Max-Plus Convex Geometry

Stéphane Gaubert[1] and Ricardo Katz[2]

[1] INRIA, Domaine de Voluceau, 78153 Le Chesnay Cédex, France
Stephane.Gaubert@inria.fr
[2] CONICET, Instituto de Matemática "Beppo Levi", Universidad Nacional de Rosario, Av. Pellegrini 250, 2000 Rosario, Argentina
rkatz@fceia.unr.edu.ar

Abstract. Max-plus analogues of linear spaces, convex sets, and polyhedra have appeared in several works. We survey their main geometrical properties, including max-plus versions of the separation theorem, existence of linear and non-linear projectors, max-plus analogues of the Minkowski-Weyl theorem, and the characterization of the analogues of "simplicial" cones in terms of distributive lattices.

1 Introduction

The max-plus semiring, $\mathbb{R}_{\max}$, is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the addition $(a,b) \mapsto \max(a,b)$ and the multiplication $(a,b) \mapsto a+b$. To emphasize the semiring structure, we write $a \oplus b := \max(a,b)$, $ab := a+b$, $\mathbb{0} := -\infty$ and $\mathbb{1} := 0$.

Many classical notions have interesting max-plus analogues. In particular, *semimodules* over the max-plus semiring can be defined essentially like linear spaces over a field. The most basic examples consist of *subsemimodules of functions* from a set X to $\mathbb{R}_{\max}$, which are subsets $\mathscr{V}$ of $\mathbb{R}_{\max}^X$ that are stable by max-plus linear combinations, meaning that:

$$\lambda u \oplus \mu v \in \mathscr{V} \tag{1}$$

for all $u, v \in \mathscr{V}$ and for all $\lambda, \mu \in \mathbb{R}_{\max}$. Here, for all scalars λ and functions u, λu denotes the function sending x to the max-plus product $\lambda u(x)$, and the max-plus sum of two functions is defined entrywise. Max-plus semimodules have many common features with convex cones. This analogy leads to define max-plus *convex* subsets $\mathscr{V}$ of $\mathbb{R}_{\max}^X$ by the requirement that (1) holds for all $u, v \in \mathscr{V}$ and for all $\lambda, \mu \in \mathbb{R}_{\max}$ such that $\lambda \oplus \mu = \mathbb{1}$. The finite dimensional case, in which $X = \{1, \ldots, n\}$, is already interesting.

Semimodules over the max-plus semiring have received much attention [1], [2], [3], [4], [5]. They are of an intrinsic interest, due to their relation with lattice and Boolean matrix theory, and also with abstract convex analysis [6]. They arise in the geometric approach to discrete event systems [7], and in the study of solutions of Hamilton-Jacobi equations associated with deterministic optimal control problems [8,4,9,10]. Recently, relations with phylogenetic analysis have been pointed out [11].

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In this paper, we survey the basic properties of max-plus linear spaces, convex sets, and polyhedra, emphasizing the analogies with classical convex geometry. We shall present a synopsis of the results of [5,12], including separation theorems, as well as new results, mostly taken from the recent works [13,14]. Some motivations are sketched in the next section. The reader interested specifically in applications to computer science might look at the work on fixed points problems in static analysis of programs by abstract interpretation [28], which is briefly discussed at the end of Section 2.3.

2 Motivations

2.1 Preliminary Definitions

Before pointing out some motivations, we give preliminary definitions. We refer the reader to [5] for background on semirings with an idempotent addition (idempotent semirings) and semimodules over idempotent semirings. In particular, the standard notions concerning modules, like linear maps, are naturally adapted to the setting of semimodules.

Although the results of [5] are developed in a more general setting, we shall here only consider semimodules of functions. A *semimodule of functions* from a set X to a semiring $\mathscr{K}$ is a subset $\mathscr{V} \subset \mathscr{K}^X$ satisfying (1), for all $u, v \in \mathscr{V}$ and $\lambda, \mu \in \mathscr{K}$. When $X = \{1, \dots, n\}$, we write $\mathscr{K}^n$ instead of $\mathscr{K}^X$, and we denote by u_i the i-th coordinate of a vector $u \in \mathscr{K}^n$.

We shall mostly restrict our attention to the case where $\mathscr{K}$ is the *max-plus semiring*, $\mathbb{R}_{\max}$, already defined in the introduction, or the *completed max-plus semiring*, $\overline{\mathbb{R}}_{\max}$, which is obtained by adjoining to $\mathbb{R}_{\max}$ a $+\infty$ element, with the convention that $(-\infty) + (+\infty) = -\infty$. Some of the results can be stated in a simpler way in the completed max-plus semiring.

The semirings $\mathbb{R}_{\max}$ and $\overline{\mathbb{R}}_{\max}$ are equipped with the usual order relation. Semimodules of functions with values in one of these semirings are equipped with the product order.

We say that a set of functions with values in $\overline{\mathbb{R}}_{\max}$ is *complete* if the supremum of an arbitrary family of elements of this set belongs to it. A *convex* subset $\mathscr{V}$ of $\overline{\mathbb{R}}_{\max}^X$ is defined like a convex subset of $\mathbb{R}_{\max}$, by requiring that (1) holds for all $u, v \in \mathscr{V}$ and $\lambda, \mu \in \mathbb{R}_{\max}$ such that $\lambda \oplus \mu = \mathbb{1}$.

If $\mathscr{X}$ is a set of functions from X to $\mathbb{R}_{\max}$, we define the semimodule that it generates, $\operatorname{span} \mathscr{X}$, to be the set of max-plus linear combinations of a finite number of functions of $\mathscr{X}$. In other words, every function f of $\operatorname{span} \mathscr{X}$ can be written as

$$f(x) = \max_{i \in I} \lambda_i + g_i(x) \enspace , \qquad (2)$$

where I is a finite set, g_i belongs to $\mathscr{X}$, and λ_i belongs to $\mathbb{R} \cup \{-\infty\}$.

If $\mathscr{X}$ is a set of functions from X to $\overline{\mathbb{R}}_{\max}$, we define the complete semimodule that it generates, $\overline{\operatorname{span}}\, \mathscr{X}$, to be the set of arbitrary max-plus linear combinations of functions of $\mathscr{X}$, or equivalently, the set of arbitrary suprema of elements of span $\mathscr{X}$. Thus, every function of $\overline{\text{span}}\,\mathscr{X}$ can be written in the form (2), if we allow I to be infinite, with $\lambda_i \in \mathbb{R} \cup \{\pm\infty\}$, and if replace the "max" by a "sup". Then, we say that f is an infinite linear combination of the functions g_i.

2.2 Solution Spaces of Max-Plus Linear Equations

An obvious motivation to introduce semimodules over $\mathbb{R}_{\max}$ or $\overline{\mathbb{R}}_{\max}$ is to study the spaces of solutions of max-plus linear equations. Such equations arise naturally in relation with discrete event systems and dynamic programming.

For instance, let $A = (A_{ij})$ denote a $p \times q$ matrix with entries in $\mathbb{R}_{\max}$, and consider the relation

$$y = Ax \enspace .$$

Here, Ax denotes the max-plus product, so that $y_i = \max_{1 \leq k \leq q} A_{ik} + x_k$. This can be interpreted as follows. Imagine a system with q initial events (arrival of a part in a workshop, entrance of a customer in a network, etc.), and p terminal events (completion of a task, exit of a customer, etc.). Assume that the terminal event i cannot be completed earlier than A_{ij} time units after the initial event j has occurred. Then, the vector $y = Ax$ represents the earliest completion times of the terminal events, as a function of the vector x of occurrence times of the initial events. The image of the max-plus linear operator A, $\mathscr{V} := \{Ax \mid x \in \mathbb{R}_{\max}^q\}$ is a semimodule representing all the possible completion times. More sophisticated examples, relative to the dynamical behavior of discrete event systems, can be found in [7,15].

Other interesting semimodules arise as *eigenspaces*. Consider the eigenproblem

$$Ax = \lambda x \enspace ,$$

that is, $\max_{1 \leq j \leq q} A_{ij} + x_j = \lambda + x_i$. We assume here that A is square. We look for the *eigenvectors* $x \in \mathbb{R}_{\max}^q$ and the *eigenvalues* $\lambda \in \mathbb{R}_{\max}$. The eigenspace of λ, which is the set of all x such that $Ax = \lambda x$, is obviously a semimodule. In dynamic programming, A_{ij} represents the reward received when moving from state i to state j. If $Ax = \lambda x$ for some vector x with finite entries, it can be checked that the eigenvalue λ gives the maximal mean reward per move, taken over all infinite trajectories. The eigenvector x can be interpreted as a fair relative price vector for the different states. See [16,10,17] for more details on the eigenproblem. The extreme generators of the eigenspace (to be defined in Section 5) correspond to optimal stationary strategies or infinite "geodesics" [10].

The infinite dimensional version of the equation $y = Ax$ and of the spectral equation $Ax = \lambda x$ respectively arise in large deviations theory [18] and in optimal control [10]. When the state space is non compact, the representation of max-plus eigenvectors is intimately related with the compactification of metric spaces in terms of horofunctions [10,19].

2.3 From Classical Convexity to Max-Plus Convexity

The most familiar examples of semimodules over the max-plus semiring arise in classical convex analysis. In this section, unlike in the rest of the paper, the words "convex", "linear", "affine", and "polyhedra", and the notation "·" for the scalar product of $\mathbb{R}^n$, have their usual meaning.

Recall that the *Legendre-Fenchel transform* of a map f from $\mathbb{R}^n$ to $\mathbb{R}\cup\{\pm\infty\}$ is the map $f^\star$ from $\mathbb{R}^n$ to $\mathbb{R}\cup\{\pm\infty\}$ defined by:

$$f^\star(p) = \sup_{x\in\mathbb{R}^n} p\cdot x - f(x) \ . \tag{3}$$

Legendre-Fenchel duality [20, Cor. 12.2.1] tells that $(f^\star)^\star = f$ if f is convex, lower semicontinuous and if $f(x)\in\mathbb{R}\cup\{+\infty\}$ for all $x\in\mathbb{R}^n$. Making explicit the identity $(f^\star)^\star = f$, we get $f(x) = \sup_{p\in\mathbb{R}^n} p\cdot x - f^\star(p)$. This classical result can be restated as follows, in max-plus terms.

Property 1 (Semimodule of convex functions). The set of convex lower semicontinuous convex functions from $\mathbb{R}^n$ to $\mathbb{R}\cup\{+\infty\}$ is precisely the set of infinite max-plus linear combinations of (conventional) linear forms on $\mathbb{R}^n$.

The numbers $-f^\star(p)$, for $p\in\mathbb{R}^n$, may be thought of as the "coefficients", in the max-plus sense, of f with respect to the "basis" of linear forms $x\mapsto p\cdot x$. These coefficients are not unique, since there may be several functions g such that $f = g^\star$. However, the map g giving the "coefficients" is unique if it is required to be lower semicontinuous and if f is *essentially smooth*, see [21, Cor. 6.4]. The semimodule of finite max-plus linear combinations of linear forms is also familiar: it consists of the convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ that are polyhedral [20], together with the identically $-\infty$ map.

By changing the set of generating functions, one obtains other spaces of functions. In particular, an useful space consists of the maps f from $\mathbb{R}^n$ to $\mathbb{R}$ that are order preserving, meaning that $x\leq y \implies f(x)\leq f(y)$, where $\leq$ denotes the standard product ordering of $\mathbb{R}^n$, and commute with the addition with a constant, meaning that $f(\lambda+x_1,\ldots,\lambda+x_n) = \lambda + f(x_1,\ldots,x_n)$. These maps play a fundamental role in the theory of Markov decision processes and games: they arise as the coordinate maps of dynamic programming operators. They are sometimes called *topical maps* [22]. Topical maps include min-plus linear maps sending $\mathbb{R}^n$ to $\mathbb{R}$, which can be written as

$$x\mapsto \min_{1\leq j\leq n} a_j + x_j \ , \tag{4}$$

where $a_1,\ldots,a_n$ are numbers in $\mathbb{R}\cup\{+\infty\}$ that are not all equal to $+\infty$. Of course, topical maps also include max-plus linear maps sending $\mathbb{R}^n$ to $\mathbb{R}$, which can be represented in a dual way. The following observation was made by Rubinov and Singer [23], and, independently by Gunawardena and Sparrow (personal communication).

Property 2 (Semimodule of topical functions). The set of order preserving maps from $\mathbb{R}^n$ to $\mathbb{R}$ that commute with the addition of a constant coincides, up to the functions identically equal to $-\infty$ or $+\infty$, with the set of infinite max-plus linear combinations of the maps of the form (4).

A map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is called a *min-max function* if each of its coordinates is a *finite* max-plus linear combination of maps of the form (4). Min-max functions arise as dynamic operators of zero-sum deterministic two player games with finite state and action spaces, and also, in the performance analysis of discrete event systems [24,25]. The decomposition of a min-max function as a supremum of min-plus linear maps (or dually, as an infimum of max-plus linear maps) is used in [26,27,25] to design *policy iteration algorithms*, allowing one to solve fixed points problems related to min-max functions. These techniques are applied in [28] to the static analysis of programs by abstract interpretation.

Another application of semimodules of functions, to the discretization of Hamilton-Jacobi equations associated with optimal control problems, can be found in [29].

3 Projection on Complete Semimodules

We now survey some of the main properties of the max-plus analogues of modules (or cones) and convex sets. In the case of Hilbert spaces, a possible approach is to define first the projection on a closed convex set, and then to show the separation theorem. We shall follow here a similar path.

Definition 1 (Projector on a complete semimodule). *If $\mathscr{V}$ is a complete semimodule of functions from X to $\overline{\mathbb{R}}_{\max}$, for all functions u from X to $\overline{\mathbb{R}}_{\max}$, we define:*

$$P_{\mathscr{V}}(u) := \sup\{v \in \mathscr{V} \mid v \leq u\} \enspace .$$

Since $\mathscr{V}$ is complete, $P_{\mathscr{V}}(u) \in \mathscr{V}$, and obviously, $P_{\mathscr{V}}$ has all elements of $\mathscr{V}$ as fixed points. It follows that

$$P_{\mathscr{V}} \circ P_{\mathscr{V}} = P_{\mathscr{V}} \enspace .$$

The projector $P_{\mathscr{V}}$ can be computed from a generating family of $\mathscr{V}$. Assume first that $\mathscr{V}$ is generated only by one function $v \in \overline{\mathbb{R}}_{\max}^X$, meaning that $\mathscr{V} = \overline{\mathbb{R}}_{\max} v := \{\lambda v \mid \lambda \in \overline{\mathbb{R}}_{\max}\}$. Define, for $u \in \overline{\mathbb{R}}_{\max}^X$,

$$u/v := \sup\{\lambda \in \overline{\mathbb{R}}_{\max} \mid u \geq \lambda v\} \enspace .$$

One can easily check that

$$u/v = \inf\{u(x) - v(x) \mid x \in X\} \enspace ,$$

with the convention that $(+\infty) - (+\infty) = (-\infty) - (-\infty) = +\infty$. Of course, $P_{\mathscr{V}}(u) = (u/v)v$. More generally, we have the following elementary result.

Proposition 1 ([5]). *If $\mathscr{V}$ is a complete subsemimodule of $\overline{\mathbb{R}}_{\max}^X$ generated by a subset $\mathscr{X} \subset \overline{\mathbb{R}}_{\max}^X$, we have*

$$P_{\mathscr{V}}(u) = \sup_{v \in \mathscr{X}} (u/v)v \enspace .$$

When $\mathscr{X}$ is finite and $X = \{1, \ldots, n\}$, this provides an algorithm to decide whether a function u belongs to $\mathscr{V}$: it suffices to check whether $P_{\mathscr{V}}(u) = u$.

Example 1. We use here the notation of Section 2.3. When $\mathscr{V}$ is the complete semimodule generated by the set of conventional linear maps $x \mapsto p \cdot x$, $P_{\mathscr{V}}(u)$ can be written as

$$[P_{\mathscr{V}}(u)](x) = \sup_{p \in \mathbb{R}^n} \Big(\inf_{y \in \mathbb{R}^n} u(y) - p \cdot y \Big) + p \cdot x = (u^\star)^\star(x) \enspace ,$$

where $u^\star$ is the Legendre-Fenchel transform of u. Hence, $P_{\mathscr{V}}(u)$ is the lower-semicontinuous convex hull of u ([20, Th. 12.2]).

When $\mathscr{V}$ is the complete semimodule generated by the set of functions of the form $x \mapsto -\|x - a\|_\infty$, with $a \in \mathbb{R}^n$, it can be checked that

$$[P_{\mathscr{V}}(u)](x) = \sup_{a \in \mathbb{R}^n} \big((\inf_{y \in \mathbb{R}^n} u(y) + \|y - a\|_\infty) - \|x - a\|_\infty \big) = \inf_{a \in \mathbb{R}^n} u(a) - \|x - a\|_\infty \enspace .$$

This is the "1-Lipschitz regularization" of u. More generally, one may consider semimodules of maps with a prescribed continuity modulus, like Hölder continuous maps, see [21].

The projection of a vector of a Hilbert space on a (conventional) closed convex set minimizes the Euclidean distance of this vector to it. A similar property holds in the max-plus case, but the Euclidean norm must be replaced by the Hilbert seminorm (the additive version of Hilbert's projective metric). For any scalar $\lambda \in \overline{\mathbb{R}}_{\max}$, define $\lambda^- := -\lambda$. For all vectors $u, v \in \overline{\mathbb{R}}_{\max}^X$, we define

$$\delta_H(u, v) := ((u/v)(v/u))^- \enspace ,$$

where the product is understood in the max-plus sense. When $X = \{1, \ldots, n\}$ and u, v take finite values, $\delta_H(u, v)$ can be written as

$$\delta_H(u, v) = \sup_{1 \le i, j \le n} (u_i - v_i + v_j - u_j) \enspace ,$$

with the usual notation.

Theorem 1 (The projection minimizes Hilbert's seminorm, [5]). *If $\mathscr{V}$ is a complete semimodule of functions from a set X to $\overline{\mathbb{R}}_{\max}$, then, for all functions u from X to $\overline{\mathbb{R}}_{\max}$, and for all $v \in \mathscr{V}$,*

$$\delta_H(u, P_{\mathscr{V}}(u)) \le \delta_H(u, v) \enspace .$$

This property does not uniquely define $P_{\mathscr{V}}(u)$, even up to an additive constant, because the balls in Hilbert's projective metric are not "strictly convex".

Example 2. Consider

$$A = \begin{bmatrix} 0 & 0 & 0 & -\infty & 0.5 \\ 1 & -2 & 0 & 0 & 1.5 \\ 0 & 3 & 2 & 0 & 3 \end{bmatrix} \enspace , \qquad u = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} \enspace . \tag{5}$$

The semimodule $\mathscr{V}$ generated by the columns of the matrix A is represented in Figure 1 (left). A non-zero vector $v \in \mathbb{R}^3_{\max}$ is represented by the point that is the barycenter with weights $(\exp(\beta v_i))_{1\leq i\leq 3}$ of the vertices of the simplex, where $\beta > 0$ is a fixed scaling parameter. Observe that vectors that are proportional in the max-plus sense are represented by the same point. Every vertex of the simplex represents one basis vector e_i. The point p_i corresponds to the i-th column of A. The semimodule $\mathscr{V}$ is represented by the closed region in dark grey and by the bold segments joining the points p_1, p_2, p_4 to it.

We deduce from Proposition 1 that

$$P_{\mathscr{V}}(u) = (-1)\begin{bmatrix}0\\1\\0\end{bmatrix} \oplus (-2.5)\begin{bmatrix}0\\-2\\3\end{bmatrix} \oplus (-1.5)\begin{bmatrix}0\\0\\2\end{bmatrix} \oplus (0)\begin{bmatrix}-\infty\\0\\0\end{bmatrix} \oplus (-2.5)\begin{bmatrix}0.5\\1.5\\3\end{bmatrix}$$
$$= \begin{bmatrix}-1\\0\\0.5\end{bmatrix} .$$

Since $P_{\mathscr{V}}(u) < u$, u does not belong to $\mathscr{V}$. The vector u and its projection $P_{\mathscr{V}}(u)$ are represented in Figure 1 (right). The ball in Hilbert's metric centered at point u the boundary of which contains $P_{\mathscr{V}}(u)$ is represented in light grey. The fact that $P_{\mathscr{V}}(u)$ is one of the points of $\mathscr{V}$ that are the closest to u (Theorem 1) is clear from the figure.

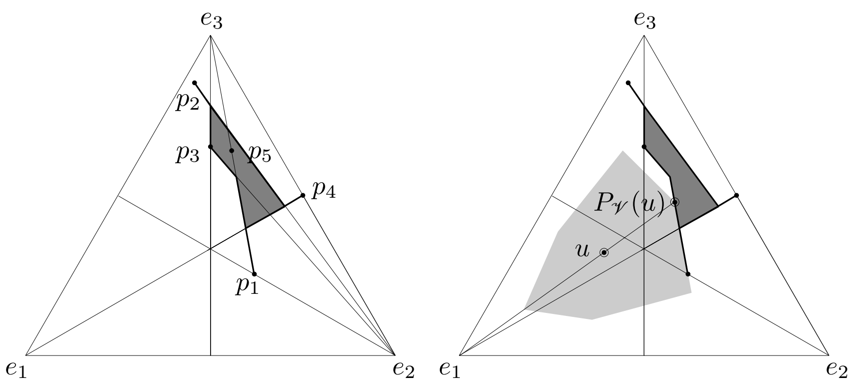

Fig. 1. A max-plus semimodule (left). A point u, its projection $P_{\mathscr{V}}(u)$, and the corresponding ball in Hilbert's projective metric (right).

4 Separation Theorems

We first state separation theorems for complete subsemimodules and complete convex subsets of $\overline{\mathbb{R}}^X_{\max}$, since the results are simpler in this setting. Then, we shall see how the completeness assumptions can be dispensed with.

Several max-plus separation theorems have appeared in the literature: the first one is due to Zimmermann [2]. Other separation theorems appeared in [30], in [5,12], and, in the polyhedral case, in [11,31]. We follow here the approach of [5,12], in which the geometrical interpretation is apparent.

We call *half-space* of $\overline{\mathbb{R}}_{\max}^X$ a set of the form

$$\mathscr{H} = \{v \in \overline{\mathbb{R}}_{\max}^X \mid a \cdot v \leq b \cdot v\} \enspace , \tag{6}$$

where $a, b \in \overline{\mathbb{R}}_{\max}^X$ and $\cdot$ denotes here the max-plus scalar product:

$$a \cdot v := \sup_{x \in X} a(x) + v(x) \enspace .$$

We extend the notation $\cdot^-$ to functions $v \in \overline{\mathbb{R}}_{\max}^X$, so that v^- denotes the function sending $x \in X$ to $-v(x)$. The following theorem is proved using residuation (or Galois correspondence) techniques.

Theorem 2 (Universal Separation Theorem, [5, Th. 8]). *Let $\mathscr{V} \subset \overline{\mathbb{R}}_{\max}^X$ denote a complete subsemimodule, and let $u \in \overline{\mathbb{R}}_{\max}^X \setminus \mathscr{V}$. Then, the half-space*

$$\mathscr{H} = \{v \in \overline{\mathbb{R}}_{\max}^X \mid (P_{\mathscr{V}}(u))^- \cdot v \leq u^- \cdot v\} \tag{7}$$

contains $\mathscr{V}$ and not u.

Since $P_{\mathscr{V}}(u) \leq u$, the inequality can be replaced by an equality in (7). A way to remember Theorem 2 is to interpret the equality

$$(P_{\mathscr{V}}(u))^- \cdot v = u^- \cdot v$$

as the "orthogonality" of v to the direction $(u, P_{\mathscr{V}}(u))$. This is analogous to the Hilbert space case, where the difference between a vector and its projection gives the direction of a separating hyperplane.

Example 3. Let $\mathscr{V}$, A, and u be as in Example 2. The half-space separating u from $\mathscr{V}$ is readily obtained from the value of u and $P_{\mathscr{V}}(u)$:

$$\mathscr{H} = \{v \in \overline{\mathbb{R}}_{\max}^3 \mid 1v_1 \oplus v_2 \oplus (-0.5)v_3 \leq (-1)v_1 \oplus v_2 \oplus (-0.5)v_3\} \enspace .$$

This half-space is represented by the zone in medium gray in Figure 2.

An *affine half-space* of $\overline{\mathbb{R}}_{\max}^X$ is by definition a set of the form

$$\mathscr{H} = \{v \in \overline{\mathbb{R}}_{\max}^X \mid a \cdot v \oplus c \leq b \cdot v \oplus d\} \enspace , \tag{8}$$

where $a, b \in \overline{\mathbb{R}}_{\max}^X$ and $c, d \in \overline{\mathbb{R}}_{\max}$. For any complete convex subset $\mathscr{C}$ of $\overline{\mathbb{R}}_{\max}^X$ and $u \in \overline{\mathbb{R}}_{\max}^X$, we define

$$\nu_{\mathscr{C}}(u) := \sup_{v \in \mathscr{C}} (u/v \wedge \mathbb{1}), \qquad Q_{\mathscr{C}}(u) := \sup_{v \in \mathscr{C}} (u/v \wedge \mathbb{1})v \enspace ,$$

where $\wedge$ denotes the pointwise minimum of vectors.

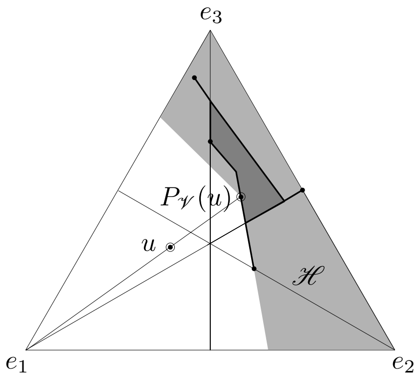

Fig. 2. Separating half-space

Corollary 1 ([5, Cor. 15]). *If $\mathscr{C}$ is a complete convex subset of $\overline{\mathbb{R}}_{\max}^X$, and if $u \in \overline{\mathbb{R}}_{\max}^X \setminus \mathscr{C}$, then the affine half-space*

$$\mathscr{H} = \{v \in \overline{\mathbb{R}}_{\max}^X \mid (Q_{\mathscr{C}}(u))^- \cdot v \oplus (\nu_{\mathscr{C}}(u))^- \leq u^- \cdot v \oplus \mathbb{1}\} \tag{9}$$

contains $\mathscr{C}$ and not u.

This corollary is obtained by projecting the vector $(u, \mathbb{1})$ on the complete subsemimodule of $\overline{\mathbb{R}}_{\max}^X \times \overline{\mathbb{R}}_{\max}$ generated by the vectors $(v\lambda, \lambda)$, where $v \in \mathscr{C}$ and $\lambda \in \overline{\mathbb{R}}_{\max}$. The projection of this vector is precisely $(Q_{\mathscr{C}}(u), \nu_{\mathscr{C}}(u))$. The operator $u \mapsto (\nu_{\mathscr{C}}(u))^- Q_{\mathscr{C}}(u)$ defines a projection on the convex set $\mathscr{C}$ [12]. (We note that the scalar $\nu_{\mathscr{C}}(u)$ is invertible, except in the degenerate case where u cannot be bounded from below by a non-zero scalar multiple of an element of $\mathscr{C}$.)

We deduce as an immediate corollary of Theorem 2 and Corollary 1.

Corollary 2. *A complete subsemimodule (resp. complete convex subset) of $\overline{\mathbb{R}}_{\max}^X$ is the intersection of the half-spaces (resp. affine half-spaces) of $\overline{\mathbb{R}}_{\max}^X$ in which it is contained.* □

We now consider subsemimodules and convex subsets arising from the max-plus semiring $\mathbb{R}_{\max}$, rather than from the completed max-plus semiring $\overline{\mathbb{R}}_{\max}$. Results of the best generality are perhaps still missing, so we shall restrict our attention to subsemimodules and convex subsets of $\mathbb{R}_{\max}^n$. By analogy with convex analysis, we call *cone* a subsemimodule of $\mathbb{R}_{\max}^n$.

We equip $\mathbb{R}_{\max}^n$ with the usual topology, which can be defined by the metric

$$d(u,v) := \max_{1\leq i\leq n} |\exp(u_i) - \exp(v_i)|, \qquad \forall u, v \in (\mathbb{R} \cup \{-\infty\})^n \enspace .$$

A *half-space* of $\mathbb{R}_{\max}^n$ is a set of the form $\mathscr{H} = \{v \in \mathbb{R}_{\max}^n \mid a \cdot v \leq b \cdot v\}$, where $a, b \in \mathbb{R}_{\max}^n$. An *affine half-space* of $\mathbb{R}_{\max}^n$ is a set of the form $\mathscr{H} = \{v \in \mathbb{R}_{\max}^n \mid a \cdot v \oplus c \leq b \cdot v \oplus d\}$, where $a, b \in \mathbb{R}_{\max}^n$ and $c, d \in \mathbb{R}_{\max}$. Note that the restriction to $\mathbb{R}_{\max}^n$ of an (affine) half-space of $\overline{\mathbb{R}}_{\max}^n$ need not be an (affine) half-space of $\mathbb{R}_{\max}^n$, because the vectors a, b in (6) and (8) can have entries equal to $+\infty$, and the scalars c, d in (8) can be equal to $+\infty$. However, we have the following refinement of Theorem 2 in the case of closed cones of $\mathbb{R}^n_{\max}$, which is slightly more precise than the result stated in [12], and can be proved along the same lines.

Theorem 3. *Let $\mathscr{V}$ be a closed cone of $\mathbb{R}^n_{\max}$ and let $u \in \mathbb{R}^n_{\max} \setminus \mathscr{V}$. Then, there exist $a \in \mathbb{R}^n_{\max}$ and disjoint subsets I and J of $\{1, \ldots, n\}$ such that the half-space of $\mathbb{R}^n_{\max}$*

$$\mathscr{H} = \{v \in \mathbb{R}^n_{\max} \mid \oplus_{i \in I} a_i v_i \leq \oplus_{j \in J} a_j v_j\} \tag{10}$$

contains $\mathscr{V}$ and not u.

Further information on half-spaces can be found in [31].

Example 4. The restriction to $\mathbb{R}^3_{\max}$ of the separating half-space constructed in Example 3 can be rewritten as:

$$\mathscr{H} = \{v \in \mathbb{R}^3_{\max} \mid 1v_1 \leq v_2 \oplus (-0.5)v_3\} \ ,$$

which is clearly of the form (10). To illustrate the technical difficulty concerning supports, which is solved in [12] and in Theorem 3 above, let us separate the point $u = [-\infty, 1, 0]^T$ from the semimodule $\mathscr{V}$ of Example 2. We have $P_{\mathscr{V}}(u) = [-\infty, 0, 0]^T$, and the half-space of $\overline{\mathbb{R}}^3_{\max}$ defined in Theorem 2 is

$$\{v \in \overline{\mathbb{R}}^3_{\max} \mid (+\infty)v_1 \oplus v_2 \oplus v_3 \leq (+\infty)v_1 \oplus (-1)v_2 \oplus v_3\} \ .$$

Note that due to the presence of the $+\infty$ coefficient, the restriction of this half-space to $\mathbb{R}^3_{\max}$ is not closed. The proof of [12] and of Theorem 3 introduces a finite perturbation of u, for instance, $w = [\epsilon, 1, 0]^T$, where ϵ is a finite number sufficiently close to $-\infty$ (here, any $\epsilon < 0$ will do), and shows that the restriction to $\mathbb{R}^n_{\max}$ of the half-space of $\overline{\mathbb{R}}^n_{\max}$ constructed in the universal separation theorem (Theorem 2), which is a half-space of $\mathbb{R}^n_{\max}$, separates u from $\mathscr{V}$. For instance, when $\epsilon = -1$, we obtain $P_{\mathscr{V}}(w) = [-1, 0, 0]^T$, which gives the half-space of $\mathbb{R}^3_{\max}$

$$\mathscr{H} = \{v \in \mathbb{R}^3_{\max} \mid 1v_1 \oplus v_3 \geq v_2\}$$

containing $\mathscr{V}$ and not u.

Corollary 3. *Let $\mathscr{C} \subset \mathbb{R}^n_{\max}$ be a closed convex set and let $u \in \mathbb{R}^n_{\max} \setminus \mathscr{C}$. Then, there exist $a \in \mathbb{R}^n_{\max}$, disjoint subsets I and J of $\{1, \ldots, n\}$ and $c, d \in \mathbb{R}_{\max}$, with $cd = \mathbb{0}$, such that the affine half-space of $\mathbb{R}^n_{\max}$*

$$\mathscr{H} = \{v \in \mathbb{R}^n_{\max} \mid \oplus_{i \in I} a_i v_i \oplus c \leq \oplus_{j \in J} a_j v_j \oplus d\}$$

contains $\mathscr{C}$ and not u.

This is proved by applying the previous theorem to the point $(u, \mathbb{1}) \in \mathbb{R}^{n+1}_{\max}$ and to the following closed cone:

$$\mathscr{V} := \mathrm{clo}(\{(v\lambda, \lambda) \mid v \in \mathscr{C}, \lambda \in \mathbb{R}_{\max}\}) \subset \mathbb{R}^{n+1}_{\max} \ .$$

We deduce as an immediate corollary of Theorem 3 and Corollary 3.

Corollary 4. *A closed cone of $\mathbb{R}^n_{\max}$ is the intersection of the half-spaces of $\mathbb{R}^n_{\max}$ in which it is contained. A closed convex subset of $\mathbb{R}^n_{\max}$ is the intersection of the affine half-spaces of $\mathbb{R}^n_{\max}$ in which it is contained.*

5 Extreme Points of Max-Plus Convex Sets

Definition 2. *Let $\mathscr{C}$ be a convex subset of $\mathbb{R}^n_{\max}$. An element $v \in \mathscr{C}$ is an* extreme point *of $\mathscr{C}$, if for all $u, w \in \mathscr{V}$ and $\lambda, \mu \in \mathbb{R}_{\max}$ such that $\lambda \oplus \mu = \mathbb{1}$, the following property is satisfied*

$$v = \lambda u \oplus \mu w \implies v = u \text{ or } v = w \enspace .$$

The set of extreme points of $\mathscr{C}$ will be denoted by $\mathrm{ext}(\mathscr{C})$.

Definition 3. *Let $\mathscr{V} \subset \mathbb{R}^n_{\max}$ be a cone. An element $v \in \mathscr{V}$ is an* extreme generator *of $\mathscr{V}$ if the following property is satisfied*

$$v = u \oplus w,\ u, w \in \mathscr{V} \implies v = u \text{ or } v = w \enspace .$$

We define an extreme ray *of $\mathscr{V}$ to be a set of the form $\mathbb{R}_{\max} v = \{\lambda v \mid \lambda \in \mathbb{R}_{\max}\}$ where v is an extreme generator of $\mathscr{V}$. The set of extreme generators of $\mathscr{V}$ will be denoted by* $\text{ext-g}\,(\mathscr{V})$.

Note that extreme generators correspond to *join irreducible* elements in the lattice theory literature.

We denote by $\mathrm{cone}\,(\mathscr{X})$ the smallest cone containing a subset $\mathscr{X}$ of $\mathbb{R}^n_{\max}$, and by $\mathrm{co}(\mathscr{X})$ the smallest convex set containing it. So $\mathrm{cone}\,(\mathscr{X})$ coincides with $\mathrm{span}\,\mathscr{X}$, if the operator "span" is interpreted over the semiring $\mathbb{R}_{\max}$.

Theorem 4. *Let $\mathscr{V} \subset \mathbb{R}^n_{\max}$ be a non-empty closed cone. Then $\mathscr{V}$ is the cone generated by the set of its extreme generators, that is,*

$$\mathscr{V} = \mathrm{cone}\,(\text{ext-g}\,(\mathscr{V})) \enspace .$$

The proof of Theorem 4, and of Corollary 5 and Theorem 5 below, can be found in [13]. After the submission of the present paper, a preprint of Buktovič, Schneider, and Sergeev has appeared [32], in which Theorem 4 is established independently. Their approach also yields informations on non-closed cones.

Corollary 5 (Max-Plus Minkowski's Theorem). *Let $\mathscr{C}$ be a non-empty compact convex subset of $\mathbb{R}^n_{\max}$. Then $\mathscr{C}$ is the convex hull of the set of its extreme points, that is,*

$$\mathscr{C} = \mathrm{co}(\mathrm{ext}(\mathscr{C})) \enspace .$$

This is more precise than Helbig's max-plus analogue of Krein-Milman's theorem [33], which only shows that a non-empty compact convex subset of $\mathbb{R}^n_{\max}$ is the *closure* of the convex hull of its set of extreme points. Unlike Helbig's proof, our proof of Theorem 4 and Corollary 5 does not use the separation theorem.

If v is a point in a convex set $\mathscr{C}$, we define the *recession cone* of $\mathscr{C}$ at point v to be the set:

$$\operatorname{rec}(\mathscr{C}) = \{u \in \mathbb{R}_{\max}^n \mid v \oplus \lambda u \in \mathscr{C} \text{ for all } \lambda \in \mathbb{R}_{\max}\} \ .$$

If $\mathscr{C}$ is a closed convex subset of $\mathbb{R}_{\max}^n$, it can be checked that the recession cone is independent of the choice of $v \in \mathscr{C}$, and that it is closed.

Theorem 5. *Let $\mathscr{C} \subset \mathbb{R}_{\max}^n$ be a closed convex set. Then,*

$$\mathscr{C} = \operatorname{co}(\operatorname{ext}(\mathscr{C})) \oplus \operatorname{rec}(\mathscr{C}) \ .$$

Corollary 4 suggests the following definition.

Definition 4. *A max-plus polyhedron is an intersection of finitely many affine half-spaces of $\mathbb{R}_{\max}^n$.*

Theorem 6 (Max-Plus Minkowski-Weyl Theorem). *The max-plus polyhedra are precisely the sets of the form*

$$\operatorname{co}(\mathscr{X}) \oplus \operatorname{cone}(\mathscr{Y})$$

where $\mathscr{X}, \mathscr{Y}$ are finite subsets of $\mathbb{R}_{\max}^n$.

Note that our notion of max-plus polyhedra is more general than the notion of tropical polyhedra which is considered in [11]: tropical polyhedra can be identified with sets of the form $\operatorname{cone}(\mathscr{Y})$ where $\mathscr{Y}$ is a finite set of vectors with only *finite* entries.

Finally, we shall consider the max-plus analogues of simplicial convex cones, which are related to the important notion of regular matrix. We need to work again in the completed max-plus semiring, $\overline{\mathbb{R}}_{\max}$, rather than in $\mathbb{R}_{\max}$. We say that a matrix $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$ is *regular* if it has a generalized inverse, meaning that there exists a matrix $X \in \overline{\mathbb{R}}_{\max}^{p\times n}$ such that $A = AXA$. Regularity is equivalent to the existence of a linear projector onto the cone generated by the columns (or the rows) of A, see [34,35].

A finitely generated subsemimodule $\mathscr{V}$ of $\overline{\mathbb{R}}_{\max}^n$ is a complete lattice, in which the supremum coincides with the supremum in $\overline{\mathbb{R}}_{\max}^n$, and the infimum of any subset of $\mathscr{V}$ is the greatest lower bound of this subset that belongs to $\mathscr{V}$. The following result extends a theorem proved by Zaretski [36] (see [37, Th. 2.1.29] for a proof in English) in the case of the Boolean semiring.

Theorem 7 ([14]). *A matrix $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$ is regular if and only if the subsemimodule of $\overline{\mathbb{R}}_{\max}^n$ generated by its columns is a completely distributive lattice.*

Of course, a dual statement holds for the rows of A. In fact, we know that the semimodule generated by the rows of A is anti-isomorphic to the semimodule generated by its columns [5].

As an illustration of Theorem 5, consider the closed convex set $\mathscr{C} \subset \mathbb{R}_{\max}^2$ depicted in Figure 5. We have $\operatorname{ext}(\mathscr{C}) = \{a, b, c, d, e\}$, where $a = [5, 2]^T$, $b =$

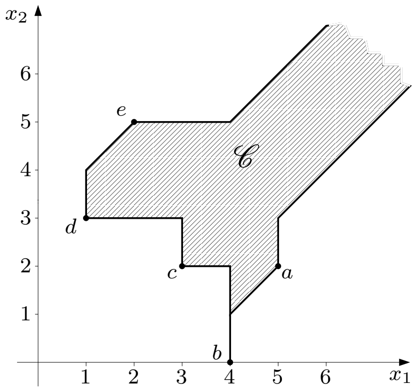

Fig. 3. An unbounded max-plus convex set

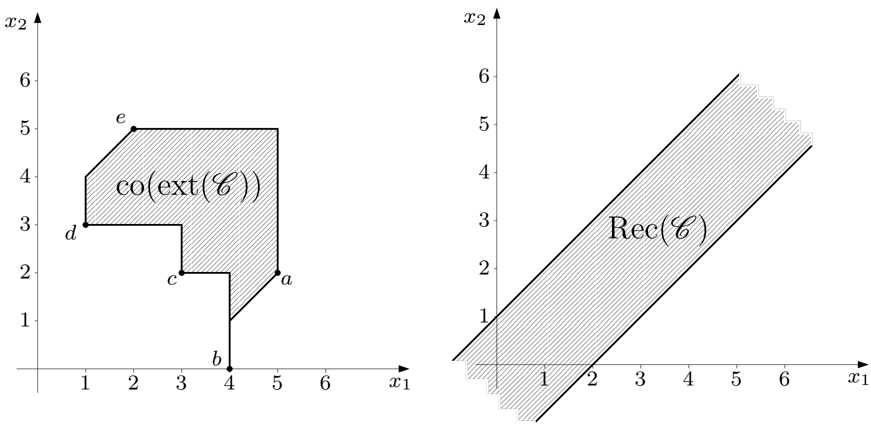

Fig. 4. The sets $\mathrm{co}(\mathrm{ext}(\mathscr{C}))$ and $\mathrm{rec}(\mathscr{C})$ of Theorem 5 for the unbounded convex set depicted in Figure 5

$[4,0]^T$, $c = [3,2]^T$, $d = [1,3]^T$, $e = [2,5]^T$, and $\mathrm{rec}(\mathscr{C}) = \mathrm{cone}\,\left\{[0,1]^T, [2,0]^T\right\}$. Then,

$$\mathscr{C} = \mathrm{co}\,\{a,b,c,d,e\} \oplus \mathrm{cone}\,\left\{[0,1]^T, [2,0]^T\right\}$$

by Theorem 5. The sets $\mathrm{co}(\mathrm{ext}(\mathscr{C}))$ and $\mathrm{rec}(\mathscr{C})$ are depicted in Figure 4. The cone $\mathrm{rec}(\mathscr{C})$ is a distributive lattice, since the infimum and supremum laws coincide with those of $\overline{\mathbb{R}}^2_{\max}$. Note that any $n \times 2$ or $2 \times n$ matrix is regular, in particular, finitely generated cones which are not distributive lattices cannot be found in dimension smaller than 3, see [34].

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