The correspondence between tropical convexity and mean payoff games

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***Abstract*— We show that several decision problems originating from max-plus or tropical convexity are equivalent to zero-sum, two player game problems. In particular, we set up an equivalence between the external representation of tropical convex sets and zero-sum stochastic games, in which tropical polyhedra correspond to deterministic games with finite action spaces. Then, we show that the winning initial positions can be determined from the associated tropical polyhedron. We obtain as a corollary a game theoretical proof of the fact that the tropical rank of a matrix, defined as the maximal size of a submatrix for which the optimal assignment problem has a unique solution, coincides with the maximal number of rows (or columns) of the matrix which are linearly independent in the tropical sense. Our proofs rely on techniques from non-linear Perron-Frobenius theory.**

I. INTRODUCTION

A. Statement of the problems and main results

The three following problems are basic in max-plus or tropical algebra.

Problem 1.1 (Is a tropical polyhedral cone non-trivial?): Given $m\times n$ matrices $A=(A_{ij})$ and $B=(B_{ij})$ with entries in $\mathbb{R}\cup\{-\infty\}$, does there exist a vector $x\in(\mathbb{R}\cup\{-\infty\})^n$ non-identicaly $-\infty$ such that the inequality "$Ax\leq Bx$" holds in the tropical sense, i.e.,

$$\max_{j\in[n]}\big(A_{ij}+x_j\big)\leq\max_{j\in[n]}\big(B_{ij}+x_j\big),\qquad\forall i\in[m]\ \ ?\qquad(1)$$

Here and in the sequel, we use the notation $[n]:=\{1,\dots,n\}$.

Problem 1.2 (Is a tropical polyhedron empty?): Given $m\times n$ matrices $A=(A_{ij})$ and $B=(B_{ij})$ with entries in $\mathbb{R}\cup\{-\infty\}$, and two vectors c,d of dimension n with entries in $\mathbb{R}\cup\{-\infty\}$, does there exist a vector $x\in(\mathbb{R}\cup\{-\infty\})^n$ such that the inequality "$Ax+c\leq Bx+d$" holds in the tropical sense, i.e., for all $i\in[m]$

$$\max\big(\max_{j\in[n]}(A_{ij}+x_j),c_i\big)\leq\max\big(\max_{j\in[n]}(B_{ij}+x_j),d_i\big)\ \ ?\quad(2)$$

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Problem 1.3 (Is a family of vectors tropically dependent?): Given $m\geq n$ and an $m\times n$ matrix $A=(A_{ij})$ with entries in $\mathbb{R}\cup\{-\infty\}$, are the columns of A tropically linearly dependent? I.e., can we find scalars $x_1,\dots,x_n\in\mathbb{R}\cup\{-\infty\}$, not all equal to $-\infty$, such that the equation "$Ax=0$" holds in the tropical sense, meaning that for every value of $i\in[m]$, when evaluating the expression

$$\max_{j\in[n]}(A_{ij}+x_j)$$

the maximum is attained by at least two values of j?

The representation of a tropical polyhedral cone by inequalities turns out to be equivalent to the description of a mean payoff game by a bipartite directed graph in which the weights indicate the payments (the weighted graph is coded by the matrices A and B). More generally, we consider an infinite system of inequalities, the set $[m]$ being replaced by an infinite set in (1). The set P of solutions of this system is a now tropical convex cone (not necessarily polyhedral), and we associate to it a mean payoff game with infinite action spaces.

Our main results set up a correspondence between the external representation (by inequalities) of a tropical convex cone P, and mean payoff games, in which

$\exists u\in P,\ u\not\equiv-\infty\Leftrightarrow$ there is at least one winning initial state

and when P is polyhedral,

$$\exists u\in P,\ u_i\neq-\infty\Leftrightarrow\quad i\text{ is a winning initial state,}$$

see Theorems 3.1 and 3.2. This shows that Problem 1.1 and its affine version, Problem 1.2, are equivalent to mean payoff game problems. We show by the same techniques that Problem 1.3 reduces to a mean payoff game problem, and derive theoretical results concerning tropical linear dependence by game techniques.

B. Motivation

The first two problems concern max-plus or tropical *convex sets*. The latter are subsets C of $(\mathbb{R}\cup\{-\infty\})^n$ such that for $u,v\in C,\ \lambda,\mu\in\mathbb{R}\cup\{-\infty\}$, the equality $\max(\lambda,\mu)=0$ implies that $(\lambda+u)\vee(\mu+v)\in C$ where "$\vee$" is the supremum operator for the partial order of $\mathbb{R}\cup\{-\infty\}$, that is the "max" applied entrywise, and where $\lambda+u$ denotes the vector obtained by adding the scalar λ to every entry of u.

Max-plus or tropical convexity has been developed by several researchers under different names. It goes back at least to the work of Zimmermann [47]. It was studied by Litvinov, Maslov, and Shpiz [38], in relation to problems of calculus of variations, and by Cohen, Gaubert, and Quadrat [10], [11],

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motivated by discrete event system problems [13]. Max-plus polyhedra have also appeared in tropical geometry, after the work of Develin and Sturmfels [16], followed by several works including the ones of Joswig and Yu, see [33], [34].

As it is shown in [23] (see also [25], [24]) max-plus polyhedra can be defined equivalently in terms of generators (extreme points and rays) or relations (linear or affine inequalities). In particular, a max-plus polyhedral cone can be defined by systems of the form "$Ax \leq Bx$", whereas max-plus polyhedra can be defined by their affine analogues, "$Ax + c \leq Bx + d$". Max-plus polyhedra have been used in particular in [36], [18] to solve controllability and observability problems for discrete event systems, and they have been used in [3] as a new domain in static analysis by abstract interpretation, allowing one to express disjunctive constraints. The question of solving "$Ax \leq Bx$" over (finite) relative integers has also been considered in [5] with motivations from SMT (SAT-modulo theory) solving. In this context, Problems 1.1 and 1.2 are the most basic ones: checking triviality or emptyness.

The third problem, concerning linear dependence, is motivated by tropical geometry. In this setting, the *tropical hyperplane* [44] determined by a vector $u \in (\mathbb{R} \cup \{-\infty\})^n$ is defined as the set of points $x \in (\mathbb{R} \cup \{-\infty\})^n$ such that the maximum in the expression $\max_{i \in [n]}(u_i + x_i)$ is attained at least twice. This arises naturally when considering amoebas, which are the images of algebraic varieties over a valued field by the map which takes the valuation entrywise, in particular, tropical hyperplanes turn out to be amoebas of classical hyperplanes when the valuation is non archimedean (this is a special case of Kapranov's theorem, see [20]).

C. Discussion of the result

One interest of the transformation to mean payoff games that we describe here is of an algorithmic nature. Mean payoff games have been well studied, since the work of Gurvich, Karzanov, and Khachiyan [29]. Since that time, the existence of a polynomial time algorithm has been an open problem (results of Condon [14] and Zwick and Paterson [48] show that these problems are in NP $\cap$ CO-NP). Pseudo-polynomial algorithms of value iteration type are known [48], and other types algorithms have been developed [9], [17], [6], [35]. These include policy iteration algorithms, which are experimentally fast on typical inputs, although a comonly used class of policy improvement rules was recently shown to lead to a worst case exponential execution time [21]. Hence, the present transformations allow one to apply any of these algorithms to solve Problems 1.1–1.3.

If one requires the vector x to be finite, Problem 1.1 becomes simpler. In this special case, a reduction which inspired the present one was made by Dhingra and Gaubert, who showed in [17, § IV,C] (Corollary 3.3 below) that "$Ax \leq Bx$" has a finite solution if and only if all the initial states of an associated mean payoff game are winning. A related result was established previously by Mohring, Skutella and Stork [39], who studied a scheduling problem with and/or precedence constraints, leading to a feasibility problem which is equivalent to finding a finite vector in a tropical polyhedron. They showed that the latter problem is polynomial time equivalent to deciding whether a mean payoff game has a winning state. In [39] as well as in [17] and the present work, a mean payoff game is canonically associated (by a syntaxic construction) to the feasibility problem. Then, the approach of [39] requires an additional transformation, adding some auxiliary states, with special weights (determined by a value iteration argument), and the proof is combinatorial.

The present work relies on a different approach (based on non-linear Perron-Frobenius theory). It allows us to deal with infinite coordinates (i.e., tropically zero coordinates). This extension is an essential matter, both in applications and for theoretical reasons, and it leads to simpler results (even in the case of finite coordinate), since there is no need to transform the game as in [39]. In particular, our first result shows that the system "$Ax \leq Bx$" has a tropically nonzero solution (possibly with infinite entries) if and only if the associated game has at least one winnning initial state. In case of Problem 1.1, our proof relies on a non-linear extension due to Nussbaum [41] of the Collatz-Wielandt characterization of the spectral radius of a matrix with nonnegative entries and the latter allows us to establish more generally a correspondence between the external representation of closed tropical (not necessarily polyhedral) convex cones and a class of mean payoff games (possibly with infinite action spaces on one side). Our approach to Problem 1.2 relies on Kohlberg's theorem and is therefore in the more special setting of polyhedra. Finally, let us note that a tropical analogue of the Farkas lemma has been recently obtained [4], based on the present results.

D. A theorem concerning the tropical rank

It is natural to look for characterizations of tropical linear independence in terms of determinants. The tropical analogue of the determinant of an $n \times n$ matrix B (with entries in $\mathbb{R} \cup \{-\infty\}$) is the value of the optimal assignment problem

$$\max_{\sigma} \left(\sum_{i \in [n]} B_{i\sigma(i)} \right) \tag{3}$$

where the maximum is taken over all the permutations σ of the set $[n]$. Following Develin, Santos, and Sturmfels [15], we say that a matrix B with entries in $\mathbb{R}$ is *tropically singular* if the above maximum is attained by at least two permutations. The same notion was first considered by Butkovič in [8], [7] (tropically nonsingular matrices being qualified there as *strong regular* matrices). We shall indeed use the following extension of the above definition to the case of matrices B with entries in $\mathbb{R} \cup \{-\infty\}$: B is *tropically singular* if the above maximum is either attained by at least two permutations, or equal to $-\infty$. As a corollary of our game reduction of Problem 1.3, we obtain the following result (see Theorem 4.9).

Theorem 1.4: Let A be an $m \times n$ matrix with entries in $\mathbb{R} \cup \{-\infty\}$, with $m \geq n$. Then, the columns of A are tropically linearly independent if and only if A has a tropically non-singular $n \times n$ submatrix.

This was first stated by Izhakian, and proved in the square case, in [30]. The proof of the rectangular case given in the present paper, relying on mean payoff games, was announced in [1]. Meanwhile, Izhakian and Rowen completed the proof in the rectangular case [32], using a different approach. In fact, as shown in [1], the "if" part of the result can be deduced from the max-plus Cramer theory [43] (see also [44], [1]), and the square case is related to a result of Gondran and Minoux [26]. It should also be noted that the special case of Theorem 1.4 in which the entries of the matrix A are finite can be derived alternatively from a result of Develin, Santos, and Sturmfels [15, Theorem 5.5], showing that the Kapranov rank of a matrix is maximal if and only if its tropical rank is maximal. However, the present game approach (Theorem 4.7 below), yields a pseudo-polynomial algorithm and implying that the corresponding decision problem (checking whether the tropical rank is maximal) is in NP$\cap$CO-NP.

What is surprising is that Theorem 1.4 still holds in the rectangular case, since the analogue of this result in the "signed" case, in which tropical hyperplanes are replaced by sets of the form

$$H = \{x \in (\mathbb{R} \cup \{-\infty\})^n \mid \max_{i \in I}(u_i + x_i) = \max_{i \in J}(u_i + x_i)\}$$

where I and J are disjoint non-empty subsets of $[n]$, and the definition of tropical singularity is modified accordingly, turns out to be non valid in the rectangular case, as shown by the counter example of [1]. This shows that some tropical linear algebra issues are better behaved when thinking of max-plus numbers as images by the valuation of complex Puiseux series rather than real ones.

II. Preliminary results

A. Mean payoff games arising from tropical cones

We first recall some basic definitions concerning mean payoff games and associate a mean payoff game to a tropical cone. The state space of the game will turn out to be finite precisely when the cone is polyhedral.

The max-plus semiring $\mathbb{R}_{\max}$ is the set of real numbers, completed by $-\infty$, equipped with the addition $(a,b) \mapsto \max(a,b)$ and the multiplication $(a,b) \mapsto a+b$. The name "tropical" will be used in the sequel as a synonym of "max-plus".

The reader is referred to [11], [16] for more background on max-plus or tropical convexity, and in particular to [12], [23] for the results on external representation.

A tropical *closed convex cone* can be defined externally by a system of linear tropical inequalities of the form

$$\max_{j \in [n]}(A_{ij} + x_j) \leq \max_{j \in [n]}(B_{ij} + x_j), \qquad i \in I \qquad (4)$$

Here, I is a possibly infinite set, recall that $[n] := \{1, \dots, n\}$, and A_{ij}, B_{ij} belong to $\mathbb{R}_{\max}$. When the previous system consists of finitely many inequalities, i.e., when $I = [m]$ for some integer m, we obtain a tropical *polyhedral cone*. Then, A and B will be thought of as $m \times n$ matrices with entries in $\mathbb{R}_{\max}$. In the sequel, we shall denote by $\mathscr{M}_{m,n}(\mathbb{R}_{\max})$ the set of these matrices.

We look for a non-trivial element of the cone, i.e., for a solution $x = (x_j) \in \mathbb{R}^n_{\max}$ of the above system, not identically $-\infty$. From the algorithmic point of view, the polyhedral case is of primary interest. However, some of our results will turn out to hold as well in the case of infinite systems of inequalities, and their relation with non-linear Perron-Frobenius theory will be more apparent in this wider setting.

To study this satisfiability problem, we define the following zero-sum game, in which there are two players, "Max" and "Min" (the maximizer and the minimizer). The state space consists of the disjoint union of the set I and the set $[n]$. The two players alternate their moves. When the current position is $i \in I$, Player Max must choose the next state $j \in [n]$ in such a way that B_{ij} is finite, and receives B_{ij} from Player Min. If Player Max does not have any available action, i.e., if $B_{ij} = -\infty$ holds for all $j \in [n]$, then Player Max pays an infinite amount to player Min and the game terminates. Similarly, when the current state is $j \in [n]$, Player Min must choose the next state $i \in I$ in such a way that A_{ij} is finite, and receives A_{ij} from Player Max. If $A_{ij} = -\infty$ holds for all $i \in I$, then, player Min pays an infinite amount to player Max and the game terminates.

When $I = [m]$, the game may be represented by a bipartite (di)graph, with two classes of nodes, $[m]$ and $[n]$. The players move alternatively a token on the graph, following the arcs of the graph, which represent the possible moves. The weight of an arc represents the associated payment.

We shall often need the following assumptions, which require every player to have at least one available action in every state.

Assumption 2.1: For all $j \in [n]$, there exists $i \in I$ such that $A_{ij} \neq -\infty$.

Assumption 2.2: For all $i \in I$, there exists $j \in [n]$ such that $B_{ij} \neq -\infty$.

Systems of the form (4) can always be transformed to enforce these assumptions, see [2] for the details.

Given an initial state i and a horizon (number of turns) N, we define v_i^N to be the value of the corresponding finite horizon game for player Max. The existence of the value is immediate when the horizon is finite (but the value may be infinite if the set I is infinite, or if Assumption 2.1 or Assumption 2.2 does not hold).

When both Assumptions 2.1 and 2.2 are fulfilled, we shall also consider the "mean payoff" game, in which the payoff of an infinite trajectory is defined as the average payment per turn received by player Max. Formally, we define this average payment as the limsup as the number N of turns goes to infinity of the payments received plus the opposite of the payments made by Player Max up to turn N divided by N. (When Assumptions 2.1 or 2.2 do not hold, the payments must be counted up to the termination time if the latter occurs before time N.) The value of such games was shown to exist by Ehrenfeucht and Mycielski [19], assuming that the state space is finite. This can also be deduced from a theorem of Kohlberg, see [37].

B. Properties of order preserving and additively homogeneous maps

We now recall some basic properties of the dynamic programming operators arising from the previous games.

We shall think of the collection of rewards $(B_{ij})_{i\in I,\ j\in[n]}$ as a kernel, to which we associate the max-plus linear operator $B:(\mathbb{R}\cup\{-\infty\})^n\to(\mathbb{R}\cup\{-\infty\})^I$,

$$(Bx)_i := \max_{j\in[n]}(B_{ij}+x_j), \qquad \forall i\in I \ .$$

When $I=[m]$, $(B_{ij})_{i\in[m],j\in[n]}$ will be thought of as a matrix in $\mathscr{M}_{m,n}(\mathbb{R}_{\max})$, and Bx is the product in the tropical sense of the matrix B and the vector x.

When Assumption 2.2 holds, this operator sends $\mathbb{R}^n$ to $\mathbb{R}^I$. We define the operator A in the same way. The *residuated operator* $A^\sharp$ from $(\mathbb{R}\cup\{\pm\infty\})^I$ to $(\mathbb{R}\cup\{\pm\infty\})^n$ is defined by

$$(A^\sharp y)_j = \inf_{i\in I}(-A_{ij}+y_i) \ , \tag{5}$$

with the convention $(+\infty)+(-\infty)=+\infty$. This operator sends $(\mathbb{R}\cup\{-\infty\})^I$ to $(\mathbb{R}\cup\{-\infty\})^n$ whenever Assumption 2.1 holds, it sends $\mathbb{R}^I$ to $\mathbb{R}^n$ when in addition I is finite.

The term residuated refers to the property

$$Ax\le y \iff x\le A^\sharp y \ , \tag{6}$$

where $\le$ is the partial order on $(\mathbb{R}\cup\{\pm\infty\})^I$ or $(\mathbb{R}\cup\{\pm\infty\})^n$. Hence, System (4), which can be rewritten as $Ax\le Bx$, is equivalent to $x\le f(x)$ where $f:(\mathbb{R}\cup\{-\infty\})^n\to(\mathbb{R}\cup\{-\infty\})^n$ is defined by

$$f(x) := A^\sharp Bx \ ,$$

denoting by concatenation the composition of operators. The map f sends $(\mathbb{R}\cup\{-\infty\})^n$ to itself whenever Assumption 2.1 holds. It sends $\mathbb{R}^n$ to $\mathbb{R}^n$ when in addition Assumption 2.2 holds and I is finite.

The map f is the dynamic programming operator of the previous game, meaning that the vector $v^N := (v^N_j)_{j\in[n]}$ of values of the game in finite horizon can be computed recursively as follows $v^N = f(v^{N-1})$, $v^0=0$. More generally, setting $v^0 := x$ for some $x\in\mathbb{R}^n$ determines the value function of a variant of the game, in which after the last step, Player Min pays to Player Max a final amount x_j depending on the final state j.

We shall call *min-max functions* the self-maps of $(\mathbb{R}\cup\{-\infty\})^n$ that are of the form $A^\sharp B$, when $A,B\in\mathscr{M}_{m,n}(\mathbb{R}_{\max})$. This terminology goes back to Olsder [42] and Gunawardena [27]. However, unlike in the latter reference, we do not require a min-max function to send $\mathbb{R}^n$ to $\mathbb{R}^n$. This generality will be needed in Section IV-B, in which the games arising from the tropical linear independence problem will turn out to have occasionally empty sets of actions for Player Max.

Any min-max function f from $(\mathbb{R}\cup\{-\infty\})^n$ to itself satisfies the following properties:

1) f is *order-preserving*: $x\le y\Rightarrow f(x)\le f(y) \quad \forall x,y\in(\mathbb{R}\cup\{-\infty\})^n$,
2) f is *additively homogeneous*: $f(\lambda+x)=\lambda+f(x) \quad \forall\lambda\in\mathbb{R}\cup\{-\infty\},\ x\in(\mathbb{R}\cup\{-\infty\})^n$,
3) f is continuous.

Here, $\mathbb{R}\cup\{-\infty\}$ is equipped with the usual topology, defined for instance by the distance $(x,y)\mapsto|\exp(x)-\exp(y)|$, and $(\mathbb{R}\cup\{-\infty\})^n$ is equipped with the product topology.

When an order-preserving and additively homogeneous map f preserves $\mathbb{R}^n$, it is easily seen to be sup-norm *nonexpansive*, meaning that

$$\|f(x)-f(y)\|\le\|x-y\|, \qquad \forall x,y\in\mathbb{R}^n \ ,$$

where $\|x\|=\max_{i\in[n]}|x_i|$. A min-max function that preserves $\mathbb{R}^n$ is piecewise affine (we can cover $\mathbb{R}^n$ by finitely many polyhedra in such a way that the restriction of the function to each polyhedron is affine). It follows that general results, such as Kohlberg's theorem, see [37], are valid for such min-max functions. In the study of mean payoff games, an important issue is to determine the limit

$$\chi(f) := \lim_{N\to\infty} f^N(0)/N = \lim_{N\to\infty} v^N/N \ ,$$

which gives the additive growth rate of the value of the finite horizon game as a function of the horizon N. Kohlberg's theorem implies that the limit $\chi(f)$ does exist.

Corollary 2.3: Assume that every player has at least one available action in every state (Assumptions 2.1-2.2) and that the state space is finite. Then, $\chi(f)=\eta$, where (v,η) is an arbitrary invariant half-line of f.

Remark 2.4: If f is an order-preserving and additively homogeneous map preserving $\mathbb{R}^n$, then, it was observed independently by Gunawardena and Sparrow (see [28]) and by Rubinov and Singer [46] that

$$f(x)=\inf_{y\in\mathbb{R}^n}\left(f(y)+\max_{j\in[n]}(x_j-y_j)\right) \quad \forall x\in\mathbb{R}^n \ .$$

This shows that f can be represented in the form $f=A^\sharp B$.

C. The Collatz-Wielandt property

Some of the main results of this paper rely on a non-linear version of the Collatz-Wielandt characterization of the spectral radius which appears in Perron-Frobenius theory.

Given any self-map f of $(\mathbb{R}\cup\{-\infty\})^n$ that is order-preserving, additively homogeneous, and continuous, we define the *Collatz-Wielandt number* of f to be

$$\mathrm{cw}(f)=\inf\{\mu\in\mathbb{R}\mid\exists w\in\mathbb{R}^n, f(w)\le\mu+w\} \ . \tag{7}$$

A vector $u\not\equiv-\infty$ is a (non-linear) eigenvector of f for the eigenvalue $\lambda\in\mathbb{R}\cup\{-\infty\}$ if

$$f(u)=\lambda+u \ .$$

The (non-linear) *spectral radius* of f is defined as the supremum of its eigenvalues

$$\rho(f)=\sup\left\{\lambda\in\mathbb{R}\cup\{-\infty\}\ \middle|\ \begin{array}{c}\exists u\in(\mathbb{R}\cup\{-\infty\})^n,\\ u\not\equiv-\infty,\ f(u)=\lambda+u\end{array}\right\},$$

and is itself an eigenvalue of f. The following is a dual version of the Collatz-Wielandt number

$$\mathrm{cw}'(f) := \sup\left\{\lambda\in\mathbb{R}\cup\{-\infty\}\ \middle|\ \begin{array}{c}\exists u\in(\mathbb{R}\cup\{-\infty\})^n,\\ u\not\equiv-\infty,\ f(u)\ge\lambda+u\end{array}\right\}.$$

We shall also need a last quantity.

Proposition 2.5: If f is an order-preserving additively homogeneous self-map of $(\mathbb{R}\cup\{-\infty\})^n$, then, for all $x\in\mathbb{R}^n$, the following limit exists and is independent of the choice of x:

$$\bar{\chi}(f) := \lim_{N\to\infty} \max_{j\in[n]} f_j^N(x)/N \ . \tag{8}$$

Of course, when $\chi(f)$ exists, we readily deduce from the definitions that

$$\bar{\chi}(f) = \max_{j\in[n]} \chi_j(f) \ .$$

Lemma 2.6 (Collatz-Wielandt property, see [41], [22]): Let f denote a map from $(\mathbb{R}\cup\{-\infty\})^n$ to itself, that is order-preserving, additively homogeneous, and continuous. Then,

$$\mathrm{cw}'(f) = \rho(f) = \mathrm{cw}(f) = \bar{\chi}(f) \ . \tag{9}$$

Moreover, there is at least one coordinate $j\in[n]$ such that $\chi_j(f) := \lim_{N\to\infty} f_j^N(x)/N$ exists and is equal to $\bar{\chi}(f)$.

D. From spectral theory to mean payoff games

We now interpret the previous results in terms of games. Whereas the "nonexpansive maps" approach of zero-sum games is well known [40], [45], the significance in terms of games of the Collatz-Wielandt property that we show in Proposition 2.7 does not seem to have been noted previously (it shows that the value is always well defined if one player is allowed to select the initial state, without the usual compactness and regularity assumptions).

We call *positional strategy* of Player Max a map $\sigma : I\to[n]$ such that $B_{i\sigma(i)}$ is finite for all $i\in I$ (so σ is a rule, telling to Player Max to move to state $\sigma(i)$ when the current state is i). Similarly, we call *positional strategy* of Player Min a map $\pi : [n]\to I$ such that $A_{\pi(j)j}$ is finite for all $j\in[n]$.

The following proposition, which is a consequence of the Collatz-Wielandt property, shows that $\bar{\chi}(f)$ can be interpreted as the *value* of a variant of the mean payoff game in which the choice of the initial state belongs to Player Max. The lack of symmetry between both players is due to the fact that the set of states in which Max plays, i.e., the set I, can be infinite.

Proposition 2.7: Make Assumptions 2.1, 2.2. Then, Player Max can choose an initial node $j\in[n]$, together with a positional strategy, so that he wins a mean payoff of at least $\bar{\chi}(f)$, whatever strategy Player Min chooses. Moreover, for all $\lambda > \bar{\chi}(f)$, Player Min can choose a positional strategy so that she looses a mean payoff no greater than λ for all initial nodes $j\in[n]$ and for all strategies of Player Max.

III. THE CORRESPONDENCE BETWEEN TROPICAL POLYHEDRA AND MEAN PAYOFF GAMES

A. The reductions

We now come back to our original system of inequalities (4), which we write as $Ax\le Bx$ for brevity. We associated to this system the mean payoff game with dynamic programming operator $f = A^\sharp B$.

Our first result does not require the number of inequalities to be finite.

Theorem 3.1: Under Assumption 2.1, the system of linear tropical inequalities $Ax\le Bx$ has a solution $x\in\mathbb{R}^n_{\max}$ non-identically $-\infty$ if and only if Player Max has a winning state in the mean payoff game with dynamic programming operator $f(x) = A^\sharp Bx$.

Actually we can arrive at the following more precise result when the number of inequalities is finite.

Theorem 3.2: Let Assumptions 2.1 and 2.2 be satisfied, and suppose that the system $Ax\le Bx$ consists of finitely many inequalities ($I=[m]$). Consider the polyhedral cone $P := \{x\in\mathbb{R}^n_{\max};\, Ax\le Bx\}$, and define the support S of P to be the union of the supports of the elements of P:

$$S := \{j\in[n];\, \exists u\in P, u_j\neq-\infty\} \ .$$

Then S is also the maximal support of an element of P, that is there exists $u\in P$ such that $S=\{j\in[n];\, u_j\neq-\infty\}$. Moreover, S coincides with the set of initial states with a nonnegative value for the associated mean payoff game, that is:

$$S = \{j\in[n];\, \chi_j(f)\ge 0\} \ , \tag{10}$$

where $f : (\mathbb{R}\cup\{-\infty\})^n \to (\mathbb{R}\cup\{-\infty\})^n$ is such that $f(x) = A^\sharp Bx$.

The case of a full support in Theorem 3.2 leads to the following result, which was already pointed out by Dhingra and Gaubert in [17].

Corollary 3.3 ([17, §IV, C]): Make Assumptions 2.1 and 2.2, and suppose that the system $Ax\le Bx$ consists of finitely many inequalities. Then, this system has a solution $x\in\mathbb{R}^n$ if and only if all the initial states of the associated game have a nonnegative value, i.e., $\chi(f)\ge 0$.

Rather than a tropical polyhedral cone, we now consider a *tropical polyhedron* P, which is defined by systems of affine tropical inequalities of the form

$$\max(\max_{j\in[n]}(A_{ij}+x_j), c_i) \le \max(\max_{j\in[n]}(B_{ij}+x_j), d_i),\ i\in[m] \tag{11}$$

where the matrices A,B are as above and $c_i, d_i\in\mathbb{R}_{\max}$.

As in the case of classical convexity, polyhedra can be represented by polyhedral cones, the latter being the projective analogues of the former affine objects. So, we construct new matrices $\hat{A}$ and $\hat{B}$ by completing the matrices A and B by an $(n+1)$th column, in such a way that $\hat{A}_{i,n+1} = c_i$ and $\hat{B}_{i,n+1} = d_i$, for all $i\in[m]$.

We now define the map $\hat{f}(y) := \hat{A}^\sharp\hat{B}y$ for all $y\in(\mathbb{R}\cup\{-\infty\})^{n+1}$.

Theorem 3.4: The tropical polyhedron P defined by (11) is nonempty if and only if the value of the mean payoff game with dynamic programming operator $\hat{f}$, starting from the initial state $n+1$, is nonnegative, i.e., $\chi_{n+1}(\hat{f})\ge 0$.

Note that this theorem shows that the emptyness problem for (affine) tropical polyhedra reduces to checking whether a mean payoff game has a specific winning state.

The next theorem yields the converse reduction.

Theorem 3.5: Let $f = A^\sharp B$, with $A, B \in \mathscr{M}_{m,n}(\mathbb{R}_{\max})$, denote the dynamic programming operator of a mean payoff game (thus Assumptions 2.1 and 2.2 are satisfied). Then, for every $r \in [n]$ and $\lambda \in \mathbb{R}$, the inequality $\chi_r(f) \geq \lambda$ holds if and only if the following tropical polyhedron is non-empty: $P_r := \{y \in \mathbb{R}^J_{\max} \mid \lambda + \max(\max_{j\in J}(A_{ij} + y_j), A_{ir}) \leq \max(\max_{j\in J}(B_{ij} + y_j), B_{ir}),\ \forall i \in [m]\}$, where $J := [n] \setminus \{r\}$.

Corollary 3.6: Each of the following problems:

1) Is an (affine) tropical polyhedron empty?
2) Is a prescribed initial state in a mean payoff game winning?

can be transformed in linear time to the other one.

When $f = A^\sharp B$ is the dynamic programming operator of a zero-sum deterministic game, the spectral radius $\rho(f)$ can be computed in pseudo-polynomial time algorithm, by value iteration, along the lines of [48, Theorem 2.3] . See [2] for details.

IV. Mean payoff games expressing tropical linear independence

A. Extension of the tropical semiring and linear independence

In tropical algebra, roots are defined by the requirement that a certain maximum is attained at least twice. Hence, the notation "$* = 0$" is often used informally. This notation can in fact be given a formal meaning, by using an extension of the tropical semiring, which was introduced by Izhakian [30]. The latter may be thought of as the "complex" analogue of the "real" (signed) extension of the tropical semiring introduced by M. Plus [43]. In a nutshell, the "numbers" of the extension of Izhakian carry an information reminding whether the maximum of an expression is attained twice, whereas the "numbers" of the extension of M. Plus carry a sign information, reminding whether the maximum of a signed formal expression is attained by a positive term, by a negative one, or both. The approach of [30] has been pursued in several works of Izhakian and Rowen like [31], whereas the authors have studied in [1] semirings with an abstract involution, in order to unify both approaches. Such extensions provide a convenient notation, and, as shown in [43], [1], they allow one to perform elimination arguments, as in the Gauss algorithm, while staying at the tropical level, and to obtain automatically polynomial identities over semirings.

In the present section we shall establish our results in the framework of the extended tropical semiring.

We referee the reader to the works [1], [2], [30] for the definition and basic properties of the extended semiring $\mathbb{T}_{\rm e}$.

The semiring $\mathbb{T}_{\rm e}$ is not idempotent, but is ordered naturally. The map $\pi : \mathbb{T}_{\rm e} \to \mathbb{R}_{\max}$, $(a,b) \mapsto \pi(a,b) := b$ is a surjective morphism, thus it is order preserving. However the natural injection from $\mathbb{R}_{\max}$ to $\mathbb{T}_{\rm e}$, which sends $b \in \mathbb{R}^*_{\max}$ to $(1,b)$ and $-\infty$ to $(0,-\infty)$ is not a morphism. Nevertheless, it is a multiplicative morphism, it is order preserving and we denote by $b^\vee$ the image of $b \in \mathbb{R}_{\max}$ by this injection.

The following notations are defined in [1] for more general classes of semirings. We recall them here.

Definition 4.1: For any $a \in \mathbb{T}_{\rm e}$, we set $a^\circ := a \oplus a$, and we denote

$\mathbb{T}^\circ_{\rm e} := \{a^\circ; a \in \mathbb{T}_{\rm e}\}$, $\mathbb{T}^\vee_{\rm e} := (\mathbb{T}_{\rm e} \setminus \mathbb{T}^\circ_{\rm e}) \cup \{(0,-\infty)\}$, and we define on $\mathbb{T}_{\rm e}$ the *balance* relation ∇ by $a \nabla b$ iff $a \oplus b \in \mathbb{T}^\circ_{\rm e}$. The balance relation is reflexive, symmetric but not transitive. Denoting $b^\circ := (b^\vee)^\circ$ for $b \in \mathbb{R}_{\max}$, we get that

$$\mathbb{T}^\circ_{\rm e} = \{b^\circ; b \in \mathbb{R}_{\max}\}, \qquad \mathbb{T}^\vee_{\rm e} = \{b^\vee; b \in \mathbb{R}_{\max}\} \ .$$

We shall say that an element of $\mathbb{T}_{\rm e}$ is of type *real* if it belongs to $\mathbb{T}^\vee_{\rm e}$ and of type *ghost* if it belongs to $\mathbb{T}^\circ_{\rm e}$ (thus, the zero element of the semiring has both types). An element a of $\mathbb{T}_{\rm e}$ is determined by its projection $\pi(a) \in \mathbb{R}_{\max}$ and by its type. The elements of $\mathbb{T}^\vee_{\rm e} \setminus \{\mathbb{0}\}$ are precisely the invertible elements of $\mathbb{T}_{\rm e}$.

The previous notations will be extended to vectors and matrices, entrywise. For instance, if $x, y \in \mathbb{T}^n_{\rm e}$, we shall write $x \nabla y$ if $x_j \nabla y_j$ for all $j \in [n]$.

Definition 4.2: If A is a matrix in $\mathscr{M}_{m,n}(\mathbb{T}_{\rm e})$, we shall say that the columns of A are tropically linearly dependent if there exists a vector $x \in (\mathbb{T}^\vee_{\rm e})^n$, different from the zero vector $\mathbb{0}$, such that $Ax \nabla \mathbb{0}$.

Note that this notion naturally extends the notion of tropical linear dependence over $\mathbb{R}_{\max}$ given in the introduction (statement of Problem 1.3).

Tropical linear independence turns out to be controlled by permanents, which are defined in a usual way: Let $A = (A_{ij}) \in \mathscr{M}_{n,n}(\mathbb{T}_{\rm e})$, then the permanent per$A$ of A is the element of $\mathbb{T}_{\rm e}$ defined by $\mathrm{per} A = \bigoplus_{\sigma\in\mathfrak{S}_n} A_{1\sigma(1)} \cdot \cdots \cdot A_{n\sigma(n)}$, where $\mathfrak{S}_n$ denotes the set of all permutations of the set $[n]$. Note that if $A_{ij} = B^\vee_{ij}$ for some $B_{ij} \in \mathbb{R}_{\max}$, then, per$A$ is invertible if and only if B is tropically nonsingular as defined in the introduction (see Section I-D). This suggests the following definition.

Definition 4.3: We shall say that the matrix $A \in \mathscr{M}_{n,n}(\mathbb{T}_{\rm e})$ is *tropically nonsingular* if per$A$ is invertible in $\mathbb{T}_{\rm e}$.

Also if $A_{ij} = B^\vee_{ij}$ for some $B_{ij} \in \mathbb{R}_{\max}$, then the projection onto $\mathbb{R}_{\max}$ of the permanent of A, $\pi(\mathrm{per} A)$ is the value of the optimal assignment problem with weights B_{ij}.

In the sequel, we shall establish results for matrices with entries in the extended tropical semiring $\mathbb{T}_{\rm e}$. Then, we shall derive the analogous results for matrices with entries in the tropical semiring as immediate corollaries.

B. Reducing tropical linear independence to mean payoff games

We denote by A an $m \times n$ matrix with entries in $\mathbb{T}_{\rm e}$, and we shall assume:

Assumption 4.4: The matrix A has no column consisting only of elements of $\mathbb{T}^\circ_{\rm e}$.

This assumption is not restrictive, see [2].

We set

$$E = \{(i,j); A_{ij} \in \mathbb{T}^\vee_{\rm e} \setminus \{\mathbb{0}\}\} \ . \tag{12a}$$

Thanks to Assumption 4.4, for all $j \in [n]$, there is at least one index $i \in [m]$ such that $(i,j) \in E$.

We define the min-max function $f:(\mathbb{R}\cup\{-\infty\})^n \to (\mathbb{R}\cup\{-\infty\})^n$ given by

$$f_j(x) = \min_{i\in[m],\,(i,j)\in E}(-B_{ij} + \max_{k\in[n],\,k\neq j}(B_{ik}+x_k)) \enspace , \qquad (12b)$$

where

$$B_{ij} := \pi A_{ij} \in \mathbb{R}_{\max} \enspace . \qquad (12c)$$

This function can be interpreted as the dynamic programming operator of the following combinatorial game, which is played on a bipartite digraph with n column nodes and m row nodes. When in column node j, player Min chooses a row node i such that $(i,j)\in E$, and moves to node i receiving B_{ij}. Then, player Max must move to some column node k which is different from the previously visited column node j, and he receives B_{ik}. Thus, when all entries of A are in $\mathbb{T}_e^\vee$ (that is $A = B^\vee$), player Min is advantaged, because she can always come back to the state from which player Max just came, ensuring her a 0 loss. In that case, it follows that $\bar{\chi}(f) \leq 0$.

Such a game may be put in the form studied in Section II-A, in which the available actions only depend on the current state, by adding the previously visited node to the state. Formally, the map f may be written as $f(x) = C^\sharp Dx$, where C, D are $(mn)\times n$ matrices, with

$$\begin{aligned} C_{(i,j),k} &= \begin{cases} B_{ij} & \text{if } k=j \text{ and } (i,j)\in E \\ -\infty & \text{otherwise,} \end{cases} \\ D_{(i,j),k} &= \begin{cases} B_{ik} & \text{if } k\neq j \\ -\infty & \text{otherwise.} \end{cases} \end{aligned} \qquad (13)$$

Due to Assumption 4.4, no column of C is identically $-\infty$, hence f sends $(\mathbb{R}\cup\{-\infty\})^n$ to itself. However, some rows of D may be identically $-\infty$, as soon as A has a row with at most one element not equal to $-\infty$. In that case the map f may not send $\mathbb{R}^n$ to itself. But one can apply Proposition 2.5, Lemma 2.6 and Theorem 3.1.

Theorem 4.5: Let A be an $m\times n$ matrix with entries in $\mathbb{T}_e$, satisfying Assumption 4.4. Let E, B and f be defined as in (12). Then a vector $u\in\mathbb{R}^n_{\max}$ is such that $Au^\vee \nabla \mathbb{0}$ if and only if $u \leq f(u)$.

We get as an immediate consequence.

Corollary 4.6: Let B be an $m\times n$ matrix with entries in $\mathbb{R}_{\max}$ which has no column consisting only of elements $-\infty$. Denote $E = \{(i,j); B_{ij}\neq -\infty\}$, and define f by (12b). Let u be a vector in $(\mathbb{R}\cup\{-\infty\})^n$, not identically $-\infty$. Then, the following conditions are equivalent:

1) $u \leq f(u)$;
2) The equation "$Bu = 0$" holds in the tropical sense, meaning that in every expression $\max_{j\in[n]}(B_{ij}+u_j)$, $i\in[m]$ the maximum is attained at least twice or is equal to $-\infty$;
3) All the rows of the matrix B are contained in the tropical hyperplane consisting of those vectors $x\in\mathbb{R}^n_{\max}$ such that the maximum in $\max_{j\in[n]}(x_j+u_j)$ is attained at least twice or is equal to $-\infty$.

The following theorem provides an expression of tropical linear independence in terms of mean payoff games.

Theorem 4.7: Let B be an $m\times n$ matrix with entries in $\mathbb{R}_{\max}$ which has no column consisting only of elements $-\infty$. Denote $E = \{(i,j); B_{ij}\neq-\infty\}$, and define f by (12b). The following assertions are equivalent.

1) The columns of the matrix B are tropically independent;
2) Player Max has no winning state in the mean payoff game with dynamic programming operator f, i.e., $\bar{\chi}(f) < 0$;
3) there exists a vector $w\in\mathbb{R}^n$ and a scalar $\lambda < 0$ such that $f(w) \leq \lambda + w$;
4) there is no vector $u\in(\mathbb{R}\cup\{-\infty\})^n$, non-identically $-\infty$, such that $u\leq f(u)$.

In fact, we shall prove the following more general result, in the setting of the extended tropical semiring,

Theorem 4.8: Let A be an $m\times n$ matrix with entries in $\mathbb{T}_e$, satisfying Assumption 4.4. Let E, B and f be defined as in (12). Then, the columns of the matrix A are tropically linearly independent if and only if the map f satisfies one of the three equivalent conditions (2,3,4) of Theorem 4.7.

C. *Characterizations of the tropical rank*

We shall now derive Theorem 1.4 and related results concerning rank of matrices in the more general framework of matrices with entries in $\mathbb{T}_e$. As a consequence of the game formulation and of the theorem [1, Theorem 6.6], we obtain the following result.

Theorem 4.9: Let $A\in\mathscr{M}_{m,n}(\mathbb{T}_e)$ with $m\geq n$. Then, the columns of A are tropically linearly independent if and only if A has an $n\times n$ submatrix that is tropically nonsingular.

Izhakian and Rowen obtained independently the same result in a recent work [32], by a different method.

Corollary 4.10: Any $m+1$ vectors of $\mathbb{T}_e^m$ are tropically linearly dependent.

Corollary 4.11: Let $A\in\mathscr{M}_{m,n}(\mathbb{T}_e)$. Then, the maximal number of tropically linearly independent rows of A, the maximal number of tropically linearly independent columns of A, and the maximal size of a tropically nonsingular submatrix of A coincide.

We next give several corollaries of these results for $\mathbb{R}_{\max}$. Till the end of this section we shall consider tropical linear dependence and tropical nonsingularity in $\mathbb{R}_{\max}$, i.e., in the sense given in the introduction.

Corollary 4.12: Let $A\in\mathscr{M}_{m,n}(\mathbb{R}_{\max})$ with $m\geq n$. Then, the columns of A are tropically linearly independent if and only if A has an $n\times n$ submatrix, which is tropically nonsingular.

Recall that the *tropical rank* of a matrix $A\in\mathscr{M}_{m,n}(\mathbb{R}_{\max})$ is defined as the maximal size of a tropically non-singular submatrix. In [1], we also defined the *maximal row (resp. column) rank* of a matrix A with entries in $\mathbb{R}_{\max}$ as the maximal number of tropically linearly independent rows (resp. columns) of A. We get as an immediate corollary of Corollary 4.11 the equivalence between all these rank notions.

Corollary 4.13: Let $A \in \mathscr{M}_{m,n}(\mathbb{R}_{\max})$. Then, the maximal row rank of A, the maximal column rank of A and the tropical rank of A coincide.

Corollary 4.14: Checking whether a matrix $A \in \mathscr{M}_{m,n}(\mathbb{R}_{\max})$, with $m \geq n$, has tropical rank at least $n-k$, reduces to solving $\binom{n}{k}$ mean payoff game problems associated to $m \times (n-k)$ matrices, and can therefore be done in pseudo-polynomial time for a fixed value of k.

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