SIAM J. MATRIX ANAL. APPL.
Vol. 31, No. 3, pp. 1412–1431

REDUCIBLE SPECTRAL THEORY WITH APPLICATIONS TO THE ROBUSTNESS OF MATRICES IN MAX-ALGEBRA*

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Abstract. Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. By max-algebra we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors. The symbol A^k stands for the kth max-algebraic power of a square matrix A. Let us denote by ε the max-algebraic "zero" vector, all the components of which are $-\infty$. The max-algebraic eigenvalue-eigenvector problem is the following: Given $A \in \overline{\mathbb{R}}^{n\times n}$, find all $\lambda \in \overline{\mathbb{R}}$ and $x \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$, such that $A \otimes x = \lambda \otimes x$. Certain problems of scheduling lead to the following question: Given $A \in \overline{\mathbb{R}}^{n\times n}$, is there a k such that $A^k \otimes x$ is a max-algebraic eigenvector of A? If the answer is affirmative for every $x \neq \varepsilon$, then A is called robust. First, we give a complete account of the reducible max-algebraic spectral theory, and then we apply it to characterize robust matrices.

Key words. max-algebra, reducible matrix, eigenspace

AMS subject classification. 15A18

DOI. 10.1137/080731232

1. Introduction.

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. Obviously, $-\infty$ plays the role of a neutral element for $\oplus$. Throughout the paper we denote $-\infty$ by ε and for convenience we also denote by the same symbol the max-algebraic "zero" vector, whose components are all $-\infty$ or a matrix whose components are all $-\infty$. If $a \in \mathbb{R}$, then the symbol a^{-1} stands for $-a$.

By *max-algebra* we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors. That is, if $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j, and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. If $\alpha \in \overline{\mathbb{R}}$, then $\alpha \otimes A = (\alpha \otimes a_{ij})$. If A is a square matrix, then the iterated product $A \otimes A \otimes \cdots \otimes A$, in which the symbol A appears k-times, will be denoted by A^k. By definition $A^0 = I$, where I is the matrix with diagonal entries 0 and off-diagonal entries ε. Obviously, $A \otimes I = I \otimes A = A$ whenever A and I are of compatible sizes.

The max-algebraic *eigenvalue-eigenvector problem* (briefly *eigenproblem*) is the following:

Given $A \in \overline{\mathbb{R}}^{n\times n}$, *find all* $\lambda \in \overline{\mathbb{R}}$ *(eigenvalues) and* $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ *(eigenvectors) such that*

$$A \otimes x = \lambda \otimes x.$$

*Received by the editors July 25, 2008; accepted for publication (in revised form) by H. J. Woerdeman October 5, 2009; published electronically December 4, 2009.
http://www.siam.org/journals/simax/31-3/73123.html

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This problem has been studied since the work of Cuninghame-Green [15]. One of the motivations was the analysis of the steady-state behavior of the following multimachine interactive production systems: Suppose that machines $M_1, \dots, M_n$ work interactively and in stages. In each stage all machines simultaneously produce components necessary for the next stage of some or all other machines. Let $x_i(k)$ denote the starting time of the kth stage on machine i $(i = 1, \dots, n)$, and let a_{ij} denote the duration of the operation at which machine M_j prepares the component necessary for machine M_i in the $(k + 1)$st stage $(i, j = 1, \dots, n)$. Then

$$x_i(k+1) = \max(x_1(k) + a_{i1}, \dots, x_n(k) + a_{in}) \ (i = 1, \dots, n; k = 0, 1, \dots)$$

or, in max-algebraic notation,

$$x(k+1) = A \otimes x(k) \ (k = 0, 1, \dots),$$

where $A = (a_{ij})$ is called a *production matrix*. More generally, systems of this kind are known to represent a class of discrete event systems [4]. We say that the system reaches a *steady state* if it eventually moves forward in regular steps, that is, if for some λ and k_0 we have $x(k + 1) = \lambda \otimes x(k)$ for all $k \geq k_0$. Obviously, a steady state is reached immediately if $x(0)$ is an eigenvector of A corresponding to an eigenvalue λ, which can be interpreted as the time between consecutive events. However, if the choice of a start-time vector is restricted, we may need to find out for which vectors a steady state will eventually be reached. Since $x(k) = A^k \otimes x(0)$ for every natural k, this question reads as follows:

Given $A \in \overline{\mathbb{R}}^{n \times n}$ and $x \in \overline{\mathbb{R}}^n$ is there a natural number k such that $A^k \otimes x$ is an eigenvector of A?

In particular, it may be of practical interest to characterize matrices for which a steady state is reached with any start-time vector, that is, matrices $A \in \overline{\mathbb{R}}^{n \times n}$ for which the following is true:

For every $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$, there is a natural number k such that *$A^k \otimes x$ is an eigenvector of A.*

This property has been considered by Butkovič and Cuninghame-Green [11], who called it *robustness*. Indeed, the system is robust when the existence of an ultimate stationary regime is insensitive to the choice of initial conditions. This is in accordance with the use of this term in control theory where "robustness" generally indicates the insensitivity of certain performance measures or qualitative properties to various types of perturbations or uncertainties. Butkovič and Cuninghame-Green [11] characterized robust matrices in the important case of irreducible matrices (for the definition of irreducible matrices, see section 2). The main aim of the current paper is to extend these results to general (reducible) matrices.

In the language of dynamical systems, the robustness property requires every orbit of $x(k) = A \otimes x(k - 1)$ to converge to a fixed point, modulo the addition of a constant. Besides the motivation from discrete event systems, the study of this property is motivated by basic questions in the theory of nonexpansive mappings, in which the structure of the periodic orbits has received considerable attention; see, in particular, [28], [30], [2], [27]. Max-algebraic linear maps are special cases of nonexpansive mappings in Hilbert's projective metric, and one may try, more generally, to find conditions which guarantee that every orbit of a nonexpansive mapping converges to a fixed point. In the present paper, we address this problem in the special case of max-algebraic linear maps, which is of particular interest since it may be thought of as the simplest case in which strict contraction techniques can be applied. We note that it might be interesting to generalize the present results to other classes of nonexpansive mappings.

The robustness problem is also of interest in relation to the power algorithm introduced by Braker and Olsder. This algorithm computes an orbit $A^k \otimes x, k = 0, 1, \ldots,$ for a given initial vector x and at each step checks whether $A^k \otimes x$ is proportional in the max-algebraic sense to some $A^m \otimes x$ with $m < k$. The robustness property identifies a situation in which the power algorithm does terminate, and then the latter test can be simplified by considering only $m = k - 1$.

Note that when a reducible matrix is not robust, its orbit can have a more complex behavior with interleaving arithmetical sequences [20], [18].

The characterization of robustness in [11] substantially relies on the max-algebraic spectral theory. A full solution of the eigenproblem in the case of irreducible matrices has been presented by Cuninghame-Green [16], [17] and Gondran and Minoux [24]; see also Vorobyov [33].

The general (reducible) case considered in the present paper requires detailed information about the spectral problem for reducible matrices. A general spectral theorem for reducible matrices was presented by Gaubert [19] and Bapat, Stanford, and van den Driessche [6]. Some of the results of [6] were stated (without proofs) by Bapat, Stanford, and van den Driessche in [7]. Additional results can be found in the work of Akian, Gaubert, and Walsh [3], where the emphasis is on the denumerable case. A survey, again without proofs, appeared in [1].

Since there is currently no complete account of the reducible spectral theory in journals or books, we give in section 3 a systematic presentation of this theory. The ideas of the proofs in this theory are instrumental for the proofs of the results on robustness of reducible matrices, which constitute the main aim of the paper. It should be noted, however, that reducible spectral theory is of general interest (independently of the application that we consider here) due to its remarkable connections to the Perron–Frobenius theory of reducible (nonnegative) matrices. It is also of importance in the analysis of discrete-event systems [4], [14], [21].

In section 4 we give answers to some specific questions related to the finiteness of the eigenvectors. A comparison with the theory of nonnegative matrices in conventional linear algebra is made (Remark 4.2). We also show how to efficiently find a basis of the eigenspace corresponding to an eigenvalue. These results are used in section 5 to provide a characterization of robustness for reducible matrices, thus completing the solution of this question for all $A \in \overline{\mathbb{R}}^{n\times n}$. This characterization is presented in the main result of the paper, Theorem 5.5.

2. Notation, definitions, and preliminary results. Unless stated otherwise, we assume everywhere in this paper that $n \geq 1$ is an integer, $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, and $\lambda \in \overline{\mathbb{R}}$. We denote by $V(A, \lambda)$ the set containing ε and all eigenvectors of A, corresponding to $\lambda \in \mathbb{R}$, by $V(A)$ the set containing ε and all eigenvectors of A, and by $\Lambda(A)$ the set of all eigenvalues of A. The sets $V^+(A, \lambda)$ and $V^+(A)$ contain finite eigenvectors of A corresponding to $\lambda \in \mathbb{R}$ and all finite eigenvectors of A, respectively.

Note that if $A = \varepsilon$, then $\Lambda(A) = \{\varepsilon\}$ and $V(A) = \overline{\mathbb{R}}^n$.

An ordered pair $D = (N, F)$ is called a *digraph* if N is a nonempty set (of *nodes*) and $F \subseteq N \times N$ (the set of *arcs*). A sequence $\pi = (v_1, \ldots, v_p)$ of nodes is called a *path* (in D) if $p = 1$ or $p > 1$ and $(v_i, v_{i+1}) \in F$ for all $i = 1, \ldots, p - 1$. The node v_1 is called the *starting node* and v_p the *end node* of π, respectively. If there is a path in D with starting node u and end node v, then we say that v is *reachable* from u, notation $u \to v$. Thus $u \to u$ for any $u \in N$. As usual a digraph D is called *strongly connected* if $u \to v$ and $v \to u$ for any nodes u, v in D. A path $(v_1, \dots, v_p)$ is called a *cycle* if $v_1 = v_p$ and $p > 1$ and it is called an *elementary cycle* if, moreover, $v_i \neq v_j$ for $i, j = 1, \dots, p-1, i \neq j$.

In the rest of the paper $N = \{1, \dots, n\}$. The digraph *associated* with $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is

$$D_A = (N, \{(i,j); a_{ij} > \varepsilon\}).$$

The matrix A is called *irreducible* if D_A is strongly connected, *reducible* otherwise. Thus, every 1×1 matrix is irreducible.

If $\pi = (i_1, \dots, i_p)$ is a path in D_A, then the *weight* of π is $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{p-1} i_p}$ if $p > 1$ and ε if $p = 1$. The symbol $\lambda(A)$ stands for the *maximum cycle mean* of A, that is, if D_A has at least one cycle, then

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{1}$$

where the maximization is taken over all cycles in D_A and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{k} \tag{2}$$

denotes the *mean* of the cycle $\sigma = (i_1, \dots, i_k, i_1)$. Note that $\lambda(A)$ remains unchanged if the maximization in (1) is taken over all elementary cycles. If D_A is acyclic, we set $\lambda(A) = \varepsilon$. Various algorithms for finding $\lambda(A)$ exist. One of them is that of Karp [26] of computational complexity $O(nm)$, where m is the number of finite entries in A (or, equivalently, the number of arcs in D_A).

We say that A is *definite* if $\lambda(A) = 0$. It is easily seen that $V(\alpha \otimes A) = V(A)$ and $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$ for any $\alpha \in \mathbb{R}$. Hence $\lambda(A)^{-1} \otimes A$ is definite whenever $\lambda(A) > \varepsilon$.

In order to construct eigenvectors explicitly, it is convenient to define the *metric matrix*

$$\Gamma(A) = A \oplus A^2 \oplus \cdots .$$

The matrix $\Gamma(A)$ is sometimes denoted by A^+; see, e.g., [4].

LEMMA 2.1 (see [16]). *If $\lambda(A) \leq 0$, in particular when A is definite, then $\Gamma(A)$ finitely converges and is equal to $A \oplus A^2 \oplus \cdots \oplus A^n$. If $\lambda(A) > 0$, then the value of at least one position in A^k is unbounded as $k \longrightarrow \infty$ and, consequently, at least one entry of $\Gamma(A)$ is $+\infty$.*

Note that the (i, j) entry of $\Gamma(A)$ yields the maximum weight of a path with a starting node i and end node j in D_A. The metric matrix of a matrix with $\lambda(A) \leq 0$ can be computed using the Floyd–Warshall algorithm in $O(n^3)$ time [17].

We also denote $E(A) = \{i \in N; \exists \sigma = (i = i_1, \dots, i_k, i_1) : \mu(\sigma, A) = \lambda(A)\}$. The elements of $E(A)$ are called *eigennodes (of A)*, or *critical nodes.* A cycle σ is called *critical* if $\mu(\sigma, A) = \lambda(A)$. *The critical digraph* of A is the digraph $C(A)$ with the set of nodes N; the set of arcs is the union of the sets of arcs of all critical cycles. It is well known that all cycles in a critical digraph are critical [4]. Two nodes i and j in $C(A)$ are called *equivalent* (notation $i \sim j$) if i and j belong to the same critical cycle of A. Clearly, $\sim$ constitutes a relation of equivalence in N.

Note that if $\lambda(A) = \varepsilon$, then $\Lambda(A) = \{\varepsilon\}$ and the eigenvectors of A are exactly the vectors $(x_1, \dots, x_n)^T \in \overline{\mathbb{R}}^n$ such that $x_j = \varepsilon$ whenever the jth column of A is not ε

(clearly in this case at least one column of A is ε). We will therefore usually assume that $\lambda(A) > \varepsilon$.

The following proposition presents elementary properties relating metric matrices and critical digraphs.

PROPOSITION 2.1 (see [16]). *Suppose that $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}, \lambda(A) > \varepsilon$, and let $g_1, \dots, g_n$ be the columns of $\Gamma((\lambda(A))^{-1} \otimes A) = (g_{ij})$. Then we have the following:*

- $i \in E(A) \iff g_{ii} = 0$.
- If $i, j \in E(A)$, then $g_i = \alpha \otimes g_j$ for some $\alpha \in \mathbb{R}$ if and only if $i \sim j$.

The following early version of the spectral theorem was proved in [16]. Related results can be found in [24], [12], [4], [25] and in the case of a denumerable state space in [3].

THEOREM 2.1. *Suppose $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}, A \neq \varepsilon$. Then the following hold:*

1. $V^+(A) \subseteq V(A, \lambda(A))$.
2. $V^+(A) \neq \emptyset$ if and only if $\lambda(A) > \varepsilon$ and in D_A we have

$$(\forall j \in N)(\exists i \in E(A)) j \to i.$$

3. If, moreover, $V^+(A) \neq \emptyset$, then

$$V^+(A) = \left\{ \sum_{i\in E(A)}^{\oplus} \alpha_i \otimes g_i; \alpha_i \in \mathbb{R} \right\},$$

where $g_1, \dots, g_n$ are the columns of $\Gamma(\lambda(A)^{-1} \otimes A)$.

COROLLARY 2.1 (see [17]). *A irreducible $\Rightarrow V^+(A) \neq \emptyset$.*

As we will see later (Proposition 3.1), $V(A) = V^+(A) \cup \{\varepsilon\} = V(A, \lambda(A))$, and thus $\Lambda(A) = \{\lambda(A)\}$ if A is irreducible. The fact that $\lambda(A)$ is the unique eigenvalue of an irreducible matrix A was proved in [15] and then independently in [33]. The description of $V^+(A)$ for irreducible matrices as given in part 3 of Theorem 2.1 was also proved in [24].

A set $S \subseteq \overline{\mathbb{R}}^n$ is called a *(max-algebraic) subspace* if $u, v \in S, \alpha, \beta \in \overline{\mathbb{R}}$ imply $\alpha\otimes u\oplus\beta\otimes v \in S$. It is easily seen that $V(A, \lambda)$ (the set containing ε and all eigenvectors of A corresponding to λ, if any) is a subspace for all $\lambda \in \overline{\mathbb{R}}$. We will therefore call $V(A, \lambda)$ the *eigenspace* of A corresponding to the eigenvalue λ.

Let $S \subseteq \overline{\mathbb{R}}^n$ be a subspace. A vector $v \in \overline{\mathbb{R}}^n$ is called an *extremal in* S if $v = u \oplus w$ for $u, v \in S$ implies $v = u$ or $v = w$. We say that $v_1, \dots, v_m \in S$ is a *basis* of S if

1. $v_1, \dots, v_m$ are extremals in S and
2. for every $v \in S$ we have $v = \sum_i^{\oplus} \alpha_i \otimes v_i$ for some $\alpha_1, \dots, \alpha_m \in \overline{\mathbb{R}}$.

The following fundamental spectral theorem determines a basis of the eigenspace of A corresponding to the eigenvalue $\lambda(A)$.

THEOREM 2.2 (see [3]). *Suppose that $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}, \lambda(A) > \varepsilon$, and let $g_1, \dots, g_n$ be the columns of $\Gamma((\lambda(A))^{-1} \otimes A)$. Then we obtain a basis of $V(A, \lambda(A))$ by taking exactly one g_i for each equivalence class.*

The vectors g_i, $i \in E(A)$, are called the *fundamental eigenvectors* (FEV) of A [16].

Obviously

$$V^+(A) \cup \{\varepsilon\} = \left\{ \Gamma((\lambda(A))^{-1} \otimes A) \otimes z; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \quad \forall j \notin E(A) \right\},$$

and also, if nonempty,

$$V^+(A) = \left\{ \sum_{i \in E^*(A)}^{\oplus} \alpha_i \otimes g_i; \alpha_i \in \mathbb{R} \right\},$$

where $E^*(A)$ is any maximal set of nonequivalent eigennodes of A.

Finally, we introduce some notation that will be used in the rest of the paper.

If

$$1 \le i_1 < i_2 < \cdots < i_k \le n, K = \{i_1, \ldots, i_k\} \subseteq N,$$

then $A[K]$ denotes the principal submatrix

$$\begin{pmatrix} a_{i_1 i_1} & \ldots & a_{i_1 i_k} \\ \ldots & \ldots & \ldots \\ a_{i_k i_1} & \ldots & a_{i_k i_k} \end{pmatrix}$$

of the matrix $A = (a_{ij})$ and $x[K]$ denotes the subvector $(x_{i_1}, \ldots, x_{i_k})^T$ of the vector $x = (x_1, \ldots, x_n)^T \in \overline{\mathbb{R}}^n$.

If $D = (N, E)$ is a digraph and $K \subseteq N$, then $D[K]$ denotes the *induced subgraph* of D, that is,

$$D[K] = (K, E \cap (K \times K)).$$

Obviously, $D_{A[K]} = D[K]$.

3. Finding all eigenvalues. The symbol $A \sim B$ for matrices A and B means that A can be obtained from B by a simultaneous permutation of rows and columns. It follows that D_A can be obtained from D_B by a renumbering of the nodes if $A \sim B$. Hence if $A \sim B$, then A is irreducible if and only if B is irreducible.

It is obvious that if $A \otimes x = \lambda \otimes x$ and a matrix B is obtained from A by a simultaneous permutation of the rows and columns, then the same permutation applied to the components of x yields a vector y such that $B \otimes y = \lambda \otimes y$. Hence we have the following lemma.

LEMMA 3.1. *If $A \sim B$, then $\Lambda(A) = \Lambda(B)$ and there is a bijection between $V(A)$ and $V(B)$.*

The following lemma is of special significance for the rest of the paper.

LEMMA 3.2. *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda \in \Lambda(A)$. If $x \in V(A, \lambda) - V^+(A, \lambda), x \neq \varepsilon$, then $n > 1$,*

$$A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix},$$

$\lambda = \lambda(A^{(22)})$, and hence A is reducible.

Proof. Permute the rows and columns of A simultaneously so that the vector obtained from x by the same permutation of its components is

$$x' = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix},$$

where $x^{(1)} = \varepsilon \in \overline{\mathbb{R}}^p$ and $x^{(2)} \in \mathbb{R}^{n-p}$ for some p $(1 \leq p < n)$. Denote the obtained matrix by A', and let us write blockwise

$$A' = \begin{pmatrix} A^{(11)} & A^{(12)} \\ A^{(21)} & A^{(22)} \end{pmatrix},$$

where $A^{(11)}$ is $p \times p$. The equality $A' \otimes x' = \lambda \otimes x'$ now yields blockwise:

$$A^{(12)} \otimes x^{(2)} = \varepsilon,$$
$$A^{(22)} \otimes x^{(2)} = \lambda \otimes x^{(2)}.$$

Since $x^{(2)}$ is finite, it follows from Theorem 2.1 that $\lambda = \lambda(A^{(22)})$; also clearly $A^{(12)} = \varepsilon$. ∎

PROPOSITION 3.1. *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. Then $V(A) = V^+(A)$ if and only if A is irreducible.*

Proof. It remains to prove the "only if" part since the "if" part follows from Lemma 3.2 immediately. If A is reducible, then $n > 1$ and

$$A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix},$$

where $A^{(22)}$ is irreducible. By setting

$$\lambda = \lambda(A^{(22)}), \quad x^{(2)} \in V^+(A_{22}), \quad x = \begin{pmatrix} \varepsilon \\ x^{(2)} \end{pmatrix} \in \overline{\mathbb{R}}^n$$

we see that $x \in V(A) - V^+(A), x \neq \varepsilon$. ∎

Every matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ can be transformed in linear time by simultaneous permutations of the rows and columns to a *Frobenius normal form* (FNF) [31]

$$\begin{pmatrix} A_{11} & \varepsilon & \dots & \varepsilon \\ A_{21} & A_{22} & \dots & \varepsilon \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}, \tag{3}$$

where $A_{11}, \dots, A_{rr}$ are irreducible square submatrices of A. If A is in an FNF, then the corresponding partition of the node set N of D_A will be denoted as $N_1, \dots, N_r$ and these sets will be called *classes (of A)*. It follows that each of the induced subgraphs $D_A[N_i]$ $(i = 1, \dots, r)$ is strongly connected and an arc from N_i to N_j in D_A exists only if $i \geq j$. As a slight abuse of language, we will also say for simplicity that $\lambda(A_{jj})$ is the eigenvalue of N_j.

If A is in an FNF, say (3), then the *condensation digraph*, notation C_A, is the digraph $(\{N_1, \dots, N_r\}, \{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon\})$. Observe that C_A is acyclic.

Recall that the symbol $N_i \to N_j$ means that there is a directed path from a node in N_i to a node in N_j in C_A (and therefore from each node in N_i to each node in N_j in D_A).

If there are neither outgoing nor incoming arcs from or to an induced subgraph $C_A[\{N_{i_1}, \dots, N_{i_s}\}]$ $(1 \leq i_1 < \dots < i_s \leq r)$ and no proper subdigraph has this property, then the submatrix

$$\begin{pmatrix} A_{i_1 i_1} & \varepsilon & \dots & \varepsilon \\ A_{i_2 i_1} & A_{i_2 i_2} & \dots & \varepsilon \\ \dots & \dots & \dots & \dots \\ A_{i_s i_1} & A_{i_s i_2} & \dots & A_{i_s i_s} \end{pmatrix}$$

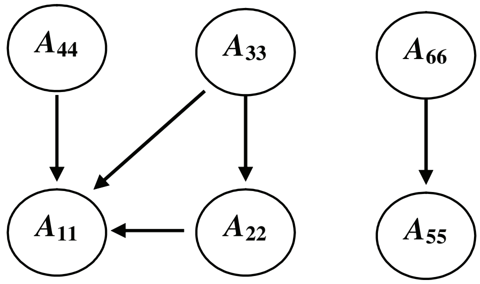

FIG. 1. *Condensation digraph for matrix* (4).

is called an *isolated superblock* (or just *superblock*). The nodes of C_A with no incoming arcs are called the *initial classes*; those with no outgoing arcs are called the *final classes*. Note that an isolated superblock may have several initial and final classes.

For instance the condensation digraph for the matrix

$$\begin{pmatrix} A_{11} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & A_{22} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & * & A_{33} & \varepsilon & \varepsilon & \varepsilon \\ * & \varepsilon & \varepsilon & A_{44} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{55} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & * & A_{66} \end{pmatrix} \tag{4}$$

can be seen in Figure 1 (note that here and elsewhere the symbols $*$ indicate submatrices different from ε) and consists of two superblocks and six classes including three initial and two final ones.

LEMMA 3.3. *If $x \in V(A), N_i \to N_j$, and $x[N_j] \neq \varepsilon$, then $x[N_i]$ is finite. In particular, $x[N_j]$ is finite.*

Proof. Suppose that $x \in V(A, \lambda)$ for some $\lambda \in \overline{\mathbb{R}}$. Fix $s \in N_j$ such that $x_s > \varepsilon$. Since $N_i \to N_j$, we have that for every $r \in N_i$ there is a positive integer q such that $b_{rs} > \varepsilon$, where $B = A^q = (b_{ij})$. Since $x \in V(B, \lambda^q)$ we also have $\lambda^q \otimes x_r \geq b_{rs} \otimes x_s > \varepsilon$. Hence $x_r > \varepsilon$. ∎

The following key result appeared in the thesis [19] and [7]. The latter work refers to the report [6] for a proof. Related results can be found in [5].

THEOREM 3.1 (spectral theorem). *Let* (3) *be an FNF of a matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. Then*

$$\Lambda(A) = \left\{ \lambda(A_{jj}); \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii}) \right\}.$$

Proof. Note first that

$$\lambda(A) = \max_{i=1,\ldots,r} \lambda(A_{ii}) \tag{5}$$

for a matrix A in FNF (3).

First, we prove the inclusion $\supseteq$. Suppose $\lambda(A_{jj}) = \max\{\lambda(A_{ii}); N_i \to N_j\}$ for some $j \in R = \{1, \ldots, r\}$. Denote $S_2 = \{i \in R; N_i \to N_j\}$ and $S_1 = R - S_2$, $M_p = \bigcup_{i \in S_p} N_i$ $(p = 1, 2)$. Then $\lambda(A_{jj}) = \lambda(A[M_2])$ and $A \sim \left(\begin{smallmatrix} A[M_1] & \varepsilon \\ * & A[M_2] \end{smallmatrix} \right)$.

If $\lambda(A_{jj}) = \varepsilon$, then at least one column, say the ℓth column in $A[M_2]$, is ε. We set x_ℓ to any real number and $x_j = \varepsilon$ for $j \neq \ell$. Then $x \in V(A, \lambda(A_{jj}))$.

If $\lambda(A_{jj}) > \varepsilon$, then $A[M_2]$ has a finite eigenvector by Theorem 2.1, say $\bar{x}$. Set $x[M_2] = \bar{x}$ and $x[M_1] = \varepsilon$. Then $x = \binom{x[M_1]}{x[M_2]} \in V(A, \lambda(A_{jj}))$.

Now we prove $\subseteq$. Suppose that $x \in V(A,\lambda), x \neq \varepsilon$, for some $\lambda \in \overline{\mathbb{R}}$.

If $\lambda = \varepsilon$, then A has an ε column, say the kth column, and thus $a_{kk} = \varepsilon$. Hence the 1×1 submatrix (a_{kk}) is a diagonal block in an FNF of A. In the corresponding decomposition of N one of the sets, say N_j, is $\{k\}$. The set $\{i; N_i \to N_j\} = \{j\}$ and the theorem statement follows.

If $\lambda > \varepsilon$ and $x \in V^+(A)$, then $\lambda = \lambda(A)$ (cf. Theorem 2.1) and the statement now follows from (5).

If $\lambda > \varepsilon$ and $x \notin V^+(A)$, then similarly, as in the proof of Lemma 3.2, permute the rows and columns of A simultaneously so that $x = \binom{x^{(1)}}{x^{(2)}}$, where $x^{(1)} = \varepsilon \in \overline{\mathbb{R}}^p$ and $x^{(2)} \in \mathbb{R}^{n-p}$ for some p $(1 \leq p < n)$. Hence $A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$, and we can assume without loss of generality that both $A^{(11)}$ and $A^{(22)}$ are in an FNF and therefore also $\begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$ is in an FNF. Let

$$A^{(11)} = \begin{pmatrix} A_{i_1 i_1} & \varepsilon & \dots & \varepsilon \\ A_{i_2 i_1} & A_{i_2 i_2} & \dots & \varepsilon \\ \dots & \dots & \dots & \dots \\ A_{i_s i_1} & A_{i_s i_2} & \dots & A_{i_s i_s} \end{pmatrix}$$

and

$$A^{(22)} = \begin{pmatrix} A_{i_{s+1} i_{s+1}} & \varepsilon & \dots & \varepsilon \\ A_{i_{s+2} i_{s+1}} & A_{i_{s+2} i_{s+2}} & \dots & \varepsilon \\ \dots & \dots & \dots & \dots \\ A_{i_q i_{s+1}} & A_{i_q i_{s+2}} & \dots & A_{i_q i_q} \end{pmatrix}.$$

We have $\lambda = \lambda(A^{(22)}) = \lambda(A_{jj}) = \max_{i=s+1,\dots,q} \lambda(A_{ii})$, where $j \in \{s+1,\dots,q\}$. It remains to say that if $N_i \to N_j$, then $i \in \{s+1,\dots,q\}$. $\square$

Note that significant correlation exists between spectral theory for matrices in max-algebra and spectral theory of nonnegative matrices in linear algebra [32], [8]; see also [31]. For instance the FNF and accessibility between classes play a key role in both theories. The maximum cycle mean corresponds to the Perron root for irreducible (nonnegative) matrices and finite eigenvectors in max-algebra correspond to positive eigenvectors in the nonnegative spectral theory. However, there are also differences; see Remark 4.2 after Theorem 4.2.

Let A be in the FNF (3). If

$$\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii}),$$

then A_{jj} (and also N_j or just j) will be called *spectral.* Thus $\lambda(A_{jj}) \in \Lambda(A)$ if j is spectral but not necessarily the other way around.

Corollary 3.1. *All initial classes of C_A are spectral.*

Proof. Initial classes have no predecessors, and so the condition of the theorem is satisfied. $\square$

Corollary 3.2. $\lambda(A) \in \Lambda(A)$ *for every matrix* A.

Proof. If A is in an FNF, say (3), then $\lambda(A) = \max_{i=1,\dots,r} \lambda(A_{ii}) = \lambda(A_{jj})$ for some j, and so the condition of the theorem is satisfied. $\square$

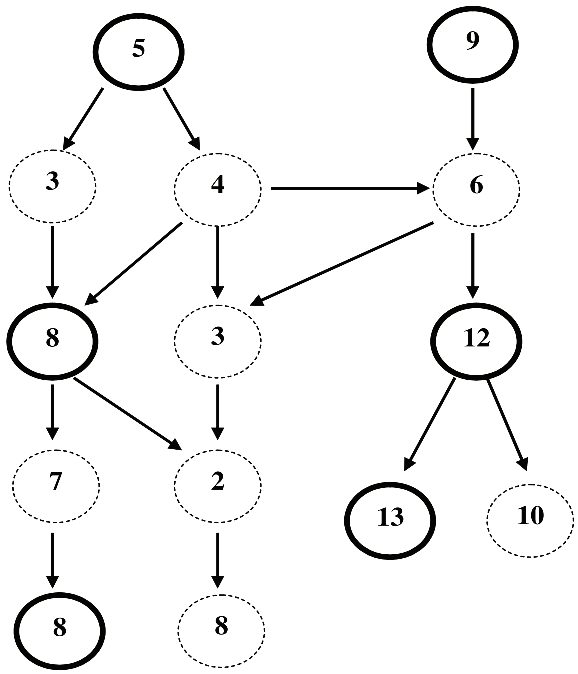

FIG. 2. *A condensation digraph with six spectral nodes.*

COROLLARY 3.3. $1 \leq |\Lambda(A)| \leq n$ *for every* $A \in \overline{\mathbb{R}}^{n\times n}$.

Proof. The proof follows from the previous corollary and from the fact that the number of classes of A is at most n. $\square$

COROLLARY 3.4. $V(A) = V(A, \lambda(A))$ *if and only if all initial classes have the same eigenvalue* $\lambda(A)$.

Proof. The eigenvalues of all initial classes are in $\Lambda(A)$ since all initial classes are spectral; hence all eigenvalues must be equal to $\lambda(A)$ if $\Lambda(A) = \{\lambda(A)\}$. On the other hand, if all initial classes have the same eigenvalue $\lambda(A)$, and λ is the eigenvalue of any spectral class, then

$$\lambda \geq \lambda(A) = \max_i \lambda(A_{ii})$$

since there is a path from some initial class to this class and thus $\lambda = \lambda(A)$. $\square$

Figure 2 shows a condensation digraph with 14 classes including two initial classes and four final ones. The numbers indicate the eigenvalues of the corresponding classes. The six bold classes are spectral; the others are not.

4. Finding all eigenvectors. Note that the unique eigenvalue of every class (that is, of a diagonal block of an FNF) can be found in $O(n^3)$ time by applying Karp's algorithm (see section 1) to each block. The condition for identifying all spectral submatrices in an FNF provided in Theorem 3.1 enables us to find them in $O(r^2) \leq O(n^2)$ time by applying standard reachability algorithms to C_A.

Let $A \in \overline{\mathbb{R}}^{n\times n}$ be in the FNF (3), $N_1, \ldots, N_r$ be the classes of A, and $R = \{1, \ldots, r\}$. Suppose $\lambda \in \Lambda(A), \lambda > \varepsilon$, and denote $I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i$ spectral$\}$. Similarly as in section 2, we denote by $g_1, \ldots, g_n$ the columns of $\Gamma(\lambda^{-1} \otimes A) = (g_{ij})$. Note that $\lambda(\lambda^{-1} \otimes A) = \lambda^{-1} \otimes \lambda(A)$ may be positive since $\lambda \leq \lambda(A)$, and thus $\Gamma(\lambda^{-1} \otimes A)$ may now include entries equal to $+\infty$ (see Lemma 2.1). Let us denote

$$E(\lambda) = \bigcup\nolimits_{i \in I(\lambda)} E(A_{ii}) = \left\{ j \in N; g_{jj} = 0, j \in \bigcup\nolimits_{i \in I(\lambda)} N_i \right\}.$$

Two nodes i and j in $E(\lambda)$ are called λ-*equivalent* (notation $i \sim_\lambda j$) if i and j belong to the same cycle of cycle mean λ.

THEOREM 4.1. *Suppose $A \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda \in \Lambda(A), \lambda > \varepsilon$. Then $g_j \in \overline{\mathbb{R}}^n$ for all $j \in E(\lambda)$ and a basis of $V(A, \lambda)$ can be obtained by taking one g_j for each $\sim_\lambda$ equivalence class.*

Proof. Let us denote $M = \bigcup_{i \in I(\lambda)} N_i$. By Lemma 3.1 we may assume without loss of generality that A is of the form

$$\begin{pmatrix} \bullet & \varepsilon \\ \bullet & A[M] \end{pmatrix}.$$

Hence $\Gamma(\lambda^{-1} \otimes A)$ is

$$\begin{pmatrix} \bullet & \varepsilon \\ \bullet & C \end{pmatrix},$$

where $C = \Gamma((\lambda(A[M]))^{-1} \otimes A[M])$, and the statement now follows by Theorems 2.1 and 2.2 since $\lambda = \lambda(A[M])$, and thus $\sim_\lambda$ equivalence for A is identical with $\sim$ equivalence for $A[M]$. ∎

COROLLARY 4.1. *A basis of $V(A, \lambda)$ for $\lambda \in \Lambda(A)$ can be found using $O(n^3)$ operations and we have*

$$V(A, \lambda) = \left\{ \Gamma(\lambda^{-1} \otimes A) \otimes z; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \quad \forall j \notin E(\lambda) \right\}.$$

Note that if the set $I(\lambda)$ consists of only one index, then it follows from the proofs of Lemma 3.2 and Theorem 3.1 that $V(A, \lambda)$ can alternatively be found as follows: If $I(\lambda) = \{j\}$, then define

$$M_2 = \bigcup_{N_i \to N_j} N_i, M_1 = N - M_2.$$

Hence

$$V(A, \lambda) = \{x; x[M_1] = \varepsilon, x[M_2] \in V^+(A[M_2])\}.$$

THEOREM 4.2. *$V^+(A) \neq \emptyset$ if and only if $\lambda(A)$ is the eigenvalue of all final classes.*

Proof. The set M_1 in the above construction must be empty to obtain a finite eigenvector; hence a class in S must be reachable from every class of its superblock. This is only possible if S is the set of all final classes since no class is reachable from a final class (other than the final class itself). Conversely, if all final classes have the same eigenvalue $\lambda(A)$, then for $\lambda = \lambda(A)$ the set S contains all the final classes, they are reachable from all classes of their superblocks, and, consequently, $M_1 = \emptyset$, yielding a finite eigenvector. ∎

COROLLARY 4.2. *$V^+(A) = \emptyset$ if and only if a final class has eigenvalue less than $\lambda(A)$.*

Remark 4.1. Note that a final class with eigenvalue less than $\lambda(A)$ may not be spectral, and so $\Lambda(A) = \{\lambda(A)\}$ is possible even if $V^+(A) = \emptyset$. For instance in the case of

$$A = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}$$

we have $\Lambda(A) = \{1\}$, but $V^+(A) = \emptyset$.

Remark 4.2. In the Perron–Frobenius theory of nonnegative matrices, a class of a nonnegative matrix is called basic if its spectral radius coincides with the spectral radius of the matrix. A classical result shows that a nonnegative matrix has a positive eigenvector if and only if its basic and final classes coincide. In the max-algebraic setting we may define a class to be basic when its eigenvalue is $\lambda(A)$. Then, Theorem 4.2 shows that the existence of a finite eigenvector requires only all final classes to be basic; unlike in the Perron–Frobenius theory, there may be nonfinal basic classes. For instance the nonnegative matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

has two basic classes $\{1\}$ and $\{2\}$ and only one final class, namely $\{1\}$, and thus it does not have a positive eigenvector. However, its max-algebraic counterpart

$$A = \begin{pmatrix} 0 & -\infty \\ 0 & 0 \end{pmatrix},$$

which satisfies the condition of Theorem 4.2, has a finite eigenvector (for instance $(0, 0)^T$). This fundamental discrepancy is due to the idempotency of $\oplus$ in max-algebra.

5. Robustness of matrices. Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. The set of vectors $x \in \overline{\mathbb{R}}^n$, such that for some r, $A^r \otimes x$ is an eigenvector of A corresponding to a finite eigenvalue, will be called the *attraction space* (of A). Obviously, if $A^r \otimes x$ is an eigenvector for some r, then $A^k \otimes x$ is an eigenvector for every $k \geq r$. Also, the attraction space of any matrix contains all eigenvectors of this matrix.

It may happen that the attraction space of A contains only eigenvectors of A, for instance when A is the irreducible matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here $\lambda(A) = 0$ and by Theorem 2.1

$$V(A) - \{\varepsilon\} = \{\alpha \otimes (0, 0)^T; \alpha \in \mathbb{R}\}.$$

Since

$$A \otimes \begin{pmatrix} a \\ b \end{pmatrix} = (\max(a-1, b), \max(a, b-1))^T,$$

we have that $A \otimes \binom{a}{b}$ is an eigenvector of A if and only if $a = b$; that is, $A \otimes x$ is an eigenvector of A if and only if x is an eigenvector of A. Hence the attraction space is $V(A) - \{\varepsilon\}$.

The attraction space of A may be different from both $V(A) - \{\varepsilon\}$ and $\overline{\mathbb{R}}^n - \{\varepsilon\}$. Consider the irreducible matrix

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Here $\lambda(A) = 0$ and $x = (-2, -2, 0)^T$ is not an eigenvector of A but $A \otimes x = (-1, -1, 0)^T$ is, showing that the attraction space also contains vectors other than eigenvectors. At the same time if $y = (0, -1, 0)^T$, then $A^k \otimes y$ is y for k even and $(-1, 0, 0)^T$ for k odd, showing that y is not in the attraction space.

A matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is called *robust* if the attraction space of A contains all vectors in $\overline{\mathbb{R}}^n$ except ε. Hence A is robust if and only if $A^k \otimes x$ is an eigenvector of A for any $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$, and large enough k. Alternatively, for every $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$, we have $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$, $A^k \otimes x \neq \varepsilon$, for some positive integer k and $\lambda \in \Lambda(A)$. The importance of robustness has been explained in section 1.

Clearly, if $A \sim B$, then A is robust if and only if B is robust. Therefore we may without loss of generality investigate robustness of matrices arising from a given matrix by a simultaneous permutation of the rows and columns.

Now we present some characterizations of robust matrices. First, we observe that matrices with an ε column are not robust. Following the terminology introduced in [16] we say that A is *column* $\mathbb{R}$-*astic* if it has no ε column. Note that every node of a nontrivial strongly connected digraph has at least one incoming arc, and so every irreducible $n \times n$ matrix ($n > 1$) is column $\mathbb{R}$-astic (but not conversely).

LEMMA 5.1. *If $A \in \overline{\mathbb{R}}^{n \times n}$ is column $\mathbb{R}$-astic and $x \neq \varepsilon$, then $A^k \otimes x \neq \varepsilon$ for every k. Hence if $A \in \overline{\mathbb{R}}^{n \times n}$ is column $\mathbb{R}$-astic, then A^k is column $\mathbb{R}$-astic for every k. This is true, in particular, when A is irreducible and $n > 1$.*

Proof. The proof is immediate from definition. ◻

We say that $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is *ultimately periodic of period* p if there is a natural number p such that the following holds for some $\lambda \in \mathbb{R}$ and k_0 natural:

$$A^{k+p} = \lambda^p \otimes A^k \quad \forall k \geq k_0.$$

If p is the smallest natural number with this property, then we call p the *period* of A and denote it as $p(A)$. If A is not ultimately periodic, then we set $p(A) = +\infty$. It is easily seen that $\lambda = \lambda(A)$ and every column of A^k is in $V(A^p, \lambda^p)$ if $p = p(A) < +\infty$ and A is irreducible. Robustness of irreducible matrices was studied in [11], and we now mention some results of that paper before we proceed with the reducible case. Note that if A is the 1×1 matrix (ε), then A is irreducible and $p(A) = 1$, but A is not robust. This is an exceptional case that has to be excluded in the statements that follow.

THEOREM 5.1 (see [11]). *Let $A \in \overline{\mathbb{R}}^{n \times n}$ be irreducible, $A \neq \varepsilon$. Then A is robust if and only if $p(A) = 1$.*

COROLLARY 5.1 (see [11]). *Let $A \in \overline{\mathbb{R}}^{n \times n}$ be irreducible, $A \neq \varepsilon$. If $p(A) = 1, x \neq \varepsilon$, then $A^k \otimes x$ is finite for all sufficiently big k.*

It was shown in [11] how the next statement follows from the results in [10, Theorem 3.4.5].

THEOREM 5.2. *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ be irreducible. Then A^k is irreducible for every $k = 1, 2, \ldots$ if and only if the lengths of all cycles in D_A are coprime.*

Previous results are closely related to the famous "cyclicity theorem", Theorem 5.3 below. For this we need to introduce a few more concepts: Let D' be a maximal strongly connected subdigraph of a digraph D. Then D' is called a *strongly connected component* of D and the greatest common divisor of all directed cycles in D' is called the *cyclicity* of D', notation $\sigma(D')$. By definition $\sigma(D') = 1$ if D' consists of only a single node. The cyclicity of D is the least common multiple of cyclicities of all strongly connected components of D.

THEOREM 5.3. *Every irreducible matrix A is ultimately periodic and $p(A) = \sigma(C(A))$.*

Note that the "if" statement of Theorem 5.1 follows immediately from Theorem 5.3.

The first part of Theorem 5.3 was proved for finite matrices in [16]. A proof of the whole statement was presented in [12]; see also [13] for an overview without proofs. A proof in a more general setting covering the case of finite matrices is given in [29]. The irreducible case is also proved in [3], [25], [4], and [22]. Note that a different generalization to the reducible case is studied in [23].

COROLLARY 5.2. *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ be irreducible and robust. Then A^k is irreducible for every $k = 1, 2, \ldots$.*

Proof. If the lengths of all critical cycles in D_A are coprime, then the lengths of all cycles are also coprime. The rest follows from Theorem 5.2. ☐

We now continue by studying the robustness of reducible matrices. Theorem 5.1 can straightforwardly be generalized to a class of reducible matrices.

THEOREM 5.4. *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be column $\mathbb{R}$-astic and $|\Lambda(A)| = 1$ (that is, $\Lambda(A) = \{\lambda(A)\}$). Then A is robust if and only if $p(A) = 1$.*

Proof. Let $p(A) = 1, x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$, and $k \geq k_0$. Then $A^k \otimes x \in \overline{\mathbb{R}}^n - \{\varepsilon\}$ by Lemma 5.1, $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$, and so $A^k \otimes x \in V(A, \lambda)$ and $\lambda = \lambda(A)$. Hence A is robust and all columns of A^k are eigenvectors of A.

Now let A be robust and all columns of A^{k_0} be eigenvectors of A corresponding to the unique eigenvalue $\lambda(A)$. Then $A \otimes A^{k_0} = \lambda(A) \otimes A^{k_0}$ and thus $A \otimes A^k = \lambda(A) \otimes A^k$ for all $k \geq k_0$. So $p(A) = 1$. ☐

We will now characterize robust reducible matrices in general—we start with two lemmas.

LEMMA 5.2. *If $A \in \overline{\mathbb{R}}^{n\times n}$ is robust, then $\varepsilon \notin \Lambda(A)$.*

Proof. If $\varepsilon \in \Lambda(A)$, then by Lemma 5.1 some column, say the kth, is ε. Take $x \in \overline{\mathbb{R}}^n$ so that $x_k = 0$ and $x_j = \varepsilon$ for $j \neq k$. Then $A^k \otimes x = \varepsilon$ for every k, and thus $A^k \otimes x$ is never an eigenvector. ☐

A class of A is called *trivial* if it contains only one index, say k, and $a_{kk} = \varepsilon$.

LEMMA 5.3. *If every nontrivial class of $A \in \overline{\mathbb{R}}^{n\times n}$ has eigenvalue 0 and period 1, then $A^{k+1} = A^k$ for some k.*

Proof. We prove the statement by induction on the number of classes.

If A has only one class, then either this class is trivial or A is irreducible. In both cases the statement follows immediately.

If A has at least two classes, then by Lemma 3.1 we can assume without loss of generality that

$$A = \begin{pmatrix} A_{11} & \varepsilon \\ A_{21} & A_{22} \end{pmatrix},$$

and thus

$$A^k = \begin{pmatrix} A_{11}^k & \varepsilon \\ B_k & A_{22}^k \end{pmatrix},$$

where

$$B_k = \sum_{i+j=k-1}^{\oplus} A_{22}^i \otimes A_{21} \otimes A_{11}^j.$$

By the induction hypothesis there are k_1 and k_2 such that

$$A_{11}^{k_1+1} = A_{11}^{k_1} \text{ and } A_{22}^{k_2+1} = A_{22}^{k_2}.$$

It is sufficient now to prove that

$$B_k = \sum\nolimits^{\oplus} \left\{ A_{22}^i \otimes A_{21} \otimes A_{11}^j; i \le k_2, j \le k_1, i = k_2 \text{ or } j = k_1 \right\} \tag{6}$$

holds for all $k \ge k_1 + k_2 + 1$.

For all i, j we have

$$A_{22}^i \otimes A_{21} \otimes A_{11}^j = A_{22}^{i'} \otimes A_{21} \otimes A_{11}^{j'},$$

where $i' = \min(i, k_2), j' = \min(j, k_1)$. If $i + j + 1 = k \ge k_1 + k_2 + 1$, then either $i \ge k_2$ or $j \ge k_1$. Hence either $i' = k_2$ or $j' = k_1$ and therefore $\le$ in (6) follows. For $\ge$ let $i = k_2$ (say) and $j \le k_1$. Since $k \ge k_1 + k_2 + 1 \ge j + i + 1$, we have $k - j - 1 \ge i = k_2$, and thus

$$A_{22}^i \otimes A_{21} \otimes A_{11}^j = A_{22}^{k-j-1} \otimes A_{21} \otimes A_{11}^j \le B_k. \qquad \square$$

Recall that if $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is in the FNF (3) and $N_1, \dots, N_r$ are the classes of A, then we have denoted $R = \{1, \dots, r\}$. If $i \in R$, then we now also denote $T_i = \{k \in R; N_k \longrightarrow N_i\}$ and $M_i = \bigcup_{j \in T_i} N_j$.

We are now ready to present the main result of this paper.

THEOREM 5.5. *Let $A \in \overline{\mathbb{R}}^{n \times n}$ be column $\mathbb{R}$-astic and in the FNF* (3). *Let $N_1, \dots, N_r$ be the classes of A and $R = \{1, \dots, r\}$. Then A is robust if and only if the following hold:*

1. *All nontrivial classes $N_1, \dots, N_r$ are spectral.*
2. *If $i, j \in R, N_i, N_j$ are nontrivial, $i \notin T_j$, and $j \notin T_i$, then $\lambda(N_i) = \lambda(N_j)$.*
3. *$p(A_{jj}) = 1$ for all $j \in R$.*

Proof. If $r = 1$, then A is irreducible and the statement follows by Theorem 5.1. We will therefore assume $r \ge 2$ in this proof.

Let A be robust.

1. Let $i \in R, A_{ii} \ne \varepsilon$, and $x \in \overline{\mathbb{R}}^n$ be defined by taking any $x_s \in \mathbb{R}$ for $s \in M_i$ and $x_s = \varepsilon$ for $s \notin M_i$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some k and $\lambda \in \Lambda(A)$. Let $z = A^k \otimes x$. Then $z[M_i]$ is finite since $A[M_i]$ has no ε row and $A[M_i] \otimes z[M_i] = (A \otimes z)[M_i] = \lambda \otimes z[M_i]$, and thus $z[M_i] \in V^+(A[M_i])$. By Lemma 5.2 $\lambda > \varepsilon$, and so by Theorem 2.1 then $\lambda(N_t) \le \lambda(N_i)$ for all $t \in T_i$. Hence N_i is spectral.
2. Suppose $i, j \in R, N_i, N_j$ are nontrivial and $i \notin T_j$, $j \notin T_i$. Let $x \in \overline{\mathbb{R}}^n$ be defined by taking any $x[N_i] \in V^+(A[N_i]), x[N_j] \in V^+(A[N_j])$, and $x_s = \varepsilon$ for $s \in N - N_i \cup N_j$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some k and $\lambda \in \Lambda(A)$.

Denote $z = A^k \otimes x$. Then $z[N_j]$ is finite. Since $i \notin T_j$ we have $a_{uv} = \varepsilon$ for all $u \in N_i$ and $v \in N_j$. Hence

$$\lambda \otimes z[N_j] = (A \otimes z)[N_j] = A[N_j] \otimes z[N_j],$$

and so by Theorem 2.1 $\lambda(N_j) = \lambda$. Similarly it is proved that $\lambda(N_i) = \lambda$.

3. Let $j \in R$ and $A[N_j] \neq \varepsilon$ (otherwise the statement follows trivially). Let $x \in \overline{\mathbb{R}}^n$ be any vector such that $x \neq \varepsilon$ and $x_s = \varepsilon$ for $s \notin N_j$. Then $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x$ for some k and $\lambda \in \Lambda(A)$. Let $z = A^k \otimes x$. Since $z[N_j] = (A[N_j])^k \otimes x[N_j]$, we may assume without loss of generality that $z[N_j] \neq \varepsilon$. At the same time $A[N_j] \otimes z[N_j] = (A \otimes z)[N_j] = \lambda \otimes z[N_j]$, and thus $z[N_j] \in V(A[N_j])$. Hence $A[N_j]$ is irreducible and robust. Thus by Theorem 5.1 $p\,(A[N_j]) = p(A_{jj}) = 1$.

Suppose now that conditions 1–3 are satisfied. We prove then that A is robust by induction on the number of classes of A. As already observed at the beginning of this proof, the case $r = 1$ follows from Theorem 5.1. Suppose now that $r \geq 2$ and let $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$. Let

$$U = \{i \in N; (\exists j)\, i \longrightarrow j, x_j \neq \varepsilon\}.$$

We have

$$\left(A^k \otimes x\right)[U] = (A[U])^k \otimes x[U]$$

and

$$\left(A^k \otimes x\right)_i = \varepsilon$$

for $i \notin U$. Therefore we may assume without loss of generality that $U = N$. Let M be a final class in C_A, then clearly $x[M] \neq \varepsilon$ by the definition of U. Let us denote

$$S = \{i \in N; (\exists j \in M)\,(i \longrightarrow j)\}$$

and

$$S' = N - S.$$

By Lemma 3.1 we may assume without loss of generality that

$$A = \begin{pmatrix} A_{11} & \varepsilon & \varepsilon \\ A_{21} & A_{22} & A_{23} \\ \varepsilon & \varepsilon & A_{33} \end{pmatrix},$$

where the individual blocks correspond (in this order) to the sets $M, S \setminus M$, and S', respectively. Let us define $x^k = A^k \otimes x$ for all integers $k \geq 0$. We also set

$$\begin{aligned} x_1^k &= x^k[M], \\ x_2^k &= x^k[S \setminus M], \\ x_3^k &= x^k[S']. \end{aligned}$$

Obviously,

$$\begin{aligned} x_1^{k+1} &= A_{11} \otimes x_1^k, \\ x_2^{k+1} &= A_{21} \otimes x_1^k \oplus A_{22} \otimes x_2^k \oplus A_{23} \otimes x_3^k, \\ x_3^{k+1} &= A_{33} \otimes x_3^k. \end{aligned}$$

Assume first that M is nontrivial. Then $\lambda(A_{11}) \neq \varepsilon$ and by taking (if necessary) $(\lambda(A_{11}))^{-1} \otimes A$ instead of A, we may assume without loss of generality that $\lambda(A_{11}) = 0$. By assumption 3 and Theorem 5.3 we have $A_{11}^{k_1+1} = A_{11}^{k_1}$ for some k_1. By assumption 2 every class of A_{33} has eigenvalue 0. Since each of these classes also has period 1 by assumption 3, it follows from Lemma 5.3 that $A_{33}^{k_3+1} = A_{33}^{k_3}$ for some k_3. We may also assume without loss of generality that

$$x_1^0 = x_1^1 = x_1^2 = \cdots$$

and

$$x_3^0 = x_3^1 = x_3^2 = \cdots .$$

Therefore

$$x_2^{k+1} = A_{21} \otimes x_1^0 \oplus A_{22} \otimes x_2^k \oplus A_{23} \otimes x_3^0.$$

Let $v = A_{21} \otimes x_1^0 \oplus A_{23} \otimes x_3^0$. We deduce that

$$x_2^k = A_{22}^k \otimes x_2^0 \oplus \left(A_{22}^{k-1} \oplus \cdots \oplus A_{22}^0\right) \otimes v \tag{7}$$

for all k. Moreover, $\lambda(A_{22}) \leq \lambda(A_{11}) = 0$ since M is spectral by assumption 1. Hence

$$A_{22}^{k-1} \oplus \cdots \oplus A_{22}^0 = \Gamma\left(A_{22}\right)$$

for all $k \geq n$. Note that x_1^0 is finite as an eigenvector of the irreducible matrix A_{11}. Also, since every node in S has access to M, the vector $\Gamma\left(A_{22}\right) \otimes A_{21} \otimes x_1^0$ is finite, and hence also $\Gamma\left(A_{22}\right) \otimes v$ is finite. If $\lambda(A_{22}) < 0$, then $A_{22}^k \otimes x_2^0 \longrightarrow -\infty$ as $k \longrightarrow \infty$, and we deduce that $x_2^k = \Gamma\left(A_{22}\right) \otimes v$ for all k big enough. If $\lambda(A_{22}) = 0$, then

$$A_{22}^{k_2+1} = A_{22}^{k_2}$$

by the induction hypothesis, and thus

$$x_2^k = A_{22}^{k_2} \otimes x_2^0 \oplus \Gamma\left(A_{22}\right) \otimes v$$

for all $k \geq \max(k_1, k_2, k_3)$.

It remains to consider the case when A_{11} is trivial. Then $x_1^k = \varepsilon$ for all $k \geq 1$ and we have

$$\begin{pmatrix} x_2^{k+1} \\ x_3^{k+1} \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ \varepsilon & A_{33} \end{pmatrix} \otimes \begin{pmatrix} x_2^k \\ x_3^k \end{pmatrix}$$

for all $k \geq 1$. We apply the induction hypothesis to the matrix

$$\begin{pmatrix} A_{22} & A_{23} \\ \varepsilon & A_{33} \end{pmatrix}$$

and deduce that $x^{k+1} = x^k$ for k sufficiently big. This completes the proof. ◻

Example 5.1. Let

$$A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix};$$

thus $r = 3, \Lambda(A) = \{0, 1, 2\}, N_j = \{j\}, j = 1, 2, 3$. If

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then $A^k \otimes x, k = 1, 2, 3, 4$, are

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}, \dots$$

which obviously will never reach an eigenvector. The reason is that $1 \notin T_2, 2 \notin T_1$ but $\lambda(N_1) \neq \lambda(N_2)$.

Example 5.2. Let

$$A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 \end{pmatrix};$$

thus $r = 3, \Lambda(A) = \{0, 2\}, N_j = \{j\}, j = 1, 2, 3$. This matrix is robust since both nontrivial classes (N_1 and N_3) are spectral, $p\,(A_{ii}) = 1$ ($i = 1, 2, 3$), and there are no nontrivial classes N_i, N_j such that $i \notin T_j$ and $j \notin T_i$. Indeed, if

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then $A^k \otimes x, k = 1, 2, 3, 4$, are

$$\begin{pmatrix} 2 \\ \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ \varepsilon \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ \varepsilon \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ \varepsilon \\ 6 \end{pmatrix}, \dots.$$

Hence an eigenvector is reached in the first step.

6. Conclusions. The primary objective of this paper was to study robustness of matrices in max-algebra. The importance of robust matrices is given by the fact that if the production matrix of a multimachine interactive production system is robust, then an ultimate stationary regime is always reached, independently of the choice of initial conditions. In addition, the problem is of an intrinsic mathematical interest, and it might be interesting to extend the present study to other classes of nonlinear maps.

In this paper (sections 3 and 4) we have first presented fundamental results on the eigenvector-eigenvalue theory for reducible matrices in max-algebra including a comparison with the classical Perron–Frobenius theory of nonnegative matrices.

In section 5 we have used these results and developed the theory of robustness of reducible matrices. The principal result of the paper, Theorem 5.5, efficiently characterizes robust matrices. It is followed by two numerical examples.

Acknowledgment. We are grateful to two referees for their comments and suggestions which improved the presentation of the paper.

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