A SPECTRAL THEOREM FOR CONVEX MONOTONE HOMOGENEOUS MAPS

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Abstract: We consider convex maps $f : \mathbb{R}^n \to \mathbb{R}^n$ that are monotone (i.e., that preserve the product ordering of $\mathbb{R}^n$), and non-expansive for the sup-norm. This includes convex monotone maps that are additively homogeneous (i.e., that commute with the addition of constants). We show that the fixed point set of f, when it is non-empty, is isomorphic to a convex inf-semilattice, whose dimension is at most equal to the number of strongly connected components of a critical graph defined from the tangent affine maps of f. This bound on the dimension is attained when f is piecewise affine. This yields in particular an uniqueness result for the bias vector of ergodic control problems. We also show that the length of periodic orbits of f is bounded by the cyclicity of its critical graph, which implies that the possible orbit lengths of f are exactly the orders of elements of the symmetric group on n letters.

1. INTRODUCTION AND STATEMENT OF RESULTS

We say that a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if for all $x, y \in \mathbb{R}^n$, $x \leq y \implies f(x) \leq f(y)$, where $\leq$ denotes the product ordering of $\mathbb{R}^n$ ($x \leq y$ if $x_i \leq y_i$, for all $1 \leq i \leq n$). We say that f is additively *homogeneous* if for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, $f(\lambda + x) = \lambda + f(x)$, where $\lambda + x = (\lambda + x_1, \dots, \lambda + x_n)$. It is easy to see that a monotone homogeneous map is non-expansive for the sup-norm: for all $x, y \in \mathbb{R}^n$, $|f(x) - f(y)| \leq |x - y|$, where $|x| = \max_{1 \leq i \leq n} |x_i|$ (see Crandall and Tartar (1980) for further properties).

Monotone homogeneous maps arise in optimal control and game theory, as dynamic programming operators. They arise, even more classically, in Perron-Frobenius theory, when considering maps F on the positive cone $(\mathbb{R}_+^*)^n$ (where $\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$), that are monotone and multiplicatively homogeneous (i.e. $F(\lambda x) = \lambda x$, for all $\lambda > 0$ and $x \in (\mathbb{R}_+^*)^n$). Both settings are equivalent, since the transformation $F \mapsto \log \circ F \circ \exp$ (where $\log : (\mathbb{R}_+^*)^n \to \mathbb{R}^n$ is the map which does log entrywise, and $\exp = \log^{-1}$) sends the set of monotone multiplicatively homogeneous maps to the set of monotone additively homogeneous maps. See for instance Bougerol (1995); Cohen *et al.* (1995); Dellacherie (1996); Gaubert and Gunawardena (1998*a*,*b*); Gunawardena (1998); Liverani and Wojtkowski (1994); Gunawardena (1999); Morishima (1964); Kolokoltsov (1992); Menon and Schneider (1969); Nussbaum (1988, 1989); Sabot (1997); Rosenberg and Sorin (1999); Vincent (1997); Wojtkowski (1985) for various studies and applications.

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A basic problem, for a monotone homogeneous map f, is the existence, and uniqueness (up to an additive constant), of the *additive eigenvectors* of f, which are the $v \in \mathbb{R}^n$ such that $f(v) = \lambda + v$, for some *additive eigenvalue* $\lambda \in \mathbb{R}$. In the sequel, we will omit the term "additive", when the additive nature of the objects will be clear from the context. When f has an eigenvector v with eigenvalue λ, by homogeneity of f, $f^k(v) = k\lambda + v$ holds for all $k \geq 0$, and by non-expansiveness of f, $|f^k(x) - k\lambda - v| = |f^k(x) - f^k(v)| \leq |x - v|$, hence,

$$f^k(x) = k\lambda + O(1) \qquad \text{when } k \to \infty, \qquad (1)$$

for all $x \in \mathbb{R}^n$ (all the orbits of f have a linear growth rate of λ). This implies in particular that the eigenvalue λ is unique. Hence, we can speak without ambiguity of the *eigenspace* of f, which is the set $\mathscr{E}(f) = \{x \in \mathbb{R}^n \mid f(x) = \lambda + x\}$.

Several Perron-Frobenius like theorems, guarantee the existence of eigenvectors of monotone (additively) homogeneous maps $\mathbb{R}^n \to \mathbb{R}^n$. Such results go back to Kreĭn and Rutman (1950), who considered more generally monotone multiplicatively homogeneous maps leaving a cone in a Banach space invariant, and to Morishima (1964), whose book contains a complete study of finite dimensional non-linear Perron-Frobenius theory. A modern overview appears in the memoirs of Nussbaum (1988, 1989), which contain in particular general existence results for eigenvectors, following Nussbaum (1986). Oshime (1984) introduced different conditions, which imply both the existence and the uniqueness of an eigenvector in the interior of the positive cone of $\mathbb{R}^n$. Recently, Gaubert and Gunawardena (1999, 2000) gave existence results which extend both Theorem 4.1 of Nussbaum (1986) and Theorem 8 of Oshime (1984). The geometry of the eigenspaces, and in particular, uniqueness conditions, seem less studied: see Oshime (1984), Nussbaum (1988), and Bruck (1973) for existing results.

Here, we shall describe the eigenspace of *convex* monotone homogeneous maps. (We say that a $\mathbb{R}^n$-valued map is convex when its coordinates are convex. We refer the reader to Rockafellar (1970) for all convexity notions used in the paper: subdifferentials, domain, Fenchel transform, etc.)

To state our main result, we need a few definitions. We first generalize the notion of subdifferential to maps $\mathbb{R}^n \to \mathbb{R}^n$ by setting, for $x \in \mathbb{R}^n$, $\partial f(x) = \{P \in \mathbb{R}^{n\times n} \mid f(y) - f(x) \geq P(y - x),\ \forall y \in \mathbb{R}^n\}$. It is easy to see that by monotonicity and homogeneity of f, the elements of $\partial f(x)$ are probability matrices.

To any $n \times n$ nonnegative matrix P, we associate the (directed) graph $\mathscr{G}(P)$ with nodes $1, \dots, n$ and arcs $i \to j$ when $P_{ij} \neq 0$. We say that i *has access* to j, and we write $i \xrightarrow{*} j$, if there is a directed path from i to j in $\mathscr{G}(P)$, or if $i = j$. We call *classes* of P (or of $\mathscr{G}(P)$) the equivalence classes for the equivalence relation "($i \xrightarrow{*} j$) and ($j \xrightarrow{*} i$)". A matrix is *irreducible* if it has only one class. A class C of P is *final* if no nodes of C have access to a node of the complement of C. If P is a stochastic matrix, C is final if, and only if, the $C \times C$ submatrix of P is stochastic.

If v is an eigenvector of f, we call *critical graph* of f, the (directed) graph $\mathscr{G}^{\mathrm{c}}(f)$ which is the union (nodes and arcs) of final classes of probability matrices $P \in \partial f(v)$. It can be checked, using the following discrete version of the maximum principle, that the graph $\mathscr{G}^{\mathrm{c}}(f)$ is independent of the choice of the eigenvector v.

Lemma 1. (Discrete Maximum Principle). If P is a $n \times n$ stochastic matrix, and if $Pv \leq v$, then, the restriction of v to any final class of P is a constant vector, and $Pv = v$ holds on the union of the final classes of P. Moreover, the minimum of v is attained on a final class.

We call *critical nodes* of f, the nodes of $\mathscr{G}^{\mathrm{c}}(f)$, and denote by $N^{\mathrm{c}}(f)$ the set of critical nodes. We call *critical classes* of f the strongly connected components $C_1, \dots, C_s$ of $\mathscr{G}^{\mathrm{c}}(f)$. We call *cyclicity* of the critical class C_i the gcd of the lengths of the circuits of C_i, and we call *cyclicity* of f the lcm of cyclicities of its critical classes. We say that a monotone homogeneous map $g : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^p$ is a monotone homogeneous *isomorphism* if it has a monotone homogeneous inverse. The goal of this short paper is to announce the following result which is proved in Akian and Gaubert (2001).

Theorem 2. (Convex Spectral Theorem). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ denote a convex monotone homogeneous map that has an eigenvector. Denote by $C = N^{\mathrm{c}}(f)$, the set of critical nodes of f, $c = \mathsf{c}(f)$ the cyclicity of f, and λ the unique eigenvalue of f. Then,

(1) the restriction $\mathsf{r} : \mathbb{R}^n \to \mathbb{R}^C, x \mapsto (x_i)_{i\in C}$, is a monotone homogeneous isomorphism from $\mathscr{E}(f)$ to its image $\mathscr{E}^{\mathrm{c}}(f)$;
(2) $\mathscr{E}^{\mathrm{c}}(f)$ is an inf-subsemilattice of $(\mathbb{R}^n, \leq)$;
(3) $\mathscr{E}^{\mathrm{c}}(f)$ is a convex set whose dimension is at most equal to the number of critical classes of f, and this bound is attained when f is piecewise affine;
(4) for all $x \in \mathbb{R}^n$, $f^{kc}(x) - kc\lambda$ has a limit when $k \to \infty$.

Since f is homogeneous, this is nothing but a theorem describing the fixed point set and the asymptotics of the iterates of the map $-\lambda + f$. The last assertion of the theorem originates from a more general theorem of Nussbaum (1990) and Sine (1990), which states that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive for the sup-norm and has a fixed point, then, for all $x \in \mathbb{R}^n$, $f^{kc}(x)$ converges when $k \to \infty$, for some minimal constant c which can be bounded by a function of n. When f is convex monotone and homogeneous, the last assertion of the convex spectral theorem shows that the possible values of c are exactly the orders of elements of the symmetric group on n letters. Equivalently, convex monotone homogeneous maps have the same orbit lengths as permutation matrices (see Nussbaum and Verduyn Lunel (1999) for a discussion on orbit lengths of various classes of nonexpansive maps).

Since piecewise affine convex monotone homogeneous maps are exactly dynamic programming operators associated to ergodic control problems with finite state and action spaces, the third assertion of the theorem gives in particular a necessary and sufficient condition for the uniqueness of the eigenvectors for such operators. In this context, an eigenvector is called a bias vector, and we are not aware of any uniqueness result of this kind in the Stochastic Control literature. (It would be interesting to extend the convex spectral theorem to infinite dimensional spectral problems, which arise for instance in ergodic control problems of diffusion processes or of denumerable Markov chains: we hope to address these issues in another paper. As a preliminary step, the first author, Sulem and Taksar (2001), inspired by the present result, proved a uniqueness result for viscosity solutions of a special ergodic Hamilton-Jacobi-Bellman equation.) The asymptotic result implied by the last assertion of our theorem seems also new. Additional assumptions on f are required to describe more precisely the behavior of f^k: for instance Schweitzer and Federgruen (1977) proved a geometric convergence result, in a special case which corresponds to piecewise affine maps f.

The convex spectral theorem extends the max-plus spectral theorem, a Perron-Frobenius like result which characterizes the eigenspaces of the subclass of convex monotone homogeneous map which are *linear* over the max-plus semiring. The first three assertions of the convex spectral theorem extend the characterization of Gondran and Minoux (1977) of the dimension of the eigenspace of a max-plus matrix as the "number of critical classes", and the last assertion extends the cyclicity theorem of Cohen, Dubois, Quadrat, and Viot (1983) (a partial account of this result also appeared in Baccelli *et al.* (1992)), which characterizes the asymptotic behavior of max-plus linear dynamics (Nussbaum (1991) proved a closely related result on periodic orbits). Initial versions of the max-plus spectral theorem appeared in Cuninghame-Green (1979) (which makes a synthesis of earlier works of the author), Romanovskiĭ (1967), and Vorobyev (1967). See the collection of articles Maslov and Samborskiĭ (1992), Kolokoltsov and Maslov (1997), and the references therein, for generalizations to infinite dimension.

It is instructive to note that the uniqueness of eigenvectors is governed by the same kind of graph properties as the existence of eigenvectors, albeit the graphs are different. For instance, a result of the second author and Gunawardena (1999, Theorem 2) guarantees the existence of an eigenvector for a monotone homogeneous map which has a strongly connected graph. Here, the *graph* $\mathscr{G}(f)$ of a monotone homogeneous map $f : \mathbb{R}^n \to \mathbb{R}^n$ has nodes $\{1, \dots, n\}$ and an arc $i \to j$ if $\lim_{\nu\to\infty} f_i(\nu e_j) = +\infty$, where e_j denotes the j-th vector of the canonical basis of $\mathbb{R}^n$. Another way to guarantee the existence of an eigenvector is to use the convex spectral theorem itself, thanks to the following observation due to the second author and Gunawardena. We denote by $\widehat{f}(x) = \lim_{\mu\to\infty} \mu^{-1} f(\mu x)$ the *recession* function of f ($\widehat{f}$ need not exist when f is monotone and homogeneous, but it does exist when f is convex). We have $\widehat{f}(0) = 0$, and when f is (additively) homogeneous, so does $\widehat{f}$, so that all points on the diagonal are (trivial) fixed points of f. It is proved in Gaubert and Gunawardena (2000) that if the recession function of a monotone homogeneous map f exists and has only fixed points on the diagonal, then, f has an eigenvector. Combining this observation with the convex spectral theorem, we obtain:

Corollary 3. A monotone homogeneous map has an eigenvector if its recession function exists, is convex, and has only one critical class.

If f is a convex monotone homogeneous map, it is not difficult to see that the recession function $\widehat{f}$ is exactly the support function of the domain of the Fenchel transform f^* of f, that is $\widehat{f}(x) = \sup_{P\in\mathrm{dom}\, f^*} Px$. In this formula, one can replace dom f^* by its closure cl(dom f^*), which is equal to $\partial\widehat{f}(0)$. The graph $\mathscr{G}(f)$ is the union of the graphs of $P \in \mathrm{dom}\, f^*$, or equivalently the union of the graphs of $P \in \mathrm{cl}(\mathrm{dom}\, f^*)$, whereas the critical graph $\mathscr{G}^{\mathrm{c}}(\widehat{f})$ is the union of the final graphs of $P \in \mathrm{cl}(\mathrm{dom}\, f^*)$. If $\mathscr{G}(f)$ is strongly connected, one can see that $\mathscr{G}^{\mathrm{c}}(\widehat{f})$ is also strongly connected, so that in the special case of convex monotone homogeneous maps, Corollary 3 is stronger than Theorem 2 of Gaubert and Gunawardena (1999) (which however holds in a more general context).

Finally, let us mention two immediate extensions of the convex spectral theorem. First, since the map $f \mapsto (x \mapsto -f(-x))$ sends convex monotone homogeneous maps to concave monotone homogeneous maps, there is of course a dual concave spectral theorem. Another, more interesting, extension, is obtained by considering *subhomogeneous* maps f, which satisfy $f(\lambda + x) \leq \lambda + f(x)$, for all $\lambda \geq 0$ and $x \in \mathbb{R}^n$. It is easy to see that a monotone map is subhomogeneous if, and only if, it is nonexpansive for the sup-norm. To a monotone subhomogeneous map $f : \mathbb{R}^n \to \mathbb{R}^n$, we associate canonically a monotone homogeneous map $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$g(x, y) = \begin{pmatrix} y + f(-y + x) \\ y \end{pmatrix} \quad , \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}$$

(this is a nonlinear extension of the classical way of passing from a submarkovian to a Markovian matrix, by adding a cemetery state, in this non-linear context, this construction is due to Gunawardena and Keane (1995)). A vector $z \in \mathbb{R}^n$ is a fixed point of f, if, and only if $(z, 0)$ is an eigenvector of g (and the eigenvalue is 0). Using this construction, one translates readily the convex spectral theorem to a theorem describing fixed point sets and the asymptotics of the iterates of convex monotone subhomogeneous maps. (We might also use this construction, with $-\lambda+f$ instead of f, to describe the eigenspace of f for an additive eigenvalue λ, but when f is only monotone and subhomogeneous, λ need not be unique, and it tells little about the asymptotics of f^k, in general.) For a convex monotone subhomogeneous map f with fixed point v, the critical graph $\mathscr{G}^c(f)$ of f is defined as the union (nodes and arcs) of the final classes C of matrices $P \in \partial f(v)$ such that the $C \times C$ submatrix of P is a probability matrix (when f is homogeneous, this property is automatically satisfied). Equivalently, $\mathscr{G}^c(f)$ (which can be empty) is the restriction of $\mathscr{G}^c(g)$ to $\{1, \dots, n\}$. The notions of critical classes and cyclicity are defined from $\mathscr{G}^c(f)$ as above. When $\mathscr{G}^c(f)$ is empty, we have $N^c(f) = \varnothing$, and we take the convention $\mathbb{R}^\varnothing = \{0\}$, and $\mathsf{c}(f) = 1$.

Corollary 4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ denote a convex monotone subhomogeneous map that has a fixed point. Then, all the conclusions of the convex spectral theorem apply to f and $\lambda = 0$. In particular, if f has no critical classes, then its fixed point is unique.

2. EXAMPLES

2.1 *A piecewise affine map*

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(x) = \begin{pmatrix} x_1 \vee \left(\frac{x_1+x_2}{2}\right) \vee (-3+x_3) \vee \left(\frac{x_1+x_3}{2}\right) \\ x_2 \vee \left(\frac{2x_1+x_2}{3}\right) \vee (-1+x_1) \\ x_3 \vee (-2+x_1) \end{pmatrix} ,$$

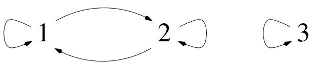
which is such that $f(0) = 0$. Let $\mathrm{co}(X)$ denote the convex hull of a set X. We have $\partial f(0) = \partial f_1(0) \times \partial f_2(0) \times \partial f_3(0)$ with

$$\partial f_1(0) = \mathrm{co}\{(1,0,0), (1/2,1/2,0), (1/2,0,1/2)\} ,$$
$$\partial f_2(0) = \mathrm{co}\{(0,1,0), (2/3,1/3,0)\} ,$$
$$\partial f_3(0) = \{(0,0,1)\} .$$

Since the identity matrix I belongs to $\partial f(0)$, and $\{1\}, \{2\}, \{3\}$ are the final classes of I, the critical graph $\mathscr{G}^c(f)$ of f contains the nodes $1, 2, 3$ and the loops $1 \to 1, 2 \to 2, 3 \to 3$. Moreover,

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 2/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \partial f(0) ,$$

which has $\{1, 2\}$ and $\{3\}$ as final classes. Hence, the complete graph with set of nodes $\{1, 2\}$ is included in $\mathscr{G}^c(f)$. It is easy to see that there are no other arcs in $\mathscr{G}^c(f)$. Thus, the critical graph of f is



and its strongly connected components are $\{1, 2\}$ and $\{3\}$. These are the critical classes of f.

Let us illustrate the fact that $\mathscr{G}^c(f)$, which is defined from $\partial f(v)$, is independent of the choice of the eigenvector v, by considering the eigenvector $v = (0, 0, -2)$. By comparison with $\partial f(0)$, only the first and third rows of $\partial f(v)$ are changed, namely

$$\partial f_1(v) = \mathrm{co}\{(1,0,0), (1/2,1/2,0)\} ,$$
$$\partial f_3(v) = \mathrm{co}\{(0,0,1), (1,0,0)\} .$$

Although $\partial f(v) \neq \partial f(0)$, the final classes of the matrices $P \in \partial f(v)$ are the same as the ones of the matrices $P \in \partial f(0)$.

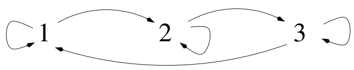
In this example, the set of critical nodes is $C = \{1, 2, 3\}$, the number of critical classes is 2, the cyclicity of f is 1 and f is piecewise affine. The convex spectral theorem says in particular that the eigenspace $\mathscr{E}(f) = \mathscr{E}^c(f)$ is a convex subset of $\mathbb{R}^3$ with dimension 2, and that for all $x \in \mathbb{R}^3$, $f^k(x)$ has a limit when $k \to \infty$. Indeed, an elementary computation shows that $\mathscr{E}(f) = \{x \in \mathbb{R}^3 \mid f(x) = x\} = \{x \in \mathbb{R}^3 \mid x_1 = x_2 \geq x_3 \geq -2 + x_1\}$.

2.2 *A uniqueness case*

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(x) = \begin{pmatrix} \frac{1}{2}(x_1 + x_2) \\ \log(e^{x_2} + 8e^{x_3}) \\ x_1 \vee (-1 + x_3) \end{pmatrix} .$$

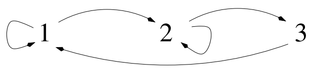
Since the graph $\mathscr{G}(f)$ of f, which is equal to



is strongly connected, we known, thanks to (Gaubert and Gunawardena, 1999, Theorem 2), that f has an eigenvector. Indeed, $f(v) = \lambda + v$, where $\lambda = \log 2$, and $v = (\log 2, 3\log 2, 0)$. We have

$$\partial f(v) = \{f'(v)\} \text{ where } f'(v) = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix} .$$

Thus, the critical graph $\mathscr{G}^c(f) = \mathscr{G}(f'(v))$ is



(only the arc $3 \to 3$ vanished, by comparison with $\mathscr{G}(f)$). Since $\mathscr{G}^c(f)$ is strongly connected, f has a unique critical class (namely $\{1, 2, 3\}$), and, by the convex spectral theorem, the eigenvector of f is unique (up to an additive constant). Equivalently, the map $F : (\mathbb{R}_+^*)^3 \to (\mathbb{R}_+^*)^3$, $F = \exp \circ f \circ \log$,

$$F(x) = \begin{pmatrix} \sqrt{x_1 x_2} \\ x_2 + 8x_3 \\ x_1 \vee (x_3/e) \end{pmatrix} ,$$

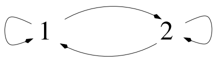
has a unique (multiplicative) eigenvector, up to a (multiplicative) constant, namely, $(2, 8, 1)$.

2.3 *Regular and singular cases*

Let h denote any convex map $\mathbb{R} \to \mathbb{R}$ whose subgradients are between 0 and 1. The map $f : \mathbb{R}^2 \to \mathbb{R}^2$,

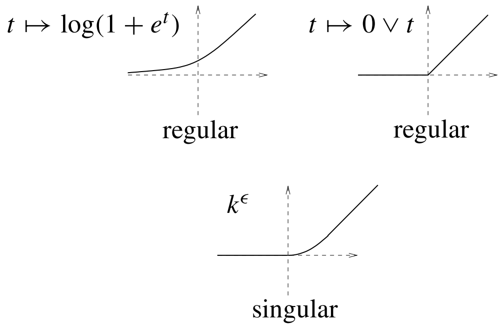
$$f(x) = \begin{pmatrix} x_1 + h(x_2 - x_1) \\ x_2 + h(x_1 - x_2) \end{pmatrix} , \qquad (2)$$

is monotone, homogeneous, convex, and $f(0) = h(0) + 0$, which means that 0 is an eigenvector of f for the eigenvalue $h(0)$. If $(h(t) = h(-t) \implies t = 0)$, then 0 is the unique eigenvector of f, up to an additive constant. This condition is satisfied by the maps $t \mapsto \log(1 + e^t)$ and $t \mapsto 0 \vee t$. It is also satisfied by the Moreau-Yoshida regularization, h_ϵ, of $t \mapsto 0 \vee t$, which, for all $\epsilon > 0$, is defined by $h_\epsilon(t) = 0$ for $t \leq 0$, $h_\epsilon(t) = t^2/2\epsilon$ for $0 \leq t \leq \epsilon$, and $h_\epsilon(t) = t - \epsilon/2$ for $t \geq \epsilon$. Let g, k, and k^ϵ denote the maps f obtained by replacing h by $t \mapsto \log(1 + e^t)$, $t \mapsto 0 \vee t$, and h_ϵ in (2), respectively. We have $\partial g_1(0) = \{(g_1)'(0)\} = \{(1/2, 1/2)\}$, $\partial k_1(0) = \mathrm{co}\{(1,0), (0,1)\}$, and $\partial k_1^\epsilon(0) = \{(k_1^\epsilon)'(0)\} = \{(1,0)\}$. Using the symmetry between the first and the second coordinate of f, we get the following critical graphs

$\mathscr{G}^c(g) = \mathscr{G}^c(k) =$  ,

$\mathscr{G}^c(k^\epsilon) =$  .

We denote by $\mathsf{m}(f)$ the number of critical classes of f. We have $\dim \mathscr{E}(g) = \mathsf{m}(g) = 1$, $\dim \mathscr{E}(k) = \mathsf{m}(k) = 1$, but $\dim \mathscr{E}(k^\epsilon) = 1 < \mathsf{m}(k^\epsilon) = 2$. This discrepancy should be intuitively clear by looking at the following graphs:



In the case of $f = k^\epsilon$, the convex spectral theorem fails to characterize the dimension of $\mathscr{E}(f)$ because the subdifferential of f does not give enough information on the local behavior of f near the eigenspace. In such cases, one needs to consider terms of higher order in the local expansion of f to establish the uniqueness of the eigenvector.

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