 3}}} \Lambda_\Gamma = a_t - a_1, \tag{1}$$

we can now proceed by induction on the number of connected components of $\mathrm{ArchTrop}(f)_{\log 3}$: We simply ignore the left-most and right-most connected components of $\mathrm{ArchTrop}(f)_{\log 3}$, and treat the *new* left-most and right-most connected components via Lemma 2.8 as in the last paragraph.

To conclude, Corollary 2.3 tells us that we can also attain Λ_Γ many log norms (counting multiplicities) within the potentially tighter interval

$$\Gamma \cap (\min \mathrm{ArchTrop}(f) - \log 2, \max \mathrm{ArchTrop}(f) + \log 2).$$

Also, Equality (1) implies that every root of f must have log norm within some Γ. So we are done. ■

We can tighten the union of intervals U_f further when $t=3$: Combining Theorem 1.7 with Assertion (2a) of Theorem 1.11 (proved independently in Section 3.3) immediately yields the following refinement.

Corollary 2.9. *Suppose* $f(x_1) = \sum_{i=1}^{3} c_i x_1^{a_i}$ *is a trinomial with* $a_1 < a_2 < a_3$. *Also let* $v_{\min} := \min \mathrm{ArchTrop}(f)$, $v_{\max} := \max \mathrm{ArchTrop}(f)$, *and assume that* $v_{\max} - v_{\min} > \log 4$. *Then there are exactly* $a_2 - a_1$ *(resp.* $a_3 - a_2$*) roots* $\zeta \in \mathbb{C}$ *of* f *with*

$$v_{\min} - \log 2 < \log|\zeta| \leq v_{\min} + \log 2$$
$$(\textit{resp. } v_{\max} - \log 2 \leq \log|\zeta| < v_{\max} + \log 2). \quad ■$$

2.2. **Classical Computational Algebra and Amoeba Membership.** Let us first recall the following results of Plaisted and Ben-Or, Kozen, and Reif.

Theorem 2.10. [Pla84] *The following problem:*

Decide whether an arbitrary input $f \in \mathbb{Z}[x_1]$ has a complex root of norm 1.

is **NP**-*hard.* ■

Theorem 2.11. [BKR86] *There is an algorithm that, given any collection of polynomials* $f_1, \ldots, f_p, g_1, \ldots, g_q, h_1, \ldots, h_r \in \mathbb{Q}[x_1, \ldots, x_n]$, *decides whether there is a* $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$ *with* $f_1(\zeta) = \cdots = f_p(\zeta) = 0$, $g_1(\zeta), \ldots, g_q(\zeta) > 0$, *and* $h_1(\zeta), \ldots, h_r(\zeta) \geq 0$, *in time*

$$\left[\sum_{i=1}^{p} \mathrm{size}(f_i)) + (\sum_{i=1}^{q} \mathrm{size}(g_i)) + (\sum_{i=1}^{r} \mathrm{size}(h_i))\right]^{O(1)},$$

using $\left[\sum_{i=1}^{p} \mathrm{size}(f_i)) + (\sum_{i=1}^{q} \mathrm{size}(g_i)) + (\sum_{i=1}^{r} \mathrm{size}(h_i))\right]^{O(1)}$ *processors.* ■

Theorem 1.15 will then follow easily from two elementary propositions. The first is a well-known trick from computational algebra for re-expressing polynomial systems in a simpler form.

Proposition 2.12. *Given any* $f_1, \ldots, f_m \in \mathbb{Q}\left[\sqrt{-1}\right]\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$, *we can find* $g_1, \ldots, g_M \in \mathbb{Q}\left[\sqrt{-1}\right]\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_N^{\pm 1}\right]$ *satisfying the following properties:*

1. $f_1 = \cdots = f_m = 0$ *has a root in* $\mathbb{C}^n \iff g_1 = \cdots = g_M = 0$ *has a root in* $\mathbb{C}^N$.
2. *Each* g_i *is either a quadratic binomial or a linear trinomial.*
3. $\sum_{i=1}^{M} \mathrm{size}(g_i) = O(\sum_{i=1}^{m} \mathrm{size}(f_i))$.

Moreover, $g_1, \ldots, g_M$ *can be found in time* $O(\sum_{i=1}^{m} \mathrm{size}(f_i))$. ■

Proposition 2.13. *Given any* $f_1, \ldots, f_m \in \mathbb{Q}\left[\sqrt{-1}\right]\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ *with each* f_i *of degree at most* 2, *we can find* $g_1, \ldots, g_M \in \mathbb{Q}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_N^{\pm 1}\right]$ *satisfying the following properties:*

1. $f_1 = \cdots = f_m = 0$ *has a root in* $\mathbb{C}^n \iff g_1 = \cdots = g_M = 0$ *has a root in* $\mathbb{R}^N$.
2. $\sum_{i=1}^{M} \mathrm{size}(g_i) = O(\sum_{i=1}^{m} \mathrm{size}(f_i))$.

Moreover, $g_1, \ldots, g_M$ *can be found in time* $O(mn)$. ■

A simple example of Proposition 2.12 is the replacement of $f(x_1) := 1 - 2x_1 + x_1^5$ by the system $G := (y_1 - x_1^2, y_2 - y_1^2, y_3 - y_2 x_1, y_4 - 1 + 2x_1, y_5 - y_4 - y_3)$: It is easy to see that at a root of G, we must have $y_5 = 1 - 2x_1 + x_1^5 = 0$. The proof of Proposition 2.12 is not much harder: One simply substitutes new variables to break down sums with more than 2 terms and (employing the binary expansions of the underlying exponents) monomials of degree more than 2. Proposition 2.13 follows easily upon expanding every complex multiplication (resp. complex addition) into 4 real multiplications (resp. 2 real additions), by introducing new variables for the real and imaginary parts of the x_i.

Proof of Theorem 1.15: First observe that $\mathrm{Log}|z| \in \mathrm{Amoeba}(f) \iff f$ has a complex root ζ with $|\zeta|=|z|$. Letting A and B denote the real and imaginary parts of f, and letting α_i and β_i denote the real and imaginary parts of ζ_i, we thus obtain that $\mathrm{Log}|z|\in\mathrm{Amoeba}(f)$ if and only if the polynomial system

$$\begin{gathered} A(\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n)=0 \\ B(\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n)=0 \\ \alpha_1^2+\beta_1^2=|z_1|^2 \\ \vdots \\ \alpha_n^2+\beta_n^2=|z_n|^2 \end{gathered}$$

has a root $(\alpha,\beta)=(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n)\in\mathbb{R}^{2n}$. Now, while the preceding system of equations has size significantly larger than $\mathrm{size}(z)+\mathrm{size}(f)$ (due to the underlying expansions of powers of $\zeta_i=\alpha_i+\sqrt{-1}\beta_i$), we can introduce new variables and equations (via Propositions 2.12 and 2.13) to obtain another polynomial system, also with a real solution if and only if $\mathrm{Log}|z|\in\mathrm{Amoeba}(f)$, with size linear in $\mathrm{size}(z)+\mathrm{size}(f)$ instead. Applying Theorem 2.11, we obtain our **PSPACE** upper bound.

Our **NP**-hardness complexity lower bound follows immediately from Theorem 2.10, since $|\zeta|=1 \iff \mathrm{Log}|\zeta|=0$. ■

Remark 2.14. *A reduction of amoeba membership to the Existential Theory of the Reals, with an* **EXPTIME** *complexity upper bound instead, was observed in* [The02, Sec. 2.2]. ⋄

3. The Proof of Theorem 1.11

The assertion that $k\leq\min\{n,t-1\}$ follows immediately since any k-dimensional polytope always has at least $k+1$ vertices, and $\mathrm{Newt}(f)\subset\mathbb{R}^n$ has at most t vertices. Further background on polytopes, fans, and triangulations (which we use below) can be found in [dLRS10].

3.1. **Proof of Assertion (0).** Note that, by definition, $\mathrm{ArchTrop}(f)$ is a linear section of the outer normal fan $\mathcal{F}\subset\mathbb{R}^{n+1}$ of $\mathrm{ArchNewt}(f)$. Each $(n-1)$-cell of $\mathrm{ArchTrop}(f)$ is a linear section of a unique n-cell of $\mathcal{F}$, dual to a unique edge of $\mathrm{ArchNewt}(f)$. Since $\mathrm{ArchNewt}(f)$ has at most t vertices, $\mathrm{ArchNewt}(f)$ has at most $\binom{t}{2}$ edges and we obtain our upper bound.

Let us call any face of $\mathrm{ArchNewt}(f)$ possessing an outer normal of the form $(v,-1)$ a *lower* face. Note that any path between points in $\mathrm{ArchTrop}(f)$ induces a sequence of relative interiors of cells of $\mathrm{ArchTrop}(f)$ (each of dimension $<n$) having a connected union. So by duality again, $\mathrm{ArchTrop}(f)$ is connected if and only if $L\setminus L_0$ is path-connected, where L (resp. L_0) is the union of all lower faces (resp. lower vertices) of $\mathrm{ArchNewt}(f)$. The set $L\setminus L_0$ is topologically a k-ball minus a finite collection of points, and is thus path-connected for $k\geq 2$. In particular, $k=1$ implies that $\mathrm{ArchNewt}(f)$ lies in a 2-plane (perpendicular to the affine hyperplane $\{x_{n+1}=-1\}$ in $\mathbb{R}^{n+1}$) and has at most $t-1$ lower edges. So we are done. ■

3.2. **Proving Assertion (1).** The follow elementary fact will be quite useful.

Proposition 3.1. *The set* $P_n:=\{(|\zeta_1|,\ldots,|\zeta_n|) \mid 1+\zeta_1+\cdots+\zeta_n=0\ ,\ \zeta_i\in\mathbb{C}\setminus\{0\}$ *for all* $i\}$ *is exactly the convex polytope* Q_n *defined by all those* $(r_1,\ldots,r_n)\in\mathbb{R}^n_+$ *with* $r_n\leq 1+\sum_{i=1}^{n-1} r_i$ *and* $r_n\geq\max\left\{1-\sum_{i=1}^{n-1} r_i, 2r_1-1-\sum_{i=1}^{n-1} r_i,\ldots,2r_{n-1}-\sum_{i=1}^{n-1} r_i\right\}$. *(We set* $Q_1:=\{1\}$*.)*

Proof: The case $n=1$ clearly holds, so we assume $n\geq 2$. Note that Q_n is indeed a convex polytope since Q_n is an intersection of finitely many (open and closed) half-spaces.

Since P_n lies in the positive orthant by definition, the containment $P_n\subseteq Q_n$ follows easily from the Triangle Inequality: $\zeta\in P_n \Longrightarrow |\zeta_n|=\left|1+\sum_{i=1}^{n-1}\zeta_i\right|$ and thus $|\zeta_n|\leq 1+\sum_{i=1}^{n-1}|\zeta_i|$. The lower bound $|\zeta_n|\geq \max\left\{1-\sum_{i=1}^{n-1}|\zeta_i|, 2|\zeta_1|-1-\sum_{i=1}^{n-1}|\zeta_i|, \ldots, 2|\zeta_{n-1}|-\sum_{i=1}^{n-1}|\zeta_i|\right\}$ follows similarly, from a simple induction argument starting from $|1+\zeta_1|\geq\max\{1-|\zeta_1|, |\zeta_1|-1\}$.

For the containment $Q_n\subseteq P_n$, we will need two constructions to extract a root of $1+x_1+\cdots+x_n$, with correct norm vector, given a point of Q_n. First, let $r:=(r_1,\ldots,r_n)\in Q_n$ and set $\zeta(\theta):=\left(r_1e^{\sqrt{-1}(\pi-\theta)},\ldots,r_{n-1}e^{\sqrt{-1}(\pi-\theta)},-1-\left(\sum_{i=1}^{n-1}r_i\right)e^{\sqrt{-1}(\pi-\theta)}\right)$ for any $\theta\in[0,\pi]$. Clearly, $(|\zeta_1(\theta)|,\ldots,|\zeta_n(\theta)|)\in P_n$ and $|\zeta_i(\theta)|=r_i$ for all $i\in[n-1]$. So r will indeed lie in P_n provided we can find a $\theta\in[0,\pi]$ with $|\zeta_n(\theta)|=r_n$.

We can accomplish this, at least in certain cases, by observing that

$$|\zeta_n(\theta)|=\sqrt{1+\left(\sum_{i=1}^{n-1}r_i\right)^2-2\left(\sum_{i=1}^{n-1}r_i\right)\cos\theta}$$

is a continuous increasing function of θ on $[0,\pi]$, with minimum $\left|1-\sum_{i=1}^{n-1}r_i\right|$ and maximum $1+\sum_{i=1}^{n-1}r_i$: If $n=2$, then all necessary values of $|\zeta_n(\theta)|$ (satisfying the constraints on r_n from the definition of Q_n) can indeed be attained for a suitable choice of θ.

So let us now assume $n\geq 3$. We need $1-\sum_{i=1}^{n-1}r_i>0$ in order to attain all necessary values of $|\zeta_n(\theta)|$, but the condition $1-\sum_{i=1}^{n-1}r_i>0$ may not always hold. So we consider one more construction: For any $j\in[n-1]$ and any $\theta\in[0,\pi]$ set $\rho_i(\theta,j):=r_i$ for all $i\in[n-1]\setminus\{j\}$, $\rho_j(\theta,j):=r_je^{\sqrt{-1}(\pi-\theta)}$, $\rho_n(\theta,j):=-1-\sum_{i=1}^{n-1}\rho_i(\theta,j)$, and $\rho(\theta,j):=(\rho_1(\theta,j),\ldots,\rho_n(\theta,j))$. Clearly, $(|\rho_1(\theta,j)|,\ldots,|\rho_n(\theta,j)|)\in P_n$ and $|\rho_i(\theta,j)|=r_i$ for all $i\in[n-1]$. Similar to $\zeta(\theta)$, we can find a $\theta\in[0,\pi]$ with $|\rho_n(\theta,j)|=r_n$: A simple calculation yields

$$|\rho_n(\theta,j)|=\sqrt{2(1+\cos\theta)r_j^2-2(1+\cos\theta)r_j\left(1+\sum_{i=1}^{n-1}r_i\right)+\left(1+\sum_{i=1}^{n-1}r_i\right)^2},$$

which is a continuous increasing function of θ on $[0,\pi]$, with minimum $\left|2r_j-1-\sum_{i=1}^{n-1}r_i\right|$ and maximum $1+\sum_{i=1}^{n-1}r_i$. So all necessary values of $|\rho_n(\theta)|$ (satisfying the constraints from the definition of Q_n) can indeed be attained for a suitable choice of θ, provided $2r_j-1-\sum_{i=1}^{n-1}r_i>0$.

Assuming at least one of the quantities $1-\sum_{i=1}^{n-1}r_i, 2r_1-1-\sum_{i=1}^{n-1}r_i,\ldots,2r_{n-1}-\sum_{i=1}^{n-1}r_i$ is positive, we can thus use $\zeta(\theta)$ or $\rho(\theta,j)$, for suitable θ and j, to certify that $r\in P_n$. Should none of the preceding quantities be positive, we conclude as follows: By assumption, $1<\sum_{i=1}^{n-1}r_i$. We then obtain that 1, $\sum_{i=1}^{n-2}r_i$, and r_{n-1} can form the side-lengths of a non-degenerate triangle, i.e., we can find $\phi_1,\phi_2\in(0,\pi)$ with $1+\left(\sum_{i=1}^{n-2}r_i\right)e^{\sqrt{-1}\phi_1}+r_{n-1}e^{\sqrt{-1}\phi_2}=0$. Picking any $\alpha_1:(0,\pi]\longrightarrow(0,\phi_1]$ and $\alpha_2:(0,\pi]\longrightarrow(0,\phi_2]$ with both functions decreasing, continuous, and surjective, we can then show that $r\in P_n$ as follows: Let $\mu_i(\theta):=r_ie^{\sqrt{-1}\alpha_1(\theta)}$ for all $i\in[n-2]$, $\mu_{n-1}(\theta):=r_ie^{\sqrt{-1}\alpha_2(\theta)}$, $\mu_n(\theta):=-1-\sum_{i=1}^{n-1}\mu_i(\theta)$, and $\mu(\theta):=(\mu_1(\theta),\ldots,\mu_n(\theta))$. Then $|\mu_n(\theta)|$ attains every value in $(0,1+r_1+\cdots+r_{n-1}]$, and $|\mu(\theta)|\in P_n$.

So any $r\in Q_n$ indeed lies in P_n and we are done. ■

Proof of Assertion (1) of Theorem 1.11: First note that the case $t=2$ was already observed in Example 1.4. So let us assume $t\geq 3$ (and thus $k\geq 2$ since $t=k+1$ here).

Note that the definitions of Amoeba(f) and ArchTrop(f) are invariant under translation of $\{a_1,\ldots,a_{k+1}\}$. So we may assume without loss of generality that a_1 is the origin $\mathbf{O}$. Furthermore, since $k=\dim\operatorname{Conv}(\{a_1,\ldots,a_{k+1}\})$, we have that $a_2,\ldots,a_{k+1}$ are linearly independent in $\mathbb{Q}^n$.

Defining the vector of monomials $x^B := \left(x_1^{b_{1,1}} \cdots x_n^{b_{n,1}}, \ldots, x_1^{b_{1,n}} \cdots x_n^{b_{n,n}}\right)$ for any $n \times n$ matrix B with (i,j)-entry $b_{i,j} \in \mathbb{Q}$, it is easily checked that the map $m(x) = x^B$ is an analytic automorphism of $(\mathbb{C} \setminus \{0\})^n$ when B is invertible. In particular, such a map induces the linear map $x \mapsto B^{-1}x$ on both Amoeba(f) and ArchTrop(f). Since invertible real linear maps are homeomorphisms, they preserve containment and contractability for Amoeba(f) and ArchTrop(f). Letting B be the matrix whose inverse has columns $a_2, \ldots, a_{k+1}, b_1, \ldots, b_{n-k}$ (for any $b_1, \ldots, b_{n-k} \in \mathbb{Q}^n$ with $\{a_2, \ldots, a_{k+1}, b_1, \ldots, b_{n-k}\}$ forming a basis for $\mathbb{Q}^n$), we may thus restrict our study of Amoeba(f) and ArchTrop(f) to the special case $k = n$ and $f(x) = c_0 + c_1x_1 + \cdots + c_nx_n$. Note also that, for any $\gamma \in \mathbb{C} \setminus \{0\}$, the scaling $x \mapsto (\gamma x_1, \ldots, \gamma x_n)$ merely affects both Amoeba(f) and ArchTrop(f) by a common translation, and the scaling $f \mapsto \gamma f$ leaves both Amoeba(f) and ArchTrop(f) unchanged. So we may further restrict to the special case $f(x) = 1 + x_1 + \cdots + x_n$.

The contractability of ArchTrop(f) follows easily: ArchTrop(f) is simply the outer normal fan of the standard n-simplex $\Delta_n := \mathrm{Conv}(\{\mathbf{O}, e_1, \ldots, e_n\})$ in $\mathbb{R}^n$ and is thus contractible.

The contractability of Amoeba(f) follows immediately from Proposition 3.1, since convex polytopes are contractible, and the function $\mathrm{Log}(x) := (\log(x_1), \ldots, \log(x_n))$ induces a homeomorphism between $\mathbb{R}^n_+$ and $\mathbb{R}^n$.

To prove containment, first note that the complement $\mathbb{R}^n \setminus$ ArchTrop(f) consists of $n+1$ open cones, each with boundary combinatorially equivalent to the boundary of the positive orthant $\mathbb{R}^n_+$. Note also that we can map any vertex of Δ_n to any other vertex of Δ_n via an invertible affine map, and this affine map also preserves any containment between Amoeba(f) and ArchTrop(f). So it suffices to work locally and prove that Amoeba$(1 + x_1 + \cdots + x_n)$ contains the boundary of the negative orthant. Furthermore, by symmetry in the variables, it in fact suffices to simply prove that Amoeba$(1 + x_1 + \cdots + x_n)$ contains the cone generated by the negatives of the first $n-1$ standard basis vectors. (Note that we've assumed $k = n \geq 2$ earlier.) Taking exponentials, this means proving that the set P_n from Proposition 3.1 contains $(0,1]^{n-1} \times \{1\}$. Thanks to the equality $P_n = Q_n$ from Proposition 3.1, we see that $P_n \cap (\mathbb{R}^{n-1}_+ \times \{1\})$ is in fact

$$\left\{(r_1, \ldots, r_n) \in \mathbb{R}^n \mid r_n = 1 \text{ and } \sum_{i=1}^{n-1} r_i \geq 2(r_j - 1) \text{ for all } j \in [n-1]\right\}.$$

So we are done. ■

3.3. **Proof of Part (a) of Assertion (2).** Let $w := (\log|\zeta_1|, \ldots, \log|\zeta_n|) \in$ Amoeba(f) and assume without loss of generality that $|c_1\zeta^{a_1}| \geq |c_2\zeta^{a_2}| \geq \cdots \geq |c_t\zeta^{a_t}|$. Since $f(\zeta) = 0$ implies that $|c_1\zeta^{a_1}| = |c_2\zeta^{a_2} + \cdots + c_t\zeta^{a_t}|$, the Triangle Inequality immediately implies that $|c_1\zeta^{a_1}| \leq (t-1)|c_2\zeta^{a_2}|$. Taking logarithms, we then obtain

$$a_1 \cdot w + \log|c_1| \geq \cdots \geq a_t \cdot w + \log|c_t| \quad \text{and} \tag{2}$$

$$a_1 \cdot w + \log|c_1| \leq \log(t-1) + a_2 \cdot w + \log|c_2| \tag{3}$$

For each $i \in \{2, \ldots, t\}$ let us then define δ_i to be the shortest vector such that

$$a_1 \cdot (w + \delta_i) + \log|c_1| = a_i \cdot (w + \delta_i) + \log|c_i|.$$

Note that $\delta_i = \lambda_i(a_i - a_1)$ for some nonnegative λ_i since we are trying to affect the dot-product $\delta_i \cdot (a_1 - a_i)$. In particular, $\lambda_i = \frac{(a_1-a_i)\cdot w + \log|c_1/c_i|}{|a_1-a_i|^2}$ so that $|\delta_i| = \frac{(a_1-a_i)\cdot w + \log|c_1/c_i|}{|a_1-a_i|}$. (Indeed, Inequality (2) implies that $(a_1 - a_i) \cdot w + \log|c_1/c_i| \geq 0$.)

Inequality (3) implies that $(a_1 - a_2) \cdot w + \log|c_1/c_2| \leq \log(t-1)$. We thus obtain $|\delta_2| \leq \frac{\log(t-1)}{|a_1-a_2|} \leq \log(t-1)$. So let $i_0 \in \{2, \ldots, t\}$ be any i minimizing $|\delta_i|$. We of course have $|\delta_{i_0}|\leq\log(t-1)$, and by the definition of δ_{i_0} we have

$$a_1\cdot(w+\delta_{i_0})+\log|c_1|=a_{i_0}\cdot(w+\delta_{i_0})+\log|c_{i_0}|.$$

Moreover, the fact that δ_{i_0} is the shortest among the δ_i implies that

$$a_1\cdot(w+\delta_{i_0})+\log|c_1|\geq a_i\cdot(w+\delta_{i_0})+\log|c_i|$$

for all i. Otherwise, we would have $a_1\cdot(w+\delta_{i_0})+\log|c_1|<a_i\cdot(w+\delta_{i_0})+\log|c_i|$ and $a_1\cdot w+\log|c_1|\geq a_i\cdot w+\log|c_i|$ (the latter following from Inequality (2)). Taking a convex linear combination of the last two inequalities, it is then clear that there must be a $\mu\in[0,1)$ such that $a_1\cdot(w+\mu\delta_{i_0})+\log|c_1|=a_i\cdot(w+\mu\delta_{i_0})+\log|c_i|$. Thus, by the definition of δ_i, we would obtain $|\delta_i|\leq\mu|\delta_{i_0}|<|\delta_{i_0}|$ — a contradiction.

We thus have the following:

$$a_1\cdot(w+\delta_{i_0})-(-\log|c_1|)=a_{i_0}\cdot(w+\delta_{i_0})-(-\log|c_{i_0}|),$$
$$a_1\cdot(w+\delta_{i_0})-(-\log|c_1|)\geq a_i\cdot(w+\delta_{i_0})-(-\log|c_i|)$$

for all i, and $|\delta_{i_0}|\leq\log(t-1)$. This implies that $w+\delta_{i_0}\in\mathrm{ArchTrop}(f)$. In other words, we've found a point in $\mathrm{ArchTrop}(f)$ sufficiently near $\mathrm{Log}|\zeta|$ to prove our desired upper bound. ■

3.4. **Proving Part (b) of Assertion (2).** We begin with a refinement of the special case $n=1$.

Theorem 3.2. *Suppose f is any univariate t-nomial with $t\geq 3$ and $s:=\#\mathrm{ArchTrop}(f)$. (So $1\leq s\leq t-1$.) Then for any $v\in\mathrm{ArchTrop}(f)$ there is a root $\zeta\in\mathbb{C}$ of f with $|v-\log|\zeta||<\log 2$, $|v-\log|\zeta||\leq\log\min\{18,t-1\}$, or $|v-\log|\zeta||<(\log 9)s-\log\frac{9}{2}<2.2s-1.5$, according as s is 1, 2, or ≥ 2. In particular, $|v-\log|\zeta||<(\log 9)t-\log\frac{81}{2}<2.2t-3.7$ for all $t\geq 3$.*

Proof: Following the notation of Theorem 1.7, let Γ be the connected component of U_f containing $v\in\mathrm{ArchTrop}(f)$ and $m:=\#(\Gamma\cap\mathrm{ArchTrop}(f))$. (So $1\leq m\leq s$.) The quantity $|v-\log|\zeta||$ is thus clearly maximized, for instance, when v is as far to the left as possible and $\log|\zeta|$ is as far to the right as possible. In other words,

$$|v-\log|\zeta||<\log(3)+(\log 9)(m-2)+\log(3)+\delta,$$

where δ is $\log 3$ or $\log 2$, according as $m<s$ or $m=s$. We thus obtain the largest possible upper bound of $(\log 9)s-\log\frac{9}{2}$ when $m=s$. Note also that $s\leq t-1$. So now we merely need to refine the cases with $s\in\{1,2\}$.

The case $s=1$ follows from Corollary 2.3 since $\min\mathrm{ArchTrop}(f)=\max\mathrm{ArchTrop}(f)$ here.

The case $s=2$ proceeds as follows: If $m=1$ then Γ is an open interval of width $2\log 3$ with v at its median, so we must have $|v-\log|\zeta||<\log 3$. If $m=2$ then Γ is an open interval of width at most $4\log 3$, but we still have

$$\min\mathrm{ArchTrop}(f)-\log 2<\log|\zeta|<\max\mathrm{ArchTrop}(f)+\log 2.$$

So $|v-\log|\zeta||$ can again be maximized by having v as far left as possible and $\log|\zeta|$ as far right as possible. In particular, $s=2$ implies that $\mathrm{ArchTrop}(f)=\{\min\mathrm{ArchTrop}(f),\max\mathrm{ArchTrop}(f)\}$. So we obtain $|v-\log|\zeta||<\log(3)+\log(3)+\log(2)=\log 18$. In addition, we can apply Corollary 2.3 to observe that there is always a root $\zeta\in\mathbb{C}$ of f with $|\min\mathrm{ArchTrop}(f)-\log|\zeta||\leq\log(t-1)$, and the same bound can be attained for $|\max\mathrm{ArchTrop}(f)-\log|\zeta||$, possibly with a different root ζ. So we obtain $|v-\log|\zeta||\leq\log\min\{18,t-1\}$. ■

We will handle the case $n\geq 2$ by showing that any point $v\in\mathrm{ArchTrop}(f)$ lies close to the intersection of $\mathrm{Amoeba}(f)$ with a specially chosen line also containing v. With some care, this enables us to reduce to the case $n=1$. In particular, intersecting a line with $\mathrm{Amoeba}(f)$ is the same as evaluating f along a monomial curve, and we'll need a technical lemma to pick exponents that permit an easy reduction to $n=1$.

Theorem 3.3. *Given any subset $\{a_1,\ldots,a_t\}\subset\mathbb{Z}^n$ of cardinality $t\geq n+1$, there exists an $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n\setminus\{\mathbf{O}\}$ such that the dot-products $\alpha\cdot a_1,\ldots,\alpha\cdot a_t$ are pair-wise distinct and, for all $i\in[n]$, $|\alpha_i|\leq\left\lceil\frac{1}{4}t(t-1)\right\rceil$ or $|\alpha_i|\leq t-2$, according as $n\geq 3$ or $n=2$.*

Proof: Observe that for the $\alpha\cdot a_i$ to remain distinct we must have α avoid a set of $\leq t(t-1)/2$ hyperplanes, depending on $\{a_1,\ldots,a_t\}$. This is equivalent to α avoiding the zero set of an n-variate polynomial of degree $t(t-1)/2$. Schwartz's Lemma (see, e.g., [Sch80]) then tells us that for *any* $S\subset\mathbb{Z}$ with $\#S>t(t-1)/2$ there is an $\alpha\in S^n$ avoiding our aforementioned set of hyperplanes. Picking $S=\left\{-\left\lceil\frac{1}{4}t(t-1)\right\rceil,\ldots,\left\lceil\frac{1}{4}t(t-1)\right\rceil\right\}$ then gives us the case $n\geq 3$.

For the case $n=2$, it is enough to prove that the set of lattice points

$$X:=\{-(t-2),\ldots,t-2\}\times\{1,\ldots,t-2\}$$

contains at least $1+t(t-1)/2$ distinct directions (and thus we can always find a suitable $\alpha\in X$). In other words, we need to prove that X has at least $1+t(t-1)/2$ points with relatively prime coordinates. Throwing out the directions $(1,0)$ and $(0,1)$, it is then enough to show that $Y:=\{1,\ldots,t-2\}^2$ contains at least $\frac{t(t-1)}{4}-\frac{1}{2}$ points with relatively prime coordinates. The number of such points, for arbitrary t, forms the sequence A018805 in Sloane's Online Encyclopedia of Integer Sequences [Slo10]. A routine, but tedious calculation then yields the $t\in\{3,\ldots,45\}$ portion of the $n=2$ case.

The remaining cases can be settled as follows: By a standard Möbius inversion argument, the number of points with relatively prime coordinates in Y is exactly $\sum\limits_{d=1}^{t-2}\mu(d)\lfloor(t-2)/d\rfloor^2$ where μ is the classical *Möbius function* (see, e.g., [HWW08]). A simple expansion then yields our desired number of points to be bounded from below by

$$A(t):=\frac{(t-2)^2}{\zeta(2)}-4(t-2)-2(t-2)\log(t-2)-2\zeta(2)(t-1).$$

A simple derivative calculation then yields that $A(t)-\frac{t(t-1)}{4}+\frac{1}{2}$ is increasing for all $t\geq 25$. So it's enough to prove that $A(46)>517$. One can check via `Maple` that $A(46)>519.9$, so we are done. ■

Proof of Part (b) of Assertion (2): Let $v=(v_1,\ldots,v_n)$ be any point of $\mathrm{ArchTrop}(f)$. If $v\in\mathrm{Amoeba}(f)$ then there is nothing to prove. So let us assume $v\notin\mathrm{Amoeba}(f)$. Since the case $n=1$ is immediate from Theorem 3.2 and Example 1.4, we will assume henceforth that $n\geq 2$.

So we can reduce to the case $k=n$, let us temporarily assume that $k<n$. Without loss of generality, we can order the variables $x_1,\ldots,x_n$ so that the image of $\mathrm{Newt}(f)$ under the coordinate projection sending $\mathbb{R}^n$ onto $\mathbb{R}^k\times\{0\}^{n-k}$ has dimension k. Define $g(x_1,\ldots,x_k):=f(x_1,\ldots,x_k,e^{v_{k+1}},\ldots,e^{v_n})$. By the definition of $\mathrm{ArchTrop}(f)$, $\max_{i\in[t]}|c_ie^{a_i\cdot v}|$ is attained for at least two distinct values of i. By our construction of g, this monomial norm condition implies that $(v_1,\ldots,v_k)\in\mathrm{ArchTrop}(g)$. Clearly then, if we can find a root $(\zeta_1,\ldots,\zeta_k)$ of g with $|(v_1,\ldots,v_k)-\mathrm{Log}|(\zeta_1,\ldots,\zeta_k)||<\varepsilon_{k,t}$, then $\zeta:=(\zeta_1,\ldots,\zeta_k,e^{v_{k+1}},\ldots,e^{v_n})$ will be a root of f with $|v-\mathrm{Log}|\zeta||<\varepsilon_{k,t}$. But finding such a $(\zeta_1,\ldots,\zeta_k)$ for g is nothing more than an instance of the case where the dimension of the underlying Newton polytope is the same as the underlying number of variables.

So we may assume $k=n\geq 2$ henceforth. Consider a monomial curve $C(t):=(\gamma_1t^{\alpha_1},\ldots,\gamma_nt^{\alpha_n})$ with $\alpha=(\alpha_1,\ldots,\alpha_n)\neq\mathbf{O}$. (Note that $\{\mathrm{Log}|C(t)|\}_{t\in\mathbb{C}}$ is always a line in $\mathbb{R}^n$.) Setting $\gamma_i=e^{v_i}$ for all i we obtain $v=\mathrm{Log}|C(1)|$, independent of α. So let us pick α via Theorem 3.3 and set $h(t):=f(C(t))$. Thanks to Theorem 3.3, h is a (univariate) t-nomial, and we can write $h(t)=\sum_{i=1}^t c_ie^{a_i\cdot v}t^{\beta_i}$ for some $\{\beta_1,\ldots,\beta_t\}\subset\mathbb{Z}$ of cardinality t. Now, by the definition of

$\mathrm{ArchTrop}(f)$, $\max_{i\in[t]}|c_ie^{a_i\cdot v}|$ is attained for at least two distinct values of i. By our construction of h, this monomial norm condition implies that $0\in\mathrm{ArchTrop}(h)$. So to find a root $\zeta\in\mathbb{C}^n$ with $\mathrm{Log}|\zeta|$ close to v, it's enough to prove that h has a root ρ close to 1. Thanks to Theorem 3.2, we can do the latter, so now we simply have to account for metric distortion from specializing f along $C(t)$.

Taking logarithms, $\mathrm{Amoeba}(h)$ containing a point at distance ε from 0 implies that $\mathrm{Amoeba}(f)$ contains a point at distance $\leq|\alpha|\varepsilon$ from v. So by the coordinate bounds of Theorem 3.3, we are done. ■

3.5. **Proof of Assertion (3).** To prove Part (a), note that $\left(\frac{-1}{t-1},\ldots,\frac{-1}{t-1}\right)$ is a root of φ and thus $p:=\left(\log\left|\frac{1}{t-1}\right|,\ldots,\log\left|\frac{1}{t-1}\right|\right)\in\mathrm{Amoeba}(\varphi)$. Also, from our proof of Assertion (1), we know that $\mathrm{ArchTrop}(\varphi)\cap\overline{\mathbb{R}^{t-1}_{-}}$ is the boundary of the negative orthant. So the distance from p to $\mathrm{ArchTrop}(\varphi)$ is $\log(t-1)$.

To prove Part (b), note that $(x_1+1)^{t-k}$ has a unique root of multiplicity $t-k$ at $x_1=-1$. Recall that the roots of a monic univariate polynomial are continuous functions of the coefficients, e.g., [RS02, Thm. 1.3.1, pg. 10].[2] So then, for any $\varepsilon>0$, we can find a $\delta_\varepsilon>0$ so that for all $\delta\in\mathbb{C}$ with $|\delta|\in[0,\delta_\varepsilon)$, *all* the roots ζ_1 of $(x_1+1)^{t-k}-\delta$ satisfy $|\zeta_1+1|<\varepsilon$. Clearly then, for any $\varepsilon'>0$, taking $|\rho_2|,\ldots,|\rho_n|$ sufficiently small (or $u_2:=\log|\rho_2|,\ldots,u_n:=\log|\rho_n|$ sufficiently negative) implies that the distance from any point $u\in\mathrm{Amoeba}(f)$ of the form $(u_1,u_2,\ldots,u_n)$ to the hyperplane $\{0\}\times\mathbb{R}^{n-1}$ is at most ε': Simply take ε so that $\varepsilon'=\log(1+\varepsilon)$ and $|x_2|+\cdots+|x_n|<\delta_\varepsilon$.

On the other hand, by the log-concavity of the binomial coefficients, $\mathrm{ArchNewt}\left((x_1+1)^{t-k}\right)$ must have an edge of slope $t-k$. This will enable us to prove that $\mathrm{ArchTrop}(\psi)$ contains a ray of the form $\{(\log(t-k),N,\ldots,N)\}_{N\rightarrow+\infty}$. and thus conclude: The points along this ray have distance to $\mathrm{Amoeba}(\psi)$ approaching $\log(t-k)$, by the preceding paragraph.

To see why such a ray lies in $\mathrm{ArchTrop}\left((x_1+1)^{t-k}\right)$ simply note that as $N\longrightarrow-\infty$, the linear form $\log(t-k)u_1+Nu_2+\cdots+Nu_n-u_{n+1}$ is maximized exactly at the vertices

$$(t-k-1,0,\ldots,0,-\log(t-k))\quad\text{and}\quad(t-k,0,\ldots,0,0)$$

of $\mathrm{ArchNewt}\left((x_1+1)^{t-k}\right)$. (Indeed, the only other possible vertices of $\mathrm{ArchNewt}\left((x_1+1)^{t-k}\right)$ are the basis vectors $e_2,\ldots,e_k$ of $\mathbb{R}^{n+1}$.) So, by Proposition 1.18, we are done. ■

4. Proving Theorem 1.17

Let us first recall the following result on comparing monomials in rational numbers.

Theorem 4.1. [BRS09, Sec. 2.4] *Suppose $\alpha_1,\ldots,\alpha_N\in\mathbb{Q}$ are positive and $\beta_1,\ldots,\beta_N\in\mathbb{Z}$. Also let A be the maximum of the numerators and denominators of the α_i (when written in lowest terms) and $B:=\max_i\{|\beta_i|\}$. Then, within*

$$O\left(N30^N\log(B)(\log\log B)^2\log\log\log(B)(\log(A)(\log\log A)^2\log\log\log A)^N\right)$$

bit operations, we can determine the sign of $\alpha_1^{\beta_1}\cdots\alpha_N^{\beta_N}-1$. ■

While the underlying algorithm is a simple application of Arithmetic-Geometric Mean Iteration (see, e.g., [Ber03]), its complexity bound hinges on a deep estimate of Nesterenko [Nes03], which in turn refines seminal work of Matveev [Mat00] and Alan Baker [Bak77] on linear forms in logarithms.

[2]The statement there excludes roots of multiplicity equal to the degree of the polynomial, but the proof in fact works in this case as well.

Proof of Theorem 1.17: From Proposition 1.18, it is clear that we merely need an efficient method to compare quantities of the form $|c_i z^{a_i}|$, and there are exactly $t-1$ such comparisons to be done. So our first complexity bound follows immediately from the case of Theorem 4.1 where $A=2^\sigma$, $B=d$, and $N=2n+2$. In particular, $30^2 \log 2 < 623.8325$ and $\sqrt{623.8325} < 25.2$.

The second assertion follows almost trivially: Thanks to the exponential form of the coefficients and the query point, one can take logarithms to reduce to comparing integer linear combinations of rational numbers of bit size linear in $\max\{\sigma, \log d\}$. So the underlying monomial norm comparisons can be reduced to standard techniques for fast integer multiplication (see, e.g., [BS96, pg. 43]). ■

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