# Performance Evaluation of (max,+) Automata

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*Abstract*—***Automata with multiplicities over the (max,+) semiring can be used to represent the behavior of timed discrete event systems. This formalism which extends both conventional automata and (max,+) linear representations covers a class of systems with synchronization phenomena and variable schedules. Performance evaluation is considered in the worst, mean, and optimal cases. A simple algebraic reduction is provided for the worst case. The last two cases are solved for the subclass of deterministic series (recognized by deterministic automata). Deterministic series frequently arise due to the finiteness properties of (max,+) linear projective semigroups. The mean performance is given by the Kolmogorov equation of a Markov chain. The optimal performance is given by a Hamilton-Jacobi-Bellman equation.***

*Keywords*—***Discrete Event Systems, (max,+) algebra, Automata, Rational Series, Performance Evaluation***

## I. Introduction

Automata with multiplicities [10] over the (max,+) or the dual (min,+) semiring (or equivalently, rational & recognizable series [5] over the (max,+) semiring) are much studied objects in language theory and combinatorics. Their applications to linguistic problems are well known (Simon, Hashiguchi, Mascle, Leung, Krob, Weber, see [17], [24], [18], [25] and the references therein). The purpose of this paper is to show that these series are also useful to model and to analyze certain timed discrete event systems (DES) which exhibit both *synchronization* features (when some task has to wait for the completion of several other tasks) and some *particular* forms of *concurrency* (when two events may occur alternatively at the same logical epoch).

The results presented here are an attempt to fill the gap between the two following popular algebraic approaches to DES. 1. The modeling of DES by conventional automata, initiated by Ramadge and Wonham [23]. In this theory (abbreviated RW in the sequel), events are represented by letters and DES are seen as finite state machines. The main results concern the logical behavior of DES under some appropriate supervision. 2. The (max,+) school (see [4], [8]) considers a much more special class of systems (which essentially coincides with timed event graphs). Contrarily to automata in which the controls (letters) allow the selection between different trajectories, (max,+)-linear stationary systems are well adapted to DES whose behavior is made deterministic by fixing the schedules. The spirit is also different since the theory basically considers certain quantitative measures (asymptotic performance, size of the stocks, earliest or latest behavior).

It is very natural to try to incorporate some time modeling in the RW framework and dually, to try to model with the (max,+) algebra the forms of undeterminism and concurrency which are easily handled with automata. Indeed, several temporal extensions of the RW modeling have already been proposed under the name of timed automata (Alur, Coucourbetis, Dill [2], Wonham and Brandin [7], Nicollin, Sifakis, Yovine …[21]). Timed automata essentially represent the *logical* behavior of systems whose transitions are constrained by some clock inequalities. The logical verification results extend to the timed case —up to an increase of complexity.

In this paper, we propose a different extension based on (max,+) automata, which generalize both conventional automata and finite dimensional causal stationary recurrent (max,+)-linear systems. Then, we extend the usual (max,+) performance evaluation results to the automata case.

In §II, we recall the basic results about automata with multiplicities and rational/recognizable series. In §III, we show that several interesting subclasses of DES are modelizable by (max,+) automata. Typically, the words can be used to represent some finite sequences of tasks (schedules) and the (max,+) automaton computes the completion time as a function of the schedule. In loose terms, the concurrency features are modeled by the possible choices between the letters, and the synchronization features are implemented by the (max,+) algebra. In §IV, we state the three basic performance evaluation problems to which the paper is mainly devoted. The *worst case* performance over a horizon $k$ consists in finding the sequence of $k$ events with the latest time of completion (worst makespan). The much more interesting *optimal case* performance consists in selecting a schedule with minimal makespan. The *mean case* performance evaluates the average time of completion of $k$ events when these events are selected with a simple (say Bernouilli) law. The worst case evaluation problem is solved in §V and §VI by appealing to the (max,+) spectral theory. Up to detail technical points, we show that the worst case performance over a horizon $k$ is asymptotically of the form $\rho k$, where $\rho$ (interpreted as the inverse of the worst case throughput) is equal to the (max,+) eigenvalue of a certain matrix. Since the optimal case and the mean case behavior turn out to be much more complex, we are led to introduce in §VII the tractable subclass of (max,+) *deterministic* series. Deterministic series admit a representation as an additive cost of the trajectory of a finite dynamical system. Thus, the optimal and mean case evaluation reduce to some classical Markovian techniques. In §VIII, we give some determinizability conditions based on finiteness properties of (max,+) linear projective semigroups. We provide an algorithm to build an additive-cost representation from a non deterministic linear representation satisfying a projective finiteness condition. In §IX, we apply this reduction to the mean case performance which is given by the Kolmogorov equation of an induced Markov Chain. This shows that for deterministic series, the mean case performance over $k$ steps is asymptotically linear in $k$, where the rate is obtained by elementary means. In §X, the optimal performance is obtained along the same lines. The Kolmogorov equation is replaced by a Hamilton-Jacobi-Bellman equation, and the Markov chain is replaced by a Bellman Chain [1]. The optimal performance also exhibits a linear growth.

It is important to notice that stochastic (max,+) automata are a finite algebraic version of random products of (max,+) matrices

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for which a precise general ergodic theory is available [3], [4], [19]. The mean case measure of performance considered here coincides with Baccelli's *Lyapunov exponent* (analogous to the Lyapunov exponents of stochastic conventional dynamical systems). The determinization procedure that we use can be seen as a finite version of the construction of 1-cocycles over the projective space on which the study *à la* Furstenberg of random products of matrices is based [6]. The possibility of using such Markovian reduction in the DES context seems to have been first noted by Olsder, Resing, de Vries, Keane and Hooghiemstra [22] [4, chap8]. Moreover, the idea of optimizing some similar finitely valued nonstationary products of (max,+) matrices is due to Olsder [4, chap9].

To conclude, let us mention that there are other important applications of (max,+) and (min,+) rational series to DES and Bellman processes. Namely: 1. as Fliess generating series for (min,+) bilinear systems (which correspond to a subclass of Timed Petri nets in which resources —tokens— can be dynamically added to the system) [11], [13], 2. as generating series of Markovian optimization problems, 3. as devices for counting the occurrences of some distinguished events in DES modeled in the conventional RW way [13], 4. as models for certain timed systems with shared resources (some variants of the dining philosophers) [13]. Note also that a similar representation has been used by Gaujal and Mairesse [15] to compare two communication protocols.

## II. (max,+) Automata, Recognizable Daters and Rational Series

Let us recall two classical ways of modeling DES.

1. In the Ramadge-Wonham (RW) theory [23], a DES is represented by a language, i.e as a subset $L$ of $\Sigma^*$, where $\Sigma$ denotes a finite alphabet whose letters are interpreted as elementary events and $\Sigma^*$ denotes the monoid of words on $\Sigma$. Usually, $w \in L$ means that the sequence of elementary events given by the successive letters of $w$ corresponds to an admissible behavior of the system.
2. In the (max,+) theory [4], [8], a system is represented by a vector $x$ of *dater functions*. I.e., the $i$-entry of $x$ is a map $x_i : \mathbb{N} \to \mathbb{R} \cup \{-\infty\}$, and $x_i(n)$ is usually interpreted as the time of the $n$-th occurrence of the event labeled $i$ (say the time of production of the $n$-th part of type $i$).

We naturally merge the two notions as follows:

*Definition 1 (Dater Function)* A *dater* is a map $y : \Sigma^* \to \mathbb{R} \cup \{-\infty\}$.

We shall write $(y|w)$ —instead of $y(w)$— for the value of $y$ at the word $w$. This scalar product notation which is standard in the rational series literature [5] will soon appear to be useful. We shall interpret $(y|w)$ as the time of completion of the sequence of events $w$, with the convention that $(y|w) = -\infty$ if $w$ does not occur. By specialization to the case of *boolean* daters (with values in $\{-\infty, 0\}$), we obtain the Ramadge-Wonham modeling. By specialization to a single letter alphabet $\Sigma = \{a\}$, we obtain the usual dater functions of the (max,+) theory. In the RW theory, the languages of interest are recognized by some finite devices (typically finite automata). Similarly, in the (max,+) theory, the daters satisfy some finite dimensional linear recurrent systems in the (max,+) algebra. Here, we shall consider the class of dater functions which are recognized by (max,+) automata.

*Definition 2 ((max,+) Automaton)*

A (finite) (max,+) automaton over an alphabet $\Sigma$ is a quadruple $\mathcal{A} = (Q, \alpha, T, \beta)$ where $Q$ is a (finite) set of states and $\alpha, T, \beta$ are maps $\alpha : Q \to \mathbb{R} \cup \{-\infty\}$, $\beta : Q \to \mathbb{R} \cup \{-\infty\}$, $T : Q \times \Sigma \times Q \to \mathbb{R} \cup \{-\infty\}$ (called respectively initial delays, final delays, transition times).

The "functioning" of the automaton is as follows. A *path* of length $n$ is a sequence of states $p = (q_1, \dots, q_{n+1}) \in Q^{n+1}$. We say that the word $w = a_1 \dots a_n$ is *accepted* or *recognized* by the path $p$ if:

$$\begin{aligned} \mathrm{weight}(p, w) &\stackrel{\text{def}}{=} \alpha(q_1) + T(q_1, a_1, q_2) + \cdots \\ &\quad + T(q_n, a_n, q_{n+1}) + \beta(q_{n+1}) \neq -\infty \ . \end{aligned} \tag{1}$$

We shall also write in a self explanatory way

$$\mathrm{weight}(p, w) = \mathrm{weight}(q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \to \cdots q_n \xrightarrow{a_n} q_{n+1}) \ .$$

The *multiplicity*[1] of the word $w = a_1 \dots a_n$ is the maximum of the weights of the paths accepting $w$, namely

$$\begin{aligned} (\mathcal{A}|w) &\stackrel{\text{def}}{=} \max_{p \in Q^{n+1}} \mathrm{weight}(p, w) \\ &= \max_{q_1 \cdots q_{n+1}} [\alpha(q_1) + T(q_1, a_1, q_2) + \cdots \\ &\qquad + T(q_n, a_n, q_{n+1}) + \beta(q_{n+1})] \ . \end{aligned} \tag{2}$$

We say that the automaton *recognizes* the *dater* $w \mapsto (\mathcal{A}|w)$. A dater $y : \Sigma^* \to \mathbb{R} \cup \{-\infty\}$ is called *recognizable* if there exists an automaton $\mathcal{A}$ such that $(y|w) = (\mathcal{A}|w)$.

There is a useful graphical representation of a (max,+) automaton $\mathcal{A}$, which can be identified to a valued multigraph, with $Q$ as set of vertices and 3 kinds of arcs:

1. the internal arcs, $i \xrightarrow{a} j$, for all $i, j \in Q$ and $a \in \Sigma$ such that $T(i, a, j) \neq -\infty$. The arc $i \xrightarrow{a} j$ is valuated by the scalar $T(i, a, j)$.
2. the input arcs $\to i$, valuated by $\alpha(i)$, for all $i$ in $Q$ such that $\alpha(i) \neq -\infty$.
3. the output arcs $i \to$, valuated by $\beta(i)$, for all $i$ in $Q$ such that $\beta(i) \neq -\infty$.

Then, $(\mathcal{A}|w)$ reads graphically as *the max of all the additive weights of the paths labeled $w$ from the input to the output of the graph associated with $\mathcal{A}$.*

*Example 1:* Let $\Sigma = \{a, b\}$. The automaton with set of states $Q = \{0, 1, 2\}$, transition times $T(0, a, 1) = 1$, $T(0, a, 2) = 3$, $T(1, a, 2) = 4$, $T(2, b, 2) = 1$, $T(2, b, 1) = 5$, $T(2, b, 0) = 7$, $T(1, b, 1) = 1$, $T(1, b, 0) = 2$, final and initial delays $\beta(0) = 2$, $\alpha(0) = 0$ (the other values of $\alpha, T, \beta$ are $-\infty$) is represented on Fig. 1. The valuations equal to $0$ will be omitted (e.g. the non valuated input arc $\to 0$ stands for $\alpha(0) = 0$). We have for instance:

$$\begin{aligned} (\mathcal{A}|ab) &= \max(\mathrm{weight}(0 \xrightarrow{a} 1 \xrightarrow{b} 0), \mathrm{weight}(0 \xrightarrow{a} 2 \xrightarrow{b} 0)) \\ &= \max(\alpha(0) + T(0, a, 1) + T(1, b, 0) + \beta(0), \\ &\qquad \alpha(0) + T(0, a, 2) + T(2, b, 0) + \beta(0)) \\ &= \max(5, 12) = 12 \ . \end{aligned} \tag{3}$$

[1] The term multiplicity is standard [10]. It is used by extension from the $(\mathbb{N}, +, \times)$ case: if we replace respectively $\max$ by $+$, $+$ by $\times$, $-\infty$ by $0$, in (1),(2), and if we assume that $\alpha, T, \beta$ are $0, 1$ valued, then the *multiplicity* $(\mathcal{A}|w)$ counts the number of paths accepting $w$.

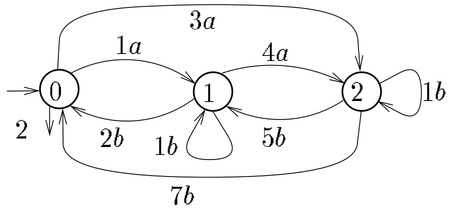


Fig. 1. A (max,+) automaton

A concrete interpretation of this automaton will be given in §III-C.2.

*Remark 1 (Underlying Conventional Automaton)* There is a natural conventional non deterministic automaton $\overline{\mathcal{A}} \stackrel{\text{def}}{=} (Q, Q_i, \delta, Q_f)$ —state $Q$, initial states $Q_i$, transition map $\delta : Q \times \Sigma \to 2^Q$, final states $Q_f$— underlying the (max,+) automaton $\mathcal{A} = (Q, \alpha, T, \beta)$. Namely:

$$\begin{aligned} \delta(q,a) &= \{q' \mid T(q,a,q') \neq -\infty\}, \\ Q_i &= \{q \mid \alpha(q) \neq -\infty\}, \\ Q_f &= \{q \mid \beta(q) \neq -\infty\} \ . \end{aligned}$$

This conventional automaton $\overline{\mathcal{A}}$ can be visualized by forgetting the time valuations on the graph of $\mathcal{A}$. The language recognized by $\overline{\mathcal{A}}$ coincides with $\{w \in \Sigma^* \mid (\mathcal{A}|w) \neq -\infty\}$. We shall say that a (max,+) automaton is *deterministic* if the underlying conventional automaton is deterministic (i.e. if $Q_i = \{q_0\}$ —a single initial state— and if $\forall q, a$, $\delta(q,a)$ has at most one element). There is a simple algebraic formulation of (2) that we next introduce. The (max,+) algebra [20], [4], [9], [16] is by definition the set $\mathbb{R} \cup \{-\infty\}$ equipped with the laws $\max$ (denoted by $\oplus$) and + (denoted by $\otimes$). E.g. $2 \otimes 1 = 3$, $2 \oplus -1 = 2$. The element $\varepsilon \stackrel{\text{def}}{=} -\infty$ satisfies $\varepsilon \oplus x = x$ and $\varepsilon \otimes x = \varepsilon$ ($\varepsilon$ acts as a zero). The element $e \stackrel{\text{def}}{=} 0$ satisfies $e \otimes x = x$ ($e$ is the unit). The main discrepancy with conventional algebra is that $x \oplus x = x$. We shall denote $\mathbb{R}_{\max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ this structure. $\mathbb{R}_{\max}$ is a special instance of *dioid* (semiring whose addition is idempotent). The (max,+) matrix product is defined in the ordinary way:

$$(A \otimes B)_{ij} \stackrel{\text{def}}{=} \bigoplus_k A_{ik} \otimes B_{kj} = \max_k [A_{ik} + B_{kj}]$$

where $A, B$ are matrices with compatible sizes. We shall write $AB$ instead of $A \otimes B$, as usual. We define the map

$$\mu : \Sigma \to \mathbb{R}_{\max}^{Q \times Q}, \mu(a)_{qq'} \stackrel{\text{def}}{=} T(q,a,q') \ .$$

Then, identifying $\alpha$ with a row vector and $\beta$ with a column vector, we get $(\mathcal{A}|a_1 \ldots a_n) = \alpha\mu(a_1)\ldots\mu(a_n)\beta$ . Since $\mu$ can be extended in a unique way to a morphism of multiplicative monoids $\Sigma^* \to \mathbb{R}_{\max}^{Q \times Q}$ by setting $\mu(a_1 \ldots a_n) = \mu(a_1)\ldots\mu(a_n)$, we get

$$(\mathcal{A}|w) = \alpha\mu(w)\beta \ . \tag{4}$$

Thus, a (max,+) automaton is equivalently defined by a triple $(\alpha, \mu, \beta)$, where $\alpha \in \mathbb{R}_{\max}^{1 \times Q}, \beta \in \mathbb{R}_{\max}^{Q \times 1}$, and $\mu$ is a morphism $\Sigma^* \to \mathbb{R}_{\max}^{Q \times Q}$. We will call such a triple a *linear representation* of the dater (4). We shall also write equivalently a dater function $y$ as a *formal series*:

$$y = \bigoplus_{w \in \Sigma^*} (y|w)\, w \ .$$

E.g, $y = 2 \oplus 4b \oplus 3a \oplus 3a^2 \oplus 3a^3 \oplus \ldots$ stands for the dater $(y|e) = 2$ ($e$ denotes the empty word) $(y|b) = 4$, $(y|a) = (y|a^2) = \ldots = 3$, and $(y|w) = \varepsilon = -\infty$ for the other words $w$. The set of formal series with coefficients in $\mathbb{R}_{\max}$ and noncommutative indeterminates $a \in \Sigma$, denoted $\mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$, is naturally equipped with a number of interesting classical operations. We shall only use here the following

$$\begin{array}{lll} \text{Sum} & & \\ (y \oplus z|w) & \stackrel{\text{def}}{=} & (y|w) \oplus (z|w) \\ & = & \max((y|w),(z|w)) \\ \text{Hadamard Product} & & \\ (y \odot z|w) & \stackrel{\text{def}}{=} & (y|w) \otimes (z|w) = (y|w) + (z|w) \\ \text{Cauchy Product} & & \\ (y \otimes z|w) & \stackrel{\text{def}}{=} & \bigoplus_{uv=w} (y|u) \otimes (z|v) \\ & = & \max_{uv=w}[(y|u) + (z|v)] \\ \text{Scalar Product} & & \\ (y|z) & \stackrel{\text{def}}{=} & \bigoplus_w (y|w) \otimes (z|w) \\ \text{Star} & & \\ (y^*|w) & \stackrel{\text{def}}{=} & \bigoplus_{n \in \mathbb{N}} (y^n|w) \end{array} \tag{5}$$

(in the definition of $y^*$, $y^n$ stands for $y \otimes y \otimes \cdots \otimes y$, i.e. the $n$-th power for the Cauchy product). It is important to note that the two latest operations are only partially defined (e.g. $y^*$ is well defined iff $(y|e) \leq e$). As it is well known [5], recognizable series are stable by the operations $\oplus, \odot, \otimes, *$. In particular (and this is the Kleene-Schützenberger theorem [5]), the dioid of rational series (defined as the closure of the dioid of polynomials by the operations $\oplus, \otimes, *$) coincides with the dioid of recognizable series, so that our object of study is nothing but rational series in several non commuting indeterminates over the (max,+) semiring.

## III. Examples of DES modelizable by (max,+) automata

We give here a few examples of DES modelizable by (max,+) automata. Some other applications are sketched in [11], [13]. The purpose of these examples is to illustrate the typical features of (max,+) automata and to discuss their relations with some existing formalisms.

### *A. Several Tasks on a Sequential Machine*

Let us consider a sequential machine processing some parts of type $a \in \Sigma$ with time $t_a \in \mathbb{R}_{\max}$. The time of completion of the sequence of tasks $w$ is $(y|w) = \sum_{a \in \Sigma} t_a \times |w|_a$, where $|w|_a$ denotes the number of occurrences of the letter $a$ in $w$. The dater $y$ is rational. Indeed, $y = (\bigoplus_{a \in \Sigma} t_a a)^*$.

### *B. Several Machines Working In Parallel*

We now assume that the parts of type $a \in \Sigma$ are processed on a dedicated machine $M_a$ with time $t_a$. Then, the time of completion of the sequence $w$ becomes $(y|w) = \max_{a\in\Sigma}(t_a \times |w|_a)$. $y$ is again rational, since $y = \bigoplus_{a\in\Sigma}(t_a a \oplus \bigoplus_{b\neq a} 0b)^*$.

*C. Storage Resource with Finite Capacity*

C.1 Deterministic Case

We consider a storage with a capacity of two units. The two following events are possible:

| | |
|---|---|
| $a$ | a part is added to the stock |
| $b$ | a part is taken out. |

This system is represented by the automaton of Figure 2 over the alphabet $\Sigma = \{a, b\}$. Node 0 represents the state "0 parts in stock", node 1 "1 part in stock", etc. First, the storage is empty. We consider the situation where the transitions of the automaton take some given times. For instance, we assume that the transition $1 \xrightarrow{a} 2$ takes 4 units of time and we set $T(1, a, 2) = 4$. The other values of $T$ are displayed on Fig. 2. We have also taken into account some initial and final delay (e.g. the arc $0 \xrightarrow{2}$ in the Figure represents a final delay of 2 units). Now, the time of completion of the admissible sequence of tasks $w = w_1 \dots w_k \in \Sigma^k$ is equal to the weight of the unique path $0 \xrightarrow{w_1} i_1 \dots i_{k-1} \xrightarrow{w_k} 0$ accepting $w$: $(y|w_1 \dots w_k) = T(0, w_1, i_1) + T(i_1, w_2, i_2) + \dots + T(i_{k-1}, w_k, 0) + \beta(0)$ . The dater $y$ admits the following linear representation

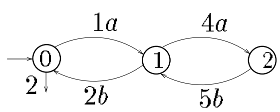


Fig. 2. Storage with Capacity of Two Units

$$\alpha = \begin{pmatrix} e & \varepsilon & \varepsilon \end{pmatrix},\ \beta = \begin{pmatrix} 2 \\ \varepsilon \\ \varepsilon \end{pmatrix}, \tag{6}$$

$$\mu(a) = \begin{pmatrix} \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \mu(b) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon \\ \varepsilon & 5 & \varepsilon \end{pmatrix} . \tag{7}$$

E.g. $(y|ab) = \alpha\mu(a)\mu(b)\beta = 5$. The above example is indeed generic of *deterministic* (max,+) automata (defined in Remark 1). More generally, *any DES whose logical behavior is represented by a conventional deterministic automaton and whose non-instantaneous actions correspond to the usage of a* single *resource is modelizable by a deterministic (max,+) automaton*. In such cases, the max structure is not really used since there is at most one path accepting a given word. The max structure becomes helpful to represent some undeterminism, as follows.

C.2 Non Deterministic Case

The automaton first shown on Fig. 1 is a non deterministic version of the preceding storage resource, in which the events $a$ and $b$ still represent arrivals and withdrawals of parts, but where the quantities are not specified in a deterministic way. E.g., at state $0$, the event $a$ may represent the arrival of either one or two parts, with respective storage times $1$ and $3$ as shown on Fig. 1. In a similar way, the event $b$ represents either the delivery of a part as before, either the simultaneous delivery of 2 parts (7 time units), either an unsuccessful attempt of delivery (e.g. if the customer refuses the part) which takes 1 unit of times and leaves the resource at the same state. We have the linear representation

$$\mu(a) = \begin{pmatrix} \varepsilon & 1 & 3 \\ \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix},\ \mu(b) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 2 & 1 & \varepsilon \\ 7 & 5 & 1 \end{pmatrix}$$

(same $\alpha, \beta$ as in (6)). The time $(y|w)$ can now be interpreted as the *maximal* duration of a successful sequence of tasks compatible with the sequence of informations $w$. More generally: *(max,+) automata naturally compute the worst case behavior of non deterministic automata with timed transitions*.

*D. Nonstationary (max,+) Linear Systems with Finitely Valued Dynamics*

The (max,+) theory [4] deals with linear recurrent systems of the form

$$x(k) = A(k)x(k-1),\ x(0) = b,\ y(k) = cx(k) \tag{8}$$

where $A(k) \in \mathbb{R}_{\max}^{n\times n}, x(k), b \in \mathbb{R}_{\max}^{n\times 1},\ c \in \mathbb{R}_{\max}^{1\times n}, y(k) \in \mathbb{R}_{\max}$. We claim that (max,+) automata represent the case where the dynamics of the system $A(k)$ only takes a finite number of values $A_1, \dots, A_p$. Indeed, if the dater $y$ is recognized by $(\alpha, \mu, \beta)$, $(y|w)$ can be computed recursively by introducing a "state vector" $(x|w) \stackrel{\text{def}}{=} \alpha\mu(w)$ and setting

$$(x|wa) = (x|w)\mu(a),\ \ (x|e) = \alpha,\ (y|w) = (x|w)\beta \tag{9}$$

which is similar to (8), up to the transposition and to the fact that the dynamic $\mu(a)$ depends on the last letter of the word $wa$ (instead of the logical time $k$). More formally, we introduce an alphabet $\Sigma = \{a_1, \dots, a_p\}$, we set $\mu(a_i) = A_i$. We represent the information $A(1) = \mu(w_1), \dots, A(k) = \mu(w_k)$ by the word $w = w_1 \dots w_k$ and we denote by $y_w$ the corresponding output $y(k)$ of (8). We denote by $\tilde{w} = w_k \dots w_1$ the mirror image of $w$. Then,

$$y_w = c\mu(\tilde{w})b\ ,$$

i.e. *nonstationary (max,+) linear systems with finitely valued dynamics are represented by (max,+) automata with reverse interpretation*. We observe that when the matrices $A(1), A(2), \dots$ are random variables, systems of the form (8) belong to the much studied class of Stochastic timed event graphs. We refer the reader to [3], [22], [4] for the important applications of these systems. We shall mention here a different one, where the word $w$ represents a schedule.

*E. Workshop with Variable Schedule*

We consider a workshop with two machines processing three types of parts. We assume that there are two working regimes $a$ and $b$ and that the workshop follows an open loop schedule $w \in \Sigma^*$. E.g. $w = aaab$ stands for "3 working periods of type a followed by 1 period of type b". The working regimes are described as follows.

*(a)* At the $n$-th working regime $a$, the $n$-th part of type 2 is processed by machine 2 during 5 units of time. Then it is

sent to machine 1 (transportation time: $1$ unit). Machine 1 processes the $n$-th type 1 part (during 3 units of time), then it is assembled with the $n$-th type 2 part received from machine 2.

*(b)* At the $p$-th working regime $b$, the $p$-th part of type 3 is processed during 10 units of time on machine 1. No additional task is imposed to Machine 2.

We assume that the workshop is initially empty, starts working at time $0$, and then operates at maximal speed provided that the precedence constraints of the schedule are satisfied. Let $(x_1|w)$ (resp. $(x_2|w)$) denote the earliest time at which all the operations on Machine 1 (resp. 2) required by the schedule $w$ are completed. Due to the initial conditions, we set $(x|e) = [e\ e] = \alpha$. We are interested in computing $(y|w) = (x_1|w)$. This is the time when the last part exits the workshop under the schedule $w$. We claim that $y$ is recognized by the following linear representation (displayed in Figure 3).

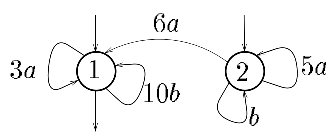


Fig. 3. Workshop with Two Working Regimes

$$\mu(a) = \begin{pmatrix} 3 & \varepsilon \\ 6 & 5 \end{pmatrix},\quad \mu(b) = \begin{pmatrix} 10 & \varepsilon \\ \varepsilon & e \end{pmatrix},$$
$$\alpha = \begin{pmatrix} e & e \end{pmatrix},\ \beta = \begin{pmatrix} e \\ \varepsilon \end{pmatrix} .$$

For instance, $((x|wa))_1 = ((x|w)\mu(a))_1$ means exactly that

$$(x_1|wa) = \max(6 + (x_2|w), 3 + (x_1|w)) \qquad (10)$$

i.e that (i) machine $1$ has to wait for the current type 2 part to arrive from machine 2 (5 +1 time units) and (ii) that machine $1$ takes 3 time units to produce the current type 1 part. The other columns of $\mu(a), \mu(b)$ are obtained by writing similar equations for $(x_2|wa), (x_i|wb)$.

Some interesting features appear when considering subbehaviors of the system, that is when we assume that the schedule $w$ belongs to a legal language $L$. Assume for instance a periodic behavior of the form $L_l = (a^l b)^*$ (that is, 1 period of type b occurs every $l$ periods of type a). $(y|(a^l b)^i)$ is equal to the maximal weight of the successful paths with label $(a^l b)^i$. It is not too difficult to see that there are only three possible successful paths with maximal weight

$$p = 2 \xrightarrow{a^{l-1}} 2 \xrightarrow{a} 1 \xrightarrow{b(a^l b)^{i-1}} 1$$
$$\text{weight}(p) = 5(l-1) + 6 + 10 + (3l+10)(i-1) \qquad (11)$$

$$p' = 2 \xrightarrow{(a^l b)^{i-1} a^{l-1}} 2 \xrightarrow{a} 1 \xrightarrow{b} 1$$
$$\text{weight}(p') = 5l(i-1) + 5(l-1) + 6 + 10 \qquad (12)$$

$$p'' = 1 \xrightarrow{(a^l b)^i} 1,\ \ \text{weight}(p'') = (3l+10)i\ . \qquad (13)$$

Hence,

$$\begin{aligned}(y|(a^l b)^i) &= \max(\text{weight}(p), \text{weight}(p'), \text{weight}(p'')) \\ &= \max(2l + 1 + (3l+10)i, 5li + 11)\ . \qquad (14)\end{aligned}$$

Let us introduce the time of completion of the first $k$ events:

$$y_k \stackrel{\text{def}}{=} (y|(a^l b)^* \cap \Sigma^k)\ .$$

We get immediately from (14) that for $k$ sufficiently large,

$$y_{k+c} = \lambda^c \otimes y_k = c \times \lambda + y_k \qquad (15)$$

with $c = l + 1$ and

$$\lambda = \max(\frac{3l+10}{l+1}, \frac{5l}{l+1})\ . \qquad (16)$$

$\lambda$ can be interpreted as the inverse of the periodic throughput. It is worth noting that the maximal term in (16) identifies the bottleneck machine. This is because $(3l+10)/(l+1)$ can be interpreted as the performance of machine 1 in isolation subject to the same schedule (and similarly, $5l/(l+1)$ corresponds to the performance of machine 2 in isolation). One of the purposes of this paper is to study such measures of performance. In particular, we shall see that periodicity properties of type (15) proceed from general properties of (max,+) rational series, which allow a direct computation of such throughputs.

## IV. Performance Evaluation of (max,+) Automata

We next state the three basic performance evaluation problems to which the remaining part of the paper is devoted. In the following, $y$ will denote a *dater* function.

### A. Worst case

For all $k \geq 0$, we consider the latest time of completion of a sequence of $k$ elementary events:

$$\ell_k^{\text{worst}}(y) \stackrel{\text{def}}{=} \sup_{w \in \Sigma^k} (y|w) = (y|\Sigma^k) \qquad (17)$$

where we have identified $\Sigma^k$ to its characteristic series $\bigoplus_{w \in \Sigma^k} w$ in order to use the scalar product notation (5). With a more conventional notation, this reads:

$$\begin{aligned}\ell_k^{\text{worst}}(y) &\stackrel{\text{def}}{=} \bigoplus_{w \in \Sigma^k} (y|w) \\ &= \max_{w_k,\dots,w_1 \in \Sigma}\ \max_{i_k,\dots,i_0} [\alpha_{i_k} + \mu(w_k)_{i_k i_{k-1}} + \cdots \\ &\qquad + \mu(w_1)_{i_1 i_0} + \beta_{i_0}]\ . \qquad (18)\end{aligned}$$

### B. Optimal case

$$\ell_k^{\text{opt}}(y) \stackrel{\text{def}}{=} \inf_{w \in \Sigma^k, (y|w) \neq \varepsilon} (y|w)\ . \qquad (19)$$

With the usual notation:

$$\begin{aligned}\ell_k^{\text{opt}}(y) &\stackrel{\text{def}}{=} \min_{\substack{w \in \Sigma^k, \\ (y|w) \neq \varepsilon}} (y|w) \\ &= \min_{\substack{w_k,\dots,w_1 \in \Sigma, \\ (y|w) \neq \varepsilon}}\ \max_{i_k,\dots,i_0} [\alpha_{i_k} + \mu(w_k)_{i_k i_{k-1}} + \cdots \\ &\qquad + \mu(w_1)_{i_1 i_0} + \beta_{i_0}]\ . \qquad (20)\end{aligned}$$

This min-max problem consists in finding a schedule minimizing the completion time of the $k$-th event. Note that the non admissible sequences $w$ such that $(y|w) = \varepsilon = -\infty$ have to be explicitly omitted in (20) since they would give a trivial infimum equal to $-\infty$.

### C. Mean case

$$\ell_k^{\text{mean}}(y) = \sum_{w \in \Sigma^k} (y|w) \times p_k(w) \tag{21}$$

where $p_k$ is a convenient probability law on $\Sigma^k$. Typically, in the Bernouilli case

$$\ell_k^{\text{mean}}(y) = \sum_{w_1 \ldots w_k} (\max_{i_k, \ldots, i_0} [\alpha_{i_k} + \mu(w_k)_{i_k i_{k-1}} + \cdots + \mu(w_1)_{i_1 i_0} + \beta_{i_0}]) p(w_1) \cdots p(w_k) \tag{22}$$

where the $p(a), a \in \Sigma$ are given probabilities. This is the average completion time of the $k$-th event under a random schedule $w$ with probability $p(w)$.

As we already noticed in Example III-E, we may consider some refinements of these measures by restricting the evaluation to a language $L$ representing a subset of admissible events. For instance, we have the following refinement of (17):

$$\sup_{w \in L \cap \Sigma^k} (y|w) = (y|L \cap \Sigma^k) \tag{23}$$

identifying as usual languages and characteristic series.

## V. Worst Case Analysis via (max,+) Spectral Theory

We shall use the (max,+) spectral theory (analogous to the Perron-Frobenius theory) [16], [4], [9], [20]. We first recall the definition and basic properties of the spectral radius.

*Lemma 1 (Spectral Radius)* Let $A \in \mathbb{R}_{\max}^{n \times n}$. The following quantities are equal:

(i) $\sup\{r \in \mathbb{R}_{\max} \mid \exists u \in \mathbb{R}_{\max}^n \setminus \{\varepsilon\}, \ Au = ru\}$
(ii) $\bigoplus_{1 \le k \le n} (\mathrm{tr} A^k)^{\frac{1}{k}} = \bigoplus_{1 \le k \le n} \bigoplus_{i_1 \ldots i_k} (A_{i_1 i_2} \ldots A_{i_k i_1})^{\frac{1}{k}} = \max_{1 \le k \le n} \max_{i_1 \ldots i_k} (A_{i_1 i_2} + \cdots + A_{i_k i_1})/k$
(iii) $\limsup_k \|A^k\|^{\frac{1}{k}}$

where

$$\|A\| \stackrel{\text{def}}{=} \sup_{ij} A_{ij} \ . \tag{24}$$

This common value will be denoted by $\rho(A)$ (spectral radius or "Perron root" of $A$).

We have the following (max,+) version of the Perron-Frobenius asymptotics.

*Theorem 1 (Cyclicity [20], [4])*

If $M$ is irreducible (i.e. $\forall ij, \exists k, M_{ij}^k \neq \varepsilon$), then the following cyclicity property holds :

$$\exists N, \exists c \ge 1, \ \forall n \ge N, \quad M^{c+n} = (\rho(M))^c M^n \ . \tag{25}$$

The least value of $c$ is called the cyclicity of $M$.

Let $y : \Sigma^* \to \mathbb{R}_{\max}$ be the dater recognized by the automaton $(\alpha, \mu, \beta)$. Then,

$$\begin{aligned} (y|\Sigma^k) &= \bigoplus_{w_1, \ldots, w_k \in \Sigma} \alpha \mu(w_1) \ldots \mu(w_k) \beta \\ &= \alpha M^k \beta \end{aligned} \tag{26}$$

where

$$M = \bigoplus_{x \in \Sigma} \mu(x) \ .$$

Recall that the representation $(\alpha, \mu, \beta)$ is *trim* if

$$\forall i, j, \ \exists k, l, (\alpha M^k)_i \neq \varepsilon, \ \ (M^l \beta)_j \neq \varepsilon$$

(i.e. if each state is both accessible and co-accessible). Then, an immediate application of (25) and of Lemma 1,(iii) to (26) gives

*Theorem 2 (Worst Case Evaluation)* (i) If $M$ is irreducible with cyclicity $c$, for $k$ large enough, we have

$$(y|\Sigma^{k+c}) = \rho(M)^c (y|\Sigma^k) \ .$$

(ii) If $(\alpha, \mu, \beta)$ is trim (but $M$ not necessarily irreducible), we have

$$\rho(M) = \limsup_k (y|\Sigma^k)^{\frac{1}{k}} \ .$$

## VI. A Refinement: Worst Case Evaluation Constrained in a Sublanguage

The rational machinery allows us to compute some more refined performance measures along the same lines. Let us consider the performance restricted to an admissible sublanguage $L$ (see Formula (23)). We assume that $L$ is recognizable (rational, regular), i.e. that there exists a linear representation $(\alpha', \mu', \beta')$ with entries in the boolean semiring $\{\varepsilon, e\}$ such that, identifying $L$ to its characteristic series $\bigoplus_{w \in L} w$,

$$L = \bigoplus_{w \in \Sigma^*} \alpha' \mu'(w) \beta' \, w \ .$$

The evaluation of (23) is equivalent to the worst case evaluation of

$$y \odot L \stackrel{\text{def}}{=} \bigoplus_{w \in L} (y|w) \, w = \bigoplus_{w \in \Sigma^*} (y|w)(L|w) \, w \tag{27}$$

since for all $A$, $\sup_{w \in L \cap A} (y|w) = \sup_{w \in A} (y \odot L|w)$. The *Hadamard product* $y \odot L$ is recognized by the "tensor product" of the linear representations of $y$ and $L$, namely, by the triple $(\alpha'', \mu'', \beta'')$:

$$\mu''(a) = \mu(a) \otimes^t \mu'(a), \quad \alpha'' = \alpha \otimes^t \alpha', \quad \beta'' = \beta \otimes^t \beta'$$

where $\otimes^t$ denotes the tensor product of matrices. We recall here that the tensor product of the $p \times r$-matrix $A$ by the $q \times s$-matrix $B$ is the $pq \times rs$-matrix $(A \otimes^t B)_{(ij)(kl)} = A_{ik} B_{jl}$.

*Proposition 1:* Let $M'' = \bigoplus_{a \in \Sigma} \mu(a) \otimes^t \mu'(a)$. Assume that $M''$ is irreducible with cyclicity $c$. Then we have for $k$ large enough:

$$(y|L \cap \Sigma^{k+c}) = \rho(M'')^c (y|L \cap \Sigma^k) \ . \tag{28}$$

*Proof:* Immediate from Theorem 2,(i). ■

*Example 2:* This allows to compute the throughput obtained by elementary means in Example III-E. For instance, let us consider $L_2 = (a^2 b)^*$ which is recognized by the automaton depicted in Fig. 4A, with boolean linear representation:

$$\alpha' = \begin{pmatrix} e & \varepsilon & \varepsilon \end{pmatrix}, \ \mu'(a) = \begin{pmatrix} \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix},$$

$$\mu'(b) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ e & \varepsilon & \varepsilon \end{pmatrix}, \ \beta' = \begin{pmatrix} e \\ \varepsilon \\ \varepsilon \end{pmatrix} \ . \tag{29}$$

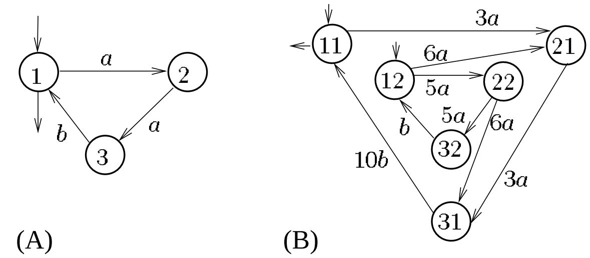


Fig. 4. (A) Automaton recognizing $L_2$, (B) Tensor product.

The automaton recognizing $L_2 \odot y$ is depicted in Fig. 4B. We have

$$M'' = \bigoplus_{a\in\Sigma} \mu(a) \otimes^t \mu'(a) = \begin{pmatrix} \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 6 & 5 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 6 & 5 \\ 10 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \epsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} .$$

Since the matrix $M''$ is reducible, we cannot apply directly Proposition 1 and we just get from Theorem 2,(ii) that $\limsup_k (y|L\cap\Sigma^k)^{\frac{1}{k}} = \rho(M'') = \frac{16}{3}$ (this value can be obtained by applying Lemma 1,(ii)). This is consistent with Formula (16) when $l = 2$. Note that (28) holds in this particular case without the irreducibility assumption.

## VII. Reduction to Deterministic Series

We next consider the asymptotic evaluation of the Optimal Case performance $\ell_k^{\text{opt}}$ and of the Mean Case performance $\ell_k^{\text{mean}}$. There is a simple important case in which this evaluation reduces to some standard Markovian techniques. We say that a series $y$ is *deterministic* if it is recognized by *a* deterministic (max,+) automaton (see Remark 1), i.e. if there exists a deterministic representation $\alpha,\mu,\beta$ of $y$ (such that there is at most one non $\varepsilon$ entry in $\alpha$ and in each row of $\mu(a), a\in\Sigma$). The following simple observation will play a crucial role in the sequel.

*Proposition 2:* A series $y$ is *deterministic* iff there exists a complete[2] deterministic conventional automaton $(Q, q_0, \delta)$ (with unspecified final states), a transition cost $\sigma : Q\times\Sigma \to \mathbb{R}_{\max}$, and a final cost $\phi : Q \to \mathbb{R}_{\max}$ such that for all $w = w_1\ldots w_k \in \Sigma^k$,

$$\begin{aligned}(y|w_1\ldots w_k) = J_k(w,q_0) &\stackrel{\text{def}}{=} \sum_{n=1}^{k} \sigma(q_{n-1},w_n) + \phi(q_k), \\ \text{where } q_n &= \delta(q_{n-1},w_n) \ . \qquad (30)\end{aligned}$$

*Remark 2:* Note that (30) is nothing but the discrete counterpart of the usual integral cost

$$\begin{aligned}J_T(u,x_0) &\stackrel{\text{def}}{=} \int_0^T L(x(t),u(t))\mathrm{d}t + \phi(x(T)), \\ \dot{x} &= f(x,u), \ \ x(0) = x_0 \text{ given.} \qquad (31)\end{aligned}$$

*Proof:* Let $(\alpha,\mu,\beta)$ be a deterministic $n$ dimensional automaton recognizing $y$. Set $Q = \{1,\ldots,n\}$, let $q_0$ be the unique index such that $\alpha_{q_0} \neq \varepsilon$, let

$$\sigma(i,a) = \bigoplus_j \mu(a)_{ij}, \ \phi(i) = \alpha_{q_0}\beta_i$$

$$\delta(i,a) = \begin{cases} \text{the unique } j \text{ such that} & \mu(a)_{ij} \neq \varepsilon \\ \text{arbitrary} & \text{if } \forall j, \mu(a)_{ij} = \varepsilon. \end{cases}$$

This provides a representation as an additive cost of the form (30). Conversely, passing from (30) to a deterministic representation is immediate. ∎

For *deterministic* automata, the evaluation of the mean case performance $\ell_k^{\text{mean}}$ is nothing but the computation of the mean additive cost $\mathbb{E}\,J_k(\cdot,q_0)$ along the trajectories of the dynamical system (30) driven by some random inputs $w_n$. The optimal case performance $\ell_k^{\text{opt}}$ coincides with the value function $\inf_w J_k(w,q_0)$ for a conventional deterministic optimal control problem. These are ordinary Markovian problems which can be solved by using some Kolmogorov and Bellman equations.

In the next section, we will show that under boundedness and integrity conditions for the linear representation, a recognizable series is deterministic. Thus, the above mentioned Markovian techniques apply to an important class of systems. In the non deterministic case, more pathological behaviors may occur. Simon [24] has exhibited a family of automata (with nonpositive (max,+) representation) which show a *sublinear* decrease, i.e. such that $\ell_{k_n}^{\text{opt}} \simeq -K \times \sqrt[p]{k_n}$ —this is the usual $p$-th root— for $p \geq 2$, $K > 0$, and $k_n \to \infty$. We refer the reader to [24] and to Weber [25] for the existing results concerning the behavior of $\ell_k^{\text{opt}}$ in the non deterministic case.

## VIII. Projective Finiteness of (max,+) Linear Semigroups

We next give some sufficient conditions of determinizability based on (max,+) linear projective semigroups. These properties can be seen as natural partial extensions of the (max,+) Perron-Frobenius theory surveyed above. We define the $(n-1)$-dimensional (max,+) projective space as the quotient of $\mathbb{R}_{\max}^n$ by the parallelism relation

$$u \simeq v \iff \exists\lambda\in\mathbb{R}_{\max}, \ \lambda\neq\varepsilon, \ u = \lambda v \ .$$

We denote by $\wp : \mathbb{R}_{\max}^n \to \mathbb{PR}_{\max}^n$ the canonical map. The linear projective monoid $\mathbb{PR}_{\max}^{n\times n}$ is defined similarly (as the quotient of the multiplicative monoid of matrices $\mathbb{R}_{\max}^{n\times n}$ by the congruence $\simeq$). We say that a subset $S \subset \mathbb{R}_{\max}^{n\times n}$ is *projectively finite* if $\wp S$ is finite, i.e. iff there are only finitely many pairwise non proportional elements in $S$. As an immediate corollary of the cyclicity result (25), we can state

*Corollary 1:* If $M \in \mathbb{R}_{\max}^{n\times n}$ is irreducible, then the semigroup generated by $M$, $S = \{M, M^2, M^3, \ldots\}$, is projectively finite.

Since a rational dater writes $(y|w) = \alpha\mu(w)\beta$, it is natural to replace the semigroup $S = \{M, M^2, \ldots\}$ by the finitely generated semigroup of matrices $\mu(\Sigma^+)$ —where $\Sigma^+$ denotes the semigroup of nonempty words. We first introduce some notation. Given $A_1,\ldots,A_p \in \mathbb{R}_{\max}^{n\times n}$, we shall denote by $\langle A_1,\ldots,A_p\rangle$ the semigroup generated by these matrices. We introduce a set of $p$ letters $\Sigma = \{a_1,\ldots,a_p\}$. Let $\mu : \Sigma^+ \to \mathbb{R}_{\max}^{n\times n}$ be the unique morphism such that $\forall i, \ \ \mu(a_i) = A_i$ (i.e. $\mu(a_{i_1}\ldots a_{i_k}) =$

[2] By complete, we mean that $\delta(q,a)$ is defined for all $q,a$.

$A_{i_1}\dots A_{i_k}$). Then $\langle A_1,\dots,A_p\rangle = \mu(\Sigma^+)$ and we say that $\Sigma$ and $\mu$ are obtained *in the canonical way* from the generators $A_1,\dots,A_p$. We say that the semigroup $S=\langle A_1,\dots,A_p\rangle$ is *primitive* if there is an integer $N$ such that for all words $w$,

$$|w|\geq N \Rightarrow \quad \forall i,j \;\; \mu(w)_{ij} > \varepsilon \;\; , \tag{32}$$

where $|w|$ denotes the length of the word $w$. This notion is independent of the finite set of generators. When $S=\{M,M^2,\dots\}$ admits a unique generator, this reduces to the primitivity of $M$ in the Perron-Frobenius sense. We set $\mathbb{Q}_{\max} \stackrel{\text{def}}{=} (\mathbb{Q}\cup\{-\infty\},\max,+)$. The following theorem extends (partially) Corollary 1 to semigroups.

*Theorem 3:* Let $A_1,\dots,A_p \in \mathbb{Q}_{\max}^{n\times n}$. If $\langle A_1,\dots,A_p\rangle$ is a primitive semigroup, then it is projectively finite.

This theorem does not hold in the irrational case. See [14] for some extensions.

*Proof:* Let $q$ be the lcm of the denominators of the entries of the matrices. Since $x\mapsto x^q$ ($x^q = x\times q$ with classical notations) is an automorphism of $\mathbb{Q}_{\max}$ which maps all the entries to integers, we shall assume that $A_1,\dots,A_p\in\mathbb{Z}_{\max}^{n\times n}$. We introduce the following "norms" for a vector $u\in\mathbb{R}_{\max}^n$.

$$||u|| = \sup_i u_i, \quad |u|_\wedge = \inf_{u_i\neq\varepsilon} u_i \tag{33}$$

with $\inf\emptyset = +\infty$. The "norms" of a matrix are defined in the same way (e.g. $||A|| = \sup_{ij} A_{ij}$). These are not norms *stricto sensu* (in particular, they can take negative values); however, they will play essentially the role of usual norms. We note that for matrices $A,B$ with compatible sizes, we have $||AB||\leq ||A||||B||$, $|AB|_\wedge \geq |A|_\wedge|B|_\wedge$. We introduce the "*projective width*"

$$\Delta u \stackrel{\text{def}}{=} \frac{||u||}{|u|_\wedge} \tag{34}$$

(i.e. $||u|| - |u|_\wedge$ in the usual algebra). The proof relies on the following Lemma.

*Lemma 2:* Let $K\in\mathbb{N}$. The set $S$ of matrices $m\in\mathbb{Z}_{\max}^{n\times n}$ such that $\Delta m\leq K$ is projectively finite.

Indeed, after normalization, we may assume that $\forall m\in S\setminus\{\varepsilon\}$, $|m|_\wedge = e$ and $||m||\leq K$. Since there are at most $(K+2)^{n^2}-1$ non $\varepsilon$ matrices of size $n\times n$ with entries in $\{\varepsilon,e,1,\dots,K\}$, the Lemma is proven.

Let

$$\begin{aligned}
\underline{a} &= \min(\,|A_1|_\wedge,,\dots,|A_p|_\wedge\,)\,,\\
\overline{a} &= \max(\,||A_1||,\dots,||A_p||\,)\;\;.
\end{aligned}$$

The primitivity assumption implies that for $w\in\Sigma^+$, $|w|\geq 3N$, we have a factorization $w=sur$ with $|s|=|r|=N$ and $\mu(s),\mu(r),\mu(u) > \varepsilon$ ($N$ is the "primitivity index" satisfying (32)). Then

$$\begin{aligned}
||\mu(w)|| &= ||\mu(sur)|| \leq ||\mu(s)||||\mu(u)||||\mu(r)||\\
&\leq \overline{a}^{2N}\mu(u)_{ij}
\end{aligned} \tag{35}$$

for some indices $ij$ belonging to the argmax in $||\mu(u)|| = \sup_{ij}\mu(u)_{ij}$. Moreover

$$\mu(sur)_{kl}\geq \mu(s)_{ki}\mu(u)_{ij}\mu(r)_{jl}\geq \underline{a}^{2N}\mu(u)_{ij}\;\;.$$

This implies that

$$\frac{||\mu(w)||}{|\mu(w)|_\wedge}\leq\left(\frac{\overline{a}}{\underline{a}}\right)^{2N}\;\;.$$

It remains to apply Lemma 2 to conclude. ∎

We shall need the following characterization of deterministic series. Another characterization in terms of Hankel matrix (involving only the values of $y$ and not a particular linear representation) is given in [13].

*Theorem 4:* The series $y$ is deterministic iff there exists a linear representation $\alpha,\mu,\beta$ of $y$ such that $\wp\alpha\mu(\Sigma^*)$ is finite.

*Proof:* For a deterministic $n$ dimensional representation, all the vectors $\alpha\mu(w)$ have at most one non-$\varepsilon$ entry. Hence $\wp\alpha\mu(\Sigma^*)$ is trivially finite (with cardinal at most $n+1$). Conversely, we assume that $\wp\alpha\mu(\Sigma^*)$ has finite cardinal $k$ and we build a $k$-dimensional representation of the form (30). Let us take $w_1,\dots,w_k$ such that $\wp\alpha\mu(\Sigma^*) = \{\wp\alpha\mu(w_1),\dots,\wp\alpha\mu(w_k)\}$. Since $\wp\alpha\mu(e) = \wp\alpha\mu(w_{i_0})$ for some $i_0$, $\exists\gamma\in\mathbb{R}_{\max}\backslash\{\varepsilon\}$ such that

$$\alpha = \alpha\mu(e) = \gamma\alpha\mu(w_{i_0})\;\;. \tag{36}$$

The same argument shows that $\forall a\in\Sigma,\forall 1\leq i\leq k$, there exists $\lambda_{i,a}\in\mathbb{R}_{\max}\backslash\{\varepsilon\}$ and $j_{i,a}$ such that

$$\alpha\mu(w_i a) = \lambda_{i,a}\alpha\mu(w_{j_{i,a}})\;\;. \tag{37}$$

We set $Q=\{w_1,\dots,w_k\}, q_0 = w_{i_0}$,

$$\sigma(w_i,a)\stackrel{\text{def}}{=}\lambda_{i,a},\;\;\delta(w_i,a)\stackrel{\text{def}}{=}w_{j_{i,a}},\;\;\phi(w_i)\stackrel{\text{def}}{=}\gamma\alpha\mu(w_i)\beta\;\;. \tag{38}$$

Then,

$$\forall q\in Q, a\in\Sigma,\quad \alpha\mu(qa) = \sigma(q,a)\mu(\delta(q,a))\;\;. \tag{39}$$

This implies that $(Q,q_0,\delta,\sigma,\phi)$ yields a representation of $y$ of the form (30). Indeed, we just prove that $(y|w)$ coincides with (30) for a word $w=a_1a_2$ of length 2, the general case being similar. We have

$$\begin{aligned}
(y|a_1a_2) &= \alpha\mu(e)\mu(a_1)\mu(a_2)\beta\\
&= \gamma\alpha\mu(q_0)\mu(a_1)\mu(a_2)\beta \quad \text{by (36)}\\
&= \gamma\sigma(q_0,a_1)\alpha\mu(q_1)\mu(a_2)\beta \quad \text{by (37),}\\
&\qquad\qquad \text{with } q_1=\delta(q_0,a_1)\\
&= \gamma\sigma(q_0,a_1)\sigma(q_1,a_2)\alpha\mu(q_2)\beta \quad \text{by (37),}\\
&\qquad\qquad \text{with } q_2=\delta(q_1,a_2)\\
&= \sigma(q_0,a_1)\sigma(q_1,a_2)\phi(q_2)\;\;.
\end{aligned}$$

∎

As an immediate application of Theorem 3 and Theorem 4, we get

*Corollary 2 (Sufficient Determinizability Condition)* Let $y$ be the series recognized by the linear representation $\alpha,\mu,\beta$ over $\mathbb{Q}_{\max}$. If the semigroup $\mu(\Sigma^+)$ is primitive, then $y$ is deterministic.

The proof of Theorem 4 allows to build effectively a representation of $y$ as an additive cost (30) under the assumption that $\wp\alpha\mu(\Sigma^*)$ is finite. It is enough to find a finite set $Q=$

$\{w_1,\dots,w_k\}$ such that any $\alpha\mu(w), w\in\Sigma^*$ is proportional to some $\alpha\mu(w_i)$. This can be done in the following way. Let $<$ denote the strict military order on $\Sigma^*$ (i.e. $e<a<b<aa<ab<ba<bb<aaa<\dots$). We set $Q_0=\{e\}$ and we define inductively

$$Q_{i+1}=\{w\in Q_i\Sigma \mid \forall z\in\Sigma^*, z<w \Rightarrow \wp\alpha\mu(w)\neq\wp\alpha\mu(z)\} \ . \quad (40)$$

When $\wp\alpha\mu(\Sigma^*)$ is finite, $Q_i=\emptyset$ for some $i$. It remains to set $Q=\bigcup_{j<i}Q_j$. We illustrate this procedure on a generic example.

*Example 3:* Let us consider the series $y$ in the indeterminates $a,b$ recognized by the linear representation

$$\mu(a)=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} \text{ and } \mu(b)=\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix} ,$$

$$\alpha=[0\ 0], \beta=\alpha^T \ .$$

Since $\mu(a),\mu(b)$ have no $\varepsilon$ entries, the semigroup $\mu(\Sigma^+)$ is trivially primitive, hence, by Corollary 2 $y$ is deterministic. Let $Q_0=\{e\}$. We have

$$\alpha=\alpha\mu(e)=[0\ 0],\ \alpha\mu(a)=[0\ 1],\ \alpha\mu(b)=[0\ 2] \ .$$

Since all these vectors are non proportional, we have $Q_1=\{a,b\}$. Moreover

$$\begin{aligned}\alpha\mu(a^2)=\alpha\mu(a),\ \alpha\mu(ab)=1\alpha\mu(a),\\ \alpha\mu(ba)=1\alpha\mu(e),\ \alpha\mu(b^2)=2\alpha\mu(e) \ .\end{aligned} \quad (41)$$

hence, $Q_2=\emptyset$ and $Q=Q_0\cup Q_1=\{e,a,b\}$. According to the proof of Theorem 4, a deterministic representation of the form (30) is obtained by setting $q_0=e$ and by taking the complete automaton displayed on Figure 5, whose transition costs $\sigma(w,a)$ are directly obtained from (41). The final cost $\phi$ is shown on the output arcs (e.g. $\phi(b)=2$). It is worth noting that these relations allow (in particular) to compute $(y|w)$ without multiplying the matrices. It is enough to take the additive weight of the corresponding path. For instance,

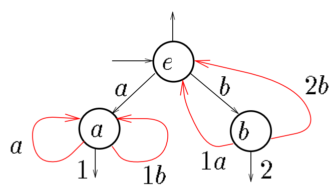

Fig. 5. Deterministic Representation

$$\begin{aligned}(y|(babb)^n) &= \alpha\mu((babb)^n)\beta\\ &= \alpha 1^n 2^n\beta = 1\times n+2\times n+\phi(e)\\ &= 3n \ .\end{aligned} \quad (42)$$

*Remark 3:* The assumptions of Theorem 3 are satisfied for a class of stochastic irreducible timed event graphs. Precisely, we consider the timed event graphs with dater vector $x(k)\in\mathbb{R}^n_{\max}$ given by (cf. [4]) $x(k+1)=A(k)x(k)$ where the $A(k)$ are i.i.d. random matrices taking only a finite number of values $A_1,\dots,A_p$ (cf. Ex. III-D). Since dater functions are nondecreasing, we may assume that the $A(k)$ only take values greater than Id (identity matrix), i.e. $\forall i,\ A_i\geq \mathrm{Id}$. Moreover, we assume that all the $A_i$ have the same pattern (i.e. the same set of positions of the non $\varepsilon$ entries) which is assumed irreducible (in other words, the durations are random but the structure of the graph is fixed and it is strongly connected). Then, the semigroup $\langle A_1,\dots,A_p\rangle$ is primitive (because a matrix with non zero diagonal entries is irreducible iff it is primitive).

*Example 4:* Although the semigroup associated with the representation of Ex. III-C.2 is nonprimitive, the algorithm (40) terminates and shows that the corresponding dater is deterministic.

*Example 5:* It can be shown that the series $(1a\oplus b)^*\oplus(a\oplus 2b)^*$ (two machines working in parallel independently with respective times 1 and 2 as in Ex. III-B) is not deterministic, even if it is the sum of two deterministic series.

## IX. Kolmogorov Equation of Deterministic Automata

In this section, we will deal with conventional Markov chains, and we will thus use the conventional notation ($xy$ will denote the usual product and not $x\otimes y$). Let $w^{(k)}\in\Sigma^k$ denote a random word of length $k$. We apply the above results to the computation of the first order asymptotics

$$\ell^{\text{mean}} \stackrel{\text{def}}{=} \lim_{k\to\infty}\frac{\ell^{\text{mean}}_k(y)}{k} = \lim_{k\to\infty}\frac{\mathbb{E}(y|w^{(k)})}{k} \ . \quad (43)$$

We assume that $w^{(k)}$ is selected with the Bernouilli measure:

$$p_k(a_1\dots a_k)=p(a_1)\cdots p(a_k) \quad (44)$$

where $p(a_i)>0, \sum_i p(a_i)=1$. We consider a deterministic series $y$ with a representation of the form (30). We associate with the deterministic representation a Markov chain.

**Induced Markov Chain** $q_k$ defined by (30) is a finite Markov chain with states $Q$ and transition matrix:

$$\mathcal{M}:\ \mathcal{M}_{qq'}=\sum_{a\in\Sigma,\ \delta(q,a)=q'} p(a)$$

with the convention $\sum_{a\in\emptyset}p(a)=0$. For instance, this Markov chain can be visualized in Figure 5 by forgetting the costs and equipping the arcs labeled $a$ with the probability $p(a)$ and the arcs labeled $b$ with the probability $p(b)$. It remains to define the mean cost at state $q$:

$$c_q \stackrel{\text{def}}{=} \sum_{a\in\Sigma}\sigma(q,a)p(a)$$

and to set

$$v^k_q=\mathbb{E}[c_{q_0}+\cdots+c_{q_{k-1}}+\phi(q_k)|q_0=q] \quad (45)$$

for the Markov chain $(q_k)$.

*Theorem 5:* For a deterministic series with representation (30),

$$\ell^{\text{mean}}_k(y)=\mathbb{E}(y|w^{(k)})=v^k_{q_0} \quad (46)$$

where $v^k$ is given by the Kolmogorov equation

$$v^k=c+\mathcal{M}v^{k-1}, v^0=\phi \ . \quad (47)$$

Note that $\phi$ and $c$ can take the value $\varepsilon = -\infty$. Formulæ(46),(47) remain valid even in this case with the convention $0 \times -\infty = 0$. Of course, if a state $m$ such that $\phi(m) = -\infty$ is reachable in $k$ steps or if a trajectory of length $k$ contains an intermediate state $m$ such that $c_m = -\infty$ then $(y|w^{(k)})$ is equal to $-\infty$ with a non zero probability, and trivially $\ell_k^{\text{mean}} = -\infty$.

The Kolmogorov equation implies that

$$v^k = (\mathrm{Id} + \ldots + \mathcal{M}^{k-1})c + \mathcal{M}^k\phi \ . \tag{48}$$

Let $\mathcal{P}$ be the spectral projector of $\mathcal{M}$ for the eigenvalue 1 (i.e. the unique matrix $\mathcal{P}$ such that $\mathcal{P} = \mathcal{MP} = \mathcal{PM} = \mathcal{P}^2$). Then, the ergodic theorem for Markov chains gives:

*Corollary 3 (Lyapunov Exponent)* For a deterministic series with finite costs $\sigma, \phi$ (see (30)), we have

$$\ell^{\text{mean}} \stackrel{\text{def}}{=} \lim_k \frac{\mathbb{E}(y|w^{(k)})}{k} = (\mathcal{P}c)_{q_0} \ . \tag{49}$$

The finiteness assumption for $\sigma, \phi$ is mentioned here for simplicity. A more precise study along the lines of [11] is possible.

*Remark 4:* The name "Lyapunov exponent" is introduced by Baccelli [3], [4] in the more general context of the first order asymptotics of random products of matrices. Given a stationary ergodic sequence of (max,+) random matrices $A(1), A(2), \ldots$, Baccelli shows —under some irreducibility and integrability assumptions— the existence of the following limit called Lyapunov exponent $\ell^{\text{mean}}$ = a.s. $\lim_{n\to\infty}(A(1) \otimes \cdots \otimes A(n))_{ij}/n = \lim_{n\to\infty} \mathbb{E}(A(1) \otimes \cdots \otimes A(n))_{ij}/n$.

*Example 6:* We consider the rational series $y$ of Ex. 3. We take $p(a) = u, p(b) = v$. We obtain the following Markov matrix

$$\mathcal{M} = \begin{array}{c c} & \begin{array}{ccc} e & a & b \end{array} \\ \begin{array}{c} e \\ a \\ b \end{array} & \begin{pmatrix} 0 & u & v \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \tag{50}$$

with

$$c = \begin{array}{c} \begin{array}{ccc} e & a & b \end{array} \\ \begin{pmatrix} 0 & v & u+2v \end{pmatrix} \end{array}^T$$

(for instance, the value $c_a$ is obtained as $c_a = 0u + 1v = v$). The unique invariant measure is the row vector $z = [0 \ 1 \ 0]$. Therefore, the spectral projector is $\mathcal{P} = 1z$, where 1 denotes the constant column vector with entries 1. Finally, the Lyapunov exponent is equal to

$$\ell^{\text{mean}} = (\mathcal{P}c)_e = zc = c_a = v \ .$$

## X. Application to the Optimal Case Performance: Hamilton-Jacobi-Bellman Equation of (max,+) Controlled Automata

The word $w \in \Sigma^k$ is now seen as a control, and we consider

$$\ell_k^{\text{opt}}(y) = \inf_{|w|=k, (y|w)\neq\varepsilon} (y|w) \ ,$$

were $y$ is a deterministic series represented by (30). Let us define

$$(y'|w) \stackrel{\text{def}}{=} \begin{cases} (y|w) & \text{if } (y|w) \neq \varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

We shall denote $\oplus'$ and $\otimes'$ the laws of the (min,+) semiring $\mathbb{R}_{\min} \stackrel{\text{def}}{=} (\mathbb{R}\cup\{+\infty\}, \min, +)$. Trivially, If $y \in \mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$ is deterministic, then $y'$ is (min,+) rational. The optimal case evaluation can now be directly obtained by appealing to the dual $\mathbb{R}_{\min}$ version of Theorem 2.

*Theorem 6:* We have in the (min,+) semiring $\ell_k^{\text{opt}} = \alpha'(\mathcal{M}')^k\beta'$, where

$$\mathcal{M}'_{qq'} = \inf_{\substack{a\in\Sigma \\ \delta(q,a)=q'}} \sigma(q,a) = \bigoplus_{\substack{a\in\Sigma \\ \delta(q,a)=q'}}{}' \sigma(q,a)$$

$$\alpha'_q = \begin{cases} \varepsilon' = +\infty & \text{if } q \neq q_0 \\ 0 & \text{if } q = q_0, \end{cases}$$

$$\beta'_q = \begin{cases} \phi(q) & \text{if } \phi(q) \neq -\infty \\ +\infty & \text{otherwise.} \end{cases}$$

The matrix $\mathcal{M}' \in \mathbb{R}_{\min}^{Q\times Q}$ can be seen as the transition matrix of an induced "Bellman Chain" [1] which plays a role analogous to the Markov chain of the preceding section. We shall denote by $\rho'(\mathcal{M}')$ the spectral radius of $\mathcal{M}'$ in the (min,+) algebra (this is the dual of the spectral radius given in Lemma 1).

*Corollary 4:* For a deterministic series $y \in \mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$ (with trim deterministic representation), we have

$$\liminf_{k\to\infty} (\ell_k^{\text{opt}}(y))^{\otimes \frac{1}{k}} = \liminf_{k\to\infty} \frac{\ell_k^{\text{opt}}(y)}{k} = \rho'(\mathcal{M}') \ ,$$

*Example 7:* For the semigroup of the example 3, we have

$$\mathcal{M}' = \begin{array}{c c} & \begin{array}{ccc} e & a & b \end{array} \\ \begin{array}{c} e \\ a \\ b \end{array} & \begin{pmatrix} \varepsilon & e & e \\ \varepsilon & e & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{pmatrix} \end{array}$$

We get $\rho'(\mathcal{M}') = e$. Indeed, $a \to a$ is the unique *critical circuit* (see e.g. [4] for the graphical interpretation of $\rho'$) i.e. $\mathcal{M}_{aa} = (\mathcal{M}_{aa})^{\frac{1}{1}}$ is the unique term attaining the bound in the dual sense, in 1,((ii)), which implies that the optimal policy which minimizes $(y|w)$ for $w \in \Sigma^k$ consists in playing $w = a^k$.

### *Concluding remarks*

In this paper, we have used automata over the (max,+) semiring as an algebraic formalism for modeling timed DES. As a byproduct of this algebraic modeling, we obtained some characterizations of the worst case, optimal case and mean case performance. From the practical point of view, the most useful result is perhaps the simplest mathematically, i.e. Proposition 2 which provides an $O(n^3)$ algorithm for the worst case analysis. The determinization procedure introduced in order to compute the optimal case and mean case performance suffers of a greater complexity, and only works for a subclass of series. This naturaly suggests some open problems. Firstly, characterizations of deterministic series more effective than Theorem 4 should be found. Indeed, it should be noted that the algorithmic translation of Theorem 4 (Eq. (40)) only yields a partial decision procedure: if the algorithm terminates, this proves that the series is deterministic, but it is not immediate to bound a priori the number of iterations of the algorithm for a deterministic series. Secondly, some more efficient alternative techniques (not using deterministic reductions) should be found.

*Acknowledgement*

The author is much indebted: to the anonymous referees for many helpful comments; to Jean-Pierre Quadrat and Michel Viot for suggesting him to use the “morphism” which sends the Bellman equation to the Kolmogorov equation and for many useful discussions; to Guy Cohen for many useful comments; to François Baccelli, Alain Jean-Marie, Bruno Gaujal and particularly to Jean Mairesse for many useful discussions on the topic.

## REFERENCES


[1] M. Akian, J.P. Quadrat, and M. Viot. Bellman processes. In *Proc. of the 11th Conf. on Anal. and Opt. of Systems: Discrete Event Systems*, number 199 in Lect. Notes. in Control and Inf. Sci, Sophia Antipolis, June 1994. Springer.

[2] R. Alur, C. Courcoubetis, and D.L. Dill. Model-checking in dense real-time. *Information and Computation*, 104(1), 1993.

[3] F. Baccelli. Ergodic theory of stochastic Petri networks. *Annals of Probability*, 20(1):375–396, 1992.

[4] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. Wiley, 1992.

[5] J. Berstel and C. Reutenauer. *Rational Series and their Languages*. Springer, 1988.

[6] P. Bougerol and J. Lacroix. *Products of Random Matrices with Applications to Schrödinger Operators*. Progress in Probability and Statistics. Birkhäuser, 1985.

[7] B.A. Brandin and W.M. Wonham. Supervisory control of timed discrete-event systems. Technical Report 9210, University of Toronto, Systems Control Group, 1992.

[8] G. Cohen. Dioids and Discrete Event Systems. In *Proc. of the 11th Conf. on Anal. and Opt. of Systems: Discrete Event Systems*, number 199 in Lect. Notes. in Control and Inf. Sci, Sophia Antipolis, June 1994. Springer.

[9] R.A. Cuninghame-Green. *Minimax Algebra*. Number 166 in Lectures notes in Economics and Mathematical Systems. Springer, 1979.

[10] S. Eilenberg. *Automata, Languages and Machines*, volume A. Acad. Press, 1974.

[11] S. Gaubert. Performance evaluation of timed automata. Rapport de recherche 1922, INRIA, May 1993.

[12] S. Gaubert. Timed automata and discrete event systems. In *Proceedings of the ECC93*, Groningen, July 1993.

[13] S. Gaubert. Discrete event systems and (max,+) automata. In *MOVEP'94, Nantes, June 21–24*, 1994.

[14] S. Gaubert. On semigroups of matrices in the $(\max, +)$ algebra. Rapport de recherche 2172, INRIA, Jan. 1994.

[15] B. Gaujal and J. Mairesse. On the comparison of the makespan of two networks of processsors. Personal communication, 1994.

[16] M. Gondran and M. Minoux. Linear algebra in dioids: a survey of recent results. *Annals of Discrete Mathematics*, 19:147–164, 1984.

[17] K. Hashiguchi. Improved limitedness theorems on finite automata with distance functions. *Theoret. Comput. Sci.*, 72:27–38, 1990.

[18] D. Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. *Int. J. of Algebra and Comput.*, 3, 1993.

[19] J. Mairesse. Products of irreducible random matrices in the (max,+) algebra - part i. Rapport de Recherche 1939, INRIA, June 1993.

[20] V. Maslov and S. Samborskiĭ, editors. *Idempotent analysis*, volume 13 of *Adv. in Sov. Math.* AMS, RI, 1992.

[21] X. Nicollin, J. Sifakis, and S. Yovine. Compiling real-time specifications into extended automata. *IEEE Transactions on Software Engineering*, 18(9), Sept. 1992.

[22] G.J. Olsder, J.A.C. Resing, R.E. de Vries, M.S. Keane, and G. Hooghiemstra. Discrete event systems with stochastic processing times. *IEEE Transactions on Automatic Control*, pages 299–302, 1990.

[23] P.J.G. Ramadge and W.M. Wonham. The control of discrete event systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1):81–97, Jan. 1989.

[24] I. Simon. The nondeterministic complexity of a finite automaton. In M. Lothaire, editor, *Mots*. Hermes, 1990.

[25] A. Weber. Finite-valued distance automata. *Theoretical Computer Science*, 1993. to appear.