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A LINEAR SYSTEM THEORY FOR SYSTEMS SUBJECT TO SYNCHRONIZATION AND SATURATION CONSTRAINTS

Max Plus*

Abstract A linear system theory is developped for a class of continuous and discrete Systems subject to Synchronization and Saturations that we call S^3. This class provides a generalization of a particular class of timed discrete event systems which can be modeled as event graphs. The development is based on a linear modeling of S^3 in the min-plus algebra. This allows us to extend elementary notions and a number of results of conventional linear systems to S^3. In particular, notions of causality, time-invariance, impulse response, convolution, transfer function and rationality are considered.

1 Introduction

In the past few decades, linear system theory has been the main topic of research among system and control engineers. Linear systems have been so popular not because there are many physical systems that can be represented by linear models but simply because they are easier to analyze. Over these years, an impressive body of knowledge about these systems has been accumulated to a point that, today, the first attempt to analyze any phenomenon is to approximately model it with a linear system through for example identification or linearization. There are of course systems that exhibit very nonlinear behaviors, such as systems subject to synchronization or saturation constraints, and these cannot in any reasonable way be approximated by linear systems. In this paper, we consider a special class of such systems and show that despite there nonlinearity, they can be described *exactly* by 'linear' equations over a different algebraic structure. Of course, because of the differences in the algebraic structures, we should not expect that all the classical results of conventional linear system theory simply carry over to this new 'linear' system theory; it turns out that some do and some don't. Our objective in this paper is to develop this new 'linear' system theory using the conventional linear system theory as a guideline. Even though our theory applies to a wider class of systems, we shall motivate its development by considering special examples of S^3 and of discrete event systems.

The outline of the paper is as follows. In § 2, we present a continuous dynamic system in which saturation and synchronization phenomena appear and we give an analogous discrete event counterpart. We introduce the algebraic structure which we shall use throughout this paper in § 3. The notion of linearity with respect to this algebraic structure is presented in § 4. In § 5, we study causal time-invariant 'linear' systems and show that, as for standard linear systems, we can completely characterize our 'linear' systems by their impulse responses. The role of convolutions in conventional sytem theory is now played by inf-convolutions. In § 6, we introduce the notion of transfer functions which are related to impulse responses by a transformation close to the Fenchel transform. In § 7, we address the problem of restraining input functions and impulse responses to nondecreasing functions of time, a constraint which appears naturally in some realistic applications of the theory. Finally, in § 8, we discuss rationality in the min-plus context, and we characterize rational elements in terms of their periodicity.

2 A Continuous System and its Discrete Counterpart

In this section, we propose a fundamental example which aims at illustrating the main ideas of this paper. Let us consider the SISO system $S : u \mapsto y$ displayed on the left-hand side of Figure 1. A fluid

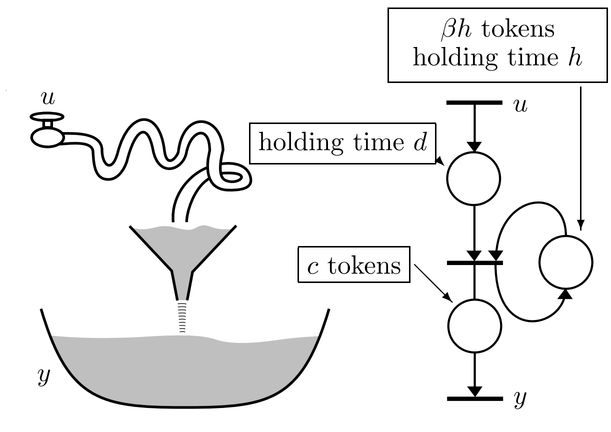

Figure 1: Continuous/discrete analogous systems

is poured through a long pipe into a first reservoir (empty at time $t = 0$). The input $u(t)$ denotes the *cumulated* flow at the inlet of the pipe up to time t (hence $u(t)$ is a nondecreasing time function and $u(t) = 0$ for $t \leq 0$). It is assumed that it takes a delay d for the fluid to travel in the pipe. From the first reservoir, the fluid drops into a second reservoir

*Collective name for a research group comprising Guy Cohen, Stéphane Gaubert, Ramine Nikoukhah, Jean-Pierre Quadrat, all with INRIA, Rocquencourt, France (Guy Cohen is also with CAS-École des Mines de Paris, Fontainebleau, France).

through an aperture which limits the instantaneous flow to a maximum value $\beta > 0$. The volume of fluid at time t in this second reservoir is denoted $y(t)$, and $y(0) = c$.

2.1 Dynamic Equations

Because the flow into the second reservoir is limited to β, we have:

$$\forall t, \forall \theta \geq 0, \quad y(t+\theta) \leq y(t) + \beta\theta \ . \tag{1}$$

On the other hand, since there is a traveling time d in the pipe, $y(t)$ should be compared with $u(t-d)$, and because there is an initial stock c in the second reservoir, we have:

$$\forall t, \quad y(t) \leq u(t-d) + c \ . \tag{2}$$

Recall that $u(t) = 0$ for $t \leq 0$, so that, for $t \leq d$, we get $y(t) \leq c$, an inequality which will become an equality later on, when taking the maximum solution of the inequalities (1)–(2). Hence the initial condition of the second reservoir is taken into account by these inequalities. We further have that, $\forall t$ and $\forall \theta \geq 0$,

$$\begin{aligned} y(t) &\leq y(t-\theta) + \beta\theta \\ &\leq u(t-d-\theta) + c + \beta\theta \ , \end{aligned}$$

hence, $\forall t$,

$$\begin{aligned} y(t) &\leq \inf_{\theta \geq 0} [u(t-d-\theta) + c + \beta\theta] \\ &= \inf_{\tau \geq d} [u(t-\tau) + c + \beta(\tau - d)] \ . \end{aligned} \tag{3}$$

Let

$$k(t) = \begin{cases} c & \text{for } t \leq d \ ; \\ c + \beta(t-d) & \text{otherwise.} \end{cases} \tag{4}$$

and consider, $\forall t$,

$$\overline{y}(t) \stackrel{def}{=} \inf_{\tau \in \mathbb{R}} [u(t-\tau) + k(\tau)] \ . \tag{5}$$

Indeed, in (5), the range of τ may be limited to $\tau \geq d$ since, for $\tau < d$, $k(\tau)$ remains equal to c whereas $u(t-\tau) \geq u(t-d)$ (remember that $u(\cdot)$ is nondecreasing). Therefore, comparing (5) with (3), it is clear that $y(t) \leq \overline{y}(t), \forall t$.

Moreover, with $\tau = d$ on the right-hand side of (5), we see that $\overline{y}$ verifies (2). On the other hand, since obviously, $\forall s$ and $\forall \theta \geq 0, k(s+\theta) \leq k(s) + \beta\theta$, then, $\forall t$ and $\forall \theta \geq 0$,

$$\begin{aligned} \overline{y}(t+\theta) &= \inf_{\tau \in \mathbb{R}} [u(t+\theta-\tau) + k(\tau)] \\ &= \inf_{s \in \mathbb{R}} [u(t-s) + k(s+\theta)] \\ &\leq \inf_{s \in \mathbb{R}} [u(t-s) + k(s)] + \beta\theta \\ &= \overline{y}(t) + \beta\theta \ . \end{aligned}$$

Thus, $\overline{y}$ verifies (1).

Finally, we have proved that $\overline{y}$ is the maximum solution of (1)–(2). It can be checked that (5) yields $\overline{y}(t) = c, \forall t \leq d$. This solution is the one which will be realized physically if we assume that, subject to (1)–(2), the fluid flows as fast as possible. We summarize this result by the following theorem.

Theorem 1 *The output $y = S(u)$ of the system shown in Figure 1 (left-hand side) is given by the inf-convolution of the input u with the function k given by (4).*

2.2 Min-Plus Linearity

Theorem 2 *The previous system S is min-plus linear, that is, if $y_i = S(u_i)$, $i = 1, 2$, then $\min(y_1, y_2) = S(\min(u_1, u_2))$ and $y_i(\cdot) + \lambda = S(u_i(\cdot) + \lambda)$, where $\lambda \in \mathbb{R}$ and $y_i(\cdot) + \lambda$ is a short-hand notation to say that we add a constant λ to a time function $y_i(\cdot)$.*

Proof The result is a direct consequence of the fact that the input-output relation is an inf-convolution.

- $$\begin{aligned} &\inf_{\tau \in \mathbb{R}} \{k(\tau) + \min[u_1(t-\tau), u_2(t-\tau)]\} \\ &= \min \Big\{ \inf_{\tau \in \mathbb{R}} [k(\tau) + u_1(t-\tau)], \\ &\qquad \min_{\tau \in \mathbb{R}} [k(\tau) + u_2(t-\tau)] \Big\} \ . \end{aligned}$$
- $$\begin{aligned} &\inf_{\tau \in \mathbb{R}} \{k(\tau) + [\lambda + u(t-\tau)]\} \\ &\qquad = \lambda + \inf_{\tau \in \mathbb{R}} [k(\tau) + u(t-\tau)] \ . \end{aligned}$$

■

2.3 Discrete Counterpart

The previous system may be considered as a continuous version of some discrete event system. Let us consider a lumped version of S:

$$S_h : \sup y \text{ s.t. } \begin{cases} y(t+h) - y(t) \leq \beta h \ , \\ y(t) \leq c + u(t-d) \ , \end{cases} \tag{6}$$

$\forall t \in h\mathbb{Z}$, and for h as small as possible but such that $\beta h \in \mathbb{N}$ (that is, $h = 1/\beta$). We also assume that $c \in \mathbb{N}$ and that $d/h \in \mathbb{N}$. The maximum solution $\overline{y}$ of (6) is given by the recursive equation:

$$\overline{y}(t) = \min(\overline{y}(t-h) + \beta h, c + u(t-d)) \ . \tag{7}$$

An interpretation of this equation in terms of a timed event graph is shown on the right-hand side of Figure 1 (see [7] for more detailed explanations). A physical interpretation in terms of a manufacturing system is as follows. Parts come into a workshop and reach a pool of machines after a traveling time d. There are βh machines working in parallel and each part spends h units of time on a machine. Initially, machines are idle (i.e. empty). Parts reaching the machines wait in a storage upstream the machines until they can be handled by some machine (which is supposed to occur as soon as possible; the discipline is FIFO). From $t = 0$ on, parts entering the system receive sequential numbers (say, the first one to enter the system receives number -10) and $u(t)$ denotes the number of the last part arrived before or at time t. Likewise, $y(t)$ denotes the number of the last part arrived at the storage located downstream the machines before or at time t. The first to arrive after time 0 is numbered $c - 10$ to take into account that c parts are already present in the storage at time 0.

2.4 Mixing and Synchronization

Suppose now that we have two continuous systems similar to the one shown in Figure 1. Say, one of the fluid is red and the other is white, and we want to produce a pink fluid [1] by mixing them in equal proportions. If $y_r(t)$ and $y_w(t)$ are the quantities available at time t in the downstream reservoirs, and if the operation of mixing takes no time, half the maximum quantity $y_p(t)$ of pink fluid one can produce up to time t is

$$y_p(t) = \min(y_r(t), y_w(t)) \ .$$

Indeed, we have just combined two SISO systems in parallel to get a new one with two inputs and one output. Obviously, this new system is again min-plus linear.

The discrete event counterpart of this mixing operation is the assembly of two kinds of parts. The equations are of course the same for this operation which would be represented as a join at a transition in the pictural language of Petri nets. Generally speaking, joins at transitions express synchronization of events.

We are now going to give a more general account of min-plus linear systems. However, we need first recall some basic facts about 'dioids' (for a deeper treatment, see [7]).

3 Dioid Structure

Definition 3 *A set endowed with two inner operations $\oplus$ (addition) and $\otimes$ (product) is called a dioid (denoted $\mathcal{D}$) if*

- *both $\oplus$ and $\otimes$ are associative;*
- *$\oplus$ is commutative;*
- *$\otimes$ is distributive with respect to $\oplus$;*
- *both $\oplus$ and $\otimes$ have neutral elements, i.e. there exist ε and e in $\mathcal{D}$ such that*

$$\forall a \in \mathcal{D}, \quad \left\{ \begin{array}{l} a \oplus \varepsilon = a \ ; \\ e \otimes a = a \otimes e = a \ ; \end{array} \right. \tag{8}$$

- *the null element ε is absorbing for $\otimes$, i.e.*

$$\forall a \in \mathcal{D}, \quad a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon \ ; \tag{9}$$

- *$\oplus$ is idempotent, i.e.*

$$\forall a \in \mathcal{D}, \quad a \oplus a = a \ . \tag{10}$$

If $\otimes$ is commutative, $\mathcal{D}$ is called a commutative dioid.

Remark 4 As usual, the multiplicative sign $\otimes$ may sometimes be omitted. ■

There is a natural order relation $\preceq$ associated with $\oplus$ and defined by:

$$a \preceq b \Leftrightarrow a \oplus b = b \ .$$

[1] or a tooth paste with red and white strips

This relation is compatible with product, that is:

$$\forall c, \quad a \preceq b \Rightarrow \left\{ \begin{array}{l} ac \preceq bc \ , \\ ca \preceq cb \ . \end{array} \right. \tag{11}$$

It can easily be checked that $a \oplus b$ is equal to the least upper bound (with respect to $\preceq$) of a and b. Hence $\mathcal{D}$ is a sup-semilattice. A *complete* sup-semilattice is a sup-semilattice for which any (finite or infinite) subset admits a least upper bound. Therefore, we adopt the following definition.

Definition 5 *A dioid is called* complete *if it is closed for all infinite sums and if $\otimes$ is distributive with respect to infinite sums.*

Given a family $\{a_i\}_{\in I} \subset \mathcal{D}$, the least upper bound is denoted

$$\bigoplus_{i \in I} a_i$$

if I is denumerable, and

$$\oint_{i \in I} a_i$$

if I is continuous. The 'idempotent integration' $\oint$ shares several features of the usual integration. In particular, the associativity of addition yields (Fubini rule):

$$\oint_{(i,j) \in \cup_{i \in I} \cup_{j \in J(i)} \{(i,j)\}} a_{ij} = \oint_{i \in I} \oint_{j \in J(i)} a_{ij} \ .$$

The distributivity of product with respect to (infinite) sum yields:

$$c \otimes \left(\oint_{i \in I} a_i \right) = \oint_{i \in I} c \otimes a_i \ ,$$

and the same for multiplication to the right.

In a complete sup-semilattice with a bottom element (here ε), there exists a greatest lower bound of two elements defined as:

$$a \wedge b = \oint_{\substack{x \preceq a \\ x \preceq b}} x \ ,$$

and more generally, there exists a lower bound of any finite or infinite family $\{a_i\}_{i \in I}$, denoted $\bigwedge_{i \in I} a_i$.

Theorem 6 *Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The set $\overline{\mathbb{R}}$ endowed with* min *as $\oplus$ and $+$ as $\otimes$ is a complete commutative dioid (denoted $\overline{\mathbb{R}}_{\min}$), in which $\varepsilon = +\infty$ and $e = 0$.*

Note that, according to (9), $-\infty + \infty = -\infty \otimes \varepsilon = \varepsilon = +\infty$ in $\overline{\mathbb{R}}_{\min}$. Moreover, according to (11), the order $\preceq$ (which is total here) is just reversed with respect of the usual order $\leq$ (i.e. $a \preceq b \Leftrightarrow a \geq b$—for example, $3 \preceq 2$ since $3 \oplus 2 = 2$). Therefore, $\wedge$ is indeed the conventional supremum.

In this paper, we will essentially study linear systems over $\overline{\mathbb{R}}_{\min}$.

4 Linear Systems

4.1 Linearity

The underlying algebraic structure used in conventional (SISO) system theory is $\mathbb{R}$ [2] endowed with the two operations $+$ and $\times$, i.e. the field $(\mathbb{R}, +, \times)$. A *signal* is defined as a real sequence indexed by time t; the indexing set is either $\mathbb{R}$ (continuous-time systems) or $\mathbb{Z}$ (discrete-time systems). A *system* is then a mapping from the set of admissible input signals to the set of output signals. The set of admissible input signals is subject to some assumptions: it should be stable by addition, multiplication by a scalar, concatenation of pieces of signals, time-shifting; it should contain constant functions, Dirac functions, etc...

A system S is then called linear if for all input functions u_1 and u_2,

$$S(u_1 + u_2) = S(u_1) + S(u_2) \ , \tag{12}$$

and for all scalars a and all inputs u,

$$S(au) = aS(u) \ . \tag{13}$$

Here linearity is defined with respect to the field $(\mathbb{R}, +, \times)$.

Let us now change this underlying algebraic structure by replacing $+$ by min and $\times$ by $+$, i.e. consider now the complete commutative dioid $\overline{\mathbb{R}}_{\min}$. By analogy with the classical case, we define signals as time functions taking their values in $\overline{\mathbb{R}}_{\min}$. A system S is again a mapping from the set of admissible input signals to the set of output signals. In this paper, we restrict ourselves to single-input single-output (SISO) systems. Moreover, to be able to develop a theory which looks very parallel to the conventional theory, *we assume that the set of admissible input signals is the whole set* $\overline{\mathbb{R}}_{\min}^{\mathbb{R}}$.

Remark 7 Indeed, the examples of continuous and discrete event systems presented earlier shows that realistic inputs are *nondecreasing* time functions (in the usual sense). The set of nondecreasing time functions is not stable by concatenation and does not contain naive counterparts to Dirac functions. It could have been possible to develop a theory which restricts the admissible input signal set to special classes of functions. In this paper, we prefer to keep the theory simple and we defer a more sophisticated treatment to a forthcoming paper. The question of nondecreasing input functions will be further addressed in § 7. ■

Definition 8 *A system S is called linear over $\overline{\mathbb{R}}_{\min}$, or min-plus linear, if*

$$S(\min(u_1, u_2)) = \min(S(u_1), S(u_2)) \ , \tag{14}$$

and, $\forall a \in \overline{\mathbb{R}}$,

$$S(a + u(\cdot)) = a + S(u(\cdot)) \ . \tag{15}$$

[2] sometimes $\mathbb{C}$

To better see the analogy with conventional linear systems, and to emphasize that we are dealing with a dioid, we will keep on denoting min by $\oplus$, addition by $\otimes$, $-\infty$ by ε and 0 by e. With this notation, Equation (5) can be expressed as follows

$$\overline{y}(t) = \oint_{\theta \in \mathbb{R}} u(t-\theta) \otimes k(\theta) \ ,$$

showing that inf-convolutions are simply convolutions of our algebra.

4.2 Continuity

In classical developments of linear system theory, an additional continuity assumption is made (sometimes implicitly):

$$S\left(\sum_{i=1}^{\infty} u_i\right) = \sum_{i=1}^{\infty} S(u_i). \tag{16}$$

Again, as in the standard case, we make the assumption that S is sufficiently smooth. Namely, we require that for any infinite collection $\{u_i\}_{i \in I}$

$$S\left(\inf_{i \in I} u_i\right) = \inf_{i \in I} S(u_i) \ .$$

Translated into the notation of $\overline{\mathbb{R}}_{\min}$ this leads to the following definition.

Definition 9 *The system S is said to be continuous if it satisfies:*

$$S\left(\oint_{i \in I} u_i\right) = \oint_{i \in I} S(u_i) \ . \tag{17}$$

This definition is meaningful as long as $\overline{\mathbb{R}}_{\min}$ is a complete dioid (see Definition 5). Linearity (Definition 8) does not imply continuity as shown by the following example.

Example 10 Consider the following non continuous time-invariant min-plus linear system:

$$u \in \overline{\mathbb{R}}^{\mathbb{R}} \mapsto S(u) \in \overline{\mathbb{R}}^{\mathbb{R}} \text{ with } [S(u)](t) = \liminf_{s \to t} u(s) \ .$$

This system verifies (14)–(15). It is obviously time-invariant, but it is not continuous. Indeed, for all $n \geq 1$, let

$$u_n(t) = \begin{cases} 0 & \text{if } t \leq 0 \ ; \\ -nt & \text{if } 0 < t < \frac{1}{n} \ ; \\ -1 & \text{if } \frac{1}{n} \leq t \ . \end{cases}$$

We have, for all $n \geq 1$, $[S(u_n)](0) = 0$, and

$$\bigoplus_{n \geq 1} u_n(t) = \begin{cases} 0 & \text{if } t \leq 0 \ ; \\ -1 & \text{otherwise.} \end{cases}$$

This yields $[S(\bigoplus_n u_n)](0) = -1$, which is different from $\bigoplus_n [S(u_n)](0) = 0$. ■

In the sequel, continuity is always assumed.

4.3 Algebra of Systems

An important feature of conventional linear systems is that we can cascade them in series, in parallel or put them in feedback, and always get a linear system. This way, from simple elementary blocks, we can construct (realize) complex linear systems. This idea can also be extended to linear systems over $\overline{\mathbb{R}}_{\min}$.

Parallel cascade: $S = S_1 \oplus S_2$ denotes the parallel cascade of S_1 and S_2 defined as follows:

$$[S(u)](t) = [S_1(u)](t) \oplus [S_2(u)](t) \ . \tag{18}$$

Serial cascade: $S = S_1 \otimes S_2$, or more briefly $S_1 S_2$, denotes the serial cascade of S_1 and S_2 defined as follows:

$$[S(u)](t) = [S_1(S_2(u))](t) \ . \tag{19}$$

Feedback: The situation with feedback is slightly more complicated. The difficulty arises from the fact that the implicit equation so obtained does not uniquely characterize the solution. Let us consider

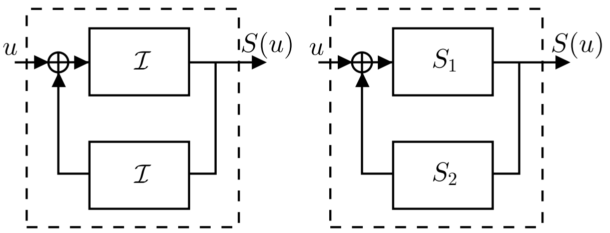

Figure 2: Feedback systems

the simple example of putting $\mathcal{I}$, the identity system [3], in feedback around another identity system (see Figure 2, left-hand side). Let us call the resulting system S. One has

$$S(u) = S(u) \oplus u \ . \tag{20}$$

This equation has clearly no unique solution since every $S(u) \succeq u$ is a solution to (20) (only if $u(t) = -\infty, \forall t$, then $[S(u)](t) = -\infty, \forall t$ is the unique solution). This means that, in order to have a well defined feedback system, we need to impose additional constraints in order to determine a unique solution to (20). A choice which in practice makes a lot of sense (as already seen in § 2) and which results in a linear system is the least solution of (20) with respect to $\preceq$ (or otherwise stated, the greatest solution with respect to $\leq$).

Let us state then the feedback operation in the general case: we define S as the result of putting S_2 in feedback around S_1 (see Figure 2, right-hand side) if

$$\forall t, \quad [S(u)](t) = \bigwedge_{y(t) = [S_1(S_2(y) \oplus u)](t)} y(t) \ , \tag{21}$$

where $\wedge$ denotes the greatest lower bound in the complete dioid $\overline{\mathbb{R}}_{\min}$ (which corresponds indeed to the supremum).

[3]For all u, $\mathcal{I}(u) = u$.

Theorem 11 *Any system obtained by cascading min-plus linear systems in series, in parallel, and by putting them in feedback is also min-plus linear.*

Linearity of cascaded linear systems in $\overline{\mathbb{R}}_{\min}$ is straightforward to show and the proof is omitted. In fact, thanks to continuity (see (17)), cascading linear systems in parallel infinitely many times also yields linear systems. Thus, we shall only prove linearity for the feedback case. For this, we need the following classical theorem which is given here for the sake of completeness.

Theorem 12 *Let $\mathcal{D}$ be a complete dioid and let $\mathcal{H}$ be a continuous mapping from $\mathcal{D}$ to $\mathcal{D}$. For $b \in \mathcal{D}$, consider the implicit equation*

$$x = \mathcal{H}(x) \oplus b \ . \tag{22}$$

Let $\mathcal{H}^0 = \mathcal{I}$ (identity),

$$\mathcal{H}^n(x) = \underbrace{\mathcal{H}(\mathcal{H}(\ldots(\mathcal{H}(x))))}_{n \text{ times}} \ ,$$

and

$$\mathcal{H}^*(x) \stackrel{def}{=} \bigoplus_{n \in \mathbb{N}} \mathcal{H}^n(x), \quad \forall x \ . \tag{23}$$

Then $\overline{x} = \mathcal{H}^*(b)$ is the least solution of (22).

Proof For any solution x of (22), by successive substitutions and continuity, we obtain, $\forall n \in \mathbb{N}$:

$$\begin{aligned} x &= \mathcal{H}^n(x) \oplus \left(\mathcal{I} \oplus \cdots \oplus \mathcal{H}^{n-1}\right)(b) \\ &\succeq \left(\mathcal{I} \oplus \cdots \oplus \mathcal{H}^{n-1}\right)(b) \ . \end{aligned}$$

It follows that $\overline{x} \stackrel{def}{=} \mathcal{H}^*(b) \preceq x$. But, using continuity,

$$\begin{aligned} \mathcal{H}(\overline{x}) \oplus b &= \mathcal{H}\left(\bigoplus_{n \in \mathbb{N}} \mathcal{H}^n(b)\right) \oplus b \\ &= \bigoplus_{n \geq 1} \mathcal{H}^n(b) \oplus b \\ &= \overline{x} \ , \end{aligned}$$

which shows that $\overline{x}$ itself verifies (22). ■

Proof of Theorem 11 continued Note that we can rewrite the constraint on y in (21) as follows:

$$y = \mathcal{H}(y) \oplus S_1(u) \ ,$$

where $\mathcal{H} \stackrel{def}{=} S_1 \otimes S_2$. Hence, according to Theorem 12, the particular solution defined by (21) can explicitly be expressed as $\mathcal{H}^*(S_1(u))$. Therefore, thanks to (23), the feedback system is obtained by cascading systems S_1 and S_2 an infinite number of times, and it is thus linear. ■

The two operators $\oplus$ (parallel cascade) and $\otimes$ (serial cascade) define a structure of complete dioid over the set Σ of continuous linear systems over $\overline{\mathbb{R}}_{\min}$.

4.4 Some Elementary Systems

We have discussed how we can combine systems using cascades and feedbacks. Here, we describe three elementary but fundamental systems from which more complex systems can be built. The examples considered in § 2 (the continuous as well as the discrete versions) were made of these three systems cascaded in series.

Stock Γ^c: This is the system $y = \Gamma^c(u)$ which maps inputs to outputs according to the equation:

$$y(t) = c \otimes u(t), \quad \forall t \ .$$

A physical interpretation is given by our previous examples: an initial stock of c units (cubic meters in a reservoir, or parts in a storage if $c \in \mathbb{N}$) introduces such a shift in 'counting' or 'numbering' between inputs and ouputs. Notice that, in our algebra, this 'initial condition' behaves like a 'gain' .

The notation Γ^c is justified by the following rule of serial cascade which should be obvious to the reader:

$$\Gamma^c \otimes \Gamma^{c'} = \Gamma^{c+c'} = \Gamma^{c \otimes c'} \ .$$

Therefore Γ^1 can be denoted Γ. Also, parallel cascade obeys the following rule:

$$\Gamma^c \oplus \Gamma^{c'} = \Gamma^{\min(c,c')} = \Gamma^{c \oplus c'} \ .$$

Delay Δ^d: This is the system $y = \Delta^d(u)$ which maps inputs to outputs according to the equation:

$$y(t) = u(t-d), \quad \forall t \ .$$

Physically, any traveling or holding time (for filtering, heating, manufacturing etc...) causes such a delay. In the context of timed event graphs, this is the general input-output relation induced by places with holding time d.

Again, the notation Δ^d is justified by the following rule of serial cascade:

$$\Delta^d \otimes \Delta^{d'} = \Delta^{d+d'} = \Delta^{d \otimes d'} \ .$$

Therefore Δ^1 can be denoted Δ. For parallel cascade, there is no obvious simplifying rule in general, unless we restrict ourselves to signals that are nondecreasing time functions (in the usual sense), in which case one has

$$\Delta^d \oplus \Delta^{d'} = \Delta^{\max(d,d')} = \Delta^{d \wedge d'} \ .$$

Flow limiter Ω_β: The system shown in Figure 1 (left-hand side) was made of three elementary systems in series. We have already discussed two of them, namely the 'stock' and the 'delay'. Therefore, we consider this example again but we assume that $c = 0$ (no initial stock) and $d = 0$ (no delay). Then, the 'flow limiter' is the system $y = \Omega_\beta(u)$ which, with an input $u(\cdot)$, associates the output $y(\cdot)$ which is the maximum—in the conventional sense—solution of the system of inequalities (1)–(2) with $c = d = 0$ (and $\beta \geq 0$). In § 2, it has been shown that $y = \Omega_\beta(u)$ is explicitely given by the right-hand side of (3) (in which $c = d = 0$). But at this point, since we are now considering general input signals, [4] the rest of our previous reasoning should be adapted. Indeed, if we keep the definition (4) of k (with $c = d = 0$), and if we consider (5), this is still a solution of (1)–(2), but not the maximum solution. To go from the right-hand side of (3) to an expression similar to (5), we must replace k by ω_β defined by:

$$\omega_\beta(t) = \begin{cases} +\infty & \text{for } t < 0 \ ; \\ \beta t & \text{otherwise.} \end{cases} \tag{24}$$

As a matter of fact, with this ω_β replacing k, it is obvious that the range of τ in (5) can again be limited to $\tau \geq 0$ as in (3) (with $c = d = 0$), whatever $u(\cdot)$ may be.

As discussed in § 2, the flow limiter behaves like a loop of an event graph with a place having $\beta \times h$ tokens in its initial marking (we assume that $\beta \times h \in \mathbb{N}$), and a holding time h (looking only at time instants which are multiples of h). That is, Ω_β is analogous to the system obtained by putting a 'stock-delay' system $\Gamma^{\beta \times h} \otimes \Delta^h$ in feedback around the identity.

Unlike Γ^c and Δ^d, we denote Ω_β with β as a subscript, because β does not behave like an exponent: indeed, the following serial cascade rule can be checked by direct calculation:

$$\Omega_\beta \otimes \Omega_{\beta'} = \Omega_{\min(\beta,\beta')} = \Omega_{\beta \oplus \beta'} \ . \tag{25}$$

Physically, two flow limiters in series behave like the single most constraining one. Therefore $\otimes$, when restricted to flow limiters, is also idempotent; e.g. $(\Omega_\beta)^2 = \Omega_\beta$. The following parallel cascade rule is easy to establish:

$$\Omega_\beta \oplus \Omega_{\beta'} = \Omega_{\min(\beta,\beta')} = \Omega_{\beta \oplus \beta'} \ . \tag{26}$$

Physically, two flow limiters in parallel obey the most constraining flow limit (remember that parallel cascade means *mixing* flows in equal proportions, not *adding* them in the usual sense).

5 Time-domain Representation of Linear Systems

In this section, we extend some fundamental notions of conventional linear system theory to linear systems over $\overline{\mathbb{R}}_{\min}$. From now on, for the sake of brievity, by 'linear systems' we mean 'continuous linear systems over $\overline{\mathbb{R}}_{\min}$'.

We recall that our admissible input signal set is $\overline{\mathbb{R}}_{\min}^{\mathbb{R}}$. If we come back to Definition 8 of linearity, it is realized that this set has been endowed with two operations, namely:

- pointwise minimum (i.e. addition in $\overline{\mathbb{R}}_{\min}$) of time functions, which plays the role of addition of signals:

$$\forall t, \quad (u \oplus v)(t) \stackrel{def}{=} u(t) \oplus v(t) = \min(u(t), v(t)) \ ;$$

[4] not only nondecreasing time functions, that is, we can also withdraw some fluid from the reservoir (without any limitation of flow)

- addition of (i.e. product in $\overline{\mathbb{R}}_{\min}$ by) a constant, which plays the role of external product of a signal with a scalar:

$$\forall t, \quad (a \cdot u)(t) \stackrel{def}{=} a \otimes u(t) = a + u(t) \ .$$

Then, the set of signals is endowed with a kind of vector space structure on which we shall not further elaborate here.

The next step is to introduce a sort of 'canonical basis' for this algebraic structure. Classically, for time functions, this basis is provided by the Dirac function at 0, and all its shifted versions at other time instants. Therefore, we now introduce:

$$e(\cdot) : t \mapsto e(t) \stackrel{def}{=} \begin{cases} e & \text{if } t = 0 \ ; \\ \varepsilon & \text{otherwise,} \end{cases} \tag{27}$$

and

$$\delta^s(\cdot) \stackrel{def}{=} \Delta^s(e(\cdot)) \ \text{ i.e. } \ \delta^s(t) = e(t-s), \quad \forall t \ . \tag{28}$$

The justification of the notation $e(\cdot)$ will come from the fact that this particular signal is the identity element for inf-convolution which will soon be considered as the internal product in the signal set. Indeed, it can be checked by direct calculation that:

$$\forall u, \forall t, \quad u(t) = \oint_s u(s) \otimes e(t-s) \ , \tag{29}$$

or otherwise stated

$$u = \oint_s u(s) \cdot \delta^s \ . \tag{30}$$

This is the decomposition of signals on the canonical basis. It is unique since, if there exist numbers v_s such that $u = \oint_s v_s \cdot \delta^s$, because of identity (29) applied to the family $\{v_s\}$, we conclude that $v_t = u(t), \forall t$. Then we can state the following theorem which introduces the notion of *impulse response*.

Theorem 13 *Let S be a linear system, then there exists a unique function $k(t,s)$ (called* impulse response*) such that $y = S(u)$ can be obtained by:*

$$\forall t, \quad y(t) = \inf_s [k(t,s) + u(s)] = \oint_s k(t,s) \otimes u(s) \ , \tag{31}$$

for all input-output pairs (u,y).

Proof From (29) and (28), it follows that

$$y(t) = [S(u)](t) = S\left(\oint_s u(s) \otimes \delta^s(t)\right) \ ,$$

which, thanks to the linearity and continuity assumptions, implies

$$y(t) = \oint_s u(s) \otimes [S(\delta^s)](t) = \oint_s k(t,s) \otimes u(s) \ ,$$

where we have set

$$k(t,s) \stackrel{def}{=} [S(\delta^s)](t) \ .$$

To prove uniqueness, suppose that there exists another function $\kappa(\cdot,\cdot)$ which verifies (31). Using inputs $u = \delta^s$, $\forall s$ and $\forall t$, we get:

$$\begin{aligned} k(t,s) &\stackrel{def}{=} [S(\delta^s)](t) \\ &= \oint_\tau \kappa(t,\tau) \otimes \delta^s(\tau) \\ &= \kappa(t,s) \ , \end{aligned}$$

where the last equality is an application of (29) to the function $\kappa(t,\cdot)$. ■

Definition 14 *A linear system S is called time-invariant if it commutes with all delay operators, that is:*

$$S(\Delta^d(u)) = \Delta^d(S(u)), \quad \forall u, \forall d \ .$$

Theorem 15 *A system S is time-invariant if and only if its impulse response $k(t,s)$ depends only on the difference $t-s$, i.e., with the usual abuse of notation:*

$$k(t,s) = k(t-s) \ ,$$

with $k(\cdot) = [S(e)](\cdot)$.

Proof

$$\begin{aligned} k(t,s) &\stackrel{def}{=} [S(\delta^s)](t) = [S(\Delta^s(e))](t) \\ &= [\Delta^s(S(e))](t) = [S(e)](t-s) \ . \end{aligned}$$

■

Consequently, in the time-invariant case, the input-output relation can be expressed as follows

$$\begin{aligned} y(t) &= (k \otimes u)(t) \\ &\stackrel{def}{=} \oint_s k(t-s) \otimes u(s) \ . \end{aligned}$$

This new operation, also denoted $\otimes$, is nothing but the *inf-convolution* [19] which plays the role of convolution in our theory. The impulse response associated with a time-invariant linear system and a signal are both time functions. Serial cascade of systems corresponds to inf-convolution, the mulptiplication of the dioid of time functions. Parallel cascade corresponds to pointwise minimum of functions, the addition of the dioid. The null element is the function $\varepsilon(\cdot) : t \mapsto \varepsilon, \forall t$ which is absorbing for multiplication (i.e. such an input yields an output equal to the input through any linear system), whereas the identity element has been described by (27). This dioid of time functions is denoted $\mathcal{S}$. It is commutative and complete.

Remark 16 So far, three different complete and commutative dioids have been considered, and consequently three different meanings of $\oplus$ and $\otimes$ have been used. As usual, the context should indicate which one is meant, according to the nature of elements on which these binary operations operate. The following table recalls these three dioids. If we restrict ourselves to time-invariant linear systems which constitute a subdioid of the dioid in the second row, there is a one-to-one correspondence between this and the dioid $\mathcal{S}$. This correspondence is compatible with the dioid structure (we say it is a dioid isomorphism). ■

Dioid	$\oplus$	$\otimes$
Scalars $\overline{\mathbb{R}}_{\min}$	min	$+$
Systems Σ	parallel cascade	serial cascade
Impulse responses Signals $\mathcal{S}$	pointwise min	inf-convolution

Table 1: Three dioids

Definition 17 *A linear system S is called causal if, for all inputs u_1 and u_2 with corresponding outputs y_1 and y_2,*

$$u_1(t) = u_2(t) \text{ for } t \le \tau \Rightarrow y_1(t) = y_2(t) \text{ for } t \le \tau \ .$$

Theorem 18 *A system S is causal if its impulse response $k(t, s) = \varepsilon$ for $s > t$.*

Proof It suffices to recall that $k(t, s) = [S(e_s)](t)$ and that, for $t < s$, $e_s(\cdot)$ coincides with $\varepsilon(\cdot)$. ∎

In the time-invariant case, the condition is simply: $k(t) = \varepsilon$ for $t < 0$.

Example 19 We consider the three elementary systems introduced at § 4.4: they are time-invariant linear systems. Let us give their impulse responses.

$$\gamma^c(t) \stackrel{def}{=} [\Gamma^c(e)](t) = \begin{cases} c & \text{if } t = 0 \ ; \\ \varepsilon & \text{otherwise.} \end{cases} \tag{32}$$

$$\delta^d(t) \stackrel{def}{=} [\Delta^d(e)](t) = \begin{cases} e & \text{if } t = d \ ; \\ \varepsilon & \text{otherwise.} \end{cases} \tag{33}$$

$$\omega_\beta(t) \stackrel{def}{=} [\Omega_\beta(e)](t) = \begin{cases} \varepsilon & \text{if } t < 0 \ ; \\ \beta^t & \text{otherwise.} \end{cases} \tag{34}$$

Of course, (33) is the same as (28), and (34) is the same as (24) but stated in the notation of $\overline{\mathbb{R}}_{\min}$. Notice also that $\gamma^0 = \delta^0 = e$ and that $\omega_\beta \xrightarrow{pointwise} e$ when $\beta \to \varepsilon = +\infty$. ∎

6 Transfer Functions

6.1 Evaluation Homomorphism

In this section, we discuss the notion of transfer function associated with time-invariant min-plus linear systems. Transfer functions are related to impulse responses by a transformation which plays the role of the Fourier transform in conventional system theory, and which is, in our case, close to the Fenchel transform of convex analysis [19]. The main discrepancy with the usual case is that transfer functions are not in a one-to-one correspondence with impulse responses: only a subclass of impulse responses, namely those which are convex lower semi-continuous (l.s.c.) time functions are fully characterized by their transfer functions.

For all signals or impulse responses, we recall the formula (30) in $\mathcal{S}$. Notice that, in general, $c \cdot f$ where $c \in \overline{\mathbb{R}}_{\min}$ and $f \in \mathcal{S}$ can also be written $\gamma^c \otimes f$. Therefore, (30) can also be written:

$$u = \oint_{s \in \mathbb{R}} \gamma^{u(s)} \otimes \delta^s \ ,$$

which is closer to the notation used in [7] in discrete time and for integer-valued functions. The dioid $\mathcal{S}$ endowed with the external multiplication by scalars is called the 'algebra of impulse responses' and δ may be viewed as the 'algebraic generator' of the algebra.

With an impulse response f, we associate a transfer function g which will be a numerical function from $\overline{\mathbb{R}}_{\min}$ to $\overline{\mathbb{R}}_{\min}$: this function is evaluated essentially by formally substituting a numerical variable in $\overline{\mathbb{R}}_{\min}$ for the generator δ, and by evaluating the resulting expression using the calculation rules of $\overline{\mathbb{R}}_{\min}$. This substitution of a numerical variable for the generator should be compared with what one does in conventional system theory when substituting numerical values in $\mathbb{C}$ for the formal operator of derivation (denoted s) in continuous time, or the shift operator (denoted z) in discrete time.

Definition 20 *For $f \in \mathcal{S}$ (written as in (30), namely $f = \oint_t f(t) \cdot \delta^t$), let*

$$g : x \in \overline{\mathbb{R}}_{\min} \mapsto \oint_t f(t) \otimes x^t \in \overline{\mathbb{R}}_{\min} \ . \tag{35}$$

Then g is called the transfer function *associated with f. The mapping*

$$\mathcal{F} : f \mapsto g$$

is called the evaluation homomorphism.

The term 'homomorphism' is justified if we endow the set of numerical functions $\overline{\mathbb{R}}_{\min}^{\overline{\mathbb{R}}_{\min}}$ with the following algebraic structure denoted $\mathcal{C}_v(\overline{\mathbb{R}}_{\min})$ (this adds a new row to Table 1):

Dioid	$\oplus$	$\otimes$
Transfer functions $\mathcal{C}_v(\overline{\mathbb{R}}_{\min})$	pointwise min	pointwise $+$

Table 2: Another dioid

We let the reader check that $\mathcal{F}$ is a continuous [5] homomorphism from the dioid $\mathcal{S}$ onto the dioid $\mathcal{C}_v(\overline{\mathbb{R}}_{\min})$.

Example 21 The following formulæ (in conventional notation) can also be established:

$$\mathcal{F}(\gamma^c)(x) = c, \quad \forall x \ ;$$

$$\mathcal{F}\left(\delta^d\right)(x) = d \times x, \quad \forall x \ ;$$

$$\mathcal{F}(\omega_\beta)(x) = \begin{cases} -\infty & \text{if } x \le -\beta \ ; \\ 0 & \text{otherwise.} \end{cases}$$

∎

[5] in the sense of Definition 9

Let us examine how $\mathcal{F}$ can be interpreted by going back to conventional notation. We have:

$$\begin{aligned} g(x) &= \inf_t [tx + f(t)] \\ &= -\sup_t [t(-x) - f(t)] \ , \end{aligned} \tag{36}$$

which shows that $[\mathcal{F}(f)](x) = -[\mathcal{F}_e(f)](-x)$, if $\mathcal{F}_e$ denotes the classical Fenchel transform. From (36), it is seen that all transfer functions are concave u.s.c. (upper-semi-continuous) (as the lower hull of a family of affine functions). We recall that the Fenchel transform converts inf-convolutions into pointwise (conventional) addition: this is consistent with the choice made for multiplication in $\mathcal{C}_v(\overline{\mathbb{R}}_{\min})$.

6.2 Convex l.s.c. Impulse Responses

It is well known that the Fenchel transform only characterizes convex l.s.c. functions, or otherwise stated, all functions having the same convex l.s.c. hull have the same Fenchel transform.

Example 22 $\delta \oplus \delta^2$ has the same transfer function as $\oint_1^2 \delta^t$, namely $x \oplus x^2$. ∎

Therefore, $\mathcal{F}$ cannot be an isomorphism; it is only an epimorphism (surjective homomorphism). Otherwise stated, the equation

$$\mathcal{F}(f) = g \ , \tag{37}$$

where the right-hand side $g \in \mathcal{C}_v(\overline{\mathbb{R}}_{\min})$ is given ($f \in \mathcal{S}$ is the unknown), always has solutions, but the solutions are in general not unique. Residuation theory [1] addresses such an issue and provides a notion of pseudoinverse for $\mathcal{F}$ denoted $\mathcal{F}^\sharp$. Such a mapping from $\mathcal{C}_v(\overline{\mathbb{R}}_{\min})$ to $\mathcal{S}$ does exist because $\mathcal{F}$ is continuous. Then

$$f_c = \mathcal{F}^\sharp(g) \tag{38}$$

is the *greatest* solution [6] of (37) (that is, the least solution with respect to conventional order).

Let us give an explicit expression for (38). A subsolution $f(\cdot)$ of (37) is defined by:

$$\begin{aligned} &\oint_t f(t) \otimes x^t \preceq g(x), \quad \forall x \\ &\Leftrightarrow \quad f(t) \preceq g(x) \otimes x^{-t}, \quad \forall x, \forall t \ . \end{aligned}$$

Therefore, the greatest subsolution f_c is given by:

$$f_c(t) = \bigwedge_x g(x) \otimes x^{-t} \ . \tag{39}$$

In conventional notation, this reads

$$f_c(t) = \sup_x [g(x) - xt] \ ,$$

from which it is clear that f_c is convex l.s.c. (as the upper hull of a family of affine functions). This is indeed the convex hull of all f in the subset $\mathcal{F}^{-1}(g)$ (the convex hull is less that all such f with respect to $\leq$, hence it is the greatest such f with respect to $\preceq$). We summarize the above considerations in the following theorem.

[6] indeed 'subsolution' in the general theory, but here 'solution' since $\mathcal{F}$ is surjective

Theorem 23 *The subset $\mathcal{F}^{-1}(g)$ admits a maximum element [7] f_c which is given by (39) and which is the convex hull of all f in the same subset.*

Remark 24 The expression x^t (power function of x) appearing in (35), where x and t are both in $\overline{\mathbb{R}}_{\min}$ means $x \times t$ in conventional notation and it can also be written t^x in $\overline{\mathbb{R}}_{\min}$ (exponential function of x). Writing it t^x in (35) shows that concave u.s.c. functions (of x) may be considered as integrals of 'weighted'—by $f(t)$—exponentials. The analogy of $\mathcal{F}$ with the Fourier or Laplace transform is then more explicit. However, in the expression of the 'inverse' transform $\mathcal{F}^\sharp$ (see (39)), for symmetry reasons, the notation x^{-t} should be kept, and convex l.s.c. functions appear as lower bounds of 'weighted' exponentials.

Indeed, the symmetry with respect to the pair of dual variables t and x—the latter being denoted $j\omega$ or s for Fourier or Laplace transform—is preserved in the usual case by the notation $\exp(t \times x)$. An analogous notation here would be $1^{t \times x}$ which is the same as x^t or t^x in $\overline{\mathbb{R}}_{\min}$. ∎

Remark 25 Observe that the subset of convex l.s.c. impulse responses (let us denote it $\mathcal{S}_{cx}$) is stable by multiplication (inf-convolutions of convex l.s.c. functions yield convex l.s.c. functions), but not by addition (the lower hull of convex functions is not in general a convex function; it is rather stable by $\wedge$, that is pointwise supremum). Therefore, this subset is not a subdioid of $\mathcal{S}$. However, since $\mathcal{F}$ is a $\otimes$-morphism, and since $\mathcal{S}_{cx}$ is in a one-to-one correspondence with $\mathcal{C}_v(\overline{\mathbb{R}}_{\min})$, we have

$$\forall f_c, h_c \in \mathcal{S}_{cx}, \quad f_c \otimes h_c = \mathcal{F}^\sharp\left(\mathcal{F}(f_c) \otimes \mathcal{F}(h_c)\right) \ ,$$

in which we recall that the sign $\otimes$ on the left-hand side means 'inf-convolution', whereas it means pointwise $+$ on the right-hand side. Using the explicit expression of $\mathcal{F}$ and $\mathcal{F}^\sharp$, the formula above may be rewritten:

$$\forall t, \quad [f_c \otimes h_c](t) = \bigwedge_x \oint_\theta \oint_\tau f_c(\theta) \otimes h_c(\tau) \otimes x^{\theta+\tau-t} \ .$$

∎

Remark 26 From residuation theory, it is known that since $\mathcal{F}$ is surjective,

$$\mathcal{F} \circ \mathcal{F}^\sharp(g) = g, \quad \forall g \in \mathcal{C}_v(\overline{\mathbb{R}}_{\min}) \ . \tag{40}$$

It means that, $\forall g \in \mathcal{C}_v(\overline{\mathbb{R}}_{\min})$ and $\forall y \in \overline{\mathbb{R}}_{\min}$,

$$\oint_t \left(\bigwedge_x g(x) \otimes x^{-t}\right) \otimes y^t = g(y) \ . \tag{41}$$

From the point of view of convexity, this can be proved by using a saddle-point argument (interchange of sup with inf for convex-concave functions). ∎

[7] i.e. a least upper bound which belongs to the subset

6.3 Concave u.s.c. Impulse Responses

Unlike $\mathcal{S}_{cx}$ which is stable for multiplication but not for addition, the subset of concave u.s.c. impulse responses—let us denote it $\mathcal{S}_{cv}$—is stable for addition and multiplication. Unfortunately, it does not contain the identity element $e(\cdot)$ which belongs to $\mathcal{S}_{cx}$ but not to $\mathcal{S}_{cv}$. Therefore, $\mathcal{S}_{cv}$ is not a subdioid of $\mathcal{S}$ either.

The intersection of $\mathcal{S}_{cx}$ and $\mathcal{S}_{cv}$ is the subset of linear—in the conventional sense—functions of t $\{\ell_a(\cdot) \mid a \in \overline{\mathbb{R}}, \ell_a : t \mapsto a \times t\}$. Referring back to Remark 24, ℓ_a is an exponential in $\mathcal{S}$ ($\ell_a(t) = a^t$). As already observed, concave functions are integrals of weighted exponentials, that is, $\forall f_v \in \mathcal{S}_{cv}$, there exist 'coordinates' $\{\widetilde{f}_v(a)\}_{a\in\overline{\mathbb{R}}}$ such that:

$$\forall t, \quad f_v(t) = \oint_a \widetilde{f}_v(a) \otimes a^t \quad . \tag{42}$$

This is a 'spectral' decomposition of concave functions on the 'basis' of exponentials. As a matter of fact, we are going to prove that exponentials, used as input signals, are 'eigenfunctions' for time-invariant linear systems, in the same way as sine functions are eigenfunctions in conventional system theory.

Theorem 27 *For all impulse responses $h \in \mathcal{S}$ and all scalars a, we have*

$$h \otimes \ell_a = \lambda \cdot \ell_a \quad \textit{with} \quad \lambda = [\mathcal{F}(h)](-a) \quad .$$

Proof In conventional notation,

$$\inf_s [h(s) + a(t-s)] = \inf_s [h(s) + (-a)s] + at \quad .$$

To complete the proof, it suffices to go back to dioid notation and to remember (36). ■

Remark 28 Concentrating our attention on the 'vector space' structure, that is forgetting the internal product $\otimes$, the identity mapping $\mathcal{I} : \mathcal{S}_{cv} \to \mathcal{C}_v(\overline{\mathbb{R}}_{\min})$ is an homomorphism for this structure. Then using identity (41) for $g = \mathcal{I}(f_v)$, for any $f_v \in \mathcal{S}_{cv}$, we obtain an explicit formula for the 'coordinates' $\widetilde{f}_v(a)$ involved in (42), namely

$$\widetilde{f}_v(a) = \bigwedge_t f_v(t) \otimes a^{-t} \quad .$$

Observe that $a \mapsto \widetilde{f}_v(a)$ is convex l.s.c. ■

Remark 29 Using distributivity of $\otimes$ (inf-convolution in $\mathcal{S}$) with respect to $\oint$, we have, $\forall f_v \in \mathcal{S}_{cv}$, and $\forall h \in \mathcal{S}$,

$$\begin{aligned} h \otimes f_v &= h \otimes \left(\oint_a \widetilde{f}_v(a) \cdot \ell_a \right) \\ &= \oint_a \widetilde{f}_v(a) \cdot (h \otimes \ell_a) \\ &= \oint_a \left(\widetilde{f}_v(a) \otimes [\mathcal{F}(h)](-a) \right) \cdot \ell_a \quad , \end{aligned}$$

which also belongs to $\mathcal{S}_{cv}$. ■

In conclusion, the subset $\mathcal{S}_{cv}$ of concave u.s.c. functions correspond to functions of L^2 in conventional system theory, which admit a decomposition over the basis of sine functions (here exponentials).

7 About monotone input time functions

In the examples considered in § 2 and serving as a motivation for this theory, it has been realized that meaningful inputs $u(\cdot)$ are nondecreasing—in the conventional sense—time functions, i.e., with the order of $\overline{\mathbb{R}}_{\min}$,

$$u(t) \succeq u(t+\theta), \quad \forall t, \forall \theta \geq 0 \quad . \tag{43}$$

To avoid ambiguity, we say 'monotone' to mean property (43) throughout this section. We are going to show that the subset of $\mathcal{S}$ of monotone functions is a kind of 'ideal' which is also a dioid, and that outputs and impulse responses of time-invariant linear systems can also naturally be constrained to lie in this ideal.

Let

$$\overline{e} = \oint_{\theta\geq 0} \delta^{-\theta} \quad .$$

Lemma 30

1. *One has that*
$$\overline{e} \otimes \overline{e} = \overline{e} \quad ; \tag{44}$$
2. *an element $u \in \mathcal{S}$ is monotone if and only if it satisfies*
$$u = \overline{e} \otimes u \quad ; \tag{45}$$
3. *given $u \in \mathcal{S}$, then $\overline{u} \overset{def}{=} \overline{e} \otimes u$ is the least* monotone *element of $\mathcal{S}$ which is larger than u (in the sense of $\preceq$);*
4. *the following defines an equivalence relation*
$$u \equiv v \Longleftrightarrow \overline{e} \otimes u = \overline{e} \otimes v \quad , \tag{46}$$
which is compatible with the dioid structure of $\mathcal{S}$ (and the external product $\cdot$ with scalars). Therefore, the quotient of $\mathcal{S}$ by this equivalence relation has the same algebraic structure as $\mathcal{S}$ and it is isomorphic to
$$\overline{e} \otimes \mathcal{S} \overset{def}{=} \{\overline{e} \otimes u \mid u \in \mathcal{S}\} \quad . \tag{47}$$

Proof

1. $\overline{e} \otimes \overline{e} = \oint_{\theta\geq 0} \oint_{\tau\geq 0} \delta^{-(\theta+\tau)} = \oint_{t\geq 0} \delta^{-t} = \overline{e}$;
2. the condition (43) is equivalent to
$$u \succeq \delta^{-\theta} \otimes u, \forall \theta \geq 0 \Leftrightarrow u = \left(\oint_{\theta\geq 0} \delta^{-\theta} \right) \otimes u \quad ;$$
3. if $\overline{u} = \overline{e} \otimes u$, then $\overline{u} \succeq u$ since $\overline{e} \succeq \delta^0 = e$, and $\overline{u}$ is monotone since it verifies (45) thanks of (44). On the other hand, for any v such that $v \succeq u$ and v monotone, one must have $v = \overline{e} \otimes v \succeq \overline{e} \otimes u = \overline{u}$;

4. this part of the proof is left to the reader who may refer to [7] in which similar results are proved.

■

Indeed, it can be checked that point 4 of the lemma, and the following considerations, extend to any situation when some $\overline{e}$ (not necessarily that related to nondecreasing functions) satisfies (44).

With this lemma at hand, it is realized that if we restrict ourselves to monotone inputs, then outputs are automatically monotone and impulse responses can also be restricted to be monotone, that is the 'world' can be restricted to the 'ideal' (47). As a matter of fact, considering any time-invariant linear system characterized by its impulse response h, and a monotone input u, we have

$$y = h \otimes u = h \otimes \overline{e} \otimes u = (\overline{e} \otimes h) \otimes u = \overline{e} \otimes y \ ,$$

which shows that y is monotone and that h can be replaced by its monotone version $\overline{e} \otimes h$.

We end this section by giving the monotone versions of (32), (33) and (34).

$$\overline{\gamma^c}(t) = \begin{cases} c & \text{if } t \le 0 \ ; \\ \varepsilon & \text{otherwise.} \end{cases}$$

$$\overline{\delta^d}(t) = \begin{cases} e & \text{if } t \le d \ ; \\ \varepsilon & \text{otherwise.} \end{cases}$$

$$\overline{\omega_\beta}(t) = \begin{cases} e & \text{if } t \le 0 \ ; \\ \beta^t & \text{otherwise.} \end{cases}$$

(recall that $\beta \ge 0$).

Remark 31 Observe that, in $\overline{e} \otimes \mathcal{S}$, impulse responses $\overline{h}(\cdot)$ of causal time-invariant linear systems are no longer characterized by the condition $\overline{h}(t) = \varepsilon$, $\forall t < 0$. ■

8 Rational Systems

In this section, we are interested in subsets of functions in $\mathcal{S}$ which can be 'finitely generated'. More precisely, we consider a subset of $\mathcal{S}$, say $\mathcal{K}$, containing ε and e and some other 'generating' elements, and we define its dioid closure and its rational closure. We give 'realizations' of rational elements which are particular representations of these rational systems by elementary (or 'generating') systems cascaded in parallel, series and feedback. In particular cases, rational systems are characterized by periodic impulse responses.

Definition 32 *The* dioid closure $\mathcal{K}^\circ$ *of* $\mathcal{K}$ *relative to* $\mathcal{S}$ *is the least subdioid of* $\mathcal{S}$ *containing* $\mathcal{K}$.

This definition is well-posed since the set of subdioids containing $\mathcal{K}$ is nonempty (it contains $\mathcal{S}$ itself) and this set has a minimum element (for the order relation $\subset$) since the intersection (greatest lower bound) of a collection of subdioids is a subdioid. The terminology 'closure' is justified because $(\mathcal{K}^\circ)^\circ = \mathcal{K}^\circ$. Clearly, $\mathcal{K}^\circ$ consists of all elements of $\mathcal{S}$ which can be obtained by a *finite* number of operations $\oplus$ and $\otimes$ involving elements of $\mathcal{K}$ only.

The idea now is to consider affine equations of the type

$$y = h \otimes y \oplus b \ , \tag{48}$$

with data h and b in $\mathcal{K}^\circ$. The least solution $h^* \otimes b$ (see Theorem 12) does not necessarily belong to $\mathcal{K}^\circ$ since the star operation involves an infinite sum. So doing, one may produce elements out of $\mathcal{K}^\circ$ from data in $\mathcal{K}^\circ$. One can then use these new elements as data of other affine equations, and so on and so forth. The 'rational closure' of $\mathcal{K}$, hereafter defined, is essentially the stable structure that contains all elements one can produce by repeating these operations a *finite* number of times.

Definition 33 *The* rational closure $\mathcal{K}^*$ *of* $\mathcal{K} \subset \mathcal{S}$, *is the least subdioid of* $\mathcal{S}$ *containing* $\mathcal{K}$ *and all* finite *sums, products and star operations over its elements.*

This definition is well-posed for the same reason as before. Moreover, it is clear that $(\mathcal{K}^\circ)^* = (\mathcal{K}^*)^* = \mathcal{K}^*$.

Before proceeding futher let us give some examples.

Example 34 Let $\mathcal{K} = \{\varepsilon\} \cup \{\gamma^c\}_{c \in \overline{\mathbb{R}}} \cup \{\delta\}$. Then $\mathcal{K}^\circ$ may be considered as the subdioid of polynomials in δ with coefficients in $\overline{\mathbb{R}}_{\min}$ since $\gamma^c \otimes \delta^n = c \cdot \delta^n$, and $\mathcal{K}^*$ is the subdioid of 'rational' power series. As functions of time, they can take any value in $\overline{\mathbb{R}}$ on $\mathbb{N}$ and the value $+\infty$ elsewhere. ■

Example 35 Here, we limit ourselves to the subdioid $\overline{e} \otimes \mathcal{S}$ of monotone functions. We take $\mathcal{K} = \{\varepsilon\} \cup \{\overline{\gamma^c}\}_{c \in \overline{\mathbb{R}}} \cup \{\overline{\delta}\}$. The subdioid $\mathcal{K}^*$ of rational power series in $\overline{\delta}$ includes time functions wich are nondecreasing, piecewise constant, continuous to the left, with values in $\overline{\mathbb{R}}$ and discontinuities on $\mathbb{N}$. ■

Example 36 Here we restrict $\mathcal{K}$ to be $\{\varepsilon, \overline{e}, \overline{\gamma}, \overline{\delta}\} \subset \overline{e} \otimes \mathcal{S}$. The main difference with the previous example is that functions are now integer valued. ■

Example 37 We take $\mathcal{K} = \{\varepsilon\} \cup \{\overline{\gamma^c}\}_{c \in \overline{\mathbb{R}}} \cup \{\overline{\omega_{\beta_i}}\}_{i \in I} \subset \overline{e} \otimes \mathcal{S}$, where I is a finite set and $\beta_i \ge 0, \forall i \in I$. According to the forthcoming theorem 40, the corresponding functions of time t are nondecreasing, they take the value $e = 0$ for $t \le 0$, they are piecewise linear and concave on $t \ge 0$, with slopes belonging to the set $\{\beta_i\}_{\in I}$. ■

Example 38 We take $\mathcal{K} = \{\varepsilon, \overline{e}, \overline{\gamma}, \overline{\delta}\} \cup \{\overline{\omega_{\beta_i}}\}_{i \in I} \subset \overline{e} \otimes \mathcal{S}$. We get nondecreasing functions, with jumps located on $\mathbb{N}$, piecewise linear elsewhere, with slopes belonging to $\{\beta_i\}_{\in I}$. ■

The following realization theorem is Theorem 19 of [7]. It states that, to obtain elements of $\mathcal{K}^*$, it suffices to arrange a finite number of systems with impulse responses in $\mathcal{K}^\circ$ in parallel and serial cascades, and with a *single* level of feedback loops (algebraically, a single level of star operations).

Theorem 39 *For all* $h \in \mathcal{K}^*$, *there exist* $n \in \mathbb{N}$ *and* $\{(a_i, b_i)\}_{i=1,\ldots,n} \subset \mathcal{K}^\circ$ *such that:*

$$h = \bigoplus_{i=1}^{n} a_i \otimes (b_i)^* \ . \tag{49}$$

The problem of 'minimal realization' is yet unsolved.

The following theorem deals with the case of Example 37.

Theorem 40 *With the choice of $\mathcal{K}$ made in Example 37, any $h \in \mathcal{K}^*$ can be written:*

$$h = \bigoplus_{i \in I} c_i \cdot \overline{\omega_{\beta_i}} \quad \textit{with} \quad c_i \in \overline{\mathbb{R}}_{\min}, \forall i \in I \ . \qquad (50)$$

Proof This is indeed a corollary of Theorem 39. First, remember that $\overline{\gamma^c} \otimes u = c \cdot u$. Second, recall the rules (25)–(26) which also apply to the $\overline{\omega_{\beta_i}}$. It is then immediate to see that expressions such as (49) reduce to (50). Moreover, $\mathcal{K}^*$ is the same as $\mathcal{K}^\circ$ in this case. ∎

The following theorem is Theorem 21 of [7]. It deals with Example 36.

Theorem 41 *With the choice of $\mathcal{K}$ made at Example 36, any $h \in \mathcal{K}^*$ can be written:*

$$h = p(\overline{\gamma}, \overline{\delta}) \oplus \left[\overline{\gamma}^\nu \otimes \overline{\delta}^\tau \otimes \left(\overline{\gamma}^r \otimes \overline{\delta}^s \right)^* \otimes q(\overline{\gamma}, \overline{\delta}) \right] \ ,$$

where $\nu, \tau, r, s \in \mathbb{N}$, $p(\overline{\gamma}, \overline{\delta})$ is a polynomial of maximum degree $\nu - 1$ in $\overline{\gamma}$ and $\tau - 1$ in $\overline{\delta}$ with coefficients in $\{\varepsilon, \overline{e}\}$, and $q(\overline{\gamma}, \overline{\delta})$ is a similar polynomial of maximum degree $r - 1$ in $\overline{\gamma}$ and $s - 1$ in $\overline{\delta}$.

The above form expresses a periodic behavior of the impulse response h: p represents the transient part and q represents a pattern of 'width' s and 'height' r which is reproduced indefinitely after the transient part of the response.

Finally, the following theorem is just the synthesis of the last two theorems and it deals with Example 38.

Theorem 42 *With the choice of $\mathcal{K}$ made at Example 38, any $h \in \mathcal{K}^*$ can be written as in Theorem 41, with polynomials p and q of the same degrees in $\overline{\gamma}$ and $\overline{\delta}$, but with coefficients linear in the $\overline{\omega_{\beta_i}}$.*

9 Conclusion

In this paper, a 'min-plus linear system' theory has been developped as an extension of that presented in [7], and it has been shown that its scope is not limited to discrete event systems. Continuous or mixed systems subject to synchronization and saturation constraints are also encompassed. Such systems can be characterized by their impulse responses, and, in the case of time-invariant systems, the responses to general input functions are obtained by inf-convolutions. A notion of transfer function has also been associated with the impulse response by a transform closely related to the Fenchel transform. However, only impulse responses which are convex l.s.c. time functions are unambiguously characterized by their associated transfer functions. Another class of impulse responses, namely concave u.s.c. time functions, can be decomposed on a basis of eigenfunctions which are exponential functions of our algebra, and which play the role played by sine functions in conventional linear system theory. Finally, issues pertaining to rationality of some subclasses of systems have been addressed.

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