Timed Event Graphs with Multipliers and Homogeneous Min-Plus Systems

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Abstract—We study fluid analogues of a subclass of Petri nets, called Fluid Timed Event Graphs with Multipliers, which are a timed extension of weighted T-Systems studied in the Petri Net literature. These event graphs can be studied naturally, with a new algebra, analogous to the min-plus algebra, but defined on piecewise linear concave increasing functions, endowed with the pointwise minimum as addition, and the composition of functions as multiplication. A subclass of dynamical systems in this algebra, which have a property of homogeneity, can be reduced to standard min-plus linear systems after a change of counting units. We give a necessary and sufficient condition under which a fluid timed event graph with multipliers can be reduced to a fluid timed event graph without multipliers. In the fluid case, this class corresponds to the so-called expansible timed event graphs with multipliers of A. Munier, or to conservative weighted T-systems. The change of variable is called here a potential. Its restriction to the transitions nodes of the event graph is a T-semiflow.

Key words. Timed Petri Nets, Timed Event Graphs, Dynamic Programming, Discrete Event Systems, Max-Plus Algebra, Potentials, Weighted T-Systems

I. INTRODUCTION

An event graph is a Petri net such that each place has only one input arc and one output arc. If the tokens have to stay a minimum amount of time in the places, we speak of Timed Event Graph (TEG). These TEGs are well adapted for modeling synchronizations. In many systems, synchronization is essential. In manufacturing, in order to start a task, a machine and a part must be both ready. In computer science, in order to achieve a computation, we need a processor and an information.

Several units of the same resource may be required to achieve a specific task. Then, the corresponding event graph consumes or produces more than one token in adjacent places, at each transition firing. The corresponding event graph is called a Timed Event Graph with Multipliers (TEGM). To assemble a bicycle, two wheels, a frame and a certain amount of manpower are needed. In a chemical process, a reaction producing a molecule consumes in general more than one atom of a given sort.

Synchronization is not specific to discrete systems, and we will consider here fluid analogues of Timed Event Graphs with Multipliers (FTEGM) in which fluids circulate instead of tokens. For instance, in chemical processes, synchronization (stoichiometry here) is essential and the products used in a chemical reaction may be fluids.

We give some mathematical tools well suited to manipulate FTEGM. In particular, very briefly, we introduce a new kind of power series, extending that considered in [1], which allow us to express the input-output relations of FTEGM (in [7], a systematic classification of all the kinds of power series that may pop up in Petri net modeling is presented). These power series are elements of a new noncommutative min-plus algebra: the set of piecewise linear concave functions, endowed with the pointwise minimum as addition, and the composition of functions as multiplication. This is the mathematical cost to pay for dealing with multipliers. This is a somewhat expensive cost and it is natural to try to find particular cases for which this can be avoided.

In [14], A. Munier introduced and studied an important subclass of TEGM, that we will call conservative, in which the product of multipliers along any circuit is equal to one. The main result of [14] reduces such a TEGM to a conventional TEG after a duplication of transitions. In the context of Petri nets where only the logical aspect is considered, this subclass is known as conservative weighted T-systems [17], [13].

When the multipliers derive from a potential (a vector indexed by both places and transitions), the dynamic of a FTEGM can be reduced to classical min-plus linear recurrent equations by a diagonal change of variables, given by the potential. The existence of a potential is equivalent to the property pointed out by A. Munier. The restriction of a potential to transitions is called a semiflow in the Petri net literature [17]. In the example of the bicycle, counting pairs of wheels instead of wheels is quite natural. It is important to remark that this change of variables, called linearization, is a min-plus algebra nonlinear operation. However with this way of counting the dynamic becomes linear.

As a by-product of the linearization, the existence of an eventual periodic regime is readily obtained, the performance being characterized in terms of invariants of the original net. We also show that linearizable FTEGM are characterized by an input-output homogeneity property which is essentially a conservation law between input and output quantities.

The fluid case, considered here, is much simpler than the discrete case considered by Munier. The linearization procedure does not increase the number of transitions of the system, while the expansion procedure of [14] results in a blow up of the number of transitions.

The fact that a fluid approximation is considered in the case of discrete systems may have an impact on the liveness of the Petri net. For instance, if a single wheel is available, in reality, the production is blocked, but the fluid model gives a production of one half of bicycle. This liveness issue is solved by Munier [14] (see also [4] for nets with a single circuit). In this paper, we show that the discrete and fluid behaviors coincide when the integer marking is a multiple of a certain minimal marking.

Finding good units for counting has nothing to do with liveness but with flows. Fluid analogues are suited to find these units but not to study liveness problems.

The paper is organized as follows. In section II, we recall classical definitions about Petri nets and introduce the subclass of TEGMs with Multipliers. In section III, we introduce an algebra of operators which yields a simple representation of FTEGM. In section IV, we state the main results of the paper, namely, the characterization of linearizable FTEGM, periodic regime, performance (periodic throughput), invariants. In Appendix VI-A, we state and prove an elementary lemma about potentials on graphs which is the algebraic core of the properties presented here. In Appendix VI-B, the main results are proved as mere consequences of this lemma. Some open ends are pointed out in conclusion.

II. RECURRENT EQUATIONS OF TIMED EVENT GRAPHS WITH MULTIPLIERS

We begin by recalling the usual definition of Timed Petri Nets and Event Graphs.
Definition II.1 (TPNM, TEGM, TEG) A Timed Petri Net with Multipliers (TPNM) is a valued bipartite graph given by a 5-tuple $N = (\mathcal{P}, \mathcal{Q}, M, m, \tau)$. 
1. The finite set $\mathcal{P}$ is called the set of places. A place may contain tokens which travel from place to place according to a firing process described later on.
2. The finite set $\mathcal{Q}$ is called the set of transitions. When firing a transition consumes tokens of the upstream places and produces tokens in the downstream places.
3. The set of nodes is $\mathcal{R} \triangleq \mathcal{P} \cup \mathcal{Q}$.
4. The matrix $M \in \mathbb{R}^{\mathcal{Q} \times \mathcal{R}}$ is called incidence matrix. The integer $M_{pj}$ (resp. $M_{pq}$) denotes the number of edges from transition $j$ to place $p$ (resp. from place $p$ to transition $q$). Since the graph is bipartite the blocks $M_{\mathcal{R}}$ and $M_{\mathcal{Q}}$ are zero. We denote by $\mathcal{R}^{\mathcal{R}}$ the set of vertices (places or transitions) downstream a vertex $r$ and $\mathcal{R}^{\mathcal{R}}$ the set of vertices upstream $r$. Formally, $\mathcal{R}^{\mathcal{R}} = \{s \mid M_{sr} \neq 0\}$, $\mathcal{R}^{\mathcal{R}} = \{s \mid M_{rs} \neq 0\}$.
5. The vector $m \in \mathbb{N}^{\mathcal{R}}$ is called initial marking. The integer $m_r$ denotes the number of tokens being initially in place $r$.
6. The vector $\tau \in \mathbb{N}^{\mathcal{R}}$ is called holding time. The integer $\tau_r$ gives the minimal time a token must spend in place $p$ before becoming available for consumption by downstream transitions.
7. A Timed Event Graph with Multipliers (TEGM) is a TPNM such that there is exactly one transition upstream and one transition downstream each place.
8. An (ordinary) Timed Event Graph (TEG) is a TEG with unit multipliers (i.e. $M_{ij} \in \{0, 1\}$).

The (earliest) functioning of a TEG is as follows. A transition $q$ fires as soon as all the places $p$ upstream $q$ contain enough tokens ($M_{pq}$) having spent at least $\tau_p$ units of time in place $p$ (earliest firing rule). When the transition $q$ fires, it consumes $M_{pq}$ tokens in each upstream place $p$ and produces $M_{pq}$ tokens in each downstream place $p'$.

Fig. 1: A timed event graph with multipliers.

Definition II.2: 1. With each place $p$ (resp. transition $q$), a counter variable $Z_p$ (resp. $Z_q$) is associated. It denotes the cumulated number of tokens that have entered place $p$ (resp. number of firings of transition $q$) up to time $t \in \mathbb{Z}$.
2. We call $\mu$ the multiplier matrix, i.e. the $\mathcal{R} \times \mathcal{R}$ matrix with values in $\mathbb{R}^+$ and entries $\mu_{pq} \triangleq M_{pq}$. $\mu_{pq} \triangleq M_{pq}^{-1}$ for $p \in \mathcal{P}$, $q \in \mathcal{Q}$, $M_{pq} \neq 0$, with $\mu_{rr} = 0$ if $M_{rr} = 0$, $r, r' \in \mathcal{R}$.

Assertion II.3: The counter variables of a TEGM\(^2\) (under the earliest firing rule) satisfy the following equations:
\[
Z_q(t) = \min \{\mu_{gp}[t] \leq M_{pq} \} \quad Z_p(t) = \mu_{gp} Z_{p'}(t) ,
\]
with $[x] = \sup \{n \in \mathbb{Z} \mid n \leq x \}$.
Eliminating $Z_p$, we get the transition-to-transition equations:
\[
Z_q(t) = \min \{\mu_{gp} Z_{p'}(t - \tau_p) \} ,
\]
with the notation $\nu_p = \mu_{gp} Z_{p'}(t)$. Dually, place-to-place equations can be obtained.

The behavior of this dynamic is extremely simple in the TEG case. Then, the multipliers are equal to one, and (1) becomes:
\[
Z_q(t) = \min \{Z_p(t - \tau_p) + m_p \} ,
\]
with $Z_p(0) = 0$. This note that the integer part has been dropped since the dynamics (3) obviously preserves integrity. Therefore, fluid TEG need not be distinguished from conventional TEG. The behavior of TEGs is well understood [6], [5], [1]. In particular, strongly connected TEG reach a periodic regime after a finite time, and the corresponding periodic throughput can be easily determined. This leads us to raise the following question: to which the remaining part of the paper is devoted: when does a TEGM reduce to a TEG by a change of variables?

We shall only consider here diagonal changes of variables (change of units): $Z = Z' D' \left(\text{where } D \text{ is a diagonal matrix with positive diagonal entries}\right)$. This restriction can be justified theoretically. We need a change of variables which preserves the min-plus structure of the equations. Thus, we need a matrix $P$ such that the changes of variables $Z = P Z'$ and $Z' = P^{-1} Z$ are order preserving, which is possible if and only if $P$ is a monomial matrix. Indeed the inverse of a nonnegative matrix is nonnegative if and only if it is a monomial matrix (see [3], Lemma 4.3, p.68).

Remark II.4: If $\tau_p = 0$ for some places, (1) becomes an implicit system and we may have difficulties in proving the existence of a finite solution. For the subclass of systems (with the potential property) discussed here, the existence will follow at once from the existing results about TEG. In general, it is not too restrictive to assume that: there are no circuits containing only places with zero holding times. We will call such graphs explicit. For explicit FTEGM, (1) has a unique solution, which can be shown by adapting the argument of [1], Lemma 2.65, p.78.

Definition II.5 (Input-Output partition) We partition the set of transitions $\mathcal{Q} = \mathcal{U}(U \cup \mathcal{Y})$, where $\mathcal{U}$ is the set of transitions with no predecessors (input transitions), $\mathcal{Y}$ is the set of transitions with no successors (output transitions), and $X = \mathcal{Q}(U \cup \mathcal{Y})$ (state transitions). We denote by $u$ (resp. $x, y$) the vector of input (resp. state, output) counters $Z_q$, $q \in \mathcal{U}$ (resp. $X, Y$).

Throughout the paper, we will study the input-output behavior of the system. That is, we will look for the trajectory $(x(t), y(t))$ corresponding to the earliest firing rule (which yields the largest possible values of counters), given an input history $u(\cdot)$. This corresponds to the autonomous regime traditionally considered in the Petri net literature, when the system is frozen at an initial condition $Z_q(t) = u_q(t) \in \mathbb{R}$ for $t < 0$ (usually $u_q(0) = 0$), and then evolves freely depending on the dynamics (1) for $t \geq 0$. This can be obtained as a specialization of the input-output case by adjoining an input transition $q'$ (with an associated empty place) upstream each original transition $q$, setting $u_{q'}(t) = u_q(t)$ for $t < 0$, $u_q(t) = +\infty$ otherwise.

\(^{2}\)Without loss of modeling power, the firing of transitions is supposed to be instantaneous (i.e. it involves no delay in consuming and producing tokens).

\(^{3}\)We consider here only $t \in \mathbb{Z}$ but the result could be generalized to $t \in \mathbb{R}$.

\(^{3}\)We shall only give here the dynamic equations of TEGM. Indeed, the general Petri net equations exhibit a higher order of complexity due to the presence of routing decisions; see [7], [2].
Example II 6: Let us consider the TEGM depicted in Fig. 1. In each place the initial marking is given by the number of tokens (dots), the timing by the number of bars (time units), and the multipliers by the numbers of arcs in parallel. The equations read
\[ x_1(t) = \min \{2 + x_1(t-1), 1 + 2x_2(t-1)\}, \quad y(t) = \lfloor 1 + x_1(t) \rfloor, \]
\[ x_2(t) = \min \{(1/2)x_1(t-1), 1 + x_2(t-3), 1/2 + (3/2)u(t)\}. \]

III. OPERATORIAL REPRESENTATION OF FTEGM

Fluid Timed Event Graphs with Multipliers (FTEGM) are defined as in Def. II.1, but the marking m and the multipliers M take real values, that is m, p ∈ R∪{+∞}, M_{i,j} ∈ R+. Counter functions are subsequently real valued, and the integer parts vanish, that is
\[ Z_{i}(t) = \min \{\nu_1 + \alpha_2 Z_{i+1}(t - t_p)\}. \]

We next introduce the algebraic tools needed to easily handle such dynamics.

1. A semiring is a set, equipped with an addition ⊕, and a product ⊗, such that: ⊕ is associative, commutative, has a zero (denoted ε); ⊗ is associative, has a unit (denoted e); ⊗ distributes over ⊕; zero is absorbing (ε ⊗ a = ε). An idempotent semiring (such that a ⊕ a = a) is called a dioid. A semifield is a semiring whose nonzero elements are invertible. E.g. (R+, ×, +) and (R+ ∪ {+∞}, max, ×) are idempotent semifields.

2. The idempotent semifield Rmin is the set R∪{+∞}, equipped with min as addition and the usual sum as multiplication, i.e. a ⊕ b = min(a, b), a ⊗ b = a + b with ε = +∞ and e = 0.

3. A signal is a map u : Z → R∪{+∞}. We denote by S the set of signals, which has a structure of min-plus semimodule (the analogue of a module but with scalars belonging to a semiring, here Rmin). Counter functions are instances of nondecreasing signals.

4. An operator is a map H : S → S. It is linear if it preserves the min-plus semimodule structure of signals, that is,
\[ H(u ⊕ v) = H(u) ⊕ H(v), \quad H(\lambda u) = \lambda H(u), \]
for all signals u, v and constants λ (we denote by λ ⊗ u the signal t ↦ λ + u(t)). An operator is additive if it only satisfies the relation (5a). The following three families of operators are central in modeling FTEGM.

\[ γ^v : γ^v x(t) = x(t) + ν (\text{shift in counting}) \]
\[ δ^v : δ^v x(t) = x(t - τ) (\text{shift in dating}) \]
\[ μ : μ x(t) = μ x(t) (\text{scaling}) \]

where ν ∈ R∪{+∞}, τ ∈ N, μ ∈ R+. We note that γ and δ are linear, while the operators μ ̸= 1 are only additive. They satisfy
\[ γ^v δ^v = δ^v γ^v, \quad μ δ^v = δ^v μ, \quad μ γ^v = γ^v μ, \]
\[ γ^v ⊕ γ^{v'} = γ^{{min}(v, v')}, \]
\[ γ^v δ^v = γ^{μv + v'}, \quad δ^v δ^{v'} = δ^{μv + μv'}, \quad μ δ^v = μ x(t), \]
\[ μ γ^v = γ^v μ. \]

5. We denote by Amin the (noncommutative) dioid of finite sums of operators μ ⊗ γv equipped with pointwise min and composition. Thus, an element of Amin is a map p = \[p(x) = \min_{1≤k≤l}(γ^v \cdot p_i + \mu_j x), \]
with ε = γ^∞ and e = γ^0.

We denote by Amin[δ] (resp. Amin[δ]) the dioid of power series (resp. polynomials) with coefficients in Amin, with the zero element γ^∞ and the identity element γ^0δ. The subset of Amin[δ], where all multipliers appearing in a power series satisfy μ = 1, is a dioid called Rmin[δ].

7. The min-plus product matrix representation notation to denote the action of matrices of operators on vectors of signals. Given a matrix A with entries in Amin[δ], and a vector of signals u (with compatible dimensions), we set \[ (A u)_i = \sum_j A_{i,j} u_j. \]

The zero (matrix whose entries are identically ε) and identity element (diagonal matrix with ε entries on the diagonal and ε elsewhere) are still denoted ε and e.

We next reproduce the dynamic equations of FTEGM algebraically.

Assertion III.1: The dynamics of a FTEGM can be written
\[ x = Ax ⊕ Bu, \quad y = Cx ⊕ Du, \]
where A, B, C, D are matrices with entries in Amin[δ]. We say that (A, B, C, D) is a representation of the system. Moreover, in the TEG case, the entries of A, B, C, D belong to Rmin[δ].

As an immediate corollary of the representation (8), one obtains the following input-output representation result, taken from [7].

Assertion III.2 (Transfer Representation) For an explicit FTEGM with representation (A, B, C, D), we have
\[ y = Hu, \]
where H = D ⊕ CAν, Aν = ε ⊕ A ⊕ A2 ⊕ · · ·

The Y × U matrix H with entries in Amin[δ] (resp. Rmin[δ]) in the TEG case) is called transfer matrix.

In other words, the input-output behavior of the system is summarized by a matrix of power series. The series obtained as en/tries of transfer matrices of systems of type (S) form the strict subclass of rational series. See [7] for more details.

Let us recall the classical characterization of these (rational) transfer series in terms of paths. The weight of a path π, denoted [π], is equal to the product of the operators of types γ, δ, μ (associated with arcs), taken along the path. We note that [π] ∈ Amin[δ] is indeed a monomial, and we write
\[ [π] = γ^α δ^β \epsilon^μ, \]
where [π]ε ∈ R+ is the multiplicative weight of the path (i.e. the product of the multipliers along the arcs). π, ∈ N is the sum of the holding times of the places of the path, and the "discounted marking" νπ ∈ R is obtained, from the original marking, by applying several times the commutation rules. The set of paths from i to j is denoted π(i,j). The following elementary fact needs no formal proof.

Assertion III.3: For all input and output transitions i ∈ U, j ∈ Y, the transfer series H_{ji} from i to j is equal to the sum of the weights of the paths π_{ji} from i to j:
\[ H_{ji} = \bigoplus_{π_{ji} \in U \xi_i} [π_{ji}] = \bigoplus_{π_{ji} \in U \xi_i} γ^α δ^β \epsilon^μ. \]

Example III.4: The TEGM depicted in Fig. 1 admits the following representation.
\[ A = \begin{pmatrix} γ^δ & γ^2 δ \\ 1/2 δ & γ^3 δ \end{pmatrix}, \]
\[ B = \begin{pmatrix} e \\ (1/2) e \end{pmatrix}, \]
\[ C = \begin{pmatrix} γ & ε \\ 0 \end{pmatrix}, \]
\[ D = \begin{pmatrix} ε \end{pmatrix}. \]

4 See Remark II.4. Without this condition, the infinite sum Aν * need not converge in Amin[δ].
The transfer will be explicitly computed in Ex. IV-5 below.

**Remark IV-1.5 (Dynamic Programming Interpretation)** We note that (4) can be interpreted as the dynamic programming equation of a deterministic Markov decision process with control dependent discount rate \(\alpha\) and cost \(\nu\). The transitions of the event graph are the states. The control chooses between the upwards arcs of the transitions. Then, \(Z\) is the Bellman function of the corresponding dynamic programming equation. In particular, the results given below characterize the subclass of discounted decision problems which reduce to the undiscounted case after a diagonal change of variables. This dynamic programming interpretation is detailed in [7].

**IV. Linearizability and Existence of Potential.**

We next introduce some elementary notions, needed to state the main result.

**Definition IV-1.1:** Let \(A\) denote an \(n \times n\) matrix, with entries in a semifield \(\mathbb{K}\).

1. We call **transition graph** of \(A\), the (directed) graph \(\Gamma(A)\) with nodes \(\{1, \ldots, n\}\), and arcs \(j \rightarrow i\) whenever \(A_{ij} \neq 0\).
2. A path from \(j\) to \(i\) in the graph \(\Gamma(A)\) is denoted \(\pi_{ij}\). Its weight is denoted \([\pi_{ij}]_{\mathbb{K}} \triangleq \prod_{k \in \pi} A_{ik}A_{kj}^{-1}\).
3. The matrix \(A\) is **conservative** if the multiplicative weight of a path depends only on the initial and final nodes of the path; i.e., if for all paths \(\pi_{ij}\) from \(j\) to \(i\) (with \(1 \leq i, j \leq n\)), \([\pi_{ij}]_{\mathbb{K}}\) depends only on \(i\) and \(j\).
4. The matrix \(A\) admits a **potential** if there exists a vector \(v \in \mathbb{K}^n\) (called potential), such that for all \(1 \leq i, j \leq n\) and for all paths \(\pi_{ij}\) from \(j\) to \(i\), \([\pi_{ij}]_{\mathbb{K}} = v_i^{-1}v_j\).
5. A FTEGM is **conservative** (resp. admits the potential \(v\)) if its multiplier matrix \(\mu\) is conservative (resp. admits the potential \(v\)).
6. A FTEGM is **linearizable** if there exists a diagonal change of variables \(Z(t) = v_t \times Z(t)\), with \(v_t \in \mathbb{K}_+ \triangleq \mathbb{K}_+ \setminus \{0\}\), such that \(Z(t)\) satisfies min-plus linear recurrent equations or equivalently and more formally, if three diagonal matrices \(V_s = \text{diag}(v_t, q \in \lambda)\) and similarly \(V_u, V_d\), are such that the entries of \(C' = (V_u)^{-1}CV_d\), \(A' = (V_d)^{-1}AV_u\), \(B' = (V_u)^{-1}BV_d\), \(D' = (V_d)^{-1}DV_u\) belong to \(\mathbb{K}_+[\delta]\).
7. A FTEGM with transfer \(H\) is **homogeneous** if there exist two vectors \(v_u, v_d \in (\mathbb{K}_+^n)^2\) such that

\[\forall \lambda \in \mathbb{K}_+, \ \ H(\lambda v_u + u) = \lambda v_y + H(u),\]

where \(\lambda v + u\) denotes the vector signal \(t \rightarrow \lambda v + u(t)\).
8. A FTEGM is **trim** if every transition is structurally controllable and observable, i.e., if for every transition, there exists a path coming from at least one input, and there exists a path going to at least one output.
9. **Remark IV-1.2:** Clearly not all matrices admit a potential.
10. A FTEGM is linearizable if it reduces to an ordinary FTEG by a change of counting units.
11. The homogeneity property (12) extends the usual linearity relation (5b) which is the specialization of (12) to vectors \(v_u, v_d\) with all entries equal to 1.
12. The additivity axiom (5a) is automatically satisfied for FTEGM, since the equations (8) involve only the shifts and scaling operators (6), which are additive.
13. The main result of this paper is the following characterization, which is proved in Appendix VI-B as a consequence of a more general lemma on potentials of matrices.

**Theorem IV-1.3:** Let \(\mathcal{E}\) denote a FTEGM. The two following assertions are equivalent.
1. \(\mathcal{E}\) is linearizable.
2. \(\mathcal{E}\) has a potential.

Moreover, the above assertions imply that
3. \(\mathcal{E}\) is homogeneous;
4. \(\mathcal{E}\) is conservative.

Conversely, if \(\mathcal{E}\) is homogeneous, trim and explicit, then it is linearizable. If \(\mathcal{E}\) is conservative and strongly connected, then it is linearizable.

From this theorem, the following result is clear.

**Corollary 1:** A FTEGM reduces to a TEG by a change of counting units iff it has a potential.

As it is well known [1], [5], autonomous TEGs reach a periodic regime after a finite time. This property being preserved by a change of counting units, we obtain as an immediate corollary of [1, Th. 3.28] and Theorem IV-3 a periodicity theorem for linearizable FTEGM.

**Corollary 2 (Cyclicity)** The counter functions of an autonomous linearizable FTEGM satisfy the following periodicity property:

\[\exists T_0, c \geq 1, \forall i \in \mathbb{K}_+ \setminus \{0\}, \exists \lambda_i, \forall t \geq T_0, \ Z_i(t + c) = \lambda_i c + Z_i(t) .\]

Moreover, for a strongly connected graph*, the periodic throughput \(\lambda_i\) at node \(i\) is given by the expression

\[v_i^{-1} \lambda_i = \sum_{c \geq 1} v_c^{-1} m_p,\]

for any potential \(v\) that fits this linearizable FTEGM, where the minimum is taken over all the circuits \(\mathcal{C}\) of the graph, and the sums are taken over all the places of the circuit \(\mathcal{C}\).

It is important to note that the terms at the right-hand side of (13) are invariants of the net. Equivalently, for all circuits \(\mathcal{C}\), the vector indexed by \(\mathcal{P}\), with entries \(v^{-1}_p m_p\) for \(p \in \mathcal{P}\), and \(v_p = 0\) otherwise, is a P-semiflow [17].

**Proposition IV-4 (Invariants)** Let \(v\) denote a potential and \(\mathcal{C}\) a circuit of a FTEGM. For all markings \(m\) reachable from the initial marking \(m\), we have

\[\sum_{p \in \mathcal{P}} v_p^{-1} m_p = \sum_{p \in \mathcal{P}} v_p^{-1} m_p .\]

**Proof:** Let \(q\) denote a transition of \(\mathcal{C}\). Let \(q^- = q^{in} \cap q \neq q^{out} \cap q \neq q^{in} \cap q \neq q^{out}\) denote respectively the places of the circuit upstream and downstream. Ignoring the trivial case where \(C\) is a loop, we assume that \(q^- \neq q^+\). After firing transition \(q\), we obtain the new marking \(m_{q^+} = m_{q^-} + \mu_{q^+} \cdot m_{q^-}, m_{q^-} = m_{q^-} - \mu_{q^-}\), the markings of the other places of \(C\) being unaltered. Thus, the sum \(\sum_{p \in \mathcal{P}} v_p^{-1} m_p^+\) increases by \(v_{q^-}^{-1} \mu_{q^-} + v_{q^+}^{-1} \mu_{q^+}\), which is zero precisely because \(v\) is a potential.

An algorithm to check the existence of a potential and to compute it, when it exists, can be easily derived from Remark (VI.2).

**Example IV-5:** We come back to the FTEGM depicted in Fig. 1. The multiplier matrix has the block partition

\[\begin{pmatrix} x_1 & x_2 & u & y \\ p_1 & \cdot & \cdot & 3 \\ p_2 & 1 & \cdot & \cdot \\ p_3 & 2 & \cdot & \cdot \\ p_4 & 1 & \cdot & \cdot \\ p_5 & 1 & \cdot & \cdot \\ p_6 & 1 & \cdot & \cdot \end{pmatrix},\]

\[\mu = \begin{pmatrix} 0 & \mu_{pq} \\ \mu_{qp} & 0 \end{pmatrix}, \quad \mu_{pq} \triangleq \mu_{pq}^{def} \begin{pmatrix} p_1 & 1 & \cdot & \cdot \\ p_2 & 2 & \cdot & \cdot \\ p_3 & 1 & \cdot & \cdot \\ p_4 & 1 & \cdot & \cdot \end{pmatrix} \]
(the zero entries are represented by dots).

Visiting the nodes in the order \( x_1, p_1, x_2, p_2, x_1, p_4, p_5, x_2, p_6, y \) and defining inductively \( v \) by \( v_i = \mu_{ij} v_j \) and \( v_e = 1/3 \) we obtain
\[
\begin{pmatrix}
x_1 & x_2 & u & y & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\
1/2 & 1 & 1/2 & & 1 & 1 & 1 & 1 & 1 & 1 \\
& & & & & & & & & 1
\end{pmatrix}
\]

for a potential possible. The three unvisited arcs \((x_2, p_1), (x_3, p_5), (x_1, p_6)\) give the three following (satisfied) compatibility conditions: \( v_{x_2} = (1/2)v_{p_1}, v_{x_3} = v_{p_5}; v_{x_1} = v_{p_6} \) to existence of a potential. Thus, \( v \) is a potential.

With the new variables \( x'_1 = x_1, x'_2 = 2x_2, u' = 3u, y' = y \), the system admits the following representation
\[
A' = \begin{pmatrix}
\gamma^2 \delta & \gamma & \gamma \\
\gamma & \gamma^2 \delta & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad B' = \begin{pmatrix}
\varepsilon \\
0 \\
0
\end{pmatrix}, \quad C' = \begin{pmatrix}
\gamma & \varepsilon
\end{pmatrix}.
\]

The corresponding linearized TEG is shown in Fig. 2. It is not difficult to compute the transfer series in the new system of coordinates:
\[
H' \overset{\text{def}}{=} C'(A')^*B' = \gamma^3 \delta (\gamma^2 \delta)^*.
\]

In the original system of coordinates:
\[
H = H' \delta = \gamma^3 \delta (\gamma^2 \delta)^* \delta = \sum_{n \in \mathbb{N}} \gamma^{3n} \delta^{1 + 2n} 3.
\]

From (15), we get the explicit input-output relation
\[
y(t) = [H u](t) = \inf_{n \in \mathbb{N}} (3 + n + 3u(t - 1 - 2n)).
\]

Finally, the application of the minimal mean-weight formula (13) gives the following expression of the periodic throughput, e.g. for transition \( x_2 \),
\[
2 \lambda_{x_2} = \min \{2, 2/3, 1/2\} = 1/2,
\]

which means that transition \( x_2 \) is asymptotically fired once every four time units in the autonomous regime.

We conclude by mentioning a case where there is no loss in considering the fluid approximation of a, originally discrete, TEG.

**Proposition IV.6:** If the minimal integer valued potential \( u \) of a TEG, admitting a potential, and the normalized initial marking \( v \) satisfy
\[
\forall q \in \mathbb{Q}, \forall p \in q^{\infty}, \quad \nu_q \in u q N,
\]

then the earliest autonomous behavior of the TEGM coincides with that of its fluid version.

**Proof:** Performing the change of variables \( Z'_i = u_i^{-1} Z_i \) in (2), we obtain
\[
Z'_q(t) = u_q^{-1} \min_{p \in q^{\infty}} [u_q (u_q^{-1} \nu_p + Z'_i (t - \tau_p))].
\]

Assuming by induction that \( Z'_q (t - \tau_p) \) takes integer values, for all \( p \in q^{\infty} \), the assumption that \( u_q^{-1} \nu_p \in N \) allows us to cancel the integer round-up at the right-hand side of (18), yielding
\[
Z'_q(t) = \min_{p \in q^{\infty}} (u_q^{-1} \nu_p + Z'_i (t - \tau_p)),
\]

which shows that \( Z'_q(t) \) is also integer. This shows by induction that \( Z' \) follows the fluid dynamics.

Apart from these exceptional cases, it is not yet clear whether the more general expansion procedure developed by Munier admits a simple operational transcription, in the spirit of the \( \mathcal{A}_{\text{min}}(\delta) \) formalism presented here.

**V. Conclusion**

To conclude, we would like to indicate the limitations of the approach presented here, and point out a few open questions. Theorem IV.3 provides a convenient way to reduce fluid TEGM admitting a potential to fluid TEG. One may therefore ask how coarse the fluid approximation can be. Since the dynamics of a (discrete) TEGM is obtained from its fluid approximation by taking down roundings, it is plain that the counter functions of the associated FTEGM dominate that of the original (discrete) TEGM. The equality case (Proposition IV.6) is exceptional. The discrete behavior may be arbitrarily far from the fluid one. Indeed a FTEGM with positive throughputs may have a deadlocked discrete version. For instance, the TEGM shown in Fig. 1 reaches a deadlock after transition \( x_1 \) is fired (since two tokens would be necessary in place \( p_1 \) to fire \( x_2 \)) while the fluid version is live. Of course, the quality of the approximation increases when the values of the initial marking becomes large.

The results given here for event graphs have been (partly) extended to general Petri nets (see [8]). In this reference one can see numerical experiments showing the quality of the fluid approximation.

Another open direction would be to treat general FTEGM, with no potentials. It is standard Bellman theory that general FTEGM recursions (see Eq. (4)) admit geometric growths or convergences. However, one cannot easily characterize the corresponding rational (transfer) series with the same degree of precision as transfer series of TEG are characterized [1].

Last, the presentation given here is not symmetric in counting and dating. A dual theory obviously exists, if one considers timing transformations of the form \( Z'_q(t) = Z_q \nu_q \times t \), rather than counting transformations \( Z'_q(t) = u_q Z_q(t) \). A more symmetric discussion, along the lines of [1, Chapter 5] in the TEG case will be considered in a forthcoming study.

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[8] All the potentials of a connected FTEGM are proportional. Moreover, the ratios \( u_i/v_i \) are rational. Hence, the existence of a real valued potential guarantees the existence of a minimal integer valued potential.
VI. APPENDIX

In this appendix, we first give a lemma of general interest about matrices, and then we use it to prove Theorem IV.3.

A. Potential Properties of Matrices

We recall that the classes of a matrix $A$ are, by definition, the strongly connected components of the transition graph of $A$, that a class is initial if there exists no other class upstream, and that it is final if there exists no other class downstream. A matrix with a single class is irreducible.

With a $n \times n$ matrix $A$ with entries in a semifield $K$, we associate the symmetrized matrix:

$$(A^{sym})_{ij} = \begin{cases} A_{ij}, & \text{if } A_{ij} \neq 0, \\ A_{ji}^{-1}, & \text{if } A_{ij} = 0 \text{ and } A_{ji} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(20)

Lemma VI.1: Let $K$ denote a semifield. Let $A \in K^{n \times n}$. The following conditions are equivalent:

1. $A^{sym}$ has a potential
2. $A$ has a potential
3. there exists a vector $v$ with non zero entries such that
   $$A_{ij} \neq 0 \Rightarrow v_i = A_{ij}v_j;$$
4. there exists a vector $v$ defined only on the initial and final classes such that for all paths $\pi_{k\ell}$ from a vertex $i$ in an initial class to a vertex $j$ in a final class, $|\pi_{k\ell}|A = v_j(v_i)^{-1};$
5. there exist an (invertible) diagonal matrix $V \in K^{n \times n}$ and a Boolean matrix $B (\in \{0,1\}^{n \times n})$ such that $A = V^{-1}BV^{-1}$. Moreover, if $A$ is irreducible, the above conditions are equivalent to any of the following statements:
6. for all circuits $c$, $|c|A = c$;
7. $A$ is conservative;
8. there exists a collection of paths $\mathbf{V}_{\delta k}$ (from 1 to $k = 2, \ldots, n$) such that
   $$\forall i,j \text{ } A_{ij} \neq 0 \Rightarrow A_{ij} [\mathbf{V}_{\delta k}]_A = |\mathbf{V}_{\delta k}|_A.$$ 

In the irreducible case, these facts are essentially classical. See [11].

Remark VI.2: The equivalence 1$\Leftrightarrow$2 allows us to consider $A^{sym}$ rather than $A$. This is a useful trick, since we may always assume that $A^{sym}$ is irreducible, and use the simpler characterizations of the second part of the lemma. Indeed, in general, $A^{sym}$ is block-diagonal, with irreducible block diagonal blocks. Thus, we have to find a potential for each irreducible block separately. Point 8 shows that it is enough to visit the graph in an arbitrary way starting from an arbitrary node (say 1); if the corresponding paths $\mathbf{V}_{\delta k}$ satisfy (21), we obtain a potential. Conversely, if this procedure fails for a special choice of paths, (9) implies that a potential does not exist.

Remark VI.3: When the semifield is idempotent, the potential $v$ is an eigenvector of the matrix $A^{sym}$ associated with the eigenvalue $c$.

Proof of Lemma VI.1: The following implications are obvious

1$\Rightarrow$7 by (20), $9 \Rightarrow 8$ $\Rightarrow 2 \Rightarrow 3$

The implication 7$\Rightarrow$9 holding only in the irreducible case.

3$\Rightarrow$5. We choose the diagonal matrix $V_{ij} = v_i$. Then $(V^{-1}AV)_{ij} = v_i^{-1}A_{ij}v_j = c$ or $c$, hence $B = V^{-1}AV$ is Boolean.

5$\Rightarrow$2. For all paths $\pi_{ij}$, $|\pi_{ij}|v = v_j^{-1}$. Hence $B = V^{-1}AV$ is Boolean.

4$\Rightarrow$2. For each vertex $i$, we choose an arbitrary vertex $\ell(i)$ in an initial class upstream $i$ and we choose an arbitrary path $\pi'_{\ell(i)}$. We set $u_i = |\pi'_{\ell(i)}|A v_{\ell(i)}$. Let $\pi'_{\ell(i)}$ denote a path from $i$ to a vertex $\ell(i)$ in a final class. The potential property of $v$ restricted to the initial and final classes yields

$v_{\ell(i)} = |\pi'_{\ell(i)}|A v_{\ell(i)} = |\pi'_{\ell(i)}|A v_{i \ell(i)}$, $v(i) = |\pi'_{\ell(i)}|A v_{\ell(i)} = |\pi'_{\ell(i)}|A u_i$, hence, canceling $|\pi'_{\ell(i)}|A$, we get $|\pi_{ij}|A u_i = u_j$. That is, $u$ is a global potential.

B. Proof of Theorem IV.3

i) Potential equivalent to linearizable. Let $v$ denote a potential of the graph. Then, the fluid version of (1) becomes

$$Z(t) = \min_{r \in \mathbb{N}} v_r v_p^{-1} (Z(t) - T_r) + m_p$$

(22)

$$Z(t) = v_i v_p^{-1} Z_{\min}(t)$$

(23)

hence the system becomes min-plus linear after the change of variables $Z_r = v_i^{-1} Z_r$. The converse implication is obtained along the same lines.

ii) Linearizable implies homogeneous. Transforming the linearity property of TEGs (5b) by a diagonal change of variables, we obtain the homogeneity property (12).

iii) Homogeneous trim and explicit implies the potential property. We only consider the single input single output case (the general case being similar). Then, the transfer series defined by (9) is scalar, i.e. $H \in \mathbb{A}_{\min}[\delta]$. Consider the expansion

$$H = \bigoplus_{r \in \mathbb{N}} H_r \delta^r; \text{ } H_r \in \mathbb{A}_{\min}.$$ 

(24)

First, we note that if the system $u \rightarrow y = Hu$ is $(v_u, v_y)$-homogeneous then each map $H_r : \mathbb{R} \rightarrow \mathbb{R}$ is $(v_u, v_y)$-homogeneous. This can be seen by introducing the "Dirac function", $\delta(t) \equiv \delta$ if $t = 0$, $\delta$ otherwise, and noting that for all $z \in \mathbb{R}$, $H(\delta(t)) = H_r(z)$.

Next, we note that the $(v_u, v_y)$-homogeneity of the map $H_r : \mathbb{R} \rightarrow \mathbb{R}$ readily implies that

$$\forall z \in \mathbb{R}, \text{ } H_r(z) = v_r + (v_y v_u)^{-1} z,$$

(25)

with $v_r = H_r(0)$.

Last, the path interpretation of the transfer (11) gives $H_r = \bigoplus_{r \in \mathbb{N}} \gamma^r[v_u]_r$, where the sum is taken over all the paths $\pi$ from the input to the output, with sum of the holding times $\tau$. Note

$$\forall z \in \mathbb{R}, \text{ } H_r(z) = v_r + (v_y v_u)^{-1} z,$$

(25)

with $v_r = H_r(0)$.
that the sum is indeed finite, since we assume that there are no circuits with zero holding times. More explicitly,

$$\forall z \in \mathbb{R}, \quad H_\pi(z) = \min_\pi (\nu_\pi + |r_\pi| z).$$

(26)

A necessary condition for (25) and (26) to coincide is obviously that, \( \forall z, |r_\pi| = v_\pi(v_\pi)^{-1}. \) Since the graph is trim, the input transition is the only initial class and the output transition is the only final class. The conclusion follows from Lemma VI.1, part 4\( \Rightarrow \)2.

ev) **Conservative and strongly connected imply linearizable.** This is the implication 7\( \Rightarrow \)2 of Lemma VI.1. This concludes the proof of Theorem IV.3.

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**References**


