Task Resource Models and (max,+) Automata

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Abstract

We show that a typical class of timed concurrent systems can be modeled as automata with multiplicities in the (max,+) semiring. This representation can be seen as a timed extension of the logical modeling in terms of trace monoids. We briefly discuss the applications of this algebraic modeling to performance evaluation.

1 Introduction

Different variations of (stochastic) queuing networks with precedence-based relations between customers have been studied for quite a long time in the performance evaluation community, see [3, 5, 20]. In the combinatorics community on the other hand, concurrent systems are usually modeled in terms of traces —elements of free partially commutative monoids—, see [8, 11]. An equivalent formalism is that of *heaps of pieces* [19].

One of the purposes of this note is to bridge the gap between the two approaches. In the first part of the paper, we establish the relations between the models. An important feature is that execution times of these models can be represented as finite dimensional (max,+) linear dynamical systems. In an essentially equivalent way, they are recognized by automata with multiplicities in the (max,+) semiring. The existence of similar (max,+) models was already noticed in the context of queuing theory [20, 7]. Their analogue for trace monoids seems to be new.

In the second part of the paper, we apply this algebraic modeling to performance evaluation problems. We present asymptotic results on the existence of mean execution time for random schedules, and for optimal and worst schedules. They are obtained by appealing to subadditive arguments borrowed from the theory of random (max,+) matrices [1].

At last, we apply the machinery of (max,+) rational series to the exact computation of the asymptotic worst case mean execution time, when the set of admissible schedules is given by a rational language.

Some generalizations of Task Resource models will be considered in a forthcoming paper [16] (heaps of pieces with arbitrary shapes, for which all the results can be extended). These models provide an algebraic framework to handle scheduling problems.

2 Basic Task Resource Model

2.1 General Presentation

Definition 2.1 (Task Resource System). *A (timed)* Task Resource *system is a 4-uple* $\mathcal{T} = (\mathcal{A}, \mathcal{R}, R, h)$ *where:*

- $\mathcal{A}$ *is a finite set whose elements are called* tasks.
- $\mathcal{R}$ *is a finite set whose elements are called resources.*
- $R : \mathcal{A} \to \mathcal{P}(\mathcal{R})$ *gives the subset of resources required by a task. We assume that each task requires at least one resource:* $\forall a \in \mathcal{A}, R(a) \neq \emptyset$.
- $h : \mathcal{A} \to \mathbb{R}^+$ *gives the execution time of a task.*

A length n *schedule* is a sequence of n tasks $a_1, \ldots, a_n$, that we will write as a word[1] $w = a_1 \ldots a_n$. The functioning of the system under the schedule w is as follows.

1. All the resources become initially available at time zero.
2. Task a_i begins as soon as all the required resources $r \in R(a_i)$ used by the earlier tasks $a_j, j < i$, become free, say, at time t_i.
3. Task a_i uses each resource $r \in R(a_i)$ during $h(a_i)$ times units. Thus, resource r is released at time $t_i + h(a_i)$.

The *execution time* or *makespan* of the schedule $w = a_1 \ldots a_n$ is the completion time of the latest task of the schedule (which is not necessarily a_n):

$$y(w) \stackrel{\text{def}}{=} \max_{1 \leqslant i \leqslant n} (t_i + h(a_i)) \ . \tag{2.1}$$

Task Resource systems are intimately related with the classical *trace monoids* that we next define.

Definition 2.2. *A* dependence alphabet *is an alphabet* $\mathcal{A}$ *equipped with a reflexive symmetric relation called* dependence relation, *denoted* D, *and written graphically* —. *We denote by* I *the complement of* D *(called* independence *relation).*

[1] We recall the following usual notation. Given a finite set (alphabet) $\mathcal{A}$, we denote by $\mathcal{A}^n$ the set of words of length n on $\mathcal{A}$. We denote by $\mathcal{A}^*$ the free monoid on $\mathcal{A}$, that is, the set of finite words equipped with concatenation. The unit (empty word) will be denoted e. We denote by $\mathcal{A}^+ = \mathcal{A}^* \setminus \{e\}$ the free semigroup on $\mathcal{A}$. The length of the word w will be denoted $|w|$. We shall write $|w|_a$ for the number of occurrences of a given letter a in w.

Definition 2.3. *The* trace monoid $\mathbb{M}(\mathcal{A}, D)$ *is the quotient of the free monoid* $\mathcal{A}^*$ *by the congruence* $\sim$ *generated by the relations* $ab = ba, \forall a\, I\, b$. *The elements of* $\mathbb{M}(\mathcal{A}, D)$ *will be called* traces*.*

Let alph(w) denote the set of letters appearing in word w. The word $\overline{w} \sim w$ is a *Cartier-Foata normal form* of w [8, 11] if we have a factorization $\overline{w} = u_1 \dots u_p$, $u_i \in \mathcal{A}^+$, such that:

$$a, b \in \text{alph}(u_i) \Rightarrow a\, I\, b, \quad a \in \text{alph}(u_i) \Rightarrow \exists b \in \text{alph}(u_{i-1}), a\, D\, b \ . \tag{2.2}$$

Such a normal form is unique up to a reordering of the letters inside factors. We shall denote by $\ell(w) = p$ the length (number of factors) of the normal form of w.

With each Task Resource system is associated a dependence relation over the alphabet $\mathcal{A}$; tasks are dependent when they share some resource:

$$a\, D\, b \Leftrightarrow R(a) \cap R(b) \neq \emptyset \ . \tag{2.3}$$

Conversely, starting from an arbitrary trace monoid $\mathbb{M}(\mathcal{A}, D)$, one can build an associated Task Resource system. For example, one can consider $\mathcal{T} = (\mathcal{A}, \mathcal{R}, R, h \equiv 1)$ with $\mathcal{R} = \{\{a, b\} \mid \ a\, D\, b\}$ and $R(a) = \{r \in \mathcal{R} \mid \ a \in r\}$.

Proposition 2.4. (i) *When* $h \equiv 1$, $y(w) = \ell(w)$*: the makespan is equal to the length of the Cartier-Foata normal form of* w*.* (ii) *For general execution times* h*,*

$$y(w) = \max \sum_{j=1}^{p} h(a_{i_j}) \ , \tag{2.4}$$

where the max is taken over the subwords $a_{i_1} \dots a_{i_p}$ *of* $w = a_1 \dots a_n$*, composed of consecutive dependent letters (i.e.* $a_{i_j}\, D\, a_{i_{j+1}}$*).*

The first assertion is classical [9]. It implies in particular that the makespan of Task-Resource systems with $h \equiv 1$ can be represented in a more intrinsic way in terms of trace monoid. The second one can easily be proved by elementary means, or deduced from the (max,+)-linear representation given below. It provides an alternative formula for (2.1).

Example 2.5. For the sequential dependence alphabet $a\, D\, b$, we have $y(w) = h(a)|w|_a + h(b)|w|_b$. For the purely parallel dependence alphabet $a\, I\, b$, we have $y(w) = \max(h(a)|w|_a, h(b)|w|_b)$.

Example 2.6 (Ring Network). Consider a ring shaped communication network with k stations $\mathcal{R} = \{r_1, \dots, r_k\}$. Messages can be sent between neighbor stations. The possible messages are $\mathcal{A} = \{a_1, \dots, a_k\}$ where a_i corresponds to a communication between r_i and r_{i+1} (with the convention $k + 1 = 1$). Therefore, we have $R(a_i) = \{r_i, r_{i+1}\}$. This system can also be viewed as a variant of the classical dining philosophers model [12] (replace stations by chopsticks, messages by philosophers). E.g., for $k = 5$, $y(a_1a_2a_4a_1a_5) = \max(2h(a_1) + h(a_2) + h(a_5), h(a_4) + h(a_5))$ (direct application of 2.4,(ii) since the maximal dependent subwords taken from $a_1a_2a_4a_1a_5$ are $a_1a_2a_1a_5$ and a_4a_5).

2.2 Linear Representation over the (max,+) Semiring

Definition 2.7. *The* (max,+) semiring $\mathbb{R}_{\max}$ *is the set* $\mathbb{R} \cup \{-\infty\}$*, equipped with* $\max$*, written additively (i.e.* $a \oplus b = \max(a, b)$*) and the usual sum, written multiplicatively (i.e.* $a \otimes b = a + b$*). We write* $\varepsilon = -\infty$ *for the zero element, and* $e = 0$ *for the unit element.*

We shall use throughout the paper the matrix and vector operations induced by the semiring structure[2]. The identity matrix ($\mathrm{I}_{ii} = e, \mathrm{I}_{ij} = \varepsilon, i \neq j$) with entries indexed by X will be denoted by I_X. The row vector with entries indexed by X and all equal to e will be denoted by e_X. We denote by $\|M\| = \bigoplus_{ij} M_{ij}$ (resp. $\|v\| = \bigoplus_i v_i$) the (max,+) *norm* of a matrix M (vector v).

A (max,+) *automaton*[3] of dimension k over the alphabet $\mathcal{A}$ is a triple $(\alpha, \mathcal{M}, \beta)$, where $\alpha \in \mathbb{R}_{\max}^{1\times k}$, $\beta \in \mathbb{R}_{\max}^{k\times 1}$, and $\mathcal{M}$ is a morphism from $\mathcal{A}^*$ to the multiplicative monoid of matrices $\mathbb{R}_{\max}^{k\times k}$. A map $y : \mathcal{A}^* \to \mathbb{R}_{\max}$ is *recognizable* if there is an automaton such that $y(w) = \alpha\mathcal{M}(w)\beta$.

In a spirit closer to discrete event systems theory, automata may be seen as (max,+) linear systems whose dynamics is indexed by letters. Indeed, introducing the "state vector" $x(w) \stackrel{\text{def}}{=} \alpha\mathcal{M}(w) \in \mathbb{R}_{\max}^{1\times k}$, we get

$$x(e) = \alpha, \quad x(wa) = x(w)\mathcal{M}(a), \quad y(w) = x(w)\beta \ , \quad \text{or} \tag{2.5}$$

$$y(a_1 \dots a_n) = \alpha\mathcal{M}(a_1)\dots\mathcal{M}(a_n)\beta \ . \tag{2.6}$$

Definition 2.8 (Task & Resource Daters). *A* dater *over the alphabet* $\mathcal{A}$ *is a scalar map* $\mathcal{A}^* \to \mathbb{R} \cup \{-\infty\}$*. With each task* $a \in \mathcal{A}$ *is associated a* task dater x_a*:* $x_a(w)$ *gives the time of completion of the last task of type* a *in the schedule* w*. With each resource* $r \in \mathcal{R}$ *is associated a* resource dater x_r*:* $x_r(w)$ *gives the last instant of release of the resource* r *under the schedule* w*. We shall denote by* $x_{\mathcal{A}}$ *and* $x_{\mathcal{R}}$ *the vectors of task and resource daters.*

Note the important duality relations

$$x_a(w) = \bigoplus_{r\in R(a)} x_r(w), \quad x_r(w) = \bigoplus_{a\in R^{-1}(r)} x_a(w) \ . \tag{2.7}$$

We identify each subset $R(a)$ with a boolean matrix of size $|\mathcal{R}| \times |\mathcal{A}|$ denoted $\mathcal{I}(a)$.

$$\forall a \in \mathcal{A}, \mathcal{I}(a)_{rb} = \begin{cases} e & \text{if } r \in R(a) \text{ and } b = a \\ \varepsilon & \text{otherwise .} \end{cases}$$

[2] I.e. for matrices A, B of appropriate sizes, $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \max(A_{ij}, B_{ij})$, $(A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj} = \max_k(A_{ik} + B_{kj})$, and for a scalar a, $(a \otimes A)_{ij} = a \otimes A_{ij} = a + A_{ij}$. We will abbreviate $A \otimes B$ to AB as usual.

[3] This is a specialization to the $\mathbb{R}_{\max}$ case of the notion of automaton with multiplicities over a semiring (or equivalently, of recognizable series over a semiring). See [13, 6].

We define the following matrices:

$$\forall a \in \mathcal{A}, \quad \mathcal{M}_{\mathcal{R}}(a) = \mathrm{I}_{\mathcal{R}} \oplus h(a)\mathcal{I}(a)\mathcal{I}(a)^T \ , \tag{2.8}$$

$$\mathcal{M}_{\mathcal{A}}(a) = \mathrm{I}_{\mathcal{A}} \oplus h(a)\left(\bigoplus_b \mathcal{I}(b)^T\right)\mathcal{I}(a) \ , \tag{2.9}$$

or more explicitly

$$\mathcal{M}_{\mathcal{R}}(a)_{rs} = \begin{cases} e & \text{if } r = s, s \notin R(a), \\ h(a) & \text{if } r \in R(a), s \in R(a), \\ \varepsilon & \text{otherwise.} \end{cases} \tag{2.10}$$

$$\mathcal{M}_{\mathcal{A}}(a)_{bc} = \begin{cases} e & \text{if } a \neq (b = c), \\ h(a) & \text{if } a = c, bDc, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{2.11}$$

We extend $\mathcal{M}_{\mathcal{A}}$ (resp. $\mathcal{M}_{\mathcal{R}}$) to a morphism $\mathcal{A}^* \to \mathbb{R}_{\max}^{\mathcal{A}\times\mathcal{A}}$ (resp. $\mathcal{A}^* \to \mathbb{R}_{\max}^{\mathcal{R}\times\mathcal{R}}$).

Theorem 2.9. *The dater functions of task resource systems admit the following linear representations over the (max,+) semiring:*

$$x_{\mathcal{R}}(wa) = x_{\mathcal{R}}(w)\mathcal{M}_{\mathcal{R}}(a), \quad x_{\mathcal{R}}(e) = e_{\mathcal{R}} \ , \tag{2.12}$$

$$x_{\mathcal{A}}(wa) = x_{\mathcal{A}}(w)\mathcal{M}_{\mathcal{A}}(a), \quad x_{\mathcal{A}}(e) = e_{\mathcal{A}} \ , \tag{2.13}$$

$$y(w) = \|x_{\mathcal{A}}(w)\| = \|x_{\mathcal{R}}(w)\| = \|\mathcal{M}_{\mathcal{A}}(w)\| = \|\mathcal{M}_{\mathcal{R}}(w)\| \ . \tag{2.14}$$

In other words, y is recognized both by the *resource automaton* $(e_{\mathcal{R}}, \mathcal{M}_{\mathcal{R}}, e_{\mathcal{R}}^T)$ and by the *task automaton* $(e_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}}, e_{\mathcal{A}}^T)$.

Proof. We have

$$x_a(wb) = \begin{cases} x_a(w) & \text{if } a \neq b, \\ \max_{r\in R(a)} x_r(w) + h(a) & \text{if } a = b, \end{cases} \tag{2.15}$$

$$x_r(e) = x_a(e) = e \ . \tag{2.16}$$

These relations are a simple translation of the functioning of the system, as described after Definition 2.1 (items 1,2,3). Eliminating x_r in (2.15) using (2.7), we get the *task equation*

$$x_a(wb) = \begin{cases} x_a(w) & \text{if } a \neq b \\ \max_{cDa} x_c(w) + h(a) & \text{if } a = b. \end{cases} \tag{2.17}$$

Dually, it is not difficult to obtain the *resource equation*

$$x_r(wa) = \begin{cases} x_r(w) & \text{if } R(a) \not\ni r \\ \max_{s\in R(a)} x_s(w) + h(a) & \text{if } R(a) \ni r. \end{cases} \tag{2.18}$$

Rewriting (2.17) and (2.18) with the semiring notations, we get (2.12),(2.13). $\square$

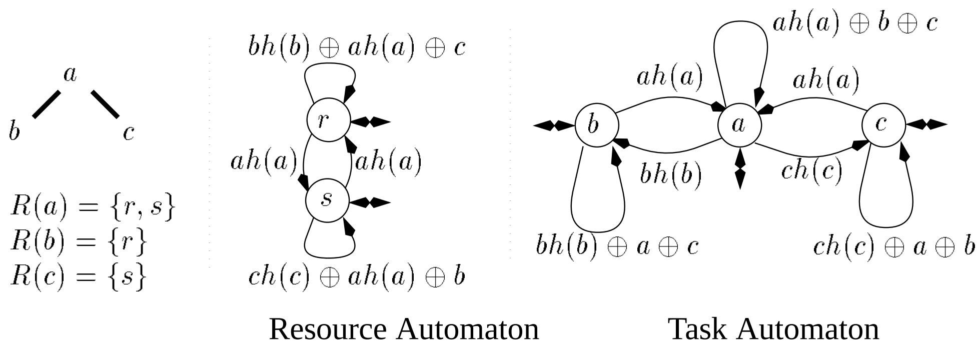

Figure 1: Task and Resource Automata for b—a—c

Example 2.10. We consider a Task Resource model with dependence alphabet b—a—c. In Fig. 1, we have represented[4] the resource automaton $(e_{\mathcal{R}}, \mathcal{M}_{\mathcal{R}}, e^T_{\mathcal{R}})$ and the task automaton $(e_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}}, e^T_{\mathcal{A}})$ associated with the dependence alphabet b—a—c. The matrices associated with the resource automaton are:

$$\mathcal{M}_{\mathcal{R}}(a) = \begin{bmatrix} h(a) & h(a) \\ h(a) & h(a) \end{bmatrix}, \mathcal{M}_{\mathcal{R}}(b) = \begin{bmatrix} h(b) & \varepsilon \\ \varepsilon & e \end{bmatrix}, \ \mathcal{M}_{\mathcal{R}}(c) = \begin{bmatrix} e & \varepsilon \\ \varepsilon & h(c) \end{bmatrix} .$$

The makespan $y(w)$ is equal to the maximal weight of a path labeled w between two arbitrary nodes of the graph. E.g., $y(cba) = \max(h(c) + h(a), h(b) + h(a))$.

2.3 Interpretation in Terms of Heaps of Pieces

There is a useful geometrical interpretation of Task Resource Models in terms of *heaps of pieces*. This interpretation was first noticed by Viennot for trace monoids. The reader is referred to [19] for a more formal presentation. Imagine an horizontal axis with as many slots as resources. With each letter a is associated a *piece*, i.e. a solid "rectangle" occupying the slots $r \in R(a)$, with height $h(a)$. The heap associated with the word $w = a_1 \dots a_n$ is built by piling up the pieces $a_1, \dots, a_n$, in this order. The makespan $y(w)$ coincides with the height of the heap. The vector $x_{\mathcal{R}}(w) = e_{\mathcal{R}}\mathcal{M}_{\mathcal{R}}(w)$ can be interpreted as the upper contour of the heap. Adding one piece above the heap amounts to right multiplication by the corresponding matrix.

Example 2.11. Consider the ring model of Example 2.6 with $k = 4$ and $h \equiv 1$. We have represented in Fig. 2.(I), the heap associated associated with the word

[4] An automaton $(\alpha, \mathcal{M}, \beta)$ of dimension k over an alphabet $\mathcal{A}$ is usually represented as a graph with nodes $1, \dots, k$, and three kinds of labeled and weighted arcs. There is an *internal arc* $i \to j$ with label $a \in \mathcal{A}$ and weight $\mathcal{M}(a)_{ij}$ whenever $\mathcal{M}(a)_{ij} = t \neq \varepsilon$. We will write $x \stackrel{at}{\to} y$ but we omit the unit valuations (when $t = e$). When there are two arcs $x \to y$ with respective labels a, b and weights t, t', we shall write $x \stackrel{at \oplus bt'}{\to} y$ as a shorthand for the two arcs $x \stackrel{at}{\to} y$, $x \stackrel{bt'}{\to} y$. There is an *input arc* at node i with weight α_i, whenever $\alpha_i \neq \varepsilon$. *Output arcs* are obtained in a dual way from β.

$a_1a_2a_3a_4a_4a_3a_2a_1$.

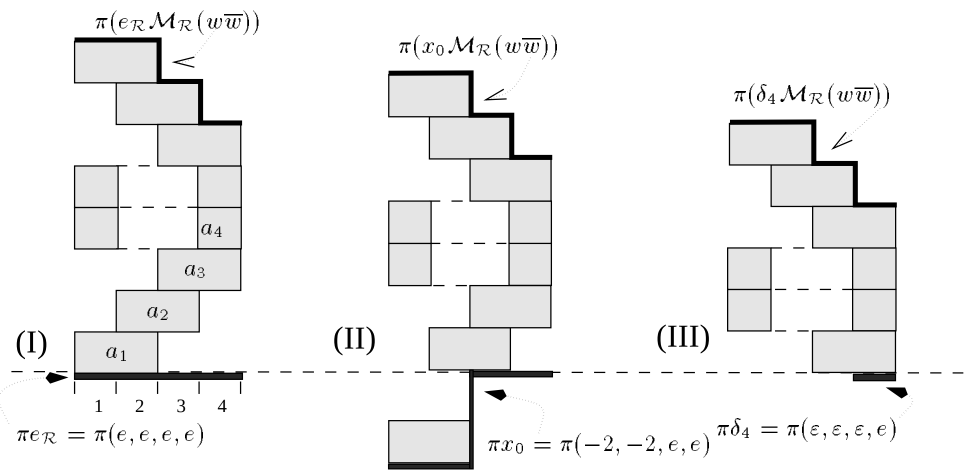

Figure 2: Heaps of pieces for a ring model.

3 Performance Evaluation

3.1 Stochastic Case

The simplest[5] stochastic extension of task resource systems arises when the sequence of tasks is given by a sequence of random variables $a(n) \in \mathcal{A}$: we get the random schedule $w_n = a(1)\dots a(n)$, and consider the asymptotics of $y(w_n), x(w_n)$, that we shall shorten to $y(n), x(n)$. For stochastic Task Resource models, we propose two types of asymptotic results.

1. First order limits or mean execution times $x(n)_i/n$.
2. Second order limits or asymptotics of relative delays $x(n)_i - x(n)_j$ (e.g. differences of last occupation times of the different resources).

Second order quantities are best defined in terms of (max,+) projective space. The (max,+) projective space $\mathbb{PR}^k_{\max}$ is the quotient of $\mathbb{R}^k$ by the parallelism relation $x \simeq y \Leftrightarrow \exists\lambda \in \mathbb{R}, x = \lambda y$. We write $\pi : \mathbb{R}^k_{\max} \to \mathbb{PR}^k_{\max}$ the canonical projection. The relative delays $x(n)_i - x(n)_j$ can be computed from $\pi x(n)$. Geometrically, $\pi x(n)$ corresponds to *the upper shape of the heap* (the quotient by $\simeq$ identifies two heaps with the same upper contour but different heights, cf. Fig. 2,(I)).

[5] In order to simplify the presentation, we shall not consider more general cases with random initial conditions, random executions times and random arrival times, which can be dealt with along the same lines.

We assume that the random variables $a(n)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$, equipped with a stationary and ergodic shift θ. We consider a *connected* Task Resource system, i.e. such that the graph of the dependence relation is connected (if it is not the case, the theorem has to be applied to each connected sub-system).

Theorem 3.1. *Let $\{a(n), n \in \mathbb{N}\}$ be a stationary and ergodic sequence (i.e. $a(n+1,\omega) = a(n,\theta(\omega))$) of integrable random variables, such that $\forall b \in \mathcal{A}, P(a(1) = b) > 0$.*

1. *There exists a constant $\lambda_E \in \mathbb{R}$ (*stochastic Lyapunov exponent*) such that, $\forall i \in \mathcal{A} \cup \mathcal{R}$,*

$$\lim_n \frac{x(n)_i}{n} = \lim_n E\left(\frac{x(n)_i}{n}\right) = \lambda_E \quad P - \text{a.s.} \tag{3.1}$$

2. *Moreover, if the sequence $\{a(n), n \in \mathbb{N}\}$ is i.i.d. then the random variable $\pi x(n)$ converges in total variation to a unique stationary distribution.*

Proof. In order to prove point 1, the main tool is the subadditivity of the sequence $\{y(w) = \|x(w)\|\}$, more precisely:

$$\forall w_1, w_2 \in \mathcal{A}^*, \quad y(w_1 w_2) \leqslant y(w_1) + y(w_2)\,. \tag{3.2}$$

This property enables to apply Kingman's subadditive ergodic theorem, see [1]. More generally, this result is just a special case of a general theorem on homogeneous and monotone operators, see [20] or [4] in this volume.

We show point 2 for the resource dater $x_{\mathcal{R}}(w) = e_{\mathcal{R}} \mathcal{M}_{\mathcal{R}}(w)$ (the behavior of $x_{\mathcal{A}}$ can be deduced easily from that of $x_{\mathcal{R}}$ by appealing to (2.7)). The following necessary and sufficient condition of existence and uniqueness of a stationary distribution for $\pi x_{\mathcal{R}}(w(n))$ is stated in [17]:

There is a word w such that the matrix $\mathcal{M}_{\mathcal{R}}(w)$ is of rank one, with non-ε entries.

The matrix $\mathcal{M}_{\mathcal{R}}(w)$ constitutes a regeneration pattern for the model. Indeed, the rank one condition is equivalent to a forgetting of the initial condition.

$$\forall x_0, x_0', \quad \pi(x_0 \mathcal{M}_{\mathcal{R}}(w)) = \pi(x_0' \mathcal{M}_{\mathcal{R}}(w))\,. \tag{3.3}$$

This pattern enables us to use regeneration theory to obtain stability of the model. The existence of the pattern is guaranteed by the following lemma.

Lemma 3.2. *Let $w = a_1 \ldots a_n$ be a path in the graph of the dependence relation (i.e $a_i\, D\, a_{i+1}$), visiting all the nodes. Let $\tilde{w} = a_n \ldots a_1$ denote the mirror image of w. The matrix $\mathcal{M}_{\mathcal{R}}(w\tilde{w})$ is of rank one with non-ε entries.*

Rather than proving formally the result (which can be done using representation (2.8), (2.12) and the fact that $\mathcal{I}(a)$ has rank one), we provide a geometrical justification using heaps of pieces. Condition (3.3) is equivalent to the following: *the upper shape of the heap is independent of the shape of the ground* (which corresponds to the initial condition). The property $a_i \, D \, a_{i+1}$ of the word $w\tilde{w}$ means that the heap is staircase shaped. It implies condition (3.3) as illustrated in the different heaps (I),(II),(III) shown on Fig. 2 (corresponding to the respective initial conditions $e_{\mathcal{R}}$, $(-2,-2,e,e)$, $(\varepsilon,\varepsilon,\varepsilon,e)$). □

Remark 3.3. A result analog to Theorem 3.1, point 2. was proved by Saheb [18] for trace monoids, using a Markovian argument. The advantage of the method presented here is that it can be applied to the various extensions mentioned in footnote 5.

3.2 Optimal Case and Worst Case

Given a language $L \subset \mathcal{A}^*$ describing the set of admissible schedules, a natural problem consists in finding an admissible schedule of length n with minimal or maximal makespan. The following theorem shows the existence of an asymptotic mean execution time, under optimal or worst case schedules. It can be seen as a (weak) analogue for optimization problems of the first order ergodic theorem 3.1,1.

Theorem 3.4. *1. For a language L such that $L^2 \subset L$, the following limit (optimal Lyapunov exponent) exists*

$$\lambda_{\min}(L) \stackrel{\text{def}}{=} \lim_{n\to\infty,\, \mathcal{A}^n\cap L\neq\emptyset} \; \min_{w\in\mathcal{A}^n\cap L} \frac{y(w)}{n} = \inf_{w\in L} \frac{y(w)}{|w|} \; . \tag{3.4}$$

2. For a bifix *language L (such that $uv \in L \Rightarrow u, v \in L$), the following limit (worst Lyapunov exponent) exists*

$$\lambda_{\max}(L) \stackrel{\text{def}}{=} \lim_{n\to\infty} \max_{w\in\mathcal{A}^n\cap L} \frac{y(w)}{n} = \inf_{n\geqslant 1} \max_{w\in\mathcal{A}^n\cap L} \frac{y(w)}{n} \; . \tag{3.5}$$

Proof. Let $m_n = \inf_{w\in\mathcal{A}^n\cap L} y(w)$. Since $L^2 \subset L$, $w \in L\cap\mathcal{A}^n, z \in L\cap\mathcal{A}^p \Rightarrow wz \in L \cap A^{n+p}$. Using the subadditivity property (3.2), we get $m_{n+p} \leqslant m_n + m_p$, from which (3.4) readily follows. The argument for $\lambda_{\max}$ is similar. □

The assumption that $L^2 \subset L$ for the optimal case is practically reasonable. For instance, for usual scheduling problems, it is natural to impose a fixed proportion of the different tasks, i.e. $L = \{w \mid \; |w|_a = r_a|w|\}$, for some fixed $r_a \in \mathbb{R}^+$, $\sum_a r_a = 1$. Such a language satisfies $L^2 \subset L$. The restriction to bifix languages for the worst case behavior is an artefact due to the subadditive argument.

The following theorem shows that the worst case performance can be exactly computed for the subclass of rational schedule languages. The reader is referred to [6] for the notation concerning series.

Theorem 3.5. *Consider the* generating series of the worst case behavior, $z = \bigoplus_{n\in\mathbb{N}} z_n x^n \in \mathbb{R}_{\max}[[x]]$, *where* $z_n = \sup_{w\in\mathcal{A}^n\cap L} y(w)$. *If the admissible language* L *is rational, the series* z *is rational.*

Proof. Let $\mathrm{char}L \in \mathbb{R}_{\max}\langle\langle\mathcal{A}\rangle\rangle$ denote the characteristic series[6] of the language L. Then, $\mathrm{char}L$ is rational. Introduce the morphism $\varphi : \mathbb{R}_{\max}\langle\langle\mathcal{A}\rangle\rangle \to \mathbb{R}_{\max}[[x]]$ such that $\forall a, \varphi(a) = x$. Recall that the *Hadamard product* of series is defined by $(s \odot t)(w) = s(w)t(w)$. Since rational series are closed under alphabetical morphisms and Hadamard product, $z = \varphi(\mathrm{char}L \odot y) \in \mathbb{R}_{\max}[[x]]$ is rational. ☐

Corollary 3.6. *Let* α, μ, β *denote a trim linear representation of* $\mathrm{char}L$. *Then,*

$$\limsup_n \frac{z_n}{n} = \rho(A), \quad A = \bigoplus_{a\in\mathcal{A}} \mu(a) \otimes^t \mathcal{M}_{\mathcal{R}}(a) \ , \tag{3.6}$$

where ρ *denotes the (max,+) maximal eigenvalue and* $\otimes^t$ *the tensor product of matrices.*

This is an immediate consequence of the (max,+) spectral theorem, together with the fact [13, 6] that $\mathrm{char}L \odot y$ is recognized by the tensor product of the representations (α, μ, β), $(e_{\mathcal{R}}, \mathcal{M}_{\mathcal{R}}, e_{\mathcal{R}}^T)$ (see [15, §3.2] for details).

Remark 3.7. More generally, Theorem 3.5 holds for an algebraic (=context-free) language L and not only for a rational one. Indeed, it is an easy extension[7] of Parikh theorem [10] that *algebraic series in several commuting indeterminates, with coefficients in* $\mathbb{R}_{\max}$, *are rational.* Since algebraic series are closed by Hadamard product with recognizable series and alphabetical morphism, the above proof shows that, when L is algebraic, the series $z = \varphi(\mathrm{char}L \odot y)$ is algebraic, hence rational. This shows that *the generating series* z *of the worst case behavior of an algebraic language* L *is rational*. In this case, the effective computation of z, along the lines of [10, Ch. XI] is less immediate, since it requires solving (max,+) commutative rational equations.

Example 3.8. Consider the dependence alphabet b—a—c, together with the set of admissible schedules $L = (a \oplus bc^*b)^*$. Its characteristic series is recognized by

$$\alpha = [e, \varepsilon], \beta = [e, \varepsilon]^T, \mu(a) = \begin{bmatrix} e & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}, \ \mu(b) = \begin{bmatrix} \varepsilon & e \\ e & \varepsilon \end{bmatrix}, \ \mu(c) = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & e \end{bmatrix} \ .$$

We get from Ex. 2.10 and (3.6),

$$A = \begin{bmatrix} h(a) & h(a) & h(b) & \varepsilon \\ h(a) & h(a) & \varepsilon & e \\ h(b) & \varepsilon & e & \varepsilon \\ \varepsilon & e & \varepsilon & h(c) \end{bmatrix} \ , \quad \rho(A) = h(a) \oplus h(c) \oplus h(b) \ ,$$

[6] The coefficient of $\mathrm{char}L$ at w is equal to e if $w \in L$, ε otherwise.

[7] By algebraic series, we mean *constructive* algebraic series as defined in [14]. The argument given in [10, Ch. XI] can be adapted to algebraic series in commuting indeterminates with coefficients in commutative idempotent semirings.

where $\rho(A)$ is obtained from its characterization as maximal mean weight of the circuits of A [2]. Note that the different terms in $\rho(A)$ are attained asymptotically for the sequences of schedules $a^n, n \in \mathbb{N}$, $bc^nb, n \in \mathbb{N}$, $b^{2n}, n \in \mathbb{N}$ (whose periodic parts correspond to circuits of A).

Remark 3.9. Cérin and Petit [9] study the absolute worst case behavior $\overline{\lambda}_{\max} \stackrel{\mathrm{def}}{=} \sup_{w \in L} |w|^{-1} \times y(w)$. This can be obtained along the same lines:

$$\overline{\lambda}_{\max} = \rho(A) \oplus \bigoplus_{1 \le i \le \dim A} cA^i b \ , \tag{3.7}$$

where $c = \alpha \otimes^t e_{\mathcal{R}}, b = \beta \otimes^t e_{\mathcal{R}}^T$. These quantities can be computed in $O((\dim A)^3)$ steps (using Karp algorithm [2] for $\rho(A)$). Observe that the dual quantity $\inf_{w \in L} y(w)/|w|$ treated in [9] cannot be obtained by such simple arguments due to its "min-max" structure.

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