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Fixed Point Sets of Payment-Free Shapley Operators and Structural Properties of Mean Payoff Games

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***Abstract*— Shapley operators are the dynamic programming operators of zero-sum stochastic games, they can be characterized as order-preserving maps commuting with the addition of a constant. We study a subclass of Shapley operators which are characterized by the property of commuting with the multiplication by a positive constant. We call them *payment-free*, as they arise in the study of *recursive games*, in which the payment only occurs when the game stops. They also arise in the study of structural properties of parametric mean payoff games (the transition probabilities are fixed, not the transition payments) with finite action spaces and perfect information: their fixed point set can be shown to be all the possible mean payoff vectors of such games. A basic problem is to check whether the fixed point set of such an operator is trivial (reduced to the multiples of the unit vector), and more precisely to determine its characteristics, for instance decide whether there is a fixed point with a prescribed $\operatorname{Arg\,min}$. Yang and Zhao showed (in Systems and Control Letters, 2004) that the former problem is co-NP-complete for deterministic games. We show that the latter problem is polynomial, and deduce that the former problem remains in co-NP for stochastic games. The proof relies on the construction of a Galois connection between faces of the hypercube that are invariant by the operator, and on a reduction to a reachability problem in a directed hypergraph.**

***Index Terms*— zero-sum games, Shapley operators, mean payoff, nonexpansive maps, fixed point.**

I. Introduction

Consider a two-person zero-sum stochastic game with finite state space $S = \{1, \dots, n\}$ and, for every state i, action spaces A_i and B_i (not necessarily finite), transition probabilities $P_{ij}^{ab} = P(j \mid i, a, b)$, and transition payments r_i^{ab}. Its study involves the Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$[T(x)]_i = \min_{a \in A_i} \max_{b \in B_i} \left(r_i^{ab} + P_i^{ab} x \right), \quad x \in \mathbb{R}^n, i \in S, \quad (1)$$

where $P_i^{ab} x = \sum_{j \in S} P_{ij}^{ab} x_j$. With a final reward $x_i \in \mathbb{R}$ at state i, the value v^k of the game after k steps is determined recursively, using dynamical programming principle: $v^0 = x$ and $v^{k+1} = T(v^k)$.

One is interested in the asymptotic behavior of the sequence $(v^k/k)_{k \in \mathbb{N}^*}$ of mean values. Its limit is known to exist under various assumptions: finite framework (Bewley, Kohlberg [1], Mertens, Neyman [2]), definable framework (Bolte, Gaubert, Vigeral [3]). In case of convergence, the *mean payoff vector* of the game $\chi(T)$ is defined by $\lim_{k \to \infty} T^k(x)/k$, the limit not depending on the final reward $x \in \mathbb{R}^n$.

In this paper, we will be interested in structural properties concerning the set of realizable mean payoff vectors.

Question 1: Given actions spaces A_i, B_i and transition probabilities P_{ij}^{ab}, does there exist payments r_i^{ab} such that the mean payoff vector $\chi(T)$ reaches a prescribed value?

We show that this problem can be studied by means of the recession function $\hat{T} : x \mapsto \lim_{\rho \to +\infty} T(\rho x)/\rho$ when it exists. It inherits the same properties as T (monotonicity and additive homogeneity), but has also the property of being positively homogeneous, that is to commute with the product by a positive scalar. We will see that these properties are sufficient to characterize $\hat{T}$ as a payment-free Shapley operator.

In regard to Question 1, it will lead us to study structural aspects of the fixed point set of such operators. The deterministic case has always been addressed by Yang and Zhao [4]. They showed that deciding whether such a fixed point set is nontrivial is equivalent to a monotone satisfiability problem and that it is NP-complete. In the stochastic case, it follows that the same problem is NP-hard, however membership to NP is no longer obvious. Indeed, deciding whether the fixed point set contains an element with prescribed $\operatorname{Arg\,min}$ and $\operatorname{Arg\,max}$, which is immediate in the deterministic case, appears to be a central issue in the stochastic case. We show that the latter problem can be solved in polynomial time. It follows that the nontriviality of the fixed point set of a payment-free Shapley operator can be solved in time $2^n \, \mathsf{Poly}(\mathsf{input\ size})$. This bound, which is exponential in the number of states but polynomial in the number of actions, should be compared with the 2^n time in the deterministic case.

II. Definition and Motivations

A. Definition of payment-free Shapley operator

In the whole paper we consider a two-person zero-sum stochastic game with finite state space $S = \{1, \dots, n\}$ and, for $i \in S$, action spaces A_i, B_i (not necessarily finite) and transition probabilities $P_{ij}^{ab} = P(j \mid i, a, b)$. No payment needs to be specified. To this game, we associate a so called *payment-free Shapley* operator $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$[T(x)]_i = \min_{a \in A_i} \max_{b \in B_i} P_i^{ab} x, \quad x \in \mathbb{R}^n, i \in S, \quad (2)$$

(operators $\min$ and $\max$ can be interchanged).

As a subclass of Shapley operators, payment-free operators are monotone and additively homogeneous. They are also

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characterized by the property of being positively homogeneous, that is to commute with the product by a positive scalar.

B. Structural properties of mean payoff of stochastic games

Proposition 2: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a Shapley operator (1). Suppose that its mean payoff vector $\chi(T)$ and its recession function $\hat{T}$ exist. Then $\hat{T}\big(\chi(T)\big) = \chi(T)$.

If T is definable in an o-minimal structure (see [3]), then its recession function $\hat{T}$ always exists. It is in particular the case of semialgebraic operators. If the payment function r is bounded, it can also be seen that the recession function exists (and appears directly as a payment-free Shapley operator).

As for the mean payoff, when the action spaces are finite, it is well defined (as well as $\hat{T}$). Indeed, in that case T is piecewise linear and so it has an invariant half-line (Kohlberg [5]), i.e. there exist two vectors $x, \nu \in \mathbb{R}^n$ such that $T(x+t\,\nu) = x+(t+1)\nu$ for every scalar t large enough, which implies that $\chi(T) = \nu$.

Conversely to Proposition 2 we have

Proposition 3: Let F be a payment-free Shapley operator, and let $\nu \in \mathbb{R}^n$ be a vector such that $F(\nu) = \nu$. Then, there exists a Shapley operator T such that $\hat{T} = F$ and $\chi(T) = \nu$.

C. Non-linear potential theory

A wide range of problems involve monotone, additively and positively homogeneous operators: given such an operator $F : \mathbb{R}^S \to \mathbb{R}^{S'}$, where $S' \subsetneq S$ are finite sets, and $w \in \mathbb{R}^{S\setminus S'}$, find $u \in \mathbb{R}^S$ such that $[F(u)]_i = u_i$ for $i \in S'$ and $u_j = w_j$ for $j \in S \setminus S'$. We show that such operators can be represented as a payment-free Shapley operator, non-linear boundary problems being then regarded as constrained problems of fixed point for payment-free operators.

As an example, Lazarus et al. study in [6], [7] a *Richman game*, played on a digraph $G = (V, E)$. They introduce a *Richman cost* function $u \in [0,1]^V$, that satisfies $u_b = 0, u_r = 1$ and $u_i = \frac{1}{2}\big(\min_{j\in\mathcal{S}(i)} u_j + \max_{j\in\mathcal{S}(i)} u_j\big)$ for $i \in V\setminus\{b, r\}$, where b and r are two absorbing vertices, and $j \in \mathcal{S}(i)$ means that $(i, j) \in E$. In [8], Peres et al. use the same non-linear boundary problem to characterize the value of a tug-of-war game and identify u as an infinity harmonic function verifying $\Delta_\infty u = 0$, where Δ_∞ is the discrete infinity Laplacian given by $[\Delta_\infty u]_i := -2u_i + \inf_{j\in\mathcal{S}(i)} u_j + \sup_{j\in\mathcal{S}(i)} u_j$.

D. Recursive games

The Shapley operator of such zero-sum stochastic games appears directly as payment-free. They were introduced by Everett [9]. In his model there is a finite set of states S, action spaces $A_i, B_i,\ i \in S$ (not necessarily finite) and transition probabilities P_i^{ab}, but there is no transition payments, and for each state i, we do not necessarily have $\sum_{j\in S} P_{ij}^{ab} = 1$. If not, the game stops with probability $1 - \sum_{j\in S} P_{ij}^{ab}$ and the maximizing player receives the final payment w_i^{ab}. If a play never stops, the payment is 0.

In [9], Everett solved this class of zero-sum games. He proved in particular that if the final payments are bounded and if every one-stage game given, for $i \in S$ and $x \in \mathbb{R}^S$ fixed, by the payments $\big(1-\sum_{j\in S} P_{ij}^{ab}\big)w_i^{ab} + \sum_{j\in S} P_{ij}^{ab} x_j$, has a value, that is $\min$ and $\max$ commute in the following value operator

$$[T(x)]_i = \min_{a\in A_i}\max_{b\in B_i}\bigg(\Big(1-\sum_{j\in S} P_{ij}^{ab}\Big)w_i^{ab} + \sum_{j\in S} P_{ij}^{ab} x_j\bigg),$$

then the recursive game has a value for every initial state. This value vector is shown to be a fixed point of T.

E. Example

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

$$T(x) = \begin{pmatrix} x_1 \wedge \frac{1}{2}(x_1+x_2) \\ \big(\frac{1}{2}(x_1+x_2) \vee x_1\big) \wedge \frac{1}{2}(x_1+x_3) \\ \frac{1}{2}(x_1+x_3) \vee \frac{1}{2}(x_2+x_4) \\ x_4 \vee \frac{1}{2}(x_3+x_4) \end{pmatrix}. \qquad (3)$$

We want to know if T has nontrivial fixed points and which structural properties they might have.

To this operator can be associated the framework of a turn-based game played on the following graph (MIN plays on circle nodes, MAX on square ones and nature decides on diamond nodes, the states are the grey nodes).

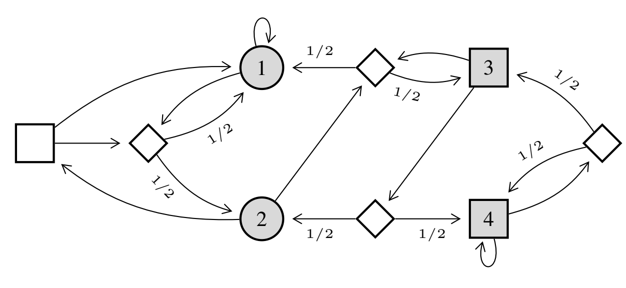

Fig. 1: the graph associated with T

III. Representation of payment-free Shapley operators

Every norm q on $\mathbb{R}^n$ (symmetric or not) is convex, continuous and positively homogeneous. Thus, according to [10, Corollary 13.2.1], it is the support function of a closed convex set defined by $\big\{p \in \mathbb{R}^n \mid \forall x \in \mathbb{R}^n,\ \langle p, x\rangle \le q(x)\big\}$.

Theorem 4: Let $F : \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous map, nonexpansive in a not necessarily symmetric norm q. Let C be the closed convex set whose support function defines q. For every $y \in \mathbb{R}^n$, let $D(y) := \{p \in C \mid \langle p, y\rangle \le F(y)\}$. Then we have
$F(x) = \inf_{y\in\mathbb{R}^n} \sup_{p\in D(y)} \langle p, x\rangle$.

Corollary 5: Every monotone, additively and positively homogeneous operator in finite dimension can be represented as a payment-free Shapley operator. Moreover, the support of each transition probability has at most cardinality 2.

IV. Decision problems concerning the fixed point sets of payment-free Shapley operators

A. Statement of the decision problems

*Problem 6 (**NonTrivialFP**):* Does a given payment-free Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ have a nontrivial fixed point.

This problem has already been addressed in the deterministic case by Yang an Zhao [4].

*Problem 7 (**MonBool**):* Does a given monotone Boolean operator have a nontrivial fixed point.

Theorem 8 (Yang, Zhao [4]): Problem **MonBool** is NP-complete.

We deduce that Problem **NonTrivialFP** is NP-hard.

*Problem 9 (**I⊂Min**):* Let $I \subset S$. Does a given payment-free Shapley operator have a nontrivial fixed point u satisfying $I \subset \operatorname{Arg\,min} u$?

*Problem 10 (***I=Min***):* Let $I \subset S$. Does a given payment-free Shapley operator have a fixed point u satisfying $I = \operatorname{Arg\,min} u$?

Theorem 8 implies that Problem **I⊂Min** is NP-hard. We will show that Problem **I=Min** is polynomial.

B. Galois connection between invariant faces of the hypercube

If K is a subset of S, denote $\mathbb{1}_K$ the vector with entries 1 on K and 0 on $S \setminus K$.

Proposition 11: Let T be a payment-free Shapley operator on $\mathbb{R}^n$. Suppose that it has a nontrivial fixed point u and denote $I := \operatorname{Arg\,min} u$ and $J := \operatorname{Arg\,max} u$. Then

$$T(\mathbb{1}_{S\setminus I}) \leq \mathbb{1}_{S\setminus I}, \tag{H1}$$

$$\mathbb{1}_J \leq T(\mathbb{1}_J). \tag{H2}$$

Remark 12: (H1) means that $[T(\mathbb{1}_{S\setminus I})]_i = 0$ for all $i \in I$ and (H2) means that $[T(\mathbb{1}_J)]_j = 1$ for all $j \in J$.

We give a geometric interpretation of (H1) and (H2). Given $I, J \subset S$, let $F_I^- := \{x \in [0,1]^n \mid \forall i \in I,\ x_i = 0\}$ and $F_J^+ := \{x \in [0,1]^n \mid \forall j \in J,\ x_j = 1\}$. We call them lower and upper faces of the hypercube $[0.1]^n$, respectively.

Proposition 13: Let T be a payment-free Shapley operator. Let I and J be two subsets of S.

$$\text{(H1)} \ \Leftrightarrow\ T(F_I^-) \subset F_I^-,$$

$$\text{(H2)} \ \Leftrightarrow\ T(F_J^+) \subset F_J^+.$$

Hence, if T has a nontrivial fixed point, then there is a lower and an upper invariant face with nonempty intersection.

We give now a game-theoretic interpretation. Consider a game with finite state space S, action spaces A_i, B_i for $i \in S$ and transition probabilities P_i^{ab}. Call T its Shapley operator. If (H1) holds true, the probability that being in $i \in I$ the state reaches $S \setminus I$ in one step is no greater than $[T(\mathbb{1}_{S\setminus I})]_i = 0$. Hence player MIN can constrain the state to stay in I. Likewise, (H2) means that MAX can force the state to stay in J

Example 14: For a repeated game played on the graph of Figure 1, check that MIN can always remain in $\{1\}$ or in $\{1,2\}$, and that MAX can always remain in $\{4\}$. No other sets have these properties. One can verify that $\{1\}$ and $\{1,2\}$ satisfy property (H1) and that $\{4\}$ satisfies property (H2) for the operator (3).

We now introduce the Galois connection. Given a payment-free Shapley operator, let $\mathcal{I}$ be the family of subsets of S verifying (H1), and let $\mathcal{J}$ be the family of those verifying (H2). These families are lattices of subsets. According to the geometric interpretation, $\mathcal{I}$ and $\mathcal{J}$ can be identified with the families of lower and upper invariant faces of the hypercube, respectively. They are nonempty since they both contain $\emptyset$ and S. Given $I \in \mathcal{I}$, we consider the greatest subsets $J \in \mathcal{J}$ satisfying $I \cap J = \emptyset$, and vice versa. This defines a Galois connection between $\mathcal{I}$ and $\mathcal{J}$, as introduced by Birkhoff [11], that we call $(\Phi, \Phi^\star)$. We denote $\bar{I} := \Phi^\star \circ \Phi(I)$ the closure of $I \in \mathcal{I}$ and $\bar{J}$ the closure of $J \in \mathcal{J}$.

Theorem 15: Let T be a payment-free Shapley operator and $I \in \mathcal{I}$. If $\Phi(I) = \emptyset$, then T has no nontrivial fixed point u satisfying $I \subset \operatorname{Arg\,min} u$.

Theorem 16: Let T be a payment-free Shapley operator and $I \in \mathcal{I} \setminus \{\emptyset\}$. If $I = \bar{I}$, then T has a fixed point u satisfying $\operatorname{Arg\,min} u = I$. In addition, if v is a nontrivial fixed point of T such that $I \subset \operatorname{Arg\,min} v$ and $\Phi(I) \subset \operatorname{Arg\,max} v$, then $I = \operatorname{Arg\,min} v$ and $\Phi(I) = \operatorname{Arg\,max} v$.

The case when, given $I \in \mathcal{I}$, we have $\Phi(I) \neq \emptyset$ and $I \subsetneq \bar{I}$ needs more attention. We treat it in a finite framework. We know that $\bar{I} \neq S$, since $\Phi(I) = \Phi(\bar{I})$ and $\Phi(S) = \emptyset$. We define thus a reduced operator $T^\vartriangle : \mathbb{R}^{\bar{I}} \to \mathbb{R}^{\bar{I}}$ as follows. According to the game-theoretic interpretation, we know that player MIN can force the state to stay in $\bar{I}$. The operator $T^\vartriangle$ is defined by considering, for each state in $\bar{I}$, the actions of MIN that achieve this goal. Formally, for each $i \in \bar{I}$, let $A_i^\vartriangle := \{a \in A_i;\ \forall b \in B_i,\ P_i^{ab}\mathbb{1}_{\bar{I}} = 1\}$ and $[T^\vartriangle(x)]_i := \min_{a\in A_i^\vartriangle} \max_{b\in B_i} P_i^{ab}\bar{x}$, where, for $x \in \mathbb{R}^{\bar{I}}$, the vector $\bar{x} \in \mathbb{R}^n$ is defined by $\bar{x}_s = x_s$ for every $s \in \bar{I}$ and $\bar{x}_s = 0$ for every $s \in S \setminus \bar{I}$. Notice that $T^\vartriangle$ is still a payment-free Shapley operator.

Theorem 17: Let T be a payment-free Shapley operator and $I \in \mathcal{I}$. Suppose that $\Phi(I) \neq \emptyset$ and $I \subsetneq \bar{I}$. Then T has a fixed point whose $\operatorname{Arg\,min}$ is I i.f.f. the same holds for $T^\vartriangle$.

Example 18: Consider the operator (3). We have already established that $\mathcal{I} = \big\{\emptyset, \{1\}, \{1,2\}, \{1,2,3,4\}\big\}$ and that $\mathcal{J} = \big\{\emptyset, \{4\}, \{1,2,3,4\}\big\}$. Thus, we have $\Phi(\{1\}) = \Phi(\{1,2\}) = \{4\}$ and $\Phi^\star(\{4\}) = \{1,2\}$. In particular, $\{1,2\}$ is closed for the Galois connection and T has a fixed point whose $\operatorname{Arg\,min}$ is $\{1,2\}$. Moreover, its $\operatorname{Arg\,max}$ can only be $\{4\}$. Check that the column vector $(0,0,1/2,1)^\mathsf{T}$ satisfies these properties. As for the set $\{1\}$, being not closed (its closure is $\{1,2\}$), T has a fixed point whose $\operatorname{Arg\,min}$ is $\{1\}$ i.f.f. this holds for the reduced operator

$$T^\vartriangle(x) = \begin{pmatrix} x_1 \wedge \frac{1}{2}(x_1+x_2) \\ x_1 \vee \frac{1}{2}(x_1+x_2) \end{pmatrix}.$$

Following the same process, we check that for $T^\vartriangle$, we have $\mathcal{J} = \big\{\emptyset, \{1,2\}\big\}$ and $\mathcal{I} = \big\{\emptyset, \{1\}, \{1,2\}\big\}$. Hence, for $T^\vartriangle$ we have $\Phi(\{1\}) = \emptyset$ and we know that no fixed point of $T^\vartriangle$ can have its $\operatorname{Arg\,min}$ equal to $\{1\}$. We conclude that T has no fixed point whose $\operatorname{Arg\,min}$ is $\{1\}$, and that a nontrivial fixed point u of T must satisfy $u_1 = u_2 < u_3 < u_4$.

Example 19: We are now able to partially answer to Question 1: given action spaces and transition probabilities, that is, given the payment-free Shapley operator (3), and given a vector $u \in \mathbb{R}^4$, does there exist a payment function such that the mean payoff vector is u? We know that a necessary condition is $u_1 = u_2 \leq u_3 \leq u_4$. We also know that if u satisfies this condition with $u_1 \neq u_4$, then taking u_i as payment for each state i, that is considering the Shapley operator $T + u$, we will get a mean payoff vector χ that verifies $\chi_1 = \chi_2 = u_1 < \chi_3 < \chi_4 = u_4$. For instance, with $u = (0, 0, 1, 1)^\mathsf{T}$ we get $\chi = (0, 0, 1/2, 1)^\mathsf{T}$. The same conclusion holds if we just have $\min u = u_1 < u_4 = \max u$. With $u = (0, 1, 1, 1)^\mathsf{T}$, we get $\chi = (0, 0, 1/2, 1)^\mathsf{T}$.

C. Boolean abstractions of payment-free Shapley operators

We assume that the action spaces are finite. Let T be a payment-free Shapley operator. We call upper and lower Boolean abstractions of T, the operators defined on $\{0, 1\}^n$ for every $i \in S$ by, respectively,

$$[T^+(x)]_i := \min_{a \in A_i} \max_{b \in B_i} \max_{s : P_{is}^{ab} > 0} x_s ,$$
$$[T^-(x)]_i := \min_{a \in A_i} \max_{b \in B_i} \min_{s : P_{is}^{ab} > 0} x_s .$$

Lemma 20: Let T be a payment-free Shapley operator and $I, J \subset S$. Then

$$\text{(H1)} \quad \Leftrightarrow \quad T^+(\mathbb{1}_{S \setminus I}) \leq \mathbb{1}_{S \setminus I} ,$$
$$\text{(H2)} \quad \Leftrightarrow \quad T^-(\mathbb{1}_J) \geq \mathbb{1}_J .$$

Lemma 21: Let T be a payment-free Shapley operator and $I \subset S$. Suppose T has a fixed point u with $\operatorname{Arg\,min} u = I$. Then $T^+(\mathbb{1}_{S \setminus I}) = \mathbb{1}_{S \setminus I}$.

Theorem 22: Let T be a payment-free Shapley operator. If $I \in \mathcal{I}$ (resp. $J \in \mathcal{J}$), then $\Phi(I)$ (resp. $\Phi^\star(J)$) is uniquely determined by the identity $\mathbb{1}_{\Phi(I)} = \lim_{k \to \infty} (T^-)^k (\mathbb{1}_{S \setminus I})$ (resp. $\mathbb{1}_{S \setminus \Phi^\star(J)} = \lim_{k \to \infty} (T^+)^k (\mathbb{1}_J)$).

We show that the sequence $\big((T^-)^k (\mathbb{1}_{S \setminus I})\big)_{k \in \mathbb{N}}$ is nonincreasing. Since it is a Boolean sequence, it is stationary in at most n steps. Hence, $\Phi(I)$ can be computed in polynomial time. The same holds for $\Phi^\star(J)$.

Example 23: Check that for the operator (3) we have $T^+(\mathbb{1}_{\{2,3,4\}}) = \mathbb{1}_{\{2,3,4\}}$ and $T^+(\mathbb{1}_{\{3,4\}}) = \mathbb{1}_{\{3,4\}}$, which means that $\{1\}$ and $\{1, 2\}$ satisfy (H1). Furthermore, we have $T^-(\mathbb{1}_{\{2,3,4\}}) = \mathbb{1}_{\{3,4\}}$, $T^-(\mathbb{1}_{\{3,4\}}) = \mathbb{1}_{\{4\}}$ and $T^-(\mathbb{1}_{\{4\}}) = \mathbb{1}_{\{4\}}$. This implies that $\Phi(\{1\}) = \{4\}$. In the same way, we have $T^+(\mathbb{1}_{\{4\}}) = \mathbb{1}_{\{3,4\}}$, from what can be deduced that $\Phi^\star(\{4\}) = \{1, 2\}$.

D. Hypergraph representation of Boolean abstractions

We can show that T^+ and T^- can be represented by hypergraphs. We deduce that computing Φ and $\Phi^\star$ is equivalent to reachability problems in these hypergraphs, hence can be done in linear time in their size (for references to hypergraphs, see the survey [12]).

E. Algorithm

We denote $(\Phi_F, \Phi_F^\star)$ the Galois connection associated with a payment-free Shapley operator F.

Theorem 24: Algorithm 1 solves Problem **I=Min** in time $O(n^3 p^2)$, where n is the number of states and p is a bound on the number of actions for each player in each state.

Corollary 25: Problem **NonTrivialFP** and **I⊂Min** are NP-complete.

Algorithm 1

Require: payment-free Shapley operator $T : \mathbb{R}^S \to \mathbb{R}^S$, nonempty subset $I \subset S$
Ensure: answer to Problem **I=Min**
$F \leftarrow T$
if $F^+(\mathbb{1}_{S \setminus I}) \neq \mathbb{1}_{S \setminus I}$ **then**
  return false
else
  loop
    if $\Phi_F(I) = \emptyset$ **then**
      return false
    else if $\Phi_F^\star \circ \Phi_F(I) = I$ **then**
      return true
    else
      $F \leftarrow F^\Delta$
    end if
  end loop
end if

F. Mixed problem

Problem 26 (**IMinJMax**)**:** Let I and J be nonempty disjoint subsets of S. Does a given payment-free Shapley operator has a fixed point u satisfying $I = \operatorname{Arg\,min} u$ and $J = \operatorname{Arg\,max} u$?

Theorem 27: Let T be a payment-free Shapley operator and let $I \in \mathcal{I}$ and $J \in \mathcal{J}$ be two nonempty disjoint subsets. Then T has a fixed point u satisfying $I = \operatorname{Arg\,min} u$ and $J = \operatorname{Arg\,max} u$ if and only if T has fixed points v and w satisfying $\operatorname{Arg\,min} v = I$ and $\operatorname{Arg\,max} w = J$.

Corollary 28: Problem **IMinJMax** can be solved in time $O(n^3 p^2)$, where n is the number of states and p is a bound on the number of actions for each player in each state.

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