Algebraic System Analysis of Timed Petri Nets

G. Cohen, S. Gaubert and J. P. Quadrat

August 2, 1995

To appear in „Idempotency", J. Gunawardena Ed.

Collection of the Isaac Newton Institute, Cambridge University Press

Abstract

We show that Continuous Timed Petri Nets (CTPN) can be modeled by generalized polynomial recurrent equations in the (min,+) semiring. We establish a correspondence between CTPN and Markov decision processes. We survey the basic system theoretical results available: behavioral (input-output) properties, algebraic representations, asymptotic regime. A particular attention is paid to the subclass of stable systems (with asymptotic linear growth).

1 Introduction

The fact that a subclass of Discrete Event Systems equations write linearly in the (min,+) or in the (max,+) semiring is now almost classical [9, 2]. The (min,+) linearity allows the presence of synchronization and saturation features but unfortunately prohibits the modeling of many interesting phenomena such as "birth" and "death" processes (multiplication of tokens) and concurrency. The purpose of this paper is to show that after some simplifications, these additional features can be represented by polynomial recurrences in the (min,+) semiring.

We introduce a fluid analogue of general Timed Petri Nets (in which the quantities of tokens are real numbers), called Continuous Timed Petri Nets (CTPN). We show that, assuming a stationary routing policy, the counter variables of a CTPN satisfy recurrent equations involving the operators $\min, +, \times$. We interpret CTPN equations as dynamic programming equations of classical Markov Decision Problems: CTPN can be seen as the dedicated hardware executing the value iteration.

We set up a hierarchy of CTPN which mirrors the natural hierarchy of optimization problems (deterministic vs. stochastic, discounted vs. ergodic). For each level and sublevel of this hierarchy, we recall or introduce the required algebraic and analytic tools, we provide input-output characterizations and give asymptotic results.

The paper is organized as follows. In §2, we give the dynamic equations satisfied by general Petri Nets under the earliest firing rule. The counter equations given here are much more tractable than the dater equations obtained previously [1]. Similar equations have been introduced by Baccelli et al. [3] in a stochastic context.

In §3, we introduce the continuous analogue of Timed Petri Nets. We discuss various natural routing policies, and show that they lead to simple recurrent equations.

In §4, we present the first level of the hierarchy: Continuous Timed Event Graphs with Multipliers (CTEGM), characterized by the absence of routing decisions. We single out several interesting subclasses. 1. Ordinary Timed Event Graphs (TEG) are probably the simplest and best understood class of Timed Discrete Event Systems. TEG are exactly causal finite dimensional recurrent linear systems over the (min,+) semiring. They correspond to deterministic decision problems with finite state and additive undiscounted cost. Their asymptotic theory is mere translation of the (min,+) spectral theory. Their input-output relations are inf-convolutions with (min,+) rational sequences. 2. We introduce the subclass of CTEGM *with potential*, which reduce to TEG after a change of units (they are linearized by a non linear change of variable in the (min,+) semiring). The importance and tractability of the (non continuous) version of these systems, called *expansible* [23] was first recognized by Munier. 3. α-discounted TEG are the TEG-analogue of uniformly discounted deterministic optimization problems. They represent systems with constant birth (or death) rate α. 4. We consider general CTEGM. Their input-output relations are affine convolutions (minima of affine functions of the delayed input). The transfer operators are rational series with coefficients in the semiring of piecewise affine concave monotone maps. To CTEGM correspond deterministic decision problems where the actualization rate (and not only the transition cost) is controlled. Last, certain routing policies, called *injective*, reduce CTPN to CTEGM. Related resource optimization problems (optimizing the allocation of the initial marking) are discussed in §4.7.

In §5, we examine the second level of the hierarchy: general CTPN, which correspond to *stochastic* decision problems. Algebraically, CTPN are (min,+) polynomial systems whose outputs admit Volterra series expansions. They are characterized by simple behavioral properties (essentially monotonicity and concavity). We focus on the following tractable subclasses. 1. *Undiscounted TPN* are the Petri Net analogue of stochastic control problems with undiscounted (ergodic) cost. They are characterized by a structural condition (as many input as output arcs at each place) plus a compatibility condition on routings. Undiscounted TPN admit an asymptotically linear growth. The asymptotic behavior can be obtained by transferring the results known for the value iteration: we give a "critical circuit" formula similar to the TEG case (the circuits have to be replaced by recurrent classes of stationary policies). 2. Similar results exist for TPN *with potential* (obtained from undiscounted TPN by diagonal change of variable). 3. CTPN with fixed birth/death rate α correspond to the well studied class of discounted Dynamic Programming recurrences.

2 Recurrent Equations of Timed Petri Nets

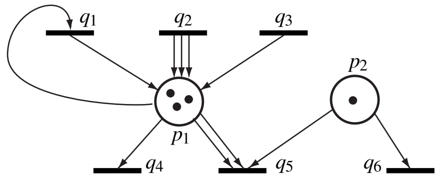

Figure 1: Notation for Petri Nets. $\mathcal{P} = \{p_1, p_2\}$, $\mathcal{Q} = \{q_1, \ldots, q_6\}$, $p_1^{\text{out}} = \{q_1, q_4, q_5\}$, $p_1^{\text{in}} = \{q_1, q_2, q_3\}$, $p_2^{\text{out}} = \{q_5, q_6\}$, $M_{q_5 p_1} = 2$, $M_{p_1 q_2} = 3$, $m_{p_1} = 3$, $m_{p_2} = 1$.

Definition 2.1 (TPNM). A Timed Petri Net with Multipliers (TPNM) is a valued bipartite graph given by a 5-tuple $\mathcal{N} = (\mathcal{P}, \mathcal{Q}, M, m, \tau)$.

1. The finite set $\mathcal{P}$ is called the set of places. *A place may contain tokens which travel from place to place according to a firing process described later on.*
2. The finite set $\mathcal{Q}$ is called the set of transitions. *A transition may fire. When it fires, it consumes and produces tokens.*
3. $M \in \mathbb{N}^{\mathcal{P}\times\mathcal{Q}\cup\mathcal{Q}\times\mathcal{P}}$. *$M_{pq}$ (resp. M_{qp}) gives the number of edges from transition q to place p (resp. from place p to transition q). In particular, the zero value for M corresponds to the absence of edge.*
4. $m \in \mathbb{N}^{\mathcal{P}}$: *$m_p$ denotes the number of tokens being initially in place p (initial marking).*
5. $\tau \in \mathbb{N}^{\mathcal{P}}$: *$\tau_p$ gives the minimal time a token must spend in place p before becoming available for consumption by downstream transitions*[1]. *It will be called* holding time *of the place throughout this paper.*

We denote by r^{out} the set of vertices (places or transitions) downstream a vertex r and r^{in} the set of vertices upstream r. Formally,

$$r^{\text{out}} = \{s \mid \; M_{sr} \neq 0\}, \quad r^{\text{in}} = \{s \mid \; M_{rs} \neq 0\} \enspace .$$

In order to specify a unique behavior of the system, we equip TPN with *routing policies.*

Definition 2.2 (Routing Policy). A routing policy at place p is a family $\{m_{qp}, \Pi^p_{qq'}\}_{q\in p^{\text{out}}, q'\in p^{\text{in}}}$, where,

1. $m_p = \sum_{q\in p^{\text{out}}} m_{qp}$ is an integer partition of the initial marking. *m_{qp} tells the number of tokens of the initial marking reserved for transition q.*

[1] Without loss of modeling power, the firing of transitions is supposed to be instantaneous (i.e. it involves no delay in consuming and producing tokens).

2. $\{\Pi^p_{qq'}\}_{q\in p^{\text{out}}}$ is a partition of the flow from q'. That is, $\Pi^p_{qq'}(n)$ *tells the number of tokens routed from q' to q via p among the first n ones.* More formally, $\Pi^p_{qq'}$ are nondecreasing maps $\mathbb{N}\to\mathbb{N}$ such that $\forall n$, $\sum_{q\in p^{\text{out}}}\Pi^p_{qq'}(n)=n$.

A routing policy for the net is a collection of routing policies for places.

Then, the *earliest behavior* of the system is defined as follows. As soon as a token enters a place, it is *reserved* for the firing of a given downstream transition according to the routing policy. A transition q must *fire* as soon as all the places p upstream q contain enough tokens (M_{qp}) reserved for transition q and having spent at least τ_p units of time in place p (by convention, the tokens of the initial marking are present since time $-\infty$, so that they are immediately available at time 0). When the transition fires, it consumes the corresponding upstream tokens and immediately produces an amount of tokens equal to M_{pq} in each place p downstream q.

We next give the dynamic equations satisfied by the Timed Petri Net. We associate *counter functions* to nodes and arcs of the graph: $Z_p(t)$ denotes the cumulated number of tokens which have entered place p up to time t, *including the initial marking*; $Z_q(t)$ denotes the number of firings of transition q having occurred up to time t; $W_{pq}(t)$ denotes the cumulated number of tokens arrived at place p from transition q up to time t; $W_{qp}(t)$ denotes the cumulated number of tokens arrived at place p up to time t (including the initial marking) reserved for the firing of transition q. We introduce the notation

$$\mu_{pq}\stackrel{\text{def}}{=}M_{pq},\quad \mu_{qp}\stackrel{\text{def}}{=}M_{qp}^{-1}\quad ,$$

and we set $\lfloor x\rfloor=\sup\{n\in\mathbb{Z}\mid\ n\leq x\}$.

Assertion 2.3. *The counter variables of a Timed Petri Net under the earliest firing rule satisfy the following equations*[2]

$$Z_q(t)=\min_{p\in q^{\text{in}}}\lfloor\mu_{qp}W_{qp}(t-\tau_p)\rfloor\quad , \tag{2.1a}$$

$$W_{pq}(t)=\mu_{pq}Z_q(t)\quad , \tag{2.1b}$$

$$Z_p(t)=m_p+\sum_{q\in p^{\text{in}}}W_{pq}(t)\quad , \tag{2.1c}$$

$$W_{qp}=m_{qp}+\sum_{q'\in p^{\text{in}}}\Pi^p_{qq'}(W_{pq'})\quad . \tag{2.1d}$$

We deduce from (2.1) the *transition-to-transition* equation

$$Z_q(t)=\min_{p\in q^{\text{in}}}\Big\lfloor\mu_{qp}\Big(m_{qp}+\sum_{q'\in p^{\text{in}}}\Pi^p_{qq'}\big(\mu_{pq'}Z_{q'}(t-\tau_p)\big)\Big)\Big\rfloor\quad . \tag{2.2}$$

[2] We adopt the convention $\sum_{q\in\emptyset}()=0$, so that (2.1c) becomes $Z_p(t)=m_p$ when $p^{\text{in}}=\emptyset$. The transitions q such that $q^{\text{in}}=\emptyset$ will be considered as *input transitions* whose behavior is given externally. Thus, Eq. (2.1a) should be ignored whenever q has no predecessors.

If $\tau_p = 0$ for some places, this equation becomes implicit and we may have difficulties in proving the existence of a finite solution. We say that the TPN is *explicit* if there is no circuits containing only places with zero holding times. This ensures the uniqueness of the solution of (2.1) and (2.2) under any routing policy Π.

Input-Output Partition We partition the set of transitions $\mathcal{Q} = \mathcal{U} \cup \mathcal{X} \cup \mathcal{Y}$ where $\mathcal{U}$ is the set of transitions with no predecessors (input transitions), $\mathcal{Y}$ is the set of transitions with no successors (output transitions) and $\mathcal{X} = \mathcal{Q} \setminus (\mathcal{U} \cup \mathcal{Y})$. We denote by u (resp. x, y) the vector of input (resp. state, output) counters $Z_q, q \in \mathcal{U}$ (resp. $\mathcal{X}$, $\mathcal{Y}$). Throughout the paper, we will study the *input-output* behavior of the system. That is, we look for the minimal trajectory (x, y) generated by the input history $u(t), t \in \mathbb{Z}$. This encompasses the *autonomous regime* traditionally considered in the Petri Net literature, when the system is frozen at an initial condition $Z_q(t) = v_q \in \mathbb{R}$ for negative t, and evolves freely according to the dynamics (2.1) for $t \geq 0$. This can be obtained as a specialization of the input-output case by adjoining an input transition q' upstream each original transition q, setting $u_{q'}(t) = v_q$ for $t < 0$, $u_{q'}(t) = +\infty$ otherwise.

3 Modeling of Continuous Timed Petri Nets

We shall address the *continuous* version of TPN (in which the number of tokens are real numbers instead of integers). Such continuous models occur naturally when fluids rather than tokens flow in networks (see [2, §1.2.7],[24] for an elementary example). They also arise as approximation of (discrete) Petri Nets since they provide an upper bound for the real behavior.

A continuous TPN (CTPN) is defined as a TPN, but the marking m, the multipliers M and the counter functions are real-valued (the multipliers must be nonnegative: $M_{rs} \in \mathbb{R}^+$). This allows one to define some simple stationary routing policies. We shall single out three classes of policies.

General Stationary Routing A *stationary* routing policy is of the form $\Pi^p_{qq'}(n) = \rho^p_{qq'} \times n$ for some constants $\rho^p_{qq'} \geq 0$ such that for all $q' \in p^{\text{in}}$, $\sum_{q \in p^{\text{out}}} \rho^p_{qq'} = 1$ That is, the flow from q' at place p goes to q with proportion $\rho^p_{qq'}$. The counter functions of a CTPN satisfy the following equations

$$Z_q(t) = \min_{p \in q^{\text{in}}} \mu_{qp} W_{qp}(t - \tau_p) \ , \tag{3.1a}$$

$$W_{qp}(t) = m_{qp} + \sum_{q' \in p^{\text{in}}} \rho^p_{qq'} W_{pq'}(t) \ , \tag{3.1b}$$

together with (2.1c), (2.1b). Eliminating W, we get a transition-to-transition equation

$$Z_q(t) = \min_{p\in q^{\text{in}}} \left(\mu_{qp} m_{qp} + \sum_{q'\in p^{\text{in}}} \mu_{qp}\rho^p_{qq'}\mu_{pq'} Z_{q'}(t-\tau_p) \right) \quad . \tag{3.2}$$

Dually, an equation involving only the variables W_{qp} can be obtained:

$$W_{qp} = m_{qp} + \sum_{q'\in p^{\text{in}}} \min_{p'\in (q')^{\text{in}}} \left(\rho^p_{qq'}\mu_{pq'}\mu_{q'p'} W_{q'p'}(t-\tau_{p'}) \right) \quad . \tag{3.3}$$

The following special cases of stationary routing are worth mentioning.

Origin Independent Routing When the routing at place p does not take into account the origin of the token but only its numbering, we get the condition

$$\forall p, q, \ \ \forall q', q'' \in p^{\text{in}}, \ \ \rho^p_{qq'} = \rho^p_{qq''}, \quad \rho^p_{qq'} m_p = m_{qp} \quad . \tag{3.4}$$

We shorten $\rho^p_{qq'}$ to ρ^p_q. The dynamics of the system (3.1) can be rewritten with the aggregated variables Z_p (instead of W_{qp}):

$$Z_q(t) = \min_{p\in q^{\text{in}}} \mu_{qp}\rho^p_q Z_p(t-\tau_p) \ , \tag{3.5a}$$

$$Z_p(t) = m_p + \sum_{q\in p^{\text{in}}} \mu_{pq} Z_q \quad . \tag{3.5b}$$

Such routing policies depending only on the numbering of tokens (and leading to similar equations) have been studied by Baccelli et al. in a stochastic context [3]. We note that when $\tau_p \equiv 1$, (3.5) reads as the coupling of a conventional linear system with a $(\min, \times)$ linear system, namely[3]

$$Z_{\mathcal{Q}}(t) = \mu'_{\mathcal{QP}} \otimes Z_{\mathcal{P}}(t-1) \ , \tag{3.6}$$

$$Z_{\mathcal{P}}(t) = m + \mu_{\mathcal{PQ}} Z_{\mathcal{Q}}(t) \ , \tag{3.7}$$

where $(A\otimes x)_i = \bigoplus_j A_{ij}\otimes x_j = \min_j A_{ij}x_j$ is the matrix product of the dioid[4] $\mathbb{R}_{\min,\times} \stackrel{\text{def}}{=} (\mathbb{R}^{+*}\cup\{+\infty\}, \min, \times)$.

Example 3.1. The origin independent routing $\rho^{p_5}_{q_3} = \rho^{p_5}_{q_4} = 1/2$ reduces the CTPN in Fig 2a to that of Fig 2d.

[3] We denote by $Z_{\mathcal{Q}}$ (resp. $Z_{\mathcal{P}}$) restriction of Z to transitions (resp. to places). The convention for μ_{pq} is similar. We have set $(\mu'_{\mathcal{QP}})_{qp} = \rho^p_q\mu_{qp}$.

[4] A *dioid* [9, 2] is a semiring whose addition is idempotent: $a\oplus a = a$.

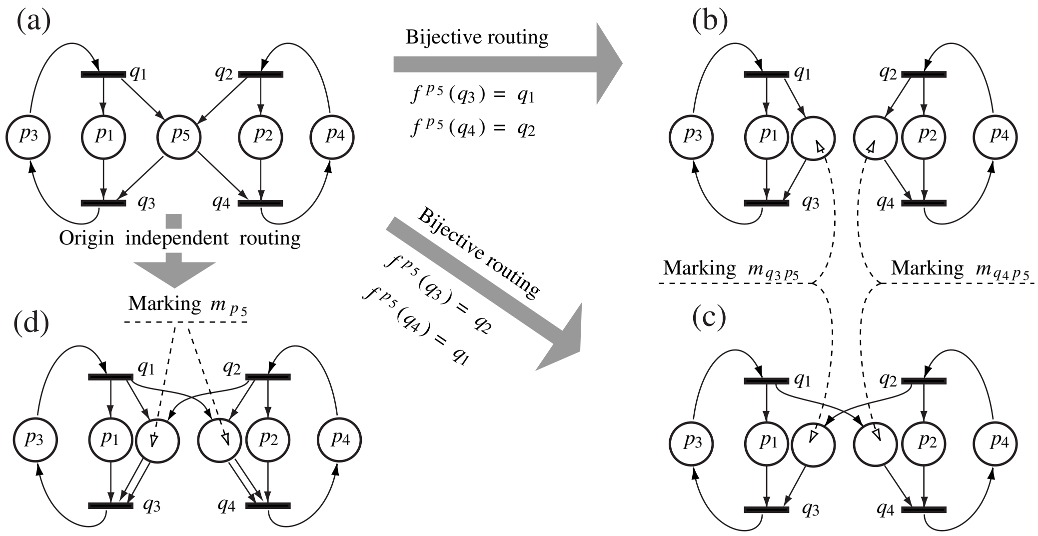

Figure 2: A Balanced Petri Net under various Routing Policies

Injective Routing We say that the routing function ρ^p at place p is *injective* if there is a map $f^p : p^{\text{out}} \to p^{\text{in}}$ such that

$$\forall q,\ \rho^p_{qq'} \neq 0 \Rightarrow q' = f^p(q)\ . \tag{3.8}$$

That is, all the tokens routed to q at place p come for a single transition $f(q)$. Such routings occur frequently when tokens correspond to *resources* (e.g. pallets) which follow some well defined physical routes. An injective routing exists iff[5] $|p^{\text{out}}| \geq |p^{\text{in}}|$. Indeed, the following stronger condition is often satisfied in practice (e.g. in Fig. 2a).

Definition 3.2 (Balanced TPN). A TPN is *balanced* if $\forall p$, $|p|^{\text{out}} = |p|^{\text{in}}$.

In this particular case, we shall speak of *bijective* routing policies (since f^p becomes a bijection $p^{\text{out}} \to p^{\text{in}}$). We shall see later on that injective and bijective routing policies lead to tractable classes of systems.

4 Timed Event Graphs and (min,+) Linear Systems

4.1 Ordinary and Generalized Timed Event Graphs

Definition 4.1 (Timed Event Graphs). A *Continuous Timed Event Graph with Multipliers* (CTEGM) is a CTPN such that there is exactly one transition upstream and one transition downstream each place. An (ordinary) *Continuous Timed Event*

[5]We denote by $|X|$ the cardinal of a set X.

Graph (CTEG) is a CTEGM such that all arcs have multiplier one: $M_{pq}, M_{qp} \in \{0,1\}$. More generally, we define the *place multipliers*[6]

$$\alpha_p \stackrel{\text{def}}{=} \mu_{p^{\text{out}}p}\mu_{pp^{\text{in}}} \quad . \tag{4.1}$$

A *(rate α)*-CTEG is a CTEGM with unit holding times and constant place multipliers. A CTEGM admits a *potential* if there exists a vector $v \in (\mathbb{R}^{+*})^{\mathcal{Q}\cup\mathcal{P}}$ (potential) such that

$$\forall r, s \in \mathcal{Q} \cup \mathcal{P}, \qquad r \in s^{\text{out}} \Rightarrow v_r = \mu_{rs} v_s \quad . \tag{4.2}$$

We set

$$\nu_p \stackrel{\text{def}}{=} \mu_{p^{\text{out}}p} m_p \quad . \tag{4.3}$$

Assertion 4.2. *The dynamics of a CTEGM writes*

$$Z_q(t) = \min_{p \in q^{\text{in}}} \left(\nu_p + \alpha_p Z_{p^{\text{in}}}(t - \tau_p)\right) \quad . \tag{4.4a}$$

We have the following specializations:

$$Z_q(t) = \min_{p \in q^{\text{in}}} \left(\nu_p + Z_{p^{\text{in}}}(t - \tau_p)\right) \qquad \text{(TEG case)}, \tag{4.4b}$$

$$Z_q(t) = \min_{p \in q^{\text{in}}} \left(\nu_p + \alpha Z_{p^{\text{in}}}(t - 1)\right) \qquad \text{(rate } \alpha \text{ case)}, \tag{4.4c}$$

$$Z_q(t) = v_q \min_{p \in q^{\text{in}}} \left(v_p^{-1} m_p + v_{p^{\text{in}}}^{-1} Z_{p^{\text{in}}}(t - \tau_p)\right) \qquad \text{(Potential case)}. \tag{4.4d}$$

The last equation shows that CTEGM with potential reduces to ordinary CTEG after the diagonal change of variable $Z_q = v_q Z'_q$. This change of variables should be interpreted as a *change of units* (v_q firings of transition q being counted as a single one).

Example 4.3. If one mixes white and red paints in equal proportions to produce pink paint, the main concern is to say that with 3 liters of red for a single liter of white, there is 2 liters of red which are useless (that is, the min is the appropriate operator) but then 2 liters of pink can be produced, hence the right thing to do is to count pink paint by *pairs* of liters.

Theorem 4.4. *CTPN under injective routing policies reduce to CTEGM. Balanced CTPN with unit multipliers reduce to (ordinary) TEG.*

Proof. Define the new set of places $\mathcal{P}' = \mathcal{Q} \times \mathcal{P}$, with the incidence relation $q^{\text{in}} = \{(qp) \mid \ p \in p^{\text{in}}\}$, $(qp)^{\text{in}} = f^p(q)$. Then, the dynamics (3.2) reduce to (4.4a), with $\alpha_{qp} = \mu_{qp}\mu_{pf^p(q)}$, The specialization to the TEG case is immediate. □

Example 4.5. The Petri Net of Figure 2a admits two possible bijective routing policies at place p_5 which lead to the two Timed Event Graphs of Fig. 2b and 2c respectively.

[6] Since p^{out} and p^{in} are singletons, the notation will be used to designate their single members.

4.2 Dynamic Programming Interpretation of CTEGM

We exhibit a correspondence between the above classes of Event Graphs and classical deterministic decision problems.

Given a CTEGM, we consider the discrete time controlled process q_n over an horizon t with

1. finite state space $\mathcal{Q}$;
2. set of admissible control histories $\mathcal{P}_{\text{ad}} = \{p_1, \dots , p_t \mid \forall n, p_n \in q_n^{\text{in}}\}$;
3. backward dynamics $q_{n-1} = p_n^{\text{in}}$ where $p_n \in q_n^{\text{in}}$.

In other words, the controlled process follows the edges of the net with the reverse orientation, backward in time. The control at state (transition) q consists in choosing a place p upstream q, which leads to the (unique) transition q' upstream p.

We shall consider the following 3 deterministic cost structures.

Additive

$$J^{\text{add}}(p,t) = Z(0)_{q_0} + \sum_{n=1}^{t} \nu_{p_n} \quad . \tag{4.5}$$

Note that the *initial cost* $Z(0)$ coincides with the initial value of the counter function of the CTEGM.

Additive with Constant Discount Rate

$$J^{\text{disc}}(p,t) = \alpha^t Z(0)_{q_0} + \sum_{n=1}^{t} \alpha^{t-n} \nu_{p_n} \quad . \tag{4.6}$$

Additive with Controlled Discount Rate

$$J^{\text{c-disc}}(p,t) = \left(\prod_{j=1}^{t} \alpha_{p_j} \right) Z(0)_{q_0} + \sum_{n=1}^{t} \left(\prod_{j=n+1}^{t} \alpha_{p_j} \right) \nu_{p_n} \quad . \tag{4.7}$$

The *value function* associated with any of the above cost functions J is the map

$$Z_q(t) = \min_{p \in \mathcal{P}_{\text{ad}},\, q_t = q} J(p,t) \quad .$$

Theorem 4.6. *When $\tau_p \equiv 1$,*

1. *The counter of a CTEG coincides with the value function for the additive cost J^{add}.*
2. *The counter of a (rate α)-CTEG coincides with the value function for the discounted cost J^{disc}.*

3. *The counter of a CTEGM coincides with the value function for the cost with controlled discount rate* $J^{\text{c-disc}}$.

Remark 4.7. Minimizing $J^{\text{c-disc}}$ is known as a problem of shortest path with gains. See [17, Chap. 3, §7] and the references therein.

4.3 Operatorial Representation of CTEGM

We introduce the set of signals $\mathcal{S} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{+\infty\})^{\mathbb{Z}}$ to represent counter functions (although this will be the case in most applications, we do not require the signals to be either positive valued or nondecreasing).

Definition 4.8. An operator $f : \mathcal{S} \to \mathcal{S}$ is

1. *additive* if it satisfies the min–superposition property
$$f(\min(x, x')) = \min(f(x), f(x')) \ ; \tag{4.8}$$
2. *linear* if it is additive and satisfies the homogeneity property
$$f(\lambda + x) = \lambda + f(x) \ .$$

Of course, "linear" refers to the (min,+) dioid $\mathbb{R}_{\text{min}} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{+\infty\}, \min, +)$. Throughout the paper, we shall freely use the dioid notation $a \oplus b$ for $\min(a, b)$, $a \otimes b$ for $a + b$, $\varepsilon = +\infty$ for the zero element, $e = 0$ for the unit.

The following 3 families of operators play a central role in CTEGM:

$$\begin{array}{rlcll} \gamma^\nu : & \gamma^\nu x(t) & \stackrel{\text{def}}{=} & x(t) + \nu & \text{(shift in counting)} \\ \delta^\tau : & \delta^\tau x(t) & \stackrel{\text{def}}{=} & x(t-\tau) & \text{(shift in dating)} \\ \mu : & \mu x(t) & \stackrel{\text{def}}{=} & \mu \times x(t) & \text{(scaling),} \end{array} \tag{4.9}$$

where $\nu \in \mathbb{R}, \tau \in \mathbb{N}, \mu \in \mathbb{R}^{+*}$. We note that γ and δ are linear while μ is only additive. We have the commutation rules:

$$\gamma^\nu \delta^\tau = \delta^\tau \gamma^\nu \ , \tag{4.10a}$$
$$\mu \delta^\tau = \delta^\tau \mu \ , \tag{4.10b}$$
$$\mu \gamma^\nu = \gamma^{\mu\nu} \mu \ . \tag{4.10c}$$

Additive operators equipped with pointwise min and composition form an idempotent semiring, that we denote by $\mathcal{O}$. The following subsemirings of $\mathcal{O}$ are central.

1. The semiring generated by $\gamma^\nu; \nu \in \mathbb{R}$ is isomorphic to $\mathbb{R}_{\text{min}}$ via the identification of ν to γ^ν.
2. The semiring generated by $\gamma^\nu, \delta^\tau; \nu \in \mathbb{R}, \tau \in \mathbb{R}$ is isomorphic to the semiring of polynomials in the indeterminate δ, $\mathbb{R}_{\text{min}}[\delta]$ (via the same identification).

3. The semiring generated by $\gamma^\nu; \nu \in \mathbb{R}^+$ and by the powers of $\alpha\delta$, where α is a given and fixed value of μ, will be denoted by $\mathbb{R}_{\min}[\alpha\delta]$. It is a particular instance of a classical structure in difference algebra: *Ore polynomials*[7] [26, 19, 13].
4. The semiring generated by $\gamma^\nu, \mu; \nu \in \mathbb{R}, \mu \in \mathbb{R}^{+*}$ is isomorphic to the semiring of *nondecreasing concave piecewise affine* maps $\mathbb{R} \cup \{+\infty\} \to \mathbb{R} \cup \{+\infty\}$, that we denote by $\mathcal{A}_{\min}$. A generic element in $\mathcal{A}_{\min}$ is a map $p = \bigoplus_{i=1}^k \mu_i \gamma^{\nu_i}$,

$$p(x) = \min_{1 \le i \le k} (\nu_i + \mu_i x) \enspace .$$

5. Finally, the semiring generated by $\gamma^\nu, \delta^\tau, \mu; \nu \in \mathbb{R}, \tau \in \mathbb{N}, \mu \in \mathbb{R}^{+*}$ is isomorphic to the semiring of polynomials $\mathcal{A}_{\min}[\delta]$.

We extend the operatorial notation to matrices by setting for $A \in \mathcal{O}^{n \times p}$ and $x \in \mathcal{S}^p$,

$$(Ax)_i \stackrel{\text{def}}{=} \min_j A_{ij}(x_j) \enspace . \tag{4.11}$$

Note that for operator matrices A, A', B and vectors of counters x, x' of appropriate sizes

$$(AB)x = A(Bx),\ (A \oplus A')x = Ax \oplus A'x,\ \ A(x \oplus x') = Ax \oplus Ax' \enspace .,$$

More formally, vectors of counter functions are a *left semimodule* under the action of additive matrix operators.

Theorem 4.9. *The counter equations of a CTEGM write*

$$x = Ax \oplus Bu,\ y = Cx \oplus Du \tag{4.12}$$

where A, B, C, D are matrices with entries in $\mathcal{O}$. More precisely,

1. *the entries of A, B, C, D belong to $\mathbb{R}_{\min}[\delta]$ for an ordinary CTEG;*
2. *the entries belong to $\mathbb{R}_{\min}[\alpha\delta]$ for a (rate α)-CTEG;*
3. *the entries belong to $\mathcal{A}_{\min}[\delta]$ for a general CTEGM.*

---

[7] We recall that given a semiring $\mathcal{S}$ equipped with an automorphism $\alpha : \mathcal{S} \to \mathcal{S}$, the semiring of *Ore polynomials* in the indeterminate X, denoted by $\mathcal{S}[X;\alpha]$, is the set of finite formal sums $\sum_n s_n X^n$ (all but a finite number of s_n are zero), equipped with the usual componentwise sum $(s \oplus s')_n \stackrel{\text{def}}{=} s_n \oplus s'_n$ and the skew Cauchy product $(s \otimes s')_n \stackrel{\text{def}}{=} \bigoplus_{p+q=n} s_p \otimes \alpha^p(s_q)$. This product is determined by the rule $Xa = \alpha(a)X$ for all $a \in \mathcal{S}$. Identifying X with $\alpha\delta$ and setting $\alpha(\nu) \stackrel{\text{def}}{=} \alpha \times \nu$ for $\nu \in \mathbb{R}_{\min}$, we see that $X\nu = \alpha(\nu)X$ is nothing but the rule $\alpha\delta\gamma^\nu = \gamma^{\alpha\nu}\alpha\delta$ which follows from (4.10).

Theorem 4.10 (Convolution Representation). *An explicit SISO*[8] *CTEGM admits an input output relation of the form*

$$y(t) = \inf_{\tau\in\mathbb{N}}[h(\tau) + u(t-\tau)] \qquad \text{(Ordinary CTEG)} \tag{4.13}$$

$$y(t) = v_y \inf_{\tau\in\mathbb{N}}[h(\tau) + v_u^{-1} u(t-\tau)] \qquad \text{(CTEG with potential)} \tag{4.14}$$

$$y(t) = \inf_{\tau\in\mathbb{N}}[h(\tau) + \alpha^\tau u(t-\tau)] \qquad \text{(CTEG with rate } \alpha\text{)} \tag{4.15}$$

$$y(t) = \inf_{i\in I}[\nu_i + \mu_i u(t-\tau_i)] \qquad \text{(General Case)} \tag{4.16}$$

where h is a map $\mathbb{N} \to \mathbb{R}\cup\{+\infty\}$, $v_u, v_y \in \mathbb{R}^{+*}$, and where the family $\{\nu_i \in \mathbb{R}, \mu_i \in \mathbb{R}^{+*}, \tau_i \in \mathbb{N}\}$ is such that there is only finitely many i such that $\tau_i = \tau$ for any $\tau \in \mathbb{N}$.

We postpone the proof: these representation results will appear as consequences of the more general behavioral properties of CTEGM operators given in §4.4.

Theorem 4.9 established a connection between various algebras of polynomial type and various classes of Event Graphs. Theorem 4.10 now establishes a similar connection between input-output representations and certain formal series algebras. Let us recall that given a semiring $\mathcal{K}$ and an indeterminate δ, we denote by $\mathcal{K}[[\delta]]$ the semiring of series with coefficients in $\mathcal{K}$ (set of formal sums $\bigoplus_{t\in\mathbb{N}} h_t\delta^t$ with $h_t \in \mathcal{K}$, equipped with pointwise sum and Cauchy product). The generic series of $\mathcal{A}_{\min}[[\delta]]$ writes

$$h = \bigoplus_{\tau\in\mathbb{N}} h_\tau \delta^\tau = \bigoplus_{\tau}\left(\bigoplus_{i\in I_\tau} \mu_{i\tau}\gamma^{\nu_{i\tau}}\right)\delta^\tau$$

where for all τ, I_τ is finite. Such series act naturally on $\mathcal{S}$ by interpreting the indeterminate δ as the shift operator

$$hu(t) = \bigoplus_{\tau\in\mathbb{N}} h_\tau(u(t-\tau)) = \inf_{\tau\in\mathbb{N}} \min_{i\in I_\tau}(\nu_{i\tau} + \mu_{i\tau}u(t-\tau)) \enspace .$$

Theorem 4.10 asserts that (i)- CTEGM operators correspond to the action of $\mathcal{A}_{\min}[[\delta]]$ on counter functions, (ii)- CTEG operators correspond to the action of $\mathbb{R}_{\min}[[\delta]]$, (iii)- α-CTEG operators correspond to the action of the dioid of *Ore series* $\mathbb{R}_{\min}[[\alpha\delta]]$ (defined as Ore polynomials, without the finiteness condition).

4.4 Behavioral Characterizations of CTEGM

Theorem 4.11. *The input-output map $\mathcal{H} : u \to y$ of a SISO explicit CTEGM satisfies the following properties.*

[8] Single Input-Single Output. The extension to the Multiple Inputs Multiple Outputs (MIMO) case is immediate.

1. *Stationarity.* $\mathcal{H}\delta^\tau = \delta^\tau\mathcal{H}$.
2. *Causality.* $u(t) = v(t), \forall t \leq \tau \Rightarrow \forall t \leq \tau,\ \mathcal{H}u(t) = \mathcal{H}v(t)$.
3. *Additivity.* $\mathcal{H}(\min(u, v)) = \min(\mathcal{H}u, \mathcal{H}v)$.
4. *Scott continuity. For any filtered*[9] *family* $\{u_i\}_{i\in I}$, $\mathcal{H}(\inf_{i\in I} u_i) = \inf_{i\in I} \mathcal{H}u_i$.
5. *Concavity.* $\mathcal{H}(\sum_{i=1}^n \lambda_i u_i) \geq \sum_i \lambda_i \mathcal{H}u_i$, $\forall \lambda_i \geq 0, \sum_i \lambda_i = 1$.

A CTEG with rate α satisfies the additional property

6. *α-homogeneity. For all constant λ, $\mathcal{H}(\lambda\alpha^t + u) = \lambda\alpha^t + \mathcal{H}u$, with an obvious convention*[10].

A CTEGM with potential satisfies the alternative additional property[11]

7. *(v_u, v_y)-homogeneity. For all $\lambda \in \mathbb{R}$, $\mathcal{H}(\lambda v_u + u) = \lambda v_y + \mathcal{H}(u)$.*

Note that the specialization of the α-homogeneity to $\alpha = 1$ gives the standard homogeneity property $\lambda + u \to \lambda + y$. So does the specialization of the (v_u, v_y)-homogeneity to the case of constant potential v.

Proof. The additivity of $\mathcal{H}$ is an immediate consequence of the additivity of A, B, C, D and the uniqueness of the solution of $x = Ax \oplus Bu, y = Cx \oplus Du$. The other properties can be proved along the same lines by transferring to $\mathcal{H}$ the properties valid for A, B, C, D. $\square$

The following converse theorem shows that the properties listed are accurate.

Theorem 4.12. *A map $\mathcal{H}$ which satisfies properties 1–5 in Theorem 4.11 is a nonincreasing limit of CTEGM operators*[12]*. An operator which satisfies 1–6 (resp. 1–5,7) is a nonincreasing limit of rate α CTEG operators (resp. with potential v).*

The main point of the proof consists in the following general "convolution" representation lemma for additive continuous stationary operators.

Lemma 4.13. *Let $\mathcal{D}$ denote a complete*[13] *dioid, $\mathcal{H} : \mathcal{D}^{\mathbb{Z}} \to \mathcal{D}^{\mathbb{Z}}$. The following assertions are equivalent. 1. $\mathcal{H}$ is stationary, causal, additive, and Scott continuous;*

[9] A family is filtered if any finite subfamily admits a lower bound in the family. Note that the Scott continuity together with additivity is equivalent to the preservation of arbitrary inf: $\mathcal{H}(\inf_i u_i) = \inf_i \mathcal{H}u_i$ for an arbitrary family. The Scott topology is presented in details in [16]. What we call here Scott continuity is in fact Scott continuity with respect to the algebraic order $\preceq$ of the (min,+) semiring, defined by $a \preceq b \iff a \oplus b = b$ (which is reversed with respect to natural order).

[10] α^t denotes the map $t \mapsto \alpha^t$.

[11] $\lambda + u$ denotes the signal $t \mapsto \lambda + u(t)$.

[12] I.e. there exists a nonincreasing sequence $\mathcal{H}_i \geq \mathcal{H}_{i+1}, i \in \mathbb{N}$ of input-output operators of CTEGM such that $\mathcal{H} = \inf_{i\in I} \mathcal{H}_i$.

[13] A dioid $\mathcal{D}$ is complete if an arbitrary subset admits a least upper bound (for the order $a \preceq b \iff a \oplus b = b$) and if the product is Scott continuous.

2. there exists a family of additive Scott continuous maps $h_\tau, \mathcal{D} \to \mathcal{D}, \tau \in \mathbb{N}$ such that

$$\mathcal{H}u(t) = \bigoplus_{\tau \in \mathbb{N}} h_\tau(u(t-\tau)) \ . \tag{4.17}$$

Proof. Clearly, 2⇒1. Conversely. We introduce the Dirac function

$$\mathsf{e} : \mathbb{Z} \to \mathcal{D}, \quad \mathsf{e}(t) = \begin{cases} e & \text{if } t = 0 \\ \varepsilon & \text{otherwise.} \end{cases}$$

We have the decomposition of an arbitrary signal $u \in \mathcal{D}^{\mathbb{Z}}$ on the basis of shifted Dirac functions:

$$u = \bigoplus_{\tau \in \mathbb{Z}} u(\tau)\delta^\tau \mathsf{e} \ .$$

The additivity, stationarity and Scott continuity assumptions yield

$$\mathcal{H}u = \bigoplus_{\tau \in \mathbb{Z}} \delta^\tau \mathcal{H}(u(\tau)\mathsf{e}) \ . \tag{4.18}$$

Now, let us decompose the output corresponding to $u = x\mathsf{e}$ (with $x \in \mathcal{D}$) on the basis $\{\delta^\tau \mathsf{e}\}_{\tau \in \mathbb{Z}}$:

$$\mathcal{H}(x\mathsf{e}) = \bigoplus_{\tau \in \mathbb{Z}} h_\tau(x)\delta^\tau \mathsf{e} \ .$$

This together with (4.18) gives

$$\mathcal{H}u = \bigoplus_{\tau, \tau' \in \mathbb{Z}} h_\tau(u(\tau))\delta^{\tau+\tau'}\mathsf{e}$$

i.e.

$$\mathcal{H}u(t) = \bigoplus_{\tau \in \mathbb{Z}} h_\tau(u(t-\tau)) \ .$$

The sum can be obviously restricted to $\tau \in \mathbb{N}$ due to causality. The additivity and continuity of h_τ are immediate. □

To complete the proof of Theorem 4.12, it suffices to observe that the additivity, concavity, and potential properties, valid for $\mathcal{H}$, transfer to each h_τ. Then, the concave monotone real valued map h_τ admits a representation as a denumerable infimum of increasing affine functions:

$$h_\tau(x) = \inf_{n \in \mathbb{N}} (\nu_{n\tau} + \mu_{n\tau} x), \quad \text{where } \ \nu_n \in \mathbb{R} \cup \{+\infty\}, \mu_{n\tau} > 0 \ . \tag{4.19}$$

The operator $\mathcal{H}^n = \bigoplus_{\tau \leq n, k \leq n} \gamma^{\nu_{k\tau}} \mu_{k\tau} \delta^\tau$ arises from a CTEGM operator (since it obtained by a finite number of parallel/series composition of elementary γ, μ, δ operators). It follows from (4.17)–(4.19) that $\lim_n \downarrow \mathcal{H}_n u = \mathcal{H}u$. This proves the first assertion of Theorem 4.12. The α-rate and potential special cases are immediate. □

Finally, we note that the construction of the above proof explicitly yields the convolution representations stated in Theorem 4.10, with the exception of the additional finiteness condition that h_τ is a finite sum of $\gamma^{\nu_i}\mu_i$. This last result stems from the *rationality* features that we next introduce.

4.5 Rational Operators

A natural problem is to characterize the subclass of series of $\mathcal{A}_{\min}[[\delta]]$ which arise as transfer operators of CTEGM (called *transfer series*). We recall that given a semiring of formal series $\mathcal{K}[[\delta]]$, the semiring of *rational* series [4] denoted by $\mathcal{K}^{\mathrm{rat}}[[\delta]]$ is the least subsemiring containing polynomials and stable by the operation $\oplus, \otimes, *$, where $a^* \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{N}} a^n$ is defined only on series with zero constant coefficient. An immediate fixed-point argument[14] shows that the input and output counters given by (4.12) satisfy $y = hu$, where $h = CA^*B \oplus D$ is the transfer series of the system. Therefore, rephrasing the Kleene-Schützenberger theorem [4], we claim that transfer series and rational series coincide.

Assertion 4.14. *The transfer series of explicit SISO CTEGM (resp. α-CTEG, CTEG) are precisely the elements of $\mathcal{A}^{\mathrm{rat}}_{\min}[[\delta]]$ (resp. $\mathbb{R}^{\mathrm{rat}}_{\min}[[\alpha\delta]]$, $\mathbb{R}^{\mathrm{rat}}_{\min}[[\delta]]$).*

One important problem is to characterize these particular classes of rational series. The answer is known in the case of $\mathbb{R}_{\min}[[\delta]]$ and $\mathbb{R}_{\min}[[\alpha\delta]]$. We say that a series is *ultimately periodic with rate α* if there exists a constant λ and a positive integer c (cyclicity) such that for t large enough

$$h_{t+c} = \lambda \frac{1-\alpha^c}{1-\alpha} + \alpha^c h_t \ . \tag{4.20}$$

When $\alpha < 1$, this periodicity property means that h_t converges towards $\lambda/(1-\alpha)$ with rate α and that the rate is attained exactly after a finite time. The specialization to $\alpha = 1$ (in fact, $\alpha = 1^-$) yields $h_{t+c} = \lambda c + h_t$. The *merge* of k series $h^{(0)}, \ldots, h^{(k-1)}$ is the series with coefficients $h_{i+nk} = h^{(i)}_n$ for $0 \leq i \leq k-1, n \in \mathbb{N}$.

Theorem 4.15. *A series in $\mathbb{R}_{\min}[[\alpha\delta]]$ is rational iff it is a merge of ultimately α-periodic series.*

[14] The unique solution of $x = Ax \oplus Bu$ is $x = A^*Bu$. The existence of A^* and the uniqueness of the solution follow from the assumption that there are no circuits with zero holding times.

The CTEG case (i.e. $\alpha = 1$) is proved in [9, 2] for the subclass of monotone[15] series $h_{n+1} \geq h_n$. It was already noticed by Moller [22] in the non monotone case. It is essentially known to the tropical community [20]. The α-generalization was announced in [13]. The proof will appear in a paper in preparation [15].

No such simple characterization seems to exist for $\mathcal{A}^{\text{rat}}_{\text{min}}[[\delta]]$: the coefficient h_τ of δ^τ in h is an element of $\mathcal{A}_{\text{min}}$, but its complexity[16] grows in general as $\tau \to \infty$.

4.6 Asymptotic Behavior of CTEGM

We consider the autonomous case $Z = AZ$ with boundary condition $\forall t \leq 0, Z(t) = v \in \mathbb{R}^{\mathcal{Q}}$, where A belongs to one of the above matrix operator algebras. We associate several additive *weights* with the circuit $\mathcal{C} = (q_1, p_1, q_2, \dots, q_k, p_k)$,

$$\begin{array}{lcll} |\mathcal{C}|_\nu & = & \sum_i \nu_{q_i p_i} & \text{Total normalized marking} \\ |\mathcal{C}|_\tau & = & \sum_i \tau_{p_i} & \text{Total holding time} \\ |\mathcal{C}|_l & = & \sum_i 1 = k & \text{Length} \\ |\mathcal{C}|_{m,v} & = & \sum_i m_{p_i} v_{p_i}^{-1} & \text{Total weighted marking} \end{array}$$

where the latest quantity will be used only when the graph admits potential v. The following periodicity theorem is central. The CTEG case is a consequence of the (max,+)-Perron Frobenius theorem [25, 8, 2, 10]. Another proof has been given by Chretienne [7]. The inequality variant below (4.24) can be found in [12, Ch. IV, Lemma 1.3.8],[14]. The α-discounted case is due to Braker and Resing [5, 6].

Theorem 4.16. *Consider a strongly connected CTEG. There exists $N \geq 0$ and $c \geq 1$ (cyclicity) such that, for all initial condition v,*

$$t \geq N \Rightarrow Z(t + c) = \lambda c + Z(t) \ , \tag{4.21}$$

where

$$\lambda = \min_{\mathcal{C}} \frac{|\mathcal{C}|_\nu}{|\mathcal{C}|_\tau} \tag{4.22}$$

(the minimum is taken over the elementary circuits of the graph). Alternatively, λ is the unique scalar for which there exists a finite vector v solution of the spectral problem[17]

$$v_q = \min_{p \in q^{\text{in}}} \left(\nu_{qp} - \lambda \tau_p + v_{p^{\text{in}}} \right) \ , \tag{4.23}$$

[15] The results are stated in the so called $\mathcal{M}^{\text{ax}}_{\text{in}}[[\gamma, \delta]]$ dioid which is isomorphic to the dioid of series in one indeterminate δ with coefficients in $\overline{\mathbb{R}}_{\text{min}} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{\pm\infty\}, \min, +)$ such that $h_{n+1} \geq h_n$.

[16] The minimal number of monomials in a sum $h_\tau = \bigoplus_i \gamma^{\nu_i} \mu_i$.

[17] With the (min,+) notation, when $\tau_p \equiv 1$, (4.23) rewrites as $Av = \lambda \otimes v$ where $A_{qq'} = \bigoplus_{p \in q^{\text{in}} \cap (q')^{\text{out}}} \nu_{qp}$.

or it is the solution of the LP problem

$$\lambda \to \max, \quad \forall p \in q^{\text{in}}, \;\; v_q \le \nu_{qp} - \lambda \tau_p + v_{p^{\text{in}}} \;. \tag{4.24}$$

For a strongly connected CTEG with potential, the periodicity property (4.21) becomes $Z_r(t+c) = \lambda_r c + Z_r(t)$*, where*

$$\lambda_r v_r^{-1} = \min_{\mathcal{C}} \frac{|\mathcal{C}|_{m,v}}{|\mathcal{C}|_\tau} \;. \tag{4.25}$$

For a strongly connected CTEG with rate α*, the periodicity property (4.21) becomes*

$$t \ge N \Rightarrow Z_q(t+c) = \lambda_q \frac{1-\alpha^c}{1-\alpha} + \alpha^c Z_q(t) \tag{4.26}$$

where $\lambda_q \in \mathbb{R}^+$ *(the dependence in* q *is essential).*

The asymptotic behavior of general CTEGM is more subtle. We shall not attempt to treat it here.

Remark 4.17. When $\alpha < 1$, from (4.26) we get $\lim_{t\to\infty} Z_q(t) = \lambda_q/(1-\alpha)$. It is well known that one obtains the average cost value as the limit of the discounted case, i.e. $\forall q, \lim_{\alpha\to 1^-} \lambda_q = \lambda$.

Remark 4.18. When the graph has a potential v, for all circuit $\mathcal{C}$, the quantity $|\mathcal{C}|_{m,v}$ used in the periodic throughput formula is an *invariant* of the net (the firing of one transition leads to a new marking m' with the same weight).

4.7 Resource Optimization Problems

As a by product of the above characterizations of the throughput λ, it is possible to address *resource optimization* problems. The most classical problem [8, 18, 21, 12] relative to TEG consists in optimizing a linear cost function $J(m)$ associated with the initial marking, under the constraint $\lambda \ge \lambda_0$. Physically, the initial marking represents resources (number of machines, pallets, processor, storage capacities), and the problem consists in minimizing the cost of the resources in order to guarantee a given throughput λ_0. By appealing to (4.24), this class of problems reduces to linear programming, with integer and real variables.

We will discuss here new resource optimization problems which arises for more general TPN due to the presence of routing decisions. We restrain to *balanced* TPN with unit multipliers. When a bijective routing f is fixed, the only remaining decision consists in the assignment of the initial marking m_p to the downstream transitions: $m_p = \sum_{q\in p^{\text{out}}} m_{qp}$. We thus consider the problem of finding the allocation of the initial marking which maximizes the performance of the system. We only consider internally *stable* systems in the sense of [2] (such that tokens do not accumulate indefinitely in places). Then, there is a single periodic throughput λ_r

associated with every simply connected component r of the graph (characterized by (4.22)). We denote by $\mathcal{R}$ the set of simply connected components. The most natural performance measure to be optimized will be a linear combination of these throughputs, $c\lambda \stackrel{\text{def}}{=} \sum_{r\in\mathcal{R}} c_r \lambda_r$ where $c_r \geq 0$ are given weights.

Theorem 4.19. *The resource assignment problem for a balanced CTPN with unit multipliers under the bijective policy f reduces to the following Linear Programming problem. Given $\{m_p, \tau_p\}_{p\in\mathcal{P}}$, c and f, denoting by $r(q)$ the simply connected component of transition q under policy f, solve*

$$\max_{v_q,\lambda_r,m_{qp}} c\lambda \ ,$$

$$\begin{cases} m_p &= \sum_{q\in p^{\text{out}}} m_{qp} \ , & \forall p \ , \\ v_q &\leq m_{qp} - \lambda_{r(q)}\tau_p + v_{fp(q)} \ , & \forall q, \forall p \in q^{\text{in}} \ , \end{cases}$$

where $\{v_q\}_{q\in\mathcal{Q}}$, $\{m_{qp}\}_{q\in p^{\text{out}}, p\in\mathcal{P}}$, and $\{\lambda_r\}_{r\in\mathcal{R}}$ are real (finitely) valued variables.

Proof. Easy consequence of the characterization (4.24). □

The same resource assignment problem for discrete (non continuous) TEG leads to a similar LP problem with mixed integer and real variables.

Example 4.20. For the routing policy of Fig. 2b, we obtain two strongly connected components with rates

$$\lambda_1 = \min\left(\frac{m_{q_3p_5} + m_{p_3}}{\tau_{p_5} + \tau_{p_3}}, \kappa_1\right), \quad \text{where } \kappa_1 = \frac{m_{p_1} + m_{p_3}}{\tau_{p_1} + \tau_{p_3}} \tag{4.27}$$

$$\lambda_2 = \min\left(\frac{m_{q_4p_5} + m_{p_4}}{\tau_{p_5} + \tau_{p_4}}, \kappa_2\right), \quad \text{where } \kappa_2 = \frac{m_{p_2} + m_{p_4}}{\tau_{p_2} + \tau_{p_4}} \ . \tag{4.28}$$

Maximizing the throughput in place p_5 reduces to

$$\max_{m_{q_3p_5} + m_{q_4p_5} = m_{p_5}} (\lambda_1 + \lambda_2) \ . \tag{4.29}$$

The bijective policy shown of Fig. 2c gives a unique strongly connected component and a throughput

$$\lambda = \min\left(\kappa_1, \kappa_2, \frac{m_{p_3} + m_{p_4} + m_{p_5}}{\tau_{p_3} + \tau_{p_4} + 2\tau_{p_5}}\right) \tag{4.30}$$

independent of the allocation of m_{p_5}.

5 Time Behavior of Continuous Timed Petri Nets

5.1 Stochastic Control Interpretation

We interpret the evolution equations of a CTPN as the dynamic programming equation of the following stochastic extension of the deterministic decision process described in §4.2. The control at state (transition) q selects an upstream place $p \in q^{\text{in}}$.

Then, q moves randomly (in backward time) to one of the upstream transitions $q' \in p^{\text{in}}$. More precisely,

1. the dynamics is given by a controlled Markov chain in backward time: the probability $P^p_{qq'}$ of the transition $q \to q'$ from time n to time $n-1$ under the decision p is given by
$$P^p_{qq'} = \alpha^{-1}_{qp}\mu_{qp}\rho^p_{qq'}\mu_{pq'}$$
where $\alpha_{qp} > 0$ is a normalization factor[18] (chosen such that $\sum_{q' \in p^{\text{in}}} P^p_{qq'} = 1$).
2. The set $\mathcal{P}_{\text{ad}}$ of admissible control histories is the set of sequences $p_1, \ldots, p_t$ such that $p_n \in q_n^{\text{in}}$ and the decision p_n is a feedback of q_n.
3. We consider a mean cost at state q of the form
$$J(p,t,q) = \mathbb{E}\Big((\prod_{j=1}^{t} \alpha_{q_j p_j}) Z(0)_{q_0} + \sum_{n=1}^{t} (\prod_{j=n+1}^{t} \alpha_{q_j p_j}) \nu_{q_n p_n} \Big| q_t = q \Big) \enspace .$$

Assertion 5.1. *For a CTPN such that $\tau_p \equiv 1$, the counter function coincides with the value function:*
$$Z_q(t) = \inf_{p \in \mathcal{P}_{\text{ad}}} J(p,t,q) \enspace . \tag{5.1}$$

As in the case of Event Graphs, we shall pay a particular attention to simple cost functions.

Definition 5.2. A CTPN is *undiscounted* if $\alpha_{qp} \equiv 1$. It is *α-discounted* if $\tau_p \equiv 1$ and $\alpha_{qp} \equiv \alpha$. It admits a *potential* if there exists a vector $v \in (\mathbb{R}^{+*})^{\mathcal{Q}}$ such that the change of variable $Z_q = v_q Z'_q$ makes the CTPN undiscounted.

Clearly, the cost function of an undiscounted (resp. α-discounted) CTPN writes
$$J(p,t,q) = \mathbb{E}\Big(Z(0)_{q_0} + \sum_{n=1}^{t} \nu_{q_n p_n} \Big| q_t = q \Big) \enspace , \tag{5.2}$$

$$\text{resp.} \enspace J(p,t,q) = \mathbb{E}\Big(\alpha^t Z(0)_{q_0} + \sum_{n=1}^{t} \alpha^{t-n} \nu_{q_n p_n} \Big| q_t = q \Big) \enspace . \tag{5.3}$$

Theorem 5.3. *1. A CTPN becomes undiscounted under a stationary routing iff it satisfies the following equilibrium condition:*
$$\forall p, \sum_{q \in p^{\text{out}}} M_{qp} = \sum_{q \in p^{\text{in}}} M_{pq} \enspace . \tag{5.4}$$

[18]Note that in the CTEGM case, for $q = p^{\text{out}}$, we have $\alpha_{qp} = \mu_{p^{\text{out}}p}\mu_{pp^{\text{in}}}$ so that α_{qp} coincides with α_p as defined in (4.1).

Then, the only origin independent routing policy which makes the net undiscounted is given by[19]*:*

$$\forall q' \in p^{\text{in}}, \qquad \rho^p_{qq'} = \frac{M_{qp}}{\sum_{q'' \in p^{\text{out}}} M_{q''p}} \quad . \tag{5.5}$$

2. A CTPN with $\tau_p \equiv 1$ becomes α-discounted under a stationary routing iff

$$\forall p, \quad \sum_{q \in p^{\text{in}}} M_{qp} = \alpha \Big(\sum_{q \in p^{\text{out}}} M_{pq} \Big) \quad . \tag{5.6}$$

3. There exists a stationary routing under potential v iff

$$\forall p, \quad \sum_{q \in p^{\text{out}}} v_q M_{qp} = \sum_{q \in p^{\text{in}}} M_{pq} v_q \quad . \tag{5.7}$$

4. A CTEGM with routing ρ admits a potential v iff for all $q \in \mathcal{Q}$, $p \in q^{\text{in}}$,

$$v_q = \sum_{q' \in p^{\text{in}}} \mu_{qp} \rho^p_{qq'} \mu_{pq'} v_{q'} \quad . \tag{5.8}$$

Proof. We prove item 3 (which contains item 1 as a special case). The CTPN has potential v iff for all p, the matrix

$$P^p_{qq'} = v_q^{-1} M_{qp}^{-1} \rho^p_{qq'} M_{pq'} v_{q'}$$

is stochastic. Summing up as $q' \in p^{\text{in}}$, we get $v_q M_{qp} = \sum_{q' \in p^{\text{in}}} \rho^p_{qq'} M_{pq'} v_{q'}$. Summing up as $q \in p^{\text{out}}$ and using the fact that the transpose of $\rho^p_{\cdot,\cdot}$ is stochastic, we get the necessary condition (5.7). Then, the origin independent routing policy

$$\rho^p_{qq'} = \frac{v_q M_{qp}}{\sum_{q'' \in p^{\text{out}}} v_{q''} M_{q''p}} \qquad \forall q' \in p^{\text{in}} \tag{5.9}$$

turns out to be admissible, which shows that the condition is also sufficient. The other points are left to the reader. $\square$

5.2 Input-Output Representation of CTPN

Pursuing the program previously illustrated with additive systems (CTEGM), we provide an algebraic input-output representation for CTPN. In view of the dynamics of CTPN (see (3.2)), we introduce (min,+) polynomials and formal series in *several* commutative indeterminates. Given a family of indeterminates $\{z_i\}_{i \in \mathcal{I}}$ (not necessarily finite), we denote by $(\mathbb{R}^+)^{(\mathcal{I})}$ the set of *almost zero* sequences

[19] This is a fairness condition which states that tokens are routed equally to the downstream arcs, counted with their multiplicities.

$\alpha_i \in \mathbb{R}^+, i \in \mathcal{I}$ (such that $I(\alpha) \stackrel{\text{def}}{=} \{i \in \mathcal{I} \mid \alpha_i \neq 0\}$ is finite). A generalized[20] formal series in the commutative indeterminates z_i with coefficients in $\mathbb{R}_{\min}$ is a sum

$$s = \bigoplus_{\alpha \in (\mathbb{R}^+)^{(\mathcal{I})}} s_\alpha \bigotimes_{i \in I(\alpha)} z_i^{\alpha_i} \ , \ \ s_\alpha \in \mathbb{R}_{\min} \ . \tag{5.10}$$

It is a polynomial whenever $s_\alpha = \varepsilon$ for all but a finite number of α. The numerical function associated with a series s is the map $S : \mathbb{R}^{\mathcal{I}} \to \mathbb{R} \cup \{\pm\infty\}$,

$$S(z) = \inf_\alpha \Big(s_\alpha + \sum_{i \in I(\alpha)} \alpha_i z_i \Big) \ . \tag{5.11}$$

When s is a nonzero polynomial, the infimum in (5.11) is finite. This defines a proper notion of finitely valued (min,+) *polynomial function*. Polynomial functions are stable by pointwise min, pointwise sum and composition. It is clear that (3.2) is nothing but a polynomial induction of the form

$$x(t) = A(x(t), \dots, x(t-\overline{\tau}), u(t), \dots, u(t-\overline{\tau})), \tag{5.12}$$
$$y(t) = C(x(t), \dots, x(t-\overline{\tau}), u(t), \dots, u(t-\overline{\tau})) \ , \tag{5.13}$$

where A, C are polynomial functions and $\overline{\tau} \stackrel{\text{def}}{=} \max_p \tau_p$. Thus, *CTPN and* $(\min, +)$ *recurrent stationary polynomial systems coincide*. For simplicity, we shall limit ourselves to SISO systems (the MIMO case is not more difficult, although the notation is more intricate). We introduce the family of indeterminates $\mathsf{u}_\tau, \tau \in \mathbb{N}$. The series s given by (5.10) is a *Volterra series* [11] if for all τ, the series is a polynomial in the indeterminate u_τ (equivalently, if the indeterminate u_τ appears in (5.10) with a finite number of exponents). The *evaluation* su of the Volterra series s at the input u is obtained by substituting $u(t-\tau)$ for the indeterminate u_τ.

Theorem 5.4 (Volterra Expansion). *The output of an explicit SISO CTPN is obtained as the evaluation of a Volterra series:*

$$y(t) = su(t) = \inf_\alpha \Big(a_\alpha + \sum_{\tau \in I(\alpha)} \alpha_\tau u(t-\tau) \Big) \ . \tag{5.14}$$

A case of particular interest arises for inputs with finite past: $u(\tau) = \varepsilon$ for $\tau \leq T_0$. Then, for all t, the Volterra expansion of $y(t)$ is obviously finite.

5.3 Behavioral Properties of CTPN

Theorem 5.5. *The input-output map $\mathcal{H}$ of a MIMO CTPN is*

1. *stationary,*

[20] We allow nonnegative real valued exponents α_i, not only integer ones.

2. *causal,*
3. *monotone:* $u \leq v \Rightarrow \mathcal{H}u \leq \mathcal{H}v$*,*
4. *Scott continuous,*
5. *concave (see Theorem 4.11 for the definitions).*

Undiscounted CTPN satisfy the following property.

6. *Homogeneity:* $\mathcal{H}(\lambda + u) = \lambda + \mathcal{H}(u)$.

CTPN with potential v *satisfy the following.*

7. (v_u, v_y)*-homogeneity:* $\mathcal{H}(\lambda v_u + u) = \lambda v_y + \mathcal{H}u$.

All these properties are immediate consequences of the (MIMO extension) of the Volterra expansion (5.14). Again, these properties are accurate: it could be shown that an map satisfying the above properties is a limit of CTPN operators, but we shall not attempt to detail this statement here.

5.4 Asymptotic Properties of Undiscounted Petri Nets

Theorem 5.6. *For a strongly connected undiscounted CTPN, we have*

$$\lim_{t\to\infty} \frac{1}{t} Z_q(t) = \lambda, \quad \forall q \ ,$$

where λ *is a constant. The* periodic throughput λ *is characterized as the unique value for which a finite vector* v *is solution of*

$$v = \min_p \left(\nu_{\cdot p} - \lambda \tau_p + P^p v\right) \ . \tag{5.15}$$

Indeed, the asymptotic behavior of $Z(t)$ is known in much more details [27]. Note that the effective computation of λ from (5.15) proceeds from standard algorithms (Policy Improvement [28], Linear Programming).

Proof. This is an adaptation of standard stochastic control results [28, Chap. 33, Th. 4.1]. The growth rate λ is independent of the initial point q for the subclass of communicating systems[21]. This assumption is equivalent to the strong connectivity of the net. □

There is an equivalent characterization of λ which exhibits the analogy with the CTEG case in a better way. A *feedback policy* (or policy[22], for short) is a map $u : \mathcal{Q} \to \mathcal{P}$. The policy is *admissible* if $u(q) \in q^{\text{in}}$, that is, if setting $p_n = u(q_n)$ yields

[21] The system is communicating if for all q, q', there is a policy u and an integer k such that $(P^u_{qq'})^k > 0$ —i.e. q has access to q'.

[22] This feedback policy has nothing to do with the *routing* policy introduced in §3.

and admissible policy for the stochastic control problem presented in §5.1. With a policy u are associated the following vectors and matrices

$$\nu^u_q \stackrel{\text{def}}{=} \nu_{qu(q)} \quad , \qquad \tau^u_q \stackrel{\text{def}}{=} \tau_{u(q)} \quad , \qquad P^u_{qq'} \stackrel{\text{def}}{=} P^{u(q)}_{qq'} \quad .$$

We denote by $\mathcal{R}(u)$ the set of final classes[23] of the matrix P^u. For each class $r \in \mathcal{R}(u)$, we have a unique invariant measure π^{ru} with support r (i.e. $\pi^{ru}P^u = \pi^{ru}$, and $\pi^{ru}_q = 0$ if $q \notin r$.)

Theorem 5.7. *For a strongly connected undiscounted CTPN, we have*

$$\lambda = \min_u \min_{r \in \mathcal{R}(u)} \frac{\pi^{ru}\nu^u}{\pi^{ru}\tau^u} \quad . \tag{5.16}$$

Thus, λ is the minimal ratio of the mean marking over the mean holding time in the places visited following a stationary policy. In the CTEG case, the final classes are precisely circuits and the invariant measures are uniform on the final classes, so that (5.16) reduces to the well known (4.22).

The proof of Theorem 5.7 uses the fact that the rate λ is obtained asymptotically for stationary policies, together with the following lemma.

Lemma 5.8. *Let u denote a policy such that P^u admits a positive invariant measure π. The unique λ such that there exists a finite vector v:*

$$v = \nu^u - \lambda\tau^u + P^u v \tag{5.17}$$

is given by

$$\lambda = \frac{\pi\nu^u}{\pi\tau^u} \quad . \tag{5.18}$$

Proof. Left multiplying (5.17) by the row vector π, we get that λ is necessarily equal to (5.18). Conversely, we are reduced to prove the existence of a solution (λ, v) when P^u is irreducible. Then 1 is a simple eigenvalue of P^u, hence, Im$(P^u - I)$ is $|\mathcal{Q}|-1$ dimensional. Moreover, $\tau^u \notin$ Im$(P^u -$I$)$ (for $\tau^u = P^u v - v \Rightarrow \pi\tau^u = \pi(P^u v - v) = 0$, a contradiction). Hence, $\mathbb{R}\tau^u +$ Im$(P^u -$ I$) = \mathbb{R}^{\mathcal{Q}}$. □

It is not surprising that the terms at the right-hand side of (5.16) are indeed *invariants* of the net.

Theorem 5.9 (Invariants). *Given an undiscounted CTPN, for all policy u and for all final class r associated with u,*

$$I^{ur} \stackrel{\text{def}}{=} \pi^{ur}\nu^u = \sum_{q \in r} \pi^{ur}\nu^u_q \tag{5.19}$$

is invariant by firing of transitions.

[23] The *classes* of a matrix A are by definition the strongly connected components of the graph of A. A class is *final* if there is no other class downstream.

Proof. After firing once transition $q \in r$ (the case when $q \notin r$ is trivial), I^{ur} increases by

$$-\pi_q^{ur} + \sum_{q' \in (q^{\text{out}})^{\text{out}} \cap r} \pi_{q'}^{ur} P_{q'q}^{u}$$

which is zero because π^{ur} is an invariant measure of P^u with support r. □

Example 5.10. The CTPN shown in Fig. 2a is equivalent to that of Fig. 2d under a fair routing policy independent of the origin of the tokens. In this particular case, we obtain the same periodic throughput λ as in the case of the bijective routing shown in Fig. 2c (see (4.30)). This can be seen from the following table and Formula (5.16).

Policy	Final classes	Invariant measures	Invariants
$u_1(q_3) = p_1$ $u_1(q_4) = p_2$	$r_1 = \{q_1, q_3\}$, $r_2 = \{q_2, q_4\}$	$\pi^{u_1 r_1} = [\frac{1}{2}, 0, \frac{1}{2}, 0]$ $\pi^{u_1 r_2} = [0, \frac{1}{2}, 0, \frac{1}{2}]$	$I^{u_1 r_1} = \frac{1}{2}(m_{p_1} + m_{p_3})$ $I^{u_1 r_2} = \frac{1}{2}(m_{p_2} + m_{p_4})$
$u_2(q_3) = p_1$ $u_2(q_4) = p_5$	r_1	$\pi^{u_1 r_1}$	$I^{u_1 r_1}$
$u_3(q_3) = p_5$ $u_3(q_4) = p_2$	r_2	$\pi^{u_1 r_2}$	$I^{u_1 r_2}$
$u_4(q_3) = p_5$ $u_4(q_4) = p_5$	$r_3 = \{q_1, q_2, q_3, q_4\}$	$\pi^{u_4 r_3} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$	$I^{u_4 r_3} = \frac{1}{4}(m_{p_3} + m_{p_2} + m_{p_5})$

Finally, we indicate how the above results can be extended to CTPN with potential. With a feedback policy u we associate the matrix R^u: $R^u_{qq'} = \mu_{qu(q)} \rho^{u(q)}_{qq'} \mu_{u(q)q'}$ if $q' \in u(q)^{\text{in}}$ ($R^u_{qq'} = 0$ otherwise); we denote by $\mathcal{R}(u)$ the set of final classes of R^u; which each final class r we associate a left eigenvector of R^u: $\pi^{ru} = \pi^{ru} R^u$ with support r; and we define ν^u, τ^u as in Theorem 5.7. We denote by diag v the diagonal matrix with diagonal entries $(\operatorname{diag} v)_{qq} = v_q$. Then, the folllowing formula is an immediate consequence of Theorem 5.7.

Corollary 5.11. *For a strongly connected CTPN with potential v, we have*

$$\lim_{t \to \infty} \frac{1}{t} Z_q(t) = \lambda_q, \quad \text{where } v_q^{-1} \lambda_q = \min_u \min_{r \in \mathcal{R}(u)} \frac{\pi^{ru} \nu^u}{\pi^{ru} (\operatorname{diag} v) \tau^u} \ . \tag{5.20}$$

The terms $\pi^{ru} \nu^u$ which determine the throughput are of course *invariants* of the net. More generally, it follows from standard dynamic programming results that the counter functions of α-discounted CTPN exhibit a geometric growth (or convergence) with rate α. The geometric growth of other classes of CTPN could be obtained by transferring existing results about non normalized dynamic programming inductions [29].

References

[1] F. Baccelli, G. Cohen, and B. Gaujal. Recursive equations and basic properties of timed Petri nets. *J. of Discrete Event Dynamic Systems*, 1(4):415–439, 1992.

[2] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. Wiley, 1992.

[3] F. Baccelli, S. Foss, and B. Gaujal. Structural, temporal and stochastic properties of unbounded free-choice petri nets. Rapport de recherche, INRIA, 1994.

[4] J. Berstel and C. Reutenauer. *Rational Series and their Languages*. Springer, 1988.

[5] H. Braker. *Algorithms and Applications in Timed Discrete Event Systems*. PhD thesis, Delft University of Technology, Dec 1993.

[6] H. Braker and J. Resing. Periodicity and critical circuits in a generalized max-algebra setting. *Discrete Event Dynamic Systems*. toappear.

[7] P. Chretienne. *Les Réseaux de Petri Temporisés*. Thèse Université Pierre et Marie Curie (Paris VI), Paris, 1983.

[8] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. Analyse du comportement périodique des systèmes de production par la théorie des dioïdes. Rapport de recherche 191, INRIA, Le Chesnay, France, 1983.

[9] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot. Algebraic tools for the performance evaluation of discrete event systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1), Jan. 1989.

[10] P. Dudnikov and S. Samborskiĭ. Endomorphisms of finitely generated free semimodules. In V. Maslov and S. Samborskiĭ, editors, *Idempotent analysis*, volume 13 of *Adv. in Sov. Math.* AMS, RI, 1992.

[11] E.Sontag. *Polynomial response maps*. Lecture notes in control and information sciences. Springer, 1979.

[12] S. Gaubert. *Théorie des systèmes linéaires dans les dioïdes*. Thèse, École des Mines de Paris, July 1992.

[13] S. Gaubert. Rational series over dioids and discrete event systems. In *Proc. of the 11th Conf. on Anal. and Opt. of Systems: Discrete Event Systems*, number 199 in Lect. Notes. in Control and Inf. Sci, Sophia Antipolis, June 1994. Springer.

[14] S. Gaubert. Resource optimization and (min,+) spectral theory. IEEE-TAC, 1995, to appear.

[15] S. Gaubert. Rational series over the (max,+) semiring, discrete event systems and Bellman processes. in preparation, 1995.

[16] G. Gierz, K.H. Hofmann, K. Keimel, J.D Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer, 1980.

[17] M. Gondran and M. Minoux. *Graphes et algorithmes*. Eyrolles, Paris, 1979. Engl. transl. *Graphs and Algorithms*, Wiley, 1984.

[18] H.P. Hillion and J.M. Proth. Performance evaluation of job-shop systems using timed event-graphs. *IEEE Trans. on Automatic Control*, 34(1):3–9, Jan 1989.

[19] D. Krob. Quelques exemples de séries formelles utilisées en algèbre non commutative. Rapport de recherche 90-2, Université de Paris 7, LITP, Jan. 1990.

[20] D. Krob and A. Bonnier-Rigny. A complete system of identities for one letter rational expressions with multiplicities in the tropical semiring. Rapport de recherche 93.07, Université de Paris 7, LITP, 1993.

[21] S. Laftit, J.M. Proth, and X.L. Xie. Optimization of invariant criteria for event graphs. *IEEE Trans. on Automatic Control*, 37(6):547–555, 1992.

[22] P. Moller. *Théorie algébrique des Systèmes à Événements Discrets*. Thèse, École des Mines de Paris, 1988.

[23] A. Munier. Régime asymptotique optimal d'un graphe d'événements temporisé

généralisé: application à un problème d'assemblage. *APII*, 27(5):487–513, 1993.

[24] M. Plus. A linear system theory for systems subject to synchronization and saturation constraints. In *Proceedings of the first European Control Conference*, Grenoble, July 1991.

[25] I.V. Romanovskiĭ. Optimization and stationary control of discrete deterministic process in dynamic programming. *Kibernetika*, 2:66–78, 1967. Engl. transl. in Cybernetics 3 (1967).

[26] L.H. Rowen. *Polynomials identities in Ring theory*. Academic Press, 1980.

[27] P.J. Schweitzer and A. Federgruen. Geometric convergence of value-iteration in multichain Markov decision problems. *Adv. Appl. Prob.*, 11:188–217, 1979.

[28] P. Whittle. *Optimization over Time*, volume II. Wiley, 1986.

[29] W.H.M. Zijm. Generalized eigenvectors and sets of nonnegative matrices. *Lin. Alg. and Appl.*, 59:91–113, 1984.