A short proof of correctness of the quasi-polynomial time algorithm for parity games

Hugo Gimbert[1] and Rasmus Ibsen-Jensen[2]

1 LaBRI, Université de Bordeaux, CNRS, France
hugo.gimbert@cnrs.fr
2 IST Austria, Vienna, Austria
ribsen@ist.ac.at

Abstract

Recently Cristian S. Calude, Sanjay Jain, Bakhadyr Khoussainov, Wei Li and Frank Stephan proposed a quasi-polynomial time algorithm for parity games [1]. This paper proposes a short proof of correctness of their algorithm.

Parity games

A parity game is given by a directed graph (V, E), a starting node $s \in V$, a function which attaches to each $v \in V$ a priority $\text{pty}(v)$ from a set $\{1, 2, ..., m\}$; the main parameter of the game is n, the number of nodes, and the second parameter is m. Two players Anke and Boris move alternately in the graph with Anke moving first. A move from a node v to another node w is valid if (v, w) is an edge in the graph; furthermore, it is required that from every node one can make at least one valid move. The alternate moves by Anke and Boris define an infinite sequence of nodes which is called a play. Anke wins a play through nodes $v_0, v_1, \cdots$ iff $\limsup_t \text{pty}(v_t)$ is even, otherwise Boris wins the play.

We say that a player *wins the parity game* if she has a strategy which guarantees the play to be winning for her. Parity games are determined [3] thus either Anke or Boris wins the parity game.

Statistics

The core of the algorithm is to keep track of statistics about the game, in the form of *partial* functions

$$f : 0 \dots k \to 1 \dots m \ .$$

The integer k is chosen such that 2^k is strictly larger than the number of vertices. The domain of f is denoted $\text{dom}(f)$ and its image $\text{im}(f)$. Statistics are assumed to be *increasing*, i.e. $\forall i, j \in \text{dom}(f), (i \leq j \implies f(i) \leq f(j))$. A statistic f can be modified by *inserting* a priority c at an index ℓ, which results in removing all pairs of index $\leq \ell$ from f and adding the pair (ℓ, c).

The initial statistic is the empty statistic $f_0 = \emptyset$, which is updated successively by all the priorities visited during the play, thus producing a sequence of statistics. The update of a statistic f by a priority c is performed by applying successively the following two rules.

- **Type I update:** If c is even then it is inserted at the highest index $j \in 0 \cdots k$ such that f is defined and even on $0 \dots j - 1$.
- **Type II update:** If $\text{im}(f)$ contains at least one value $< c$ then c is inserted at the highest index $j \in \text{dom}(f)$ such that $f(j) < c$.

Applying both rules I and II successively, in this order, gives the same result than applying only rule II. Both rules ensures that the update of an increasing statistic is increasing.

Anke (resp. Boris) *wins the statistics game* if she (resp. he) has a strategy to enforce (resp. to avoid) a visit to a statistic whose domain contains k. Similarly to the game of chess, statistics games are determined: either Anke or Boris has a winning strategy [2].

The main result of [1] is:

▶ **Theorem 1** (Calude et al.). *Anke wins the parity game iff she wins the statistics games.*

Since statistics games are determined, the direct implication follows from:

▶ **Lemma 2.** *If Boris wins the statistics games, he wins the parity game.*

Proof. We prove that any play won by Boris in the statistics game is won by Boris in the parity game, for that we show that $c = \limsup_t c_t$ is odd in every sequence of statistics update $f_0 \to_{c_0} f_1 \to_{c_1} \ldots$ such that $\forall t \geq 0, k \notin \text{dom}(f_t)$.

An easy case is when the sequence of statistics is ultimately constant equal to some f then $f \to_c f$ thus c is odd because an update by an even priority always performs an insertion. In the opposite case define (ℓ, d) the maximal pair (for the dichotomic order) inserted infinitely often. Since d is inserted infinitely often then $d \leq \limsup_t c_t = c$. And $d \geq c$ otherwise c would be inserted infinitely often at a index $\geq \ell$ which would contradict the maximality of (ℓ, d). Since (ℓ, c) is inserted infinitely often then it is removed infinitely often, let (ℓ', c') be a pair used infinitely often to remove (ℓ, c). Since (ℓ, c) is removed by the insertion of (ℓ', c') then $\ell \leq \ell'$ hence $\ell = \ell'$ by maximality of ℓ. Since $(\ell', c') = (\ell, c') \neq (\ell, c)$ then $c' < c$ by maximality of c. Since (ℓ, c) is removed by the insertion of (ℓ', c') and $c' < c$ then this insertion is necessarily performed by a type I update and since $\ell = \ell'$ then c is odd otherwise the insertion would occur at a higher index. ◀

The converse implication (Corollary 7) relies on several crucial properties of statistics.

▶ **Definition 3** (Counter value). With every statistic f is associated its *counter value*

$$\text{bin}(f) = \sum_{j \in \text{dom}(f)} 2^j \ .$$

▶ **Lemma 4** (Increments). *Let $f \to_c f'$ be a statistic update inserting c at index $\ell \in 1 \ldots k$. The following conditions are equivalent:*

- $\text{bin}(f') > \text{bin}(f)$,
- $\text{bin}(f') = \text{bin}(f) + 1$,
- $\ell \notin \text{dom}(f)$.

If any of these conditions hold then $f \to_c f'$ is a type I update called an increment.

Proof. If $\ell \in \text{dom}(f)$ then $\text{dom}(f') \subseteq \text{dom}(f)$ thus $\text{bin}(f') \leq \text{bin}(f)$ and none of the conditions hold. If $\ell \notin \text{dom}(f)$ then by definition of updates, $f \to_c f'$ is a type I update and f is defined and even on $1 \ldots \ell - 1$. Thus $\text{bin}(f') - \text{bin}(f) = 2^\ell - (2^0 + 2^1 \ldots + 2^{\ell-1}) = 1$ and the three conditions hold. ◀

In the sequel we fix a sequence $f_0 \to_{c_0} f_1 \ldots \to_{c_N} f_{N+1}$ of statistics updates.

▶ **Definition 5** (Even statistics and even factorizations). A statistic f is *even* if all values in $\text{im}(f)$ are even. An *even factorization* of length j is a sequence $0 \leq t_0 < \ldots < t_j = N + 1$ such that for every $i \in 0 \ldots j - 1$, the maximum of $c_{t_i}, c_{t_i+1}, \ldots, c_{t_j-1}$ is even.

Long even factorizations exist whenever the last statistic is even.

▶ **Lemma 6.** *Assume $f_0 = \emptyset$ and f_{N+1} is even. Then there is an even factorization of length at least* $\text{bin}(f_{N+1})$.

▶ **Corollary 7.** *If Anke wins the statistics game then she wins the parity game.*

Proof. By definition of the statistics game, Anke can enforce the play to reach a statistic f_{N+1} such that $k \in \text{dom}(f_{N+1})$. If N is chosen minimal then $f_N \to_{c_N} f_{N+1}$ is an increment by Lemma 4, thus $\text{bin}(f_{N+1}) = 2^k$. Moreover f_{N+1} is even, because f_{N+1} only contains the just inserted pair (k, c_N), by definition of bin, and c_N is even by Lemma 4. According to Lemma 6, such a play has an even factorization $t_0 < t_1 < \ldots < t_j = N + 1$ of length $j \geq 2^k$. Since 2^k is $>$ than the number of vertices, the play loops on the same vertex at some dates t_i and $t_{i'}$ with $0 \leq i < i' \leq j$. By definition of even factorizations, the maximal priority on this loop is even. Thus Boris has no positional winning strategy in the parity game, and since parity games are positional [3], Boris has no winning strategy at all in the parity game. ◀

Proof of Lemma 6. The proof is by induction on N. We first rule out two easy cases: if $\text{bin}(f_{N+1}) = 0$ then $t_0 = N + 1$ is an even factorization of length 0; and if $N = 0$ and $0 < \text{bin}(f_1)$ then according to Lemma 4, $f_0 \to_{c_0} f_1$ is an increment and c_0 is even thus $t_0 = 0, t_1 = 1$ is an even factorization of length 1.

From now on assume $N > 0$ and $0 < \text{bin}(f_{N+1})$. Then f_{N+1} is not empty, let $\ell = \min \text{dom}(f_{N+1})$. We will use the following *insertion property* several times: assume that $\ell \notin \text{dom}(f_t)$ for some t, then there exists $t' \in t+1 \ldots N$ such that $f_{t'} \to_{c_{t'}} f_{t'+1}$ is a type I insertion at index ℓ (take t' the maximal date such that $\ell \notin \text{dom}(f'_t)$).

Since $f_0 = \emptyset$ and according to the insertion property there exists a maximal date $t \in 0 \ldots N$ such that $f_t \to_{c_t} f_{t+1}$ is a type I insertion at index ℓ. We show that:

i) $\text{bin}(f_t) \geq \text{bin}(f_{t+1}) - 1$,
ii) $\text{bin}(f_{t+1}) = \text{bin}(f_{N+1})$,
iii) f_t is even,
iv) the largest priority c in $c_t, \ldots, c_N$ is $f_{N+1}(\ell)$ and thus even.

Property i) is a straightforward application of Lemma 4: in case $\ell \notin \text{dom}(f_t)$ then $f_t \to_{c_t} f_{t+1}$ is an increment and in case $\ell \in \text{dom}(f_t)$ then $\text{bin}(f_{t+1}) \leq \text{bin}(f_t)$.

An insertion of index $> \ell$ removes ℓ from the domain. According to the insertion property and by maximality of t, there is not such insertion after date t thus $\text{dom}(f_{t+1}) \cap \ell+1 \ldots N = \text{dom}(f_{N+1}) \cap \ell + 1 \ldots N$. Moreover $\ell = \min \text{dom}(f_{t+1}) = \min \text{dom}(f_{N+1})$ thus $\text{dom}(f_{t+1}) = \text{dom}(f_{N+1})$, hence ii). Another consequence is that f_{t+1} and f_{N+1} coincide on $\ell + 1 \ldots k$. Since moreover f_{N+1} is even and $f_{t+1}(\ell) = c_t$ is even then f_{t+1} is even as well. Since the update $f_t \to_{c_t} f_{t+1}$ has type I then f_t is even as well hence iii).

Now we show iv). Observe first that if $f_{t'}(\ell) = c$ for some $t' \in t + 1 \ldots N + 1$, then $f_{N+1}(\ell) = c$ because by maximality of t, all updates at index ℓ after date t have type II thus they cannot remove the maximal priority c. If $c = c_t$, then $f_{t+1}(\ell) = c$ by definition of t and thus $f_{N+1}(\ell) = c$. Otherwise, if $c \neq c_t$ let $t^* \in t + 1, \ldots N$ be the minimal date such that $c_{t^*} = c$. The update t^* is a type II update of index ℓ (according to the insertion property and the maximality of t, $\ell \in \text{dom}(f_{t^*})$ and there is no insertion at index $> \ell$ after date t, and $f_{t^*}(\ell) < c$ by definition of c and t^*). Hence, $f_{t^*+1}(\ell) = c$ and thus $f_{N+1}(\ell) = c$.

Since $t \leq N$ and according to iii) we can apply the induction hypothesis to $f_0 \to_{c_0} f_1 \ldots \to_{c_{t-2}} f_t$ and get an even factorization $0 \leq t_0 < t_1 < \ldots < t_n = t$ of length $\geq \text{bin}(f_t)$. According to iv), $0 \leq t_0 < t_1 < \ldots < t_n = t < N + 1$ is an even factorization as well, of length $\geq 1 + \text{bin}(f_t)$. According to i) and ii), $1 + \text{bin}(f_t) \geq \text{bin}(f_{t+1}) = \text{bin}(f_{N+1})$ thus the proof of the induction step is over. ◀

References

1 C. S. Calude, S. Jain, B. Khoussainov, W. Li, and F. Stephan. Deciding parity games in quasipolynomial time. Technical report, CDMTCS, October 2016. URL: https://www.cs.auckland.ac.nz/research/groups/CDMTCS/researchreports/index.php?download&paper_

2 E. Zermelo. Uber eine anwendung der mengenlehre auf die theorie des schachspiels. In *Proc. of the Fifth International Congress of Mathematicians*, volume II, pages 501–504. Cambridge University Press, 1913.

3 Wiesław Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *TCS*, 200:135–183, 1998.