Appears in the Proceedings of the Workshop on Max-Plus Algebras,
IFAC SSSC'01, Praha, August 2001.

GENERIC ASYMPTOTICS OF EIGENVALUES USING MIN-PLUS ALGEBRA

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Abstract: We consider a square matrix $\mathcal{A}_\epsilon$ whose entries have first order asymptotics of the form $(\mathcal{A}_\epsilon)_{ij} \sim a_{ij}\epsilon^{A_{ij}}$ when ϵ goes to 0, for some $a_{ij} \in \mathbb{C}$ and $A_{ij} \in \mathbb{R}$. We show that under a non-degeneracy condition, the order of magnitudes of the different eigenvalues of $\mathcal{A}_\epsilon$ are given by min-plus eigenvalues of min-plus Schur complements built from $A = (A_{ij})$, or equivalently by generalized minimal mean weights of circuits. This construction gives, in non singular cases, a graph interpretation to the slopes of the Newton polygon of the characteristic polynomial of $\mathcal{A}_\epsilon$. It explains the order of magnitudes of eigenvalues in the perturbation formula of Lidskiĭ, Višik and Ljusternik, and it allows us to solve some cases which are singular in this theory. *Copyright 2001 IFAC*

Keywords: Perturbation theory, Max-plus algebra, Spectral theory, Graphs, Schur complements.

1. INTRODUCTION

Let $\mathcal{A}_\epsilon$ denote a $n \times n$ matrix whose entries, which are continuous functions of $\epsilon > 0$, have first order asymptotics of the form $(\mathcal{A}_\epsilon)_{ij} \sim a_{ij}\epsilon^{A_{ij}}$ when ϵ goes to 0, where $a_{ij} \in \mathbb{C}$, and $A_{ij} \in \mathbb{R}$. The goal of this paper is to give first order asymptotics $\mathcal{L}^i_\epsilon \sim \lambda_i\epsilon^{\Lambda_i}$, with $\lambda_i \in \mathbb{C}$ and $\Lambda_i \in \mathbb{R}$, for each of the eigenvalues $\mathcal{L}^1_\epsilon, \dots, \mathcal{L}^n_\epsilon$ of $\mathcal{A}_\epsilon$, in some generic cases.

Computing the asymptotics of spectral elements is a central problem of perturbation theory, see Kato (1995) and Baumgärtel (1985). For instance, when the entries of $\mathcal{A}_\epsilon$ have Taylor (or, more generally, Puiseux) series expansions in ϵ, the eigenvalues $\mathcal{L}^i_\epsilon$ have Puiseux series expansions in ϵ, which can be computed by applying the Newton-Puiseux algorithm to the characteristic polynomial of $\mathcal{A}_\epsilon$. The leading exponents Λ_i of the eigenvalues of $\mathcal{A}_\epsilon$ are the slopes of the associated Newton polygon, but it is hard to guess these slopes from the dominant exponents of the entries of $\mathcal{A}_\epsilon$.

In this paper, we show that the dominant exponents of the eigenvalues are min-plus eigenvalues of min-plus Schur complements built from the matrix of dominant exponents $A = (A_{ij})$, provided that certain (conventional) Schur complements built from $a = (a_{ij})$ are invertible. This often allows us to get the first order asymptotics of the eigenvalues of $\mathcal{A}_\epsilon$, by mere inspection. For instance, if

$$\mathcal{A}_\epsilon = \begin{bmatrix} \epsilon & 1 & \epsilon^4 \\ 0 & \epsilon & \epsilon^{-2} \\ \epsilon & \epsilon^2 & 0 \end{bmatrix} , \qquad (1)$$

we get by direct application of Theorem 1 below (without any computation) that the spectrum of $\mathcal{A}_\epsilon$

⋆ This work was partially supported by the European Community Framework IV program through the research network ALAPEDES ("The Algebraic Approach to Performance Evaluation of Discrete Event Systems").

consists of three eigenvalues

$$\mathcal{L}_\epsilon^1 \sim \epsilon^{-1/3}, \mathcal{L}_\epsilon^2 \sim j\epsilon^{-1/3}, \mathcal{L}_\epsilon^3 \sim j^2\epsilon^{-1/3}, \qquad (2)$$

where $j = \exp(2i\pi/3)$. See §3 for details (and for more refined examples).

The present work is a continuation of Akian *et al.* (1998), where related max-plus formulæ were given for the Perron eigenvalue and eigenvector, when $\mathcal{A}_\epsilon$ is nonnegative. All these results are partial versions of a "matrix Puiseux theorem", which determines the asymptotic expansions of the eigenvalues of $\mathcal{A}_\epsilon$ by reasoning on $\mathcal{A}_\epsilon$, rather than on its characteristic polynomial. This is the object of a forthcoming paper Akian *et al.* (2001).

Theorem 1 below can be thought of as an extension of a theorem due to Višik and Ljusternik (1960) and Lidskiĭ (1965), which gives the first order expansions of eigenvalues and eigenvectors, in the special case where

$$\mathcal{A}_\epsilon = \mathcal{A}_0 + \epsilon b \ , \qquad (3)$$

for some $b \in \mathbb{C}^{n\times n}$, using the Jordan structure of $\mathcal{A}_0$ (see Moro *et al.* (1997) for a recent overview). Theorem 1, together with the graph interpretation of §4, shows that the dominant exponents of the eigenvalues are given by generalized mean weights of circuits, which explains the dominant exponents found by Višik, Ljusternik and Lidskiĭ, in the special case (3), and allows us to solve cases which are singular in their theory.

Finally, we note that results of max-plus spectral theory were already applied to WKB type asymptotics in Dobrokhotov *et al.* (1992). See also Kolokoltsov and Maslov (1997).

2. STATEMENT OF THE RESULT

2.1 *Preliminaries*

We first recall some classical facts of min-plus algebra. See for instance Baccelli *et al.* (1992) for more details.

The min-plus semiring, $\mathbb{R}_{\min}$, is the set $\mathbb{R}\cup\{+\infty\}$ equipped with the addition $(a,b)\mapsto a\oplus b = \min(a,b)$ and the multiplication $(a,b)\mapsto a\otimes b = a+b$. We shall denote by $\mathbb{0} = +\infty$ and $\mathbb{1} = 0$ the zero and unit elements of $\mathbb{R}_{\min}$, respectively. We shall use the familiar algebraic conventions, in the min-plus context. For instance, if A, B are matrices of compatible dimensions with entries in $\mathbb{R}_{\min}$, $(AB)_{ij} = (A\otimes B)_{ij} = \min_k(A_{ik}+B_{kj})$, $A^2 = A\otimes A$, etc.

We shall need some results from min-plus (or equivalently, max-plus) spectral theory, which can be found for instance in Baccelli *et al.* (1992) and Cuninghame-Green (1995). (The max-plus spectral theorem has been discovered by many authors, including Cuninghame-Green, Gondran and Minoux (1977), Vorobyev (1967), Romanovskiĭ (1967), Cohen *et al.* (1983). See also Maslov and Samborskiĭ (1992); Gaubert and Plus (1997); Bapat (1998).)

To a matrix $A \in (\mathbb{R}_{\min})^{n\times n}$, we associate the (directed) graph $G(A)$, which has nodes $1,\dots,n$ and an arc $(i\to j)$ if $A_{ij}\neq \mathbb{0}$. We say that A is *irreducible* if $G(A)$ is strongly connected. The min-plus spectral theorem states that an irreducible matrix $A\in(\mathbb{R}_{\min})^{n\times n}$ has a unique eigenvalue,

$$\rho_{\min}(A) = \min_{1\le k\le n}\ \min_{i_1,\dots,i_k} \frac{A_{i_1i_2}+\cdots+A_{i_ki_1}}{k} \ . \qquad (4)$$

We say that a circuit $(i_1\to i_2\to\cdots\to i_k\to i_1)$ of $G(A)$ is *critical* if $(i_1,\dots,i_k)$ attains the minimum in (4), and we call critical the nodes and arcs of this circuit. The critical nodes and critical arcs form the *critical graph*, $G^c(A)$.

The *Kleene's star* of a matrix $A\in(\mathbb{R}_{\min})^{n\times n}$ is defined by

$$A^* = A^0\oplus A\oplus A^2\oplus\cdots \ ,$$

i.e. $(A^*)_{ij} = \inf_{k\ge 0}(A^k)_{ij}$. All the entries of $(A^*)_{ij}$ are $> -\infty$ if, and only if, $\rho_{\min}(A)\ge 0$. When $\rho_{\min}(A)\ge 0$, $A^* = A^0\oplus\cdots\oplus A^{n-1}$.

The min-plus spectral theorem also states that any eigenvector of an irreducible matrix A is a linear combination of the columns $(\rho_{\min}(A)^{-1}A)^*_{\cdot,j}$ corresponding to critical nodes j. (We warn the reader that when $\alpha\in\mathbb{R}_{\min}\setminus\{\mathbb{0}\}$ and $B\in\mathbb{R}_{\min}^{n\times n}$, $\alpha^{-1}B$ should be interpreted in the min-plus sense, i.e. $(\alpha^{-1}B)_{ij} = -\alpha + B_{ij}$.)

If A is an $L\times L$ matrix with entries in $\mathbb{R}_{\min}$, for all $J,K\subset L$, we denote by A_{JK} the $J\times K$ submatrix of A. If $C\subset L$, if $\lambda\in\mathbb{R}_{\min}\setminus\{\mathbb{0}\}$, and if $\rho_{\min}(\lambda^{-1}A_{CC})\ge 0$, the (twisted, min-plus) *Schur complement* of C in A is defined by

$$\mathrm{Schur}(C,\lambda,A) = A_{NN}\oplus A_{NC}(\lambda^{-1}A_{CC})^*\lambda^{-1}A_{CN},$$

where $N = I\setminus C$. When $\lambda = \mathbb{1} = 0$, we shall simply write $\mathrm{Schur}(C,A)$ instead of $\mathrm{Schur}(C,\mathbb{1},A)$. Note that matrices are indexed by "abstract indices", not by integers $1,2,\dots,k$. For instance, if A is a $\{1,2,3\}\times\{1,2,3\}$ matrix, $B = \mathrm{Schur}(\{1,2\},A)$ is a $\{3\}\times\{3\}$ matrix, whose unique entry is denoted by B_{33} (not by B_{11}).

We shall also need conventional Schur complements. If a is a $L\times L$ matrix with entries in $\mathbb{C}$, and if a_{CC} is invertible (we use the same notations as for submatrices with entries in $\mathbb{R}_{\min}$), we define

$$\mathrm{Schur}(C,a) = a_{NN} - a_{NC}(a_{CC})^{-1}a_{CN} \ .$$

Using the same symbol, "Schur", both for conventional and min-plus Schur complements is not ambiguous: considering min-plus Schur complements of complex matrices, or conventional Schur complements of min-plus matrices, would be meaningless.

Both min-plus and conventional Schur complements satisfy

$$\mathrm{Schur}(C \cup C', a) = \mathrm{Schur}(C, \mathrm{Schur}(C', a)) \qquad (5)$$

for all $L \times L$ matrices a, and for all disjoint subsets of indices $C, C' \subset L$, provided that the Schur complements are well defined.

2.2 *Main Theorem*

It will be convenient to use an equivalence notion slightly weaker than the usual equivalence $\sim$. If $f_\epsilon \in \mathbb{C}$, $a \in \mathbb{C}$, $A \in \mathbb{R}_{\min}$, we write

$$f_\epsilon \simeq a\epsilon^A \qquad (6)$$

if either $\lim_{\epsilon \to 0} \epsilon^{-A} f_\epsilon = a$ and $A \neq +\infty$, or, if $A = +\infty$ and $f_\epsilon = 0$ for ϵ small enough. (This is consistent with the convention $\epsilon^{+\infty} = 0$.) If $a \neq 0$, $f_\epsilon \simeq a\epsilon^A \iff f_\epsilon \sim a\epsilon^A$, but $f_\epsilon \simeq 0\epsilon^A$ just means that $f_\epsilon = o(\epsilon^A)$. Of course, $a\epsilon^A$ must be viewed as a formal expression, for (6) to be meaningful when $a = 0$.

In the sequel, we shall assume that $(\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}$, for some $a \in \mathbb{C}^{n\times n}$ and for some irreducible matrix $A \in (\mathbb{R}_{\min})^{n\times n}$. (The case where A is reducible is a straightforward extension.)

We build by induction a finite sequence of min-plus square matrices A_i and scalars $\alpha_i \in \mathbb{R}$, for $1 \leq i \leq k$, together with a partition $C_1 \cup \cdots \cup C_k = \{1, \ldots, n\}$.

First, we set $A_1 = A$. For all $i \geq 1$,

$$\alpha_i = \rho_{\min}(A_i) \qquad (7)$$

and we take for C_i the set of critical nodes of A_i. We build, as long as $C_1 \cup \cdots \cup C_i \neq \{1, \ldots, n\}$, the min-plus Schur complement:

$$A_{i+1} = \mathrm{Schur}(C_i, \alpha_i, A_i) \ .$$

Due to the irreducibility of A, it is not difficult to see that A_i is irreducible, so that $C_i \neq \varnothing$. Hence, the algorithm stops at some index $k \leq n$.

We denote by D the min-plus diagonal matrix such that $D_{jj} = \alpha_i$ when $j \in C_i$, we set $\tilde{A} = D^{-1}A$, and we select an arbitrary eigenvector V of $\tilde{A}$, for instance any column of $\tilde{A}^*$ (it is not difficult to see that all the nodes $1, \ldots, n$ belong to the critical graph of $\tilde{A}$).

We define the *saturation graph*, Sat, with nodes $1, \ldots, n$, and the arcs $i \to j$ such that $V_i = \tilde{A}_{ij} + V_j$. The matrix a^{Sat} is defined by

$$(a^{\mathrm{Sat}})_{ij} = \begin{cases} a_{ij} & \text{if } (i \to j) \in \mathrm{Sat}, \\ 0 & \text{otherwise.} \end{cases} \qquad (8)$$

We finally define recursively the conventional Schur complements:

$$s^1 = a^{\mathrm{Sat}}, \ s^{i+1} = \mathrm{Schur}(C_i, s^i) \ ,$$

as long as the $C_i \times C_i$ submatrix of s^i, $s^i_{C_iC_i}$, is invertible. Using (5), we get equivalently

$$s^{i+1} = \mathrm{Schur}(C_1 \cup \cdots \cup C_i, a^{\mathrm{Sat}}) \ . \qquad (9)$$

It will be convenient to set

$$t^i = s^i_{C_iC_i} \ .$$

Theorem 1. Assume that the matrices $t^1, \ldots, t^\ell$ are invertible. Then, for $1 \leq i \leq \ell$, $\mathcal{A}_\epsilon$ has $|C_i|$ eigenvalues with asymptotics

$$\mathcal{L}_\epsilon \sim \lambda_j \epsilon^{\alpha_i}, \quad j = 1, \ldots, |C_i|$$

where the λ_j are the eigenvalues of t^i. Moreover, if $\ell < k$, to each non-zero eigenvalue λ_j of $t^{\ell+1}$ is associated an eigenvalue of $\mathcal{A}_\epsilon$ with asymptotics

$$\mathcal{L}_\epsilon \sim \lambda_j \epsilon^{\alpha_{\ell+1}}, \qquad (10)$$

and all the remaining eigenvalues of $\mathcal{A}_\epsilon$ are $o(\epsilon^{\alpha_{\ell+1}})$. (All eigenvalues are counted with multiplicities.)

In fact, the asymptotics (10) are valid as soon as the $(C_1 \cup \cdots \cup C_\ell) \times (C_1 \cup \cdots \cup C_\ell)$ submatrix of a^{Sat} is invertible, which allows us to define $s^{\ell+1}$ by (9). Theorem 1, together with such refinements, will be proved in an extended version of the present paper. Here, we only show examples (in §3), and give an intrinsic graph interpretation of the exponents α_i (in §4). (The graph interpretation of §4 implies that the asymptotics predicted by Theorem 1 are independent of the choice of the eigenvector V of $\tilde{A}$.)

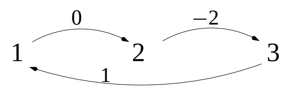
3. EXAMPLES

3.1 *A Simple Example*

Let us first apply Theorem 1 to the matrix (1). We can write $(\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}$, with

$$a = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ , \quad A = \begin{bmatrix} 1 & 0 & 4 \\ +\infty & 1 & -2 \\ 1 & 2 & +\infty \end{bmatrix} \ . \qquad (11)$$

We have $\rho_{\min}(A) = -1/3$, and $G^c(A)$ consists of the critical circuit:



so that the construction of §2.2 stops with $C_1 = \{1, 2, 3\}$. Since $G^c(A)$ is strongly connected and covers all the nodes, the saturation graph of A coincides with $G^c(A)$ (independently of the choice of the eigenvector of A), hence,

$$a^{\mathrm{Sat}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \ , \qquad (12)$$

and since the spectrum of a^{Sat} is $\{1, j, j^2\}$, Theorem 1 shows that the spectrum of $\mathcal{A}_\epsilon$ consists of the three eigenvalues (2), as announced in the introduction.

3.2 *Comparison with Lidskiĭ's theorem: Regular Case*

To see that Theorem 1 explains the dominant exponents found by Lidskiĭ, we next revisit the example of Moro *et al.* (1997) illustrating Lidskiĭ's result. We shall see in the next section that Theorem 1 solves cases which are singular in Lidskiĭ (1965).

Let $\mathcal{A}_\epsilon = \mathcal{A}_0 + \epsilon b$, where

$$\mathcal{A}_0 = \left[\begin{array}{ccc|ccc|cc|c} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \qquad (13)$$

(the dots represent 0), and $b \in \mathbb{C}^{n\times n}$. We can write $(\mathcal{A}_\epsilon)_{ij} \simeq a_{ij}\epsilon^{A_{ij}}$, with

$$A = \left[\begin{array}{ccc|ccc|cc|c} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] , \qquad (14)$$

$a_{ij} = 1$ when $A_{ij} = 0$ and $a_{ij} = b_{ij}$ when $A_{ij} = 1$. We get $\alpha_1 = \rho_{\min}(A_1) = 1/3$, and the critical graph of A_1 is composed of the circuit $(1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1)$, together with the arcs $(3 \to 1)$ and $(6 \to 4)$. Thus, $C_1 = \{1, \dots, 6\}$. A simple computation gives

$$A_2 = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 1 & 1 \end{array}\right] .$$

We have $\alpha_2 = \rho_{\min}(A_2) = 1/2$, and the critical graph of A_2 is reduced to the circuit $7 \to 8 \to 7$. Thus, $C_2 = \{7, 8\}$. The last min-plus Schur complement is $A_3 = (1)$. Thus, $\alpha_3 = 1$, and $C_3 = \{9\}$. Let us take the eigenvector $V = (\tilde{A}^*)_{\cdot,9}$:

$$V = \left[\, 0 \;\; 1/3 \;\; 2/3 \;\; 0 \;\; 1/3 \;\; 2/3 \;\; 0 \;\; 1/2 \;\; 0 \,\right]^T .$$

We get

$$a^{\mathrm{Sat}} = \left[\begin{array}{ccc|ccc|cc|c} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{31} & \cdot & \cdot & b_{34} & \cdot & \cdot & b_{37} & \cdot & b_{39} \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ b_{61} & \cdot & \cdot & b_{64} & \cdot & \cdot & b_{67} & \cdot & b_{69} \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ b_{81} & \cdot & \cdot & b_{84} & \cdot & \cdot & b_{87} & \cdot & b_{89} \\ \hline b_{91} & \cdot & \cdot & b_{94} & \cdot & \cdot & b_{97} & \cdot & b_{99} \end{array}\right] .$$

Hence, $s^1 = a^{\mathrm{Sat}}$, and

$$t^1 = \left[\begin{array}{ccc|ccc} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ b_{31} & \cdot & \cdot & b_{34} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ b_{61} & \cdot & \cdot & b_{64} & \cdot & \cdot \end{array}\right] .$$

To reobtain the results of Lidskiĭ, it is convenient to reorder the nodes of C_1, in order to put t^1 in cyclic form

$$\left[\begin{array}{cccccc} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ b_{31} & b_{34} & \cdot & \cdot & \cdot & \cdot \\ b_{61} & b_{64} & \cdot & \cdot & \cdot & \cdot \end{array}\right] .$$

Thus, t^1 is invertible if, and only if,

$$r = \begin{bmatrix} b_{31} & b_{34} \\ b_{61} & b_{64} \end{bmatrix}$$

is invertible, and the eigenvalues of t^1 are the cubic roots of the eigenvalues of r. At this point, we know already, by Theorem 1, that when r is invertible, $\mathcal{A}_\epsilon$ has 6 eigenvalues with asymptotics $\mathcal{L}_\epsilon \sim \lambda\epsilon^{1/3}$, corresponding to the different eigenvalues λ of t^1. The inverse of t^1 can be computed easily in block form from r^{-1}, which leads to:

$$\begin{aligned} s^2 &= \left[\begin{array}{cc|c} \cdot & 1 & \cdot \\ b_{87} & \cdot & b_{89} \\ \hline b_{97} & \cdot & b_{99} \end{array}\right] - \left[\begin{array}{cc} \cdot & \cdot \\ b_{81} & b_{84} \\ \hline b_{91} & b_{94} \end{array}\right] r^{-1} \left[\begin{array}{cc|c} b_{37} & \cdot & b_{39} \\ b_{67} & \cdot & b_{69} \end{array}\right] \\ &= \left[\begin{array}{cc|c} \cdot & 1 & \cdot \\ b'_{87} & \cdot & b'_{89} \\ \hline b'_{97} & \cdot & b'_{99} \end{array}\right], \quad \text{and} \quad t^2 = \begin{bmatrix} \cdot & 1 \\ b'_{87} & \cdot \end{bmatrix} , \end{aligned}$$

where

$$b'_{87} = b_{87} - \left[\, b_{81} \;\; b_{84} \,\right] r^{-1} \begin{bmatrix} b_{37} \\ b_{67} \end{bmatrix}$$

(the other entries of b' are computed in a similar way). Thus, t^2 is invertible if, and only if, $b'_{87} \neq 0$, and the eigenvalues of t^2 are the square roots of b'_{87}. We know by Theorem 1 that, when $b'_{87} \neq 0$, $\mathcal{A}_\epsilon$ has two eigenvalues with asymptotics $\mathcal{L}_\epsilon \sim \lambda\epsilon^{1/2}$, corresponding to the two square roots λ of b'_{87}. Finally,

$$s^3 = t^3 = b'_{99} - b'_{97}(b'_{87})^{-1}b'_{89} , \qquad (15)$$

which shows that, when $s^3 \neq 0$, $\mathcal{A}_\epsilon$ has one eigenvalue $\mathcal{L}_\epsilon \sim s^3\epsilon$. (The expression (15) is equal to that of Moro *et al.* (1997), due to the identity (5).)

3.3 *Singular case*

Let us now assume that $b_{61} = b_{64} = 0$. We may keep A as in (14), but this gives little information since $r = \left[\begin{smallmatrix} b_{31} & b_{34} \\ 0 & 0 \end{smallmatrix}\right]$ is not invertible (which implies that t^1 is not invertible). This case is considered as singular in Lidskiĭ (1965) and Moro *et al.* (1997). However, $(\mathcal{A}_\epsilon)_{ij} = a_{ij}\epsilon^{A_{ij}}$ still holds if we change the following values of A: $A_{61} = A_{64} = +\infty$. To make the example more interesting, we shall also assume that $A_{69} = +\infty$ (hence, $b_{69} = 0$).

We still have $\alpha_1 = 1/3$ but the critical graph now consists only of the circuit $1 \to 2 \to 3 \to 1$, so that $C_1 = \{1, 2, 3\}$. We now get

$$A_2 = \left[\begin{array}{ccc|cc|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 4/3 & 1 & 1 & 1 & 1 & 4/3 \\ \hline 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

so that $\alpha_2 = 2/5$, with a critical graph C_2 consisting of the only circuit $4 \to 5 \to 6 \to 7 \to 8 \to 4$. We leave it to the reader to check that $A_3 = (1)$. We take the eigenvector $V = (\tilde{A}^*)_{\cdot,9}$:

$$V = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 2/5 & 4/5 & 1/5 & 3/5 & 0 \end{bmatrix}^T \quad .$$

The matrix t^1 is invertible if, and only if, $b_{31} \neq 0$. In this case, $\mathcal{A}_\epsilon$ has three eigenvalues with asymptotics $\mathcal{L}_\epsilon \sim \lambda\epsilon^{1/3}$, corresponding to the different cubic roots λ of b_{31}. We have

$$s^2 = \left[\begin{array}{ccc|cc|c} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b_{67} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ b'_{84} & \cdot & \cdot & \cdot & \cdot & b'_{89} \\ \hline b'_{94} & \cdot & \cdot & \cdot & \cdot & b'_{99} \end{array}\right]$$

where for instance $b'_{84} = b_{84} - b_{81}b_{31}^{-1}b_{34}$. Thus, t^2 is invertible, if, and only if, $b'_{84}b_{67} \neq 0$. When this is the case, $\mathcal{A}_\epsilon$ has five eigenvalues with asymptotics $\mathcal{L}_\epsilon \sim \lambda\epsilon^{2/5}$, corresponding to the different quintic roots λ of $b'_{84}b_{67}$. Finally, $s^3 = b'_{99} - b'_{94}(b'_{84})^{-1}b'_{89}$, and, when $s^3 \neq 0$, $\mathcal{A}_\epsilon$ has a last eigenvalue $\mathcal{L}_\epsilon \sim s^3\epsilon$.

It is a good exercise to perform again the computations when $b_{69} \neq 0$ and $B_{69} \neq +\infty$. Then, $\alpha_3 = 4/5$, and s^3 is identically 0: in this case, the conclusions for the eigenvalues of order $\epsilon^{1/3}$ and $\epsilon^{2/5}$ are unchanged, but we can only conclude from Theorem 1 that the last eigenvalue is $o(\epsilon^{4/5})$. Such cases can be desingularized using the methods of Akian *et al.* (2001), which need higher order informations on the asymptotics of the entries of $\mathcal{A}_\epsilon$.

4. GRAPH INTERPRETATION

In this section, we give a graph interpretation of the exponents α_i of Theorem 1. If $p = (i_0 \to i_1 \to \cdots \to i_j)$ is a path of $G(A)$, we denote by $|p|_A = A_{i_0 i_1} + \cdots + A_{i_{j-1} i_j}$ the *weight* of p, and by $|p| = j$ its *length*. For all $L \subset \{1, \ldots, n\}$, we denote by $|p|_L$ the number of arcs of p with initial node in L, i.e., $|p|_L = |\{0 \leq m \leq j-1 \mid i_m \in L\}|$ (all the path interpretations below have dual versions, obtained by replacing "initial" by "final"). We also denote by $p \cap L$ the subsequence of p obtained by deleting the nodes not in L ($p \cap L$ need not be a path of $G(A)$). We get, by mere rephrasing of the definition of $\rho_{\min}(A)$:

$$\begin{gathered} \rho_{\min}(A) = \max \lambda, \text{ subject to} \\ |c|_A \geq |c|\lambda \text{ for all circuits } c \text{ of } G(A) \quad . \end{gathered} \tag{16}$$

The α_i have a similar characterization:

Proposition 2. The numbers α_i defined in (7) satisfy:

$$\begin{aligned} \alpha_i = \max \lambda,\ & \text{subject to} \\ & (|c|_A - \alpha_1|c|_{C_1} - \cdots - \alpha_{i-1}|c|_{C_{i-1}}) \geq \\ & \qquad (|c| - |c|_{C_1} - \cdots - |c|_{C_{i-1}})\lambda \quad , \end{aligned} \tag{17}$$

for all circuits c in $G(A)$.

The previous formula is equivalent to:

$$\alpha_i = \min \frac{|c|_A - \alpha_1|c|_{C_1} - \cdots - \alpha_{i-1}|c|_{C_{i-1}}}{|c| - |c|_{C_1} - \cdots - |c|_{C_{i-1}}} \quad ,$$

where the minimum is taken over all circuits c in $G(A)$ which are not included in $C_1 \cup \ldots \cup C_{i-1}$. Note that if c is included in $C_1 \cup \ldots \cup C_{i-1}$, that is if the denominator is zero, the numerator is necessarily nonnegative (by definition of α_{i-1}), so that (17) holds for all λ.

The proof of Proposition 2 is based on the following classical interpretation of Schur complements.

Lemma 3. Consider a matrix $A \in \mathbb{R}_{\min}^{n\times n}$, a partition $C \cup N$ of $\{1, \ldots, n\}$ and a real α. Then, for all paths p in $G(\mathrm{Schur}(C, \alpha, A))$, we have

$$|p|_{\mathrm{Schur}(C,\alpha,A)} = \min |p'|_A - \alpha|p'|_C \quad ,$$

where the minimum is taken over all the paths p' of $G(A)$ that have the same extremal nodes as p and satisfy $p' \cap N = p$. Moreover, $|p| = |p'| - |p'|_C$ for all paths p' with the same properties as before.

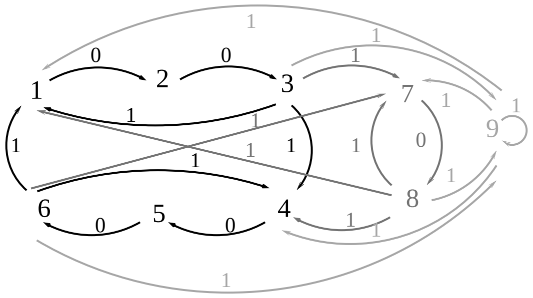
Indeed, using (16) together with the definition (7) of α_i, and using repeatedly Lemma 3, we get Proposition 2.

We say that a circuit c of $G(A)$ is a *critical circuit of order i* if the inequality (17) evaluated at $\lambda = \alpha_i$ is an equality. We call *critical graph of order i* the graph $G_i^c(A)$ whose nodes and arcs belong to critical circuits of order i. Of course, $G^c(A) = G_1^c(A)$.

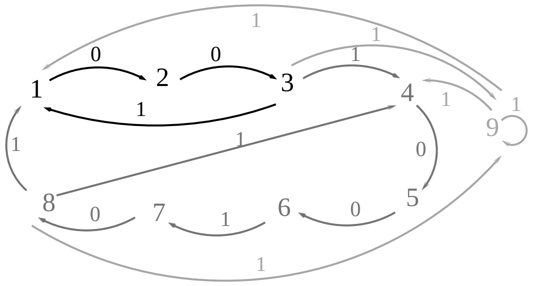
If c is a critical circuit of order i, we get by applying the equality in (17) with $\lambda = \alpha_i$ that $|c|_A - \alpha_1|c|_{C_1} - \cdots - \alpha_{i-1}|c|_{C_{i-1}} - \alpha_i|c|_{N_i} = 0$, where $N_i = \{1, \ldots, n\} \setminus (C_1 \cup \ldots \cup C_{i-1})$. After replacing c by a cyclic conjugate of c, we may assume that the initial node of c is in N_i. Then, applying repeatedly Lemma 3, we obtain that $c \cap N_i$, which is a critical circuit of A_i, is included in C_i. Therefore, c is included in $C_1 \cup \ldots \cup C_i$, $|c|_{N_i} = |c|_{C_i}$ and both terms of (17) evaluated at c, with i instead of $i-1$, are zero, which shows that c is a critical circuit of order $i+1$. Thus,

$$G_i^c(A) \subset G_{i+1}^c(A) \quad , \tag{18}$$

which means that the nodes and arcs of $G_i^c(A)$ belong to $G_{i+1}^c(A)$. For instance, the matrix A of (14) (from the example of Section 3.2) has the following graphs:



The graphs $G_i^c(A)$, for $i = 1, 2, 3$ are represented in black, magenta (medium gray), and green (light grey), respectively; for readability, a node or arc is drawn with the color of the minimal graph $G_i^c(A)$ to which it belongs. For the matrix A of the singular case of Section 3.3, the graphs $G_i^c(A)$ become:



Let D_i denote the min-plus diagonal matrix such that $(D_i)_{jj} = \alpha_m$ if $j \in C_m$ with $m \le i$, and $(D_i)_{jj} = \alpha_i$ if $j \in C_m$ with $m > i$, and let $\hat{A}_i = D_i^{-1} A$. We easily derive from Proposition 2, (16) and (18) the following lemma.

Lemma 4. For all $1 \le i \le k$, we have $G_i^c(A) = G^c(\hat{A}_i)$, $\hat{A}_i$ has min-plus eigenvalue $\mathbb{1}$, and the set of critical nodes of $\hat{A}_i$ is $C_1 \cup \ldots \cup C_i$. In particular, $G_k^c(A) = G^c(\tilde{A})$ and all the nodes of $\tilde{A} = \hat{A}_k$ are critical.

To any irreducible matrix $B \in \mathbb{R}_{\min}^{n\times n}$ with eigenvalue α and eigenvector W of B, we associate the saturation graph $\mathrm{Sat}(B, W)$, with nodes $1, \ldots, n$ and an arc $i \to j$ if $\alpha + W_i = B_{ij} + W_j$. The saturation graph Sat defined above in Section 2.2 is equal to $\mathrm{Sat}(\tilde{A}, V)$. We need the following well known (and easy) properties:

Lemma 5. Let $B \in \mathbb{R}_{\min}^{n\times n}$ be an irreducible matrix with eigenvalue α, and let $W \in \mathbb{R}_{\min}^{n}$. If $BW = \alpha W$, then the strongly connected components of $\mathrm{Sat}(B, W)$ are exactly the strongly connected components of $G^c(B)$. If $BW \ge \alpha W$, then $(BW)_i = \alpha W_i$ for all critical nodes $i \in G^c(B)$.

Hence, the strongly connected components of the graph $\mathrm{Sat} = \mathrm{Sat}(\tilde{A}, V)$ coincide with the strongly connected components of $G^c(\tilde{A})$, which shows that the irreducible blocks of the matrix a^{Sat} are independent of the choice of the eigenvector V. Moreover, for all $i = 1, \ldots, k$ and for $C = C_1 \cup \ldots \cup C_i$, the restriction V_i of V to C satisfies $\tilde{A}_{CC} V_i = (\hat{A}_i)_{CC} V_i \ge V_i$, which implies, using Lemmas 4 and 5, that V_i is an eigenvector of $(\hat{A}_i)_{CC}$, so that the strongly connected components of $\mathrm{Sat} \cap (C \times C)$ coincide with the strongly connected components of $G^c((\hat{A}_i)_{CC}) = G^c(\hat{A}_i) = G_i^c(A)$. Hence, the irreducible blocks of $(a^{\mathrm{Sat}})_{CC}$ are independent of the choice of V. This implies that the irreducible blocks of the matrices $t^1, \ldots, t^{\ell+1}$, and, a fortiori, their eigenvalues which occur in the first order asymptotics of Theorem 1, are independent of the choice of V.

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