Rational Series over Dioids and Discrete Event Systems

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Introduction

We survey the different kinds of rational series which arise in the study of Discrete Event Systems (DES) and of certain related Markov Decision Processes. The use of rational series over fields is classical, e.g. as transfer functions, generating series of finite Markov chains, skew Ore series in Difference and Differential Algebra, commutative multivariable series for linear PDE with constant coefficients, Fliess' noncommutative generating series for bilinear systems. It turns out that all these more or less familiar classes of series admit useful counterparts for DES, when the scalars belong to some *dioids* [5] such as the $(\max, +)$ semiring. The main interest of this series theoretical point of view consists in introducing some efficient algebraic techniques in the study of these dynamical systems. Since this paper is obviously too short for such a program, we have chosen to propose an introductive guided tour. A more detailed exposition will be found in our references and in a more complete paper to appear elsewhere.

1 Rational Series in a Single Indeterminate

1.1 Rational Series as Transfers of $(\max, +)$-Linear Systems

We consider systems of recurrent linear equations in the $(\max, +)$ algebra of the form

$$x(n) = Ax(n-1) \oplus Bu(n), \quad y(n) = Cx(n) \ , \tag{1}$$

where[1] $x(n) \in \mathbb{R}^p_{\max}$, $u(n) \in \mathbb{R}_{\max}$, $y(n) \in \mathbb{R}_{\max}$. As it is well known [2, 5], this class of systems encompasses in particular the dater equations of SISO Timed Event Graphs (TEG). A straightforward argument shows that the least solution[2] of (1) is given by

$$y(n) = \bigoplus_{k \in \mathbb{N}} CA^k Bu(n-k) \ , \tag{2}$$

[1] $\mathbb{R}_{\max}$ denotes the "(max,+) semiring" $(\mathbb{R} \cup \{-\infty\}, \max, +)$. The reader is referred to [5] for the general notation about dioids. In particular, $\oplus$ and $\otimes$ denote the sum and the product, ε denotes the zero and e the unit (in $\mathbb{R}_{\max}$, $a \oplus b = \max(a, b)$, $a \otimes b = a + b$, $\varepsilon = -\infty$, $e = 0$).
[2] which corresponds for a TEG to the earliest behavior [5].

hence, the input-output behavior of the system is completely determined by the following *transfer series*[3]

$$H = \bigoplus_{k \in \mathbb{N}} CA^k B X^k = C(AX)^* B \in \mathbb{R}_{\max}[[X]] \ , \tag{3}$$

where[4] $M^* \stackrel{\text{def}}{=} \bigoplus_{k \in \mathbb{N}} M^k$. A series is *realizable* if it admits a representation of the form (3) for some finite dimensional triple (A, B, C). The celebrated Kleene-Schützenberger Theorem [3] states that *realizable* series coincide with *rational* series defined as follows.

Definition 1. Let $\mathcal{S}$ denote a dioid. The dioid of *rational series* over $\mathcal{S}$ in the indeterminate X is the least subset of $\mathcal{S}[[X]]$ containing the polynomials and stable by the operations $\oplus, \otimes$ and $*$.

Thus, *the transfer series* $H = C(AX)^* B$ *of a stationary finite dimensional* $(\max, +)$ *linear system is rational*.

1.2 Generating Series of Bellman Chains

Let us consider a Markovian maximization problem with finite state $S = \{1, \ldots, n\}$, finite horizon k, final reward b and transition reward $i \stackrel{A_{ij}}{\to} j$. The value function for the initial position $i \in S$ is thus given by

$$v_i^{(k)} \stackrel{\text{def}}{=} \max_{i_1 \ldots i_k} [A_{ii_1} + \ldots + A_{i_{k-1} i_k} + b(i_k)] \ . \tag{4}$$

This is a particular case of Bellman chain (Akian, Quadrat and Viot [1]). As it is well known and obvious from (4), the value function is given by a product of matrices in the $(\max, +)$ algebra, that is, $v^{(k)} = A^k b$. Now, let us consider the vector of *generating series*

$$V_i \stackrel{\text{def}}{=} \bigoplus_k v_i^{(k)} X^k = \bigoplus_k (A^k b)_i X^k \in \mathbb{R}_{\max}[[X]] \ . \tag{5}$$

An immediate comparison with (3) shows that *the generating series of an homogeneous Bellman chain with finite state are rational*.

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[3] Given a semiring $\mathcal{S}$, $\mathcal{S}[[X]]$ denotes the semiring of formal series $H = \bigoplus_k H_k X^k$, equipped with componentwise sum and Cauchy product. $\mathcal{S}[X]$ denotes the subdioid of polynomials. We shall sometimes write $(H|X^k)$ instead of H_k, in line with the scalar product notation $(H|H') \stackrel{\text{def}}{=} \bigoplus_k H_k H'_k$.

[4] For a in a dioid $\mathcal{D}$, a^* is to be interpreted as the least upper bound of the set $\{a^0, a^1, a^2, \ldots\}$ with respect to the natural order $a \leq b \iff a \oplus b = b$. Here, $\mathcal{D} = (\mathcal{S}[[X]])^{n \times n}$ and the convergence of $(AX)^*$ is immediate, due to the fact that AX has no constant coefficient. In the scalar case ($a \in \mathcal{D} = \mathbb{R}_{\max}[[X]]$), a^* is well defined iff $a_0 = (a|X^0) \leq e$. See [12] for a more precise discussion.

1.3 Representation Theorems for Rational Series over Commutative Dioids

Definition 2 (Simple Series). A series $s \in \mathcal{S}[[X]]$ is *simple* if it writes $s = aX^k(bX^p)^*$ with $a, b \in \mathcal{S}$, $k \in \mathbb{N}$, $p \in \mathbb{N}\backslash\{0\}$.

Theorem 3. *Let $\mathcal{S}$ be a commutative dioid. A series $s \in \mathcal{S}[[X]]$ is rational iff it is a sum of simple series.*

The proof consists in reducing an arbitrary rational expression to a sum of simple series via the following classical rational identities[5]

$$\begin{array}{rr} (C) & (a \oplus b)^* = a^*b^* \\ (SC) & (ab^*)^* = e \oplus aa^*b^* \\ n \geq 1,\ (P(n)) & a^* = (e \oplus a \oplus \ldots \oplus a^{n-1})(a^n)^* \quad . \end{array}$$

Definition 4 (Weak and Strong Stabilization). The dioid $\mathcal{S}$ satisfies the weak stabilization condition[6] if

$$\forall a, b, \lambda, \mu \in \mathcal{S}, \quad \exists c, \nu \in \mathcal{S},\ \exists K \in \mathbb{N},\ \forall k \geq K, \qquad a\lambda^k \oplus b\mu^k = c\nu^k \quad . \tag{6}$$

Strong stabilization holds if the value $\nu = \lambda \oplus \mu$ is allowed (whenever $a, b \neq \varepsilon$).

We also introduce the following central notion which already appears in the theory of nonnegative rational series [3].

Definition 5 (Merge of Series). The merge of the series $s^{(0)}$,..., $s^{(c-1)}$ is the series t such that $\forall k \in \mathbb{N}, \forall i \in \{0, \ldots, c-1\}, t_{i+kc} = s_k^{(i)}$.

For instance, the series $(X^2)^* \oplus 1X(1X^2)^* = 0 \oplus 1X \oplus 0X^2 \oplus 2X^2 \oplus 0X^3 \oplus 3X^4 \oplus \ldots$ is the merge of the series $X^* = 0 \oplus 0X \oplus 0X^2 \oplus \ldots$ and $1(1X)^* = 1 \oplus 2X \oplus 3X^2 \oplus \ldots$

Definition 6 (Ultimately Geometric Series). The series s is ultimately geometric if $\exists K \in \mathbb{N}, \exists \alpha \in \mathcal{S}, \forall k \geq K, s_{k+1} = \alpha s_k$.

We have the following central characterization first noted by Moller [19] for $\mathcal{S} = \mathbb{R}_{\max}$.

Theorem 7 [12]. *Let $\mathcal{S}$ be a commutative dioid satisfying the weak stabilization condition. Then, a series is rational iff it is a merge of ultimately geometric series.*

As a corollary, we obtain a version of the classical periodicity theorem [2, 6].

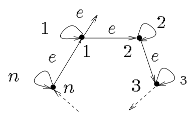
Corollary 8 (Cyclicity). *Assume that $\mathcal{S}$ is commutative without divisors of zero and satisfies the strong stabilization condition. Let $A \in \mathcal{S}^{n \times n}$ be an irreducible matrix. Then, $\exists \alpha \in \mathcal{S}$ and $k \geq 0, c \geq 1$ such that $A^{k+c} = \alpha A^k$.*

The proof [12] consists in showing that all the merged series which appear in $(AX)^*_{ij}$ have the same asymptotic rate – independent of ij – due to the irreducibility.

Example 1 (Computing the Value Function of a Bellman Chain). Let us consider the Bellman Chain (4), whose transition rewards are described on Fig. 1. For instance, the

[5] For a more complete study of rational identities, see Bonnier and Krob [16, 14]. The two labels (C) and (SC) recall that these identities are specific to commutative rational series.

[6] The strong stabilization is borrowed to Dudnikov and Samborkskiĭ [6, 18]. See [12] for a comparison of these properties.



$$V_1 = (1X)V_1 \oplus XV_2 \oplus e$$
$$V_2 = (2X)V_2 \oplus XV_3$$
$$\dots$$
$$V_n = (nX)V_n \oplus XV_1 \ .$$

Fig. 1. A Simple Bellman Chain with its Generating Equations

arc $n \xrightarrow{e} 1$ means that $A_{n1} = e$. We take as final reward a Dirac at node 1: $b_i = e$ if $i = 1$, $b_i = \varepsilon$ otherwise. Let v be the value function defined by (4) and consider the *generating series* V given by (5). V can be computed by performing a Gaussian elimination[7] on the system $V = AXV \oplus b$ displayed on Fig. 1. After some computations making an intensive use of the rational identities (C),(SC), we get eventually $V_1 = (1X)^* \oplus X^n(nX)^*$. Taking the coefficient of V_1 at X^p, we obtain the value function with initial state 1 and horizon p

$$v_1^{(p)} = \max(p, n(p-n)) \ .$$

2 Skew Rational Series

Let σ be an endomorphism of the semiring $\mathcal{S}$. The semiring of *skew series* over $\mathcal{S}$, denoted by $\mathcal{S}[[X;\sigma]]$ is by definition the set of formal sums $s = \bigoplus_{n\in\mathbb{N}} s_n X^n$ equipped with the usual componentwise sum and the skew product:

$$(s \otimes t)_n \stackrel{\text{def}}{=} \bigoplus_{p+q=n} s_p \otimes \sigma^p(t_q) \ .$$

This noncommutative product arises from the rule $Xa = \sigma(a)X$, $\forall a \in \mathcal{S}$.

2.1 Skew Series and Discounted Bellman Chains

Let $A \in \mathbb{R}_{\max}^{n\times n}$ and let us consider the following Markovian discounted optimization problem (discounted "Bellman chain") with initial state i, final state j and horizon k:

$$(A^{[k]})_{ij} \stackrel{\text{def}}{=} \max_{i_1,\dots,i_{k-1}} \left(A_{ii_1} + \beta \times A_{i_1 i_2} + \dots + \beta^{k-1} \times A_{i_{k-1}j}\right) \ , \qquad (7)$$

where $\beta \in]0,1]$ denotes the discount rate. We have the following Hamilton-Jacobi-Bellman equation

$$(A^{[k+1]})_{ij} = \max_q \left(A_{iq} + \beta \times (A^{[k]})_{qj}\right) \ . \qquad (8)$$

After introducing the automorphism of $\mathbb{R}_{\max}$ and of $\mathbb{R}_{\max}^{n\times n}$:

$$x \in \mathbb{R}_{\max},\ \sigma(x) \stackrel{\text{def}}{=} \beta \times x \ , \qquad A \in \mathbb{R}_{\max}^{n\times n},\ \sigma(A)_{ij} \stackrel{\text{def}}{=} \sigma(A_{ij}) \ ,$$

we rewrite (8) as $A^{[k+1]} = A \otimes \sigma(A^{[k]})$, hence

$$A^{[k]} = A \otimes \sigma(A) \otimes \dots \otimes \sigma^{k-1}(A) \ . \qquad (9)$$

Since $A^{[k]} = ((AX)^*|X^k)$, the asymptotic study of the sequence of "skew powers" $A^{[k]}$ reduces to the evaluation of the matrix $(AX)^*$ whose entries are skew rational series.

[7] E.g. V_2 can be eliminated by noting that $V_2 = (2X)V_2 \oplus XV_3$ is equivalent to $V_2 = (2X)^*XV_3$.

2.2 Theorems of Representation for $(\max, +)$ Rational Skew Series

We again consider *simple* series which write $s = (aX^p)^* bX^n$ or equivalently $s = bX^n(fX^p)^*$ with $a, b, f \in \mathbb{R}_{\max}$, $p \geq 1, n \geq 0$.

Theorem 9. *A skew series $s \in \mathbb{R}_{\max}[[X;\sigma]]$ is rational iff it is a sum of simple series.*

This theorem is rather surprising because rational series usually cannot be expressed with a single level of star in noncommutative structures. The proof uses the machinery of noncommutative rational identities, such as

$$(S) \qquad (a \oplus b)^* = a^*(ba^*)^*$$

together with a few specific "commutative" identities, in particular

$$\forall a, b \in \mathbb{R}_{\max}, \quad p \geq 1, \qquad ((a \oplus b)X^p)^* = (aX^p)^*(bX^p)^* \ . \tag{10}$$

Definition 10. The series $s \in \mathbb{R}_{\max}[[X;\sigma]]$ is *ultimately skew geometric* iff

$$\exists K \in \mathbb{N}, \alpha \in \mathbb{R}_{\max} \quad k \geq K \Rightarrow s_{k+1} = \alpha\sigma(s_k) \qquad (= \alpha + \beta \times s_k) \ . \tag{11}$$

Theorem 11. *A series $s \in \mathbb{R}_{\max}[[X;\sigma]]$ is rational iff it is a merge of ultimately skew geometric series.*

We obtain as a corollary the following remarkable periodicity theorem first proved by Braker and Resing [4] (for primitive matrices).

Theorem 12 (Skew Cyclicity). *For an irreducible matrix $A \in \mathbb{R}_{\max}^{n\times n}$, $\exists c \geq 1$, $\forall ij$, $\exists \alpha_{ij}$ such that $A_{ij}^{[k+c]} = \alpha_{ij}\sigma^c(A_{ij}^{[k]})$ for k large enough.*

Example 2 (Value Function of a Discounted Bellman Chain). Let us compute the value function v for the discounted version of the Bellman Chain described in Fig. 1. Using the identity (10), it is not too difficult to obtain $V_1 = (1X \oplus (\alpha X)^* X^n)^*$ with $\alpha = \bigoplus_{1\leq i\leq n-1} \sigma^i(i+1)$. An application of the identity (S) gives $V_1 = (1X)^* \oplus (\alpha X)^* X^n$, hence $v_1^{(p)} = 1^{[p]} \oplus \alpha^{[p-n]}$, which rewrites:

$$v_1^{(p)} = \max\left(\frac{1-\beta^p}{1-\beta}, \alpha\frac{1-\beta^{p-n}}{1-\beta}\right) \ \text{ with } \ \alpha = \max_{1\leq i\leq n-1}(\beta^i(i+1)) \ .$$

3 Rational Series in Several Commuting Indeterminates

3.1 Timed Event Graphs with Unknown Resources and Holding Times

As shown in [2, 5], the algebraic modelization of Timed Event Graphs uses the two operators[8]

$$\delta : u \in \mathbb{R}_{\max}^{\mathbb{Z}} \mapsto y \in \mathbb{R}_{\max}^{\mathbb{Z}},\ y(k) = 1 + u(k)$$
$$\gamma : u \in \mathbb{R}_{\max}^{\mathbb{Z}} \mapsto y \in \mathbb{R}_{\max}^{\mathbb{Z}},\ y(k) = u(k-1) \ .$$

[8] The signals u and y are *dater functions* [5, §4.1], δ and γ are the shifts in dating and in counting.

When some task has an unknown duration τ_1, we get an equation of the form $y(k) = \tau_1 + u(k)$, which suggests to introduce a new operator δ_1 : $\delta_1 u(k) \stackrel{\text{def}}{=} \tau_1 + u(k)$. In the same vein, an unknown initial marking q_1 (say an unknown number of parts, of machines) is represented by a new operator γ_1 : $\gamma_1 u(k) = u(k - q_1)$. Then, the dater functions of a TEG with unknown resources and holding times satisfy the following polynomial equations[9]:

$$x = Ax \oplus Bu,\ y = Cx,\quad A \in (\mathbb{B}[\gamma_i, \delta_i])^{n\times n}, B \in (\mathbb{B}[\gamma_i, \delta_i])^{n\times p}, C \in (\mathbb{B}[\gamma_i, \delta_i])^{r\times n}$$

(with essentially as many δ_i as unknown holding times and as many γ_i as unknown markings). The transfer $H = CA^*B$ is a rational series of $\mathbb{B}[[\gamma_i, \delta_i]]$.

Example 3 (Transfer of a Machine with Two Part Types). Consider a single machine producing 2 different parts with processing times t_1, t_2. Under a cyclic scheduling, we obtain the TEG shown on Fig. 2,(a). The input u_i represents the arrivals of row

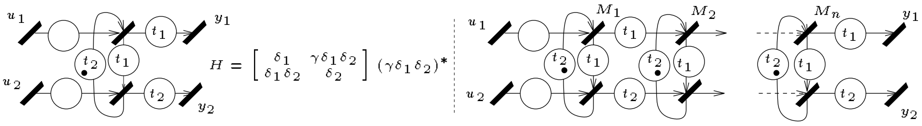


Fig. 2. (a): A Single Machine Producing 2 Types of Parts. (b): A Cascade of n Machines.

materials for part i ($u_i(k)$ = date of k-th arrival), and the output y_i represents the dates of production of the same part. For instance, the expression[10] $H_{11} = \delta_1(\gamma\delta_1\delta_2)^*$ shows that for the autonomous regime, one part of type 1 exits every $t_1 + t_2$ units of time (after time t_1).

3.2 Some Algebraic Results

Let $\mathcal{S}$ denote a (commutative) dioid. It follows from the commutative rational identities (C),(SC) that a series $s \in \mathcal{S}[[X_1, \ldots, X_n]]$ is rational iff it can be written $s = \bigoplus_{i=1}^{k} u_i v_i^*$ where u_i, v_i are polynomials. However, we have a much more precise result inspired by the theory of rational subsets of $\mathbb{N}^k$ [8].

Definition 13. A *simple* series s writes $s = aX_1^{\beta_1} \ldots X_k^{\beta_k}(\bigoplus_{i=1}^{r} a_i X_1^{\alpha_{i1}} \ldots X_k^{\alpha_{ik}})^*$, where the vectors $(\alpha_{1,i})_{1\le i\le k}, \ldots, (\alpha_{r,i})_{1\le i\le k}$ form a free family of $\mathbb{N}^k$.

Theorem 14. *Assume that $\mathcal{S}$ is totally ordered. Then, a series $s \in \mathcal{S}[[X_1, \ldots, X_m]]$ is rational iff it is a sum of simple series.*

The proof uses essentially the following kind of rational identity:

$$(aX)^*(bY)^*(cXY)^* = (aX)^*(bY)^* \oplus (aX)^*(cXY)^* \oplus (bY)^*(cXY)^* \ . \tag{12}$$

[9] $\mathbb{B} = \{\varepsilon, e\}$ denotes the boolean semiring.

[10] For the autonomous regime starting at time 0 (e.g. if an infinite quantity of inputs become available at time 0, see [2, §5.4.4.1]), the monomial transfer $\gamma^n \bigotimes_i \delta_i^{k_i}$ can be interpreted as "the event n occurs at the earliest at time $\sum_i k_i \times t_i$".

Example 4 (Transfer of a Flowshop with n Identical Machines). Let us consider a cascade of n identical machines of type shown on Fig. 2,(b). The transfer matrix of the flowshop is equal to H^n. An easy induction gives

$$H^n = \begin{bmatrix} \delta_1^n \oplus \gamma\delta_1^2\delta_2^2(\delta_1 \oplus \delta_2)^{n-2} & \gamma\delta_1\delta_2(\delta_1 \oplus \delta_2)^{n-1} \\ \delta_1\delta_2(\delta_1 \oplus \delta_2)^{n-1} & \delta_2^n \oplus \gamma\delta_1^2\delta_2^2(\delta_1 \oplus \delta_2)^{n-2} \end{bmatrix} (\gamma\delta_1\delta_2)^*$$

(modulo the licit additional simplification rules $\delta_i^s \oplus \delta_i^t = \delta_i^{\max(s,t)}$). This expression specifies the input/output behavior in function of the unknown processing times t_1, t_2. In particular, the term[10] $\delta_1^n \oplus \gamma\delta_1^2\delta_2^2(\delta_1 \oplus \delta_2)^{n-2}$ in the expression of H_{11}^n means that, for the autonomous regime, the part of type 1 numbered 0 has a transfer time of $n \times t_1$, while the part of the same type numbered 1 has a transfer time of $2(t_2 + t_1) + (n-2)\max(t_1, t_2)$.

Example 5 (Heaviside Calculus for some Special Variational Inequalities). Let us search for the least solution u, v of the system[11]:

$$0 \geq \lambda - \frac{\partial v}{\partial x} - \frac{\partial v}{\partial t},\ 0 \geq \max(\alpha - \frac{\partial u}{\partial x}, \beta - \frac{\partial u}{\partial t}),\ u \geq \max(\gamma + v, f),\ v \geq \mu + u \ . \quad (13)$$

A discretization gives with the $\mathbb{R}_{\max}$ notation:

$$\begin{aligned} u_h(x,t) &\geq \alpha^h u_h(x-h,t) \oplus \beta^h u_h(x,t-h) \oplus \gamma v_h(x,t) \oplus f(x,t) \\ v_h(x,t) &\geq \lambda^h v_h(x-h,t-h) \oplus \mu u_h(x,t) \ . \end{aligned} \quad (14)$$

Introducing the space and time shifts $Xu(x,t) \stackrel{\text{def}}{=} u(x-h,t)$ and $Tu(x,t) \stackrel{\text{def}}{=} u(x,t-h)$, we get

$$\begin{aligned} u_h &\geq (\alpha^h X \oplus \beta^h T)u_h \oplus \gamma v_h \oplus f \\ v_h &\geq \lambda^h XT v_h \oplus \mu u_h \ . \end{aligned} \quad (15)$$

A Gaussian elimination gives the least solution $u = (\alpha^h X \oplus \beta^h T \oplus (\lambda^h XT)^* \gamma\mu)^* f$. The convergence of this star in $\mathbb{R}_{\max}[[X,T]]$ yields the compatibility condition $\gamma\mu \leq e$. After some computation involving the rational identities (C), (SC), (12), we obtain

$$u = \left\{ (\alpha^h X)^*(\beta^h T)^* \oplus \gamma\mu(\alpha^h X)^*(\lambda^h XT)^* \oplus \gamma\mu(\beta^h T)^*(\lambda^h XT)^* \right\} f \ .$$

We have for the second term of this sum:

$$\begin{aligned} (\alpha^h X)^*(\lambda^h XT)^* f(x,t) &= \sup_{p,n \geq 0}[\alpha hp + \lambda hn + f(x - hp - hn, t - hn)] \\ &= \sup_{q \geq n \geq 0}[\alpha h(q-n) + \lambda hn + f(x - hq, t - hn)] \ . \end{aligned}$$

After an analogous argument for the two other terms, we introduce the three kernels

$$k_1(\xi,\tau) = \alpha\xi + \beta\tau,\ k_2(\xi,\tau) = \begin{cases} \alpha(\xi - \tau) + \lambda\tau + \gamma + \mu & \text{if } \xi \geq \tau \\ -\infty & \text{otherwise,} \end{cases}$$

$$k_3(\xi,\tau) = \begin{cases} \beta(\tau - \xi) + \lambda\xi + \gamma + \mu & \text{if } \tau \geq \xi \\ -\infty & \text{otherwise.} \end{cases}$$

Then, letting $h \to 0$, we get the explicit solution

$$u(x,t) = \sup_{\xi,\tau \geq 0} [\max(k_1, k_2, k_3)(\xi,\tau) + f(x - \xi, t - \tau)] \ .$$

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[11] u, v, f are maps $\mathbb{R}^2 \to \mathbb{R}$. $\alpha, \beta, \lambda, \gamma, \mu$ are constant. We do not address the regularity issues here.

4 Rational Series in Non Commuting Indeterminates

4.1 Definition and Basic Examples

Let Σ^* denote the free monoid over a finite alphabet Σ. The series $y = \bigoplus_{w\in\Sigma^*}(y|w)w$ (in $\mathbb{R}_{\max}\langle\langle\Sigma\rangle\rangle$) is *recognizable* iff there exists $\alpha \in \mathbb{R}_{\max}^{1\times n}, \beta \in \mathbb{R}_{\max}^{n\times 1}$, and a morphism $\mu : \Sigma^* \to \mathbb{R}_{\max}^{n\times n}$ such that $(y|w) = \alpha\mu(w)\beta$. The general version of the Kleene-Schützenberger theorem states that recognizable and rational series coincide.

Example 6 (Deterministic Cost). Let (Q, q_0, δ) denote a finite deterministic automaton, $\alpha : Q \to \mathbb{R}_{\max}$ a final cost and $\sigma : Q \times \Sigma \to \mathbb{R}_{\max}$ a transition cost. Let $w = w_k \dots w_1 \in \Sigma^k$ denote a sequence of decisions (w_i denotes the i-th letter of w, read read from right to left). We consider the cost

$$(c|w) \stackrel{\text{def}}{=} \sum_{n=1}^{k} \sigma(q_{n-1}, w_n) + \alpha(q_k), \quad \text{subject to } q_n = \delta(q_{n-1}, w_n) \tag{16}$$

(by convention, $(c|w) = \varepsilon$ if $\delta(q_0, w)$ is undefined). This is the discrete counterpart of the usual integral cost $\int_0^T \sigma(x(t), u(t))dt + \alpha(x(T))$ for the system $\dot{x} = f(x, u)$. The series $\bigoplus_w (c|w)w$ is recognizable (take β = Dirac at q_0 and $\forall a \in \Sigma, \forall p, q \in Q,\ \mu(a)_{pq} = \sigma(q, a)$ if $\delta(q, a) = p$ and ε otherwise).

Example 7 (A Workshop with Different Schedules). We consider a workshop with 2 machines M_1, M_2 and 2 different regimes of production corresponding to the processing of 2 differents parts (a) and (b), as represented[12] by the TEGs (a) and (b) displayed on Fig. 3. We assume that the workshop can switch from a regime to the other, according to

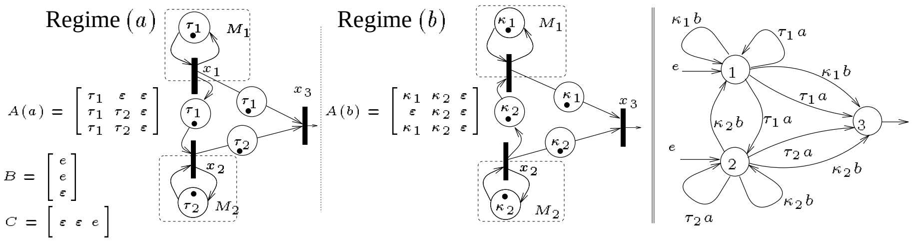

Fig. 3. A Workshop with Two Production Regimes and its Equivalent $(\max, +)$ Automaton

an open loop schedule described by a word $w \in \{a, b\}^*$. E.g. the word baa means that the two machines first follow two tasks with times and precedence constraints determined by the TEG (a) and after that one task described by the TEG (b). The behavior of the workshop under the schedule $w = w_k \dots w_1$ is determined by the system of non stationary $(\max, +)$ linear equations

$$x_w(0) = B,\ x_w(n) = A(w_n)x_w(n-1),\ y_w(n) = Cx_w(n)\ , \quad 1 \le n \le k\ , \tag{17}$$

[12] The interpretation of regime (b) is the following. A part is processed by M_2 during κ_2 time units. Then it is sent to M_1 (processing time of κ_1). For simplicity, we have neglected the transportation times. A dual interpretation can be provided for regime (a). The initial condition $B = [e, e, \varepsilon]^T$ require M_1 and M_2 to begin to work at time $e = 0$.

where $A(a), B, C$ and $A(b), B, C$ stand for the linear representations of the TEGs (a) and (b) displayed on Fig. 3. The output $y_w(n)$ represents the date of completion of the latest task of the sequence $w_{n-1}\ldots w_1$. The map $w \mapsto (y|w) \stackrel{\text{def}}{=} y_w(k)$ is recognizable (take $\alpha = C, \beta = B, \mu(a) = A(a), \mu(b) = A(b)$). The linear representation α, μ, β can be visualized by the automaton with multiplicities[13] in the $(\max, +)$ semiring displayed on Fig. 3.

4.2 Asymptotic Behavior of $(\max, +)$ Automata

The asymptotic analysis of a recognizable series y consists in estimating $(y|w)$ where w is a word of length $k \to \infty$. We consider the *worst case performance*:

$$r_k^{\text{worst}} \stackrel{\text{def}}{=} \bigoplus_{w\in\Sigma^k} (y|w) = \max_{w_k,\ldots,w_1\in\Sigma}\ \max_{i_k,\ldots,i_0} [\alpha_{i_k} + \mu(w_k)_{i_k i_{k-1}} + \ldots + \mu(w_1)_{i_1 i_0} + \beta_{i_0}] \enspace .$$

Theorem 15 [11, 10]. *Let y be the series given by the linear representation α, μ, β. Let $M = \bigoplus_{a\in\Sigma} \mu(a)$. Then $r_k^{\text{worst}} = \alpha M^k \beta$.*

This is a *superposition principle* which states that the worst case behavior is obtained by mere superposition of the TEGs with transition matrices $\mu(a), a \in \Sigma$.

Example 8. For the automaton of Ex. 7, the asymptotic behavior is determined by the eigenvalue λ of $M = A(a) \oplus A(b)$, i.e. $\lim_k (r_k^{\text{worst}})^{\otimes \frac{1}{k}} = \lim_k r_k^{\text{worst}}/k = \lambda = \max_{i=1,2} \max(\tau_i, \kappa_i)$.

We next turn to the dual *optimal case* measure of performance[14]:

$$r_k^{\text{opt.}} \stackrel{\text{def}}{=} \min_{\substack{w\in\Sigma^k,\\ (y|w)\neq\varepsilon}} (y|w) = \min_{\substack{w_k,\ldots,w_1\in\Sigma,\\ (y|w)\neq\varepsilon}}\ \max_{i_k,\ldots,i_0} [\alpha_{i_k} + \mu(w_k)_{i_k i_{k-1}} + \ldots + \mu(w_1)_{i_1 i_0} + \beta_{i_0}] \enspace .$$

This min-max problem consists in finding a schedule minimizing the transfer time of the k-th input. We remark that for the subclass of *deterministic*[15] series, $(y|w)$ does not involve maximization, so that the computation of $r^{\text{opt.}}$ and r^{worst} become dual. More formally, we introduce $y' = \bigoplus_{(y|w)\neq-\infty} (y|w)w \in \mathbb{R}_{\min}\langle\langle\Sigma\rangle\rangle$ (where $\mathbb{R}_{\min} \stackrel{\text{def}}{=} (\mathbb{R}\cup\{+\infty\}, \min, +)$) and we get:

Proposition 16. *For a deterministic series y, $r_k^{\text{opt.}} = \alpha'(M')^k\beta'$, where α', μ', β' is a $\mathbb{R}_{\min}$-linear representation of y', $M' = \bigoplus_{a\in\Sigma} \mu'(a)$ and all the operations are interpreted in $\mathbb{R}_{\min}$.*

Thus, we are reduced to characterize the series which admit deterministic representations. Under some finiteness and integrity conditions on α, μ, β, e.g. if $\forall a, i, j, \mu(a)_{ij} \in \mathbb{Z}$, the series y is deterministic [11, 10, 13]. However, the minimal dimension of a deterministic representation can be arbitrarily larger than the dimension of α, μ, β. Thus

[13] See [7]. The multiplicity of w – i.e. $(y|w)$ – is equal to the sum of the multiplicative weights of all the paths with label w from the input to the output. E.g. $(y|ba) = \tau_1\kappa_1 \oplus \tau_2\kappa_2 \oplus \tau_1\kappa_2$.

[14] The mean case performance $r_k^{\text{mean}} = \mathbb{E}(y|w^{(k)})$ – where $w^{(k)}$ is a random word of length k – is dealt with in the first order ergodic theory of TEGs [2, 17].

[15] A series is deterministic if it admits a representation of the form (16).

computing $r^{\text{opt.}}$ from Prop. 16 can be much more complex than computing r^{worst} from Th. 15. Some different techniques based on rational simplifications can sometimes be applied as in Ex. 9, but it is hopeless to obtain universal canonical forms since the equality of $(\max,+)$ rational series is undecidable [15]. It remains an open question whether the asymptotic behavior of $r^{\text{opt.}}$ can be exactly evaluated for some reasonable classes of series without appealing to the above determinization procedure.

Example 9. For the automaton of Ex. 7, $\lim_k r_k^{\text{opt.}}/k = \min(\max_i(\tau_i), \max_i(\kappa_i))$. When $\tau_2 \geq \tau_1$, $\kappa_2 \geq \kappa_1$, this follows for instance from the easily obtained expression $y = (\tau_2 a \oplus \kappa_2 b)(\tau_2 a \oplus \kappa_2 b)^*$.

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