This is a preliminary version of a note to appear in C. R. Acad. Sci. Paris, t. 335, Série I, p. 1–6, 2002
Analyse mathématique/*Mathematical analysis*

Invertibility of Functional Galois Connections

Marianne Akian [a], Stéphane Gaubert [b], Vassili Kolokoltsov [c]

[a] **INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France.**
E-mail: Marianne.Akian@inria.fr

[b] **INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex, France.**
E-mail: Stephane.Gaubert@inria.fr

[c] **Dep. of Computing and Mathematics, Nottingham Trent University, Burton Street, Nottingham, NG1 4BU, UK, and Institute for Information Transmission Problems of Russian Academy of Science, Moscow.**
E-mail: vk@maths.ntu.ac.uk

Abstract. We consider equations of the form $Bf = g$, where B is a Galois connection between lattices of functions. This includes the case where B is the Fenchel transform, or more generally a Moreau conjugacy. We characterize the existence and uniqueness of a solution f in terms of generalized subdifferentials, which extends K. Zimmermann's covering theorem for max-plus linear equations. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Inversibilité des correspondances de Galois fonctionnelles

Résumé. *On considère des équations de la forme $Bf = g$, où B est une correspondance de Galois entre des treillis de fonctions, ce qui inclut le cas où B est la transformation de Fenchel, ou plus généralement une conjugaison de Moreau. Nous caractérisons l'existence et l'unicité d'une solution f, en termes de sous-différentiels généralisés, et étendons ainsi le théorème de couverture de K. Zimmermann pour les équations linéaires max-plus.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soient $(\mathscr{F}, \leq_{\mathscr{F}})$ et $(\mathscr{G}, \leq_{\mathscr{G}})$ deux ensembles partiellement ordonnés, et $B : \mathscr{F} \to \mathscr{G}$, $C : \mathscr{G} \to \mathscr{F}$. On dit que (B, C) est une correspondance de Galois si la propriété (7b) ci-dessous est satisfaite. L'application C, qui est unique, est notée B°. On dit aussi que B et C sont des correspondances de Galois. On s'intéresse au cas où $\mathscr{F} = \mathrm{sci}(Y, \overline{\mathbb{R}})$ est l'ensemble des fonctions semi-continues inférieurement d'un espace topologique séparé Y dans $\overline{\mathbb{R}}$, et où $\mathscr{G} = \overline{\mathbb{R}}^X$, pour un espace topologique séparé X. On montre en particulier que B et B° s'écrivent sous la forme (8), où b et b° sont des applications $X \times Y \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, et où pour tout $x \in X, y \in Y$, $(b(x, y, \cdot), b^\circ(x, y, \cdot))$ est une correspondance de Galois. Quand $b(x, y, \lambda) = b^\circ(x, y, \lambda) =$

Note présentée par Pierre-Louis LIONS

$b(x,y)-\lambda$, pour une application $b : X\times Y\to\overline{\mathbb{R}}$ (avec la convention $(-\infty)+(+\infty)=(+\infty)+(-\infty)=-\infty$), on retrouve la conjuguaison de Moreau (6), dont la transformée de Fenchel est un cas spécial. Étant donné un sous-ensemble $X'\subset X$ et une application $g\in\mathscr{G}$, on considère le problème

$$(\mathcal{P}') : \qquad \text{Trouver } f\in\mathscr{F} \text{ tel que } Bf\leq g, \qquad Bf(x)=g(x),\forall x\in X' \ .$$

On supposera qu'il existe un ensemble $\mathscr{S}\subset X\times Y$ tel que : A1) $\mathscr{S}_x=\{y\in Y \mid (x,y)\in\mathscr{S}\}\neq\emptyset$, pour tout $x\in X$; A2) $\mathscr{S}^y=\{x\in X \mid (x,y)\in\mathscr{S}\}\neq\emptyset$, pour tout $y\in Y$; A3) $b(x,y,\cdot)$ est une bijection décroissante $\overline{\mathbb{R}}\to\overline{\mathbb{R}}$, pour tout $(x,y)\in\mathscr{S}$; et A4) $b(x,y,\cdot)\equiv-\infty$, pour $(x,y)\in X\times Y\setminus\mathscr{S}$. Pour toute application g d'un espace topologique Z dans $\overline{\mathbb{R}}$, on utilise les notations (9). On définit aussi les sous-différentiels (10), pour $f\in\mathscr{F}$ et $g\in\mathscr{G}$. On dit que b est coercive si pour tout $x\in X$ et $\alpha\in\mathbb{R}$, $b(x,\cdot,\alpha)$ est semi-continue supérieurement, et si pour tout $x\in X$, pour tout voisinage V de x dans X, et pour tout $\alpha\in\mathbb{R}$, la fonction $b^\alpha_{x,V}$ de (11) est telle que tous les ensembles de sous-niveau $\{y\in Y \mid b^\alpha_{x,V}(y)\leq\beta\}$, avec $\beta\in\mathbb{R}$, sont relativement compacts. Lorsque F est une application de Y dans l'ensemble $\mathscr{P}(X)$ des parties de X, on dit que $\{F(y)\}_{y\in Y}$ est un recouvrement de $X'\subset X$ si $\cup_{y\in Y}F(y)\supset X'$. Ce recouvrement est dit algébriquement minimal si pour tout $z\in Y$, $\cup_{y\in Y\setminus\{z\}}F(y)\not\supset X'$. Il est dit topologiquement minimal si pour tout ouvert non vide $U\subset Y$, $\cup_{y\in Y\setminus U}F(y)\not\supset X'$. Si G est une application de X dans $\mathscr{P}(Y)$, on note G^{-1} l'application de Y dans $\mathscr{P}(X)$, donnée par $G^{-1}(y)=\{x\in X \mid y\in G(x)\}$. Nous énonçons maintenant les résultats principaux de cette note, en suivant la numérotation de la version anglaise.

THÉORÈME 2. – *Soit $X'\subset X$, et $g\in\mathscr{G}$, et considérons les assertions suivantes :*

$$\textit{Le problème } (\mathcal{P}') \textit{ a une solution,} \tag{1}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y\in Y} \textit{ est un recouvrement de } X', \tag{2}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y\in \mathrm{ldom}(B^\circ g)} \textit{ est un recouvrement de } X'\cap\mathrm{udom}(g). \tag{3}$$

Alors, (3)⇔(2)⇒(1)*. En outre, l'implication* (1)⇒(3) *a lieu dans les cas suivants : 1) si Y est fini, ou 2) si $X'\subset\mathrm{idom}(g)$, b est coercive, et si Y est discret ou bien $B^\circ g(y)>-\infty$ pour tout $y\in Y$.*

On dit qu'une application $h : X\to Y$ est quasi-continue si pour tout ouvert G de Y, l'ensemble $h^{-1}(G)$ est semi-ouvert, ce qui signifie que $h^{-1}(G)$ est inclus dans la cloture de son interieur. Toute application s.c.i. convexe propre sur $\mathbb{R}^n$ est quasi-continue sur son domaine $\mathrm{dom}(f)$. On définit $\mathrm{dom}(\partial^\circ g)=\{x\in X \mid \partial^\circ g(x)\neq\emptyset\}=\cup_{y\in Y}(\partial^\circ g)^{-1}(y)$.

THÉORÈME 3. – *Soit $X'\subset X$, et $g\in\mathscr{G}$, et considérons les assertions suivantes :*

$$\textit{Le problème } (\mathcal{P}') \textit{ a une unique solution,} \tag{4}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y\in \mathrm{ldom}(B^\circ g)} \textit{ est un recouvrement topologiquement minimal de } X'\cap\mathrm{udom}(g). \tag{5}$$

L'équivalence (4)⇔(5) *a lieu lorsque Y est fini et est muni de la topologie discrète. Supposons en outre que b est coercive et que $B^\circ g(y)>-\infty$ pour tout $y\in Y$. Alors, l'implication* (5)⇒(4) *a lieu si $X'\subset\mathrm{idom}(g)$ et $B^\circ g$ est quasi-continu sur son domaine. L'implication* (4)⇒(5) *a lieu si $X'\subset\mathrm{dom}(\partial^\circ g)\cap\mathrm{dom}(g)$.*

Puisque (15) entraîne que le problème $(\mathcal{P})$ a au plus une solution, le théorème 3 fournit une condition suffisante d'unicité pour la solution du problème $(\mathcal{P})$.

Une fonction convexe s.c.i. propre g sur $\mathbb{R}^n$ est dite essentiellement régulière si $\mathrm{idom}(g)\neq\emptyset$, si g est différentiable dans $\mathrm{idom}(g)$, et si la norme de la différentielle de g au point x tend vers l'infini, lorsque x s'approche de la frontière de $\mathrm{dom}(g)$.

COROLLAIRE 5. – *Soit g une fonction convexe essentiellement régulière (s.c.i propre) sur $\mathbb{R}^n$. Si f est une fonction s.c.i. propre telle que $f^\star\leq g$ et $f^\star(x)=g(x)$ pour tout $x\in\mathrm{idom}(g)$, alors $f=g^\star$. En particulier, g a une unique préimage par la transformation de Fenchel.*

La condition suffisante du corollaire 5 n'est pas nécessaire.

1. Introduction

We call *functional Galois connection* a (dual) Galois connection between a sublattice $\mathscr{F}$ of $\overline{\mathbb{R}}^Y$ and a sublattice $\mathscr{G}$ of $\overline{\mathbb{R}}^X$, where X, Y are two sets and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, see §2. for definitions. An important example of functional Galois connection is the Fenchel transform, and more generally, the Moreau conjugacy associated to a kernel $b : X \times Y \to \overline{\mathbb{R}}$,

$$B : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^X, \qquad Bf(x) = \sup\{b(x,y) - f(y) \mid y \in Y\} \ , \tag{6}$$

with the convention $(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty$. Moreau conjugacies are instrumental in nonconvex duality, see [16, Ch. 11, § E],[17]. They are equivalent to max-plus linear operators with kernel. Such operators, which are of the form $f \mapsto B(-f)$, arise in deterministic optimal control and asymptotics, and have been widely studied, see in particular [7, 12, 3, 11, 1, 9]. Given a map $g \in \mathscr{G}$ and a functional Galois connection $B : \mathscr{F} \to \mathscr{G}$, we consider the problem:

$$(\mathcal{P}) : \quad \text{Find } f \in \mathscr{F} \text{ such that } Bf = g \ .$$

In particular, we look for effective conditions on g for the solution f to exist and be unique. This problem arises in large deviations: as shown in [2], the classical Gärtner-Ellis theorem (see e.g. [8, Th. 2.3.6,(c)]) is essentially a uniqueness result for the preimage of the Fenchel transform, and the results of this note yield a new proof, as well as generalizations, of this theorem. Problem $(\mathcal{P})$ is also intimately related to the Monge-Kantorovitch mass transfer problem (see e.g. [14, §3.3]). When X, Y are finite sets, $\mathscr{F} = \mathbb{R}^Y$ and $\mathscr{G} = \mathbb{R}^X$, when B is as in (6) with $b(x,y) \in \mathbb{R}$, the existence and uniqueness of the solutions of $(\mathcal{P})$ was characterized by K. Zimmermann [19] (see [5]), who gave conditions involving coverings of X by sets. In this note, we extend K. Zimmermann's theorem to the case of functional Galois connections: the existence and uniqueness of the solutions of $(\mathcal{P})$ are characterized in terms of generalized subdifferentials of g.

2. Representation of Functional Galois Connections

Let $(\mathscr{F}, \leq_{\mathscr{F}})$ and $(\mathscr{G}, \leq_{\mathscr{G}})$ be two partially ordered sets, and let $B : \mathscr{F} \to \mathscr{G}$ and $C : \mathscr{G} \to \mathscr{F}$. We say that B is *antitone* if $f \leq_{\mathscr{F}} f' \implies Bf' \leq_{\mathscr{G}} Bf$. The pair (B, C) is a *dual Galois connection* (see e.g. [4, Chapter V, Section 8]) between $\mathscr{F}$ and $\mathscr{G}$ if it satisfies one of the following equivalent conditions:

$$I_{\mathscr{F}} \geq_{\mathscr{F}} CB, \quad I_{\mathscr{G}} \geq_{\mathscr{G}} BC, \quad \text{and } B, C \text{ are antitone maps,} \tag{7a}$$

$$(g \geq_{\mathscr{G}} Bf \iff f \geq_{\mathscr{F}} Cg) \qquad \forall f \in \mathscr{F},\ g \in \mathscr{G} \ , \tag{7b}$$

$$Cg = \min\nolimits_{\mathscr{F}}\{f \mid g \geq_{\mathscr{G}} Bf\} \qquad \forall g \in \mathscr{G} \ , \tag{7c}$$

$$Bf = \min\nolimits_{\mathscr{G}}\{g \mid f \geq_{\mathscr{F}} Cg\} \qquad \forall f \in \mathscr{F} \ , \tag{7d}$$

where $\min_{\mathscr{F}}$ and $\min_{\mathscr{G}}$ denote the minimal elements for the orders $\leq_{\mathscr{F}}$ and $\leq_{\mathscr{G}}$, respectively, and where $I_{\mathscr{A}}$ denotes the identity on a set $\mathscr{A}$. Since there is at most one map C such that (B, C) is a dual Galois connection, we shall denote this C by B°, and call B or B° a dual Galois connection.

Ordinary (non dual) *Galois connections* are defined by reversing the order of $\mathscr{F}$ and $\mathscr{G}$ in (7). In the sequel, we shall only consider dual Galois connections and omit the term "dual".

We call *lattice of functions* a sublattice $\mathscr{F}$ of S^Y, where $(S, \leq)$ is a lattice, Y is a set, and S^Y is equipped with the product ordering. When S has a maximum element $\top_S$, we define the *Dirac function* at point $y \in Y$ with value $s \in S$: $\delta_y^s \in S^Y$, $\delta_y^s(y') = s$ if $y' = y$, and $\delta_y^s(y') = \top_S$ otherwise.

THEOREM 1. – *Let S, T be two lattices that have a maximum element, let X, Y be arbitrary nonempty sets and let $\mathscr{F} \subset S^Y$ (resp. $\mathscr{G} \subset T^X$) be a lattice of functions containing all the Dirac functions of S^Y*

(resp. T^X). Then, (B, B°) is a Galois connection between $\mathscr{F}$ and $\mathscr{G}$ if, and only if, there exists two maps $b : X \times Y \times S \to T$ and $b^\circ : X \times Y \times T \to S$ such that: for all $(x, y) \in X \times Y$, $(b(x, y, \cdot), b^\circ(x, y, \cdot))$ is a Galois connection between S and T; for all $(x, t) \in X \times T$, $b^\circ(x, \cdot, t) \in \mathscr{F}$; for all $(y, s) \in Y \times S$, $b(\cdot, y, s) \in \mathscr{G}$; and

$$Bf = \sup_{\mathscr{G}}\{b(\cdot, y, f(y)) \mid y \in Y\}, \quad \forall f \in \mathscr{F} \ , \tag{8a}$$

$$B^\circ g = \sup_{\mathscr{F}}\{b^\circ(x, \cdot, g(x)) \mid x \in X\}, \quad \forall g \in \mathscr{G} \ . \tag{8b}$$

When Y is a T_1 topological space and S has a maximum element, Theorem 1 can be applied to the set $\mathscr{F} = \mathrm{lsc}(Y, S)$ of lower semicontinuous maps $Y \to S$ (we say that a map $f : Y \to S$ is *lower semicontinuous*, or l.s.c., if for all $s \in S$, the sublevel set $\{y \in Y \mid f(y) \leq s\}$ is closed). In this case, $\sup_{\mathscr{F}} = \sup$ since the sup of l.s.c. maps is l.s.c.

Kolokoltsov already proved [10] (see also [11, Theorem 1.4]) a "Riesz representation theorem" similar to Theorem 1, for a continuous map B between (non-complete) lattices of *continuous* functions $\mathscr{F}$ and $\mathscr{G}$, assuming that B preserves finite sups. Singer [17, Th. 7.1] also proved a theorem similar to Theorem 1 when $\mathscr{F} = A^Y$ and A is a complete lattice included in $\overline{\mathbb{R}}$.

3. Existence and Uniqueness of Solutions of $Bf = g$

In this section, we take $S = T = \overline{\mathbb{R}}$, we assume that X and Y are Hausdorff topological spaces, and take $\mathscr{F} = \mathrm{lsc}(Y, \overline{\mathbb{R}})$, $\mathscr{G} = \overline{\mathbb{R}}^X$, together with B, B°, b, b° as in Theorem 1. The case when $\mathscr{F} = \overline{\mathbb{R}}^Y$ and X, Y are arbitrary sets can be obtained by taking discrete topologies on X and Y. The property that $b(x, y, \cdot)$, or $b^\circ(x, y, \cdot)$, is a Galois connection can be made explicit by noting that a map $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Galois connection if, and only if, h is nonincreasing, right-continuous, and $h(+\infty) = -\infty$.

We shall assume that there is a subset $\mathscr{S} \subset X \times Y$ such that the following properties hold: A1) $\mathscr{S}_x = \{y \in Y \mid (x, y) \in \mathscr{S}\} \neq \emptyset$, for all $x \in X$; A2) $\mathscr{S}^y = \{x \in X \mid (x, y) \in \mathscr{S}\} \neq \emptyset$, for all $y \in Y$; A3) $b(x, y, \cdot)$ is a bijection $\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ for all $(x, y) \in \mathscr{S}$; and A4) $b(x, y, \cdot) \equiv -\infty$, for $(x, y) \in X \times Y \setminus \mathscr{S}$. When B is the Moreau conjugacy given by (6), $b(x, y, \lambda) = b^\circ(x, y, \lambda) = b(x, y) - \lambda$, Assumptions A3),A4) are satisfied when $b(x, y) \in \mathbb{R} \cup \{-\infty\}$, for all $x \in X, y \in Y$, and Assumptions A1),A2) are satisfied when for all $x \in X, y \in Y$, $b(x, \cdot)$ and $b(\cdot, y)$ are not identically $-\infty$.

Rather than Problem $(\mathcal{P})$, we will consider the more general problem:

$$(\mathcal{P}') : \qquad \text{Find } f \in \mathscr{F} \text{ such that } Bf \leq g, \qquad Bf(x) = g(x)\ \forall x \in X',$$

where $g \in \mathscr{G}$ and $X' \subset X$ are given.

To state our results, we need some definitions. First, for any map g from a topological space Z to $\overline{\mathbb{R}}$, we define the lower domain, upper domain, domain, and inner domain:

$$\begin{aligned} &\mathrm{ldom}(g) = \{z \in Z \mid g(z) < +\infty\}, \quad &&\mathrm{udom}(g) = \{z \in Z \mid g(z) > -\infty\}, \\ &\mathrm{dom}(g) = \mathrm{ldom}(g) \cap \mathrm{udom}(g), &&\mathrm{idom}(g) = \{z \in \mathrm{dom}(g) \mid \limsup_{z' \to z} g(z') < +\infty\} \ . \end{aligned} \tag{9}$$

Next, given $f \in \mathscr{F}, y \in Y, g \in \mathscr{G}$ and $x \in X$, we define the subdifferentials:

$$\partial f(y) = \{x \in X \mid (x, y) \in \mathscr{S}, \quad b(x, y', f(y')) \leq b(x, y, f(y))\ \forall y' \in Y\}, \tag{10a}$$

$$\partial^\circ g(x) = \{y \in Y \mid (x, y) \in \mathscr{S}, \quad b^\circ(x', y, g(x')) \leq b^\circ(x, y, g(x))\ \forall x' \in X\}. \tag{10b}$$

We say that b is *coercive* if for all $x \in X$ and $\alpha \in \mathbb{R}$, $b(x, \cdot, \alpha)$ is upper semicontinuous (u.s.c.) and if for all $x \in X$, all neighborhoods V of x in X, and all $\alpha \in \mathbb{R}$, the function

$$y \in Y \mapsto b^\alpha_{x,V}(y) = \sup_{z \in V} b(z, y, b^\circ(x, y, \alpha)) \ , \tag{11}$$

has relatively compact finite sublevel sets, which means that $\{y \in Y \mid b^\alpha_{x,V}(y) \leq \beta\}$ is relatively compact for all $\beta \in \mathbb{R}$. The latest property holds trivially, and independently of the topology on X, when Y is compact (and in particular when Y is finite). If $X = Y = \mathbb{R}^n$ and $b(x, y, \alpha) = \langle x, y\rangle - \alpha$ then for any neighborhood V of x, and $\alpha \in \mathbb{R}$, $b^\alpha_{x,V}(y) \geq \varepsilon\|y\| + \alpha$, for some $\varepsilon > 0$, so that b is coercive. Similarly, if $b(x, y, \alpha) = a\|x - y\|^2 - \alpha$, where $a \in \mathbb{R} \setminus \{0\}$ and $\|\cdot\|$ is the Euclidean norm, then $b^\alpha_{x,V}(y) \geq \varepsilon\|y - x\| - 1 + \alpha$, for some $\varepsilon > 0$, so that b is coercive.

When F is a map from Y to the set $\mathscr{P}(X)$ of all subsets of X, we say that $\{F(y)\}_{y\in Y}$ is a *covering* of $X' \subset X$ if $\cup_{y\in Y} F(y) \supset X'$. This covering is *algebraically minimal* if for all $z \in Y$, $\cup_{y\in Y\setminus\{z\}} F(y) \not\supset X'$. It is *topologically minimal* if for all non-empty open sets $U \subset Y$, $\cup_{y\in Y\setminus U} F(y) \not\supset X'$. Algebraic minimality implies topological minimality. Both notions coincide if Y is a discrete topological space. If G is a map from X to $\mathscr{P}(Y)$, we denote by G^{-1} the map from Y to $\mathscr{P}(X)$ given by $G^{-1}(y) = \{x \in X \mid y \in G(x)\}$.

THEOREM 2. – *Let $X' \subset X$, and $g \in \mathscr{G}$. Consider the following statements:*

$$\textit{Problem } (\mathcal{P}') \textit{ has a solution,} \tag{12}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y\in Y} \textit{ is a covering of } X', \tag{13}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y\in \mathrm{ldom}(B^\circ g)} \textit{ is a covering of } X' \cap \mathrm{udom}(g). \tag{14}$$

Then, (14)⇔(13)⇒(12)*. Moreover, the implication* (12)⇒(14) *holds in the following cases: 1) when Y is finite, or 2) when $X' \subset \mathrm{idom}(g)$, b is coercive and either Y is discrete or $B^\circ g(y) > -\infty$ for all $y \in Y$.*

To give an uniqueness result, we need some additional definitions. We say that a map $h : X \to Y$ is *quasi-continuous* if for all open sets G of Y, the set $h^{-1}(G)$ is *semi-open*, which means that $h^{-1}(G)$ is included in the closure of its interior (see for instance [13] for definitions and properties of quasi-continuous functions and multi-applications). If $h : X \to \overline{\mathbb{R}}$ is l.s.c., then h is quasi-continuous if, and only if, $h = \mathrm{lsc}(\mathrm{usc}(h))$, where lsc (resp. usc) means the l.s.c. (resp. u.s.c.) hull. We say that B is *regular* if for all $f \in \mathscr{F}$, Bf is l.s.c. on X and quasi-continuous on its domain, which means that the restriction of Bf to its domain is quasi-continuous for the induced topology. The notion of regularity for B° is defined in the symmetric way. When X is endowed with the discrete topology, B is always regular. When $\mathscr{S} = X \times Y$ and $\{b(\cdot, y, \alpha)\}_{y\in Y,\ \alpha\in\mathbb{R}}$ is an equicontinuous family of functions, then Bf is continuous on X for any $f \in \mathscr{F}$, so B is regular. When $X = Y = \mathbb{R}^n$, the Fenchel transform is regular because any l.s.c. proper convex function f is quasi-continuous on its domain $\mathrm{dom}(f)$. We define $\mathrm{dom}(\partial^\circ g) = \{x \in X \mid \partial^\circ g(x) \neq \emptyset\} = \cup_{y\in Y}(\partial^\circ g)^{-1}(y)$.

THEOREM 3. – *Let $X' \subset X$, and $g \in \mathscr{G}$. Consider the following statements:*

$$\textit{Problem } (\mathcal{P}') \textit{ has a unique solution,} \tag{15}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y\in \mathrm{ldom}(B^\circ g)} \textit{ is a topologically minimal covering of } X' \cap \mathrm{udom}(g). \tag{16}$$

The equivalence (15)⇔(16) *holds when Y is finite (with the discrete topology). Moreover, assume that b is coercive and $B^\circ g(y) > -\infty$ for all $y \in Y$. Then, the implication* (16)⇒(15) *holds if $X' \subset \mathrm{idom}(g)$ and $B^\circ g$ is quasi-continuous on its domain. The implication* (15)⇒(16) *holds if $X' \subset \mathrm{dom}(\partial^\circ g) \cap \mathrm{dom}(g)$.*

Since (15) implies that Problem $(\mathcal{P})$ has at most one solution, Theorem 3 yields a sufficient uniqueness condition for the solution of Problem $(\mathcal{P})$. However, for Problem $(\mathcal{P})$, the necessary uniqueness condition implied by Theorem 3 only holds when Y is finite, or when $\mathrm{dom}(\partial^\circ g) \cap \mathrm{dom}(g) = X$. To give a more specific uniqueness result, we shall use the following condition:

$$(\mathcal{C}) : \quad \begin{array}{l} \text{there exists a basis } \mathscr{B} \text{ of neighborhoods such that} \\ \forall U \in \mathscr{B},\ \exists \varepsilon > 0,\quad \forall x \in X, \qquad \sup\limits_{y\in U\cap\mathscr{S}_x,\ \alpha\in\mathbb{R}} (b(x, y, \alpha) - b(x, y, \alpha + \varepsilon)) < +\infty\ . \end{array}$$

Condition $(\mathcal{C})$ is satisfied in particular when for all $x \in X$, $\{b(x,y,\cdot)\}_{y\in Y}$ is a family of α-Hölder continuous functions (for $0 < \alpha \leq 1$), uniformly in $y \in U$, for all small enough open sets U, or if $\{b(x,y,\cdot)\}_{x\in X,\, y\in U}$ is an equicontinuous family, for all small enough open sets U. In particular, condition $(\mathcal{C})$ is satisfied when $b(x,y,\alpha) = b(x,y) - \alpha$ or when $b(x,y,\alpha) = -(|\langle x,y\rangle| + 1)\alpha$.

THEOREM 4. – *Let $g \in \mathscr{G}$. Assume that b is coercive, and that $B^\circ g(y) > -\infty$ for all $y \in Y$. Then, the uniqueness of the solution of Problem $(\mathcal{P})$ implies* (16)*, when any of the following assumptions holds:*

1. *$X' = \mathrm{dom}(\partial^\circ g) \cap \mathrm{dom}(g)$, and Y is locally compact.*
2. *$X' = \mathrm{idom}(g)$, B is regular, $\mathrm{dom}(g)$ is included in the closure of $\mathrm{idom}(g)$, and either Y is locally compact or condition $(\mathcal{C})$ holds.*

The topological minimality in (16) is a relaxation of algebraic minimality, which is a generalized differentiability condition. Indeed, if $\{(\partial^\circ g)^{-1}(y)\}_{y\in \mathrm{ldom}(B^\circ g)}$ is a covering of $X' \cap \mathrm{udom}(g)$, this covering is algebraically minimal if, and only if, for all $y \in \mathrm{ldom}(B^\circ g)$, there is an $x \in X' \cap \mathrm{udom}(g)$ such that $\partial^\circ g(x) = \{y\}$. When B is the Fenchel transform, the following classical notion is intermediate between algebraic and topological minimality: a l.s.c. proper convex function g on $\mathbb{R}^n$ is *essentially smooth* if $\mathrm{idom}(g) \neq \emptyset$ (the interior of its domain is nonempty), g is differentiable in $\mathrm{idom}(g)$, and the norm of the differential of g at x tends to infinity, when x goes to the boundary of $\mathrm{dom}(g)$, see [15, §26].

COROLLARY 5. – *Let g be an essentially smooth (l.s.c. proper) convex function on $\mathbb{R}^n$. If f is a l.s.c. proper function such that $f^\star \leq g$ and $f^\star(x) = g(x)$ for all $x \in \mathrm{idom}(g)$, then $f = g^\star$. In particular, g has a unique preimage by the Fenchel transform.*

The following function g satisfies (16) but is not essentially smooth: consider $X = Y = \mathbb{R}^2$, $g = Bf = f^*$ where $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$, with $f(y) = y_1^2(y_2^2 + 3)$ if $|y_2| \leq 1$ and $f(y) = +\infty$ elsewhere.

References

[1] M. Akian. Densities of idempotent measures and large deviations. *Transactions of the American Mathematical Society*, 351(11):4515–4543, 1999.

[2] M. Akian, S. Gaubert, and V. Kolokoltsov. Invertibility of functional Galois connections and large deviations. In preparation, 2002.

[3] F. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat. *Synchronization and linearity : an algebra for discrete events systems*. John Wiley & Sons, New-York, 1992.

[4] G. Birkhoff. *Lattice Theory*, volume 25 of *Colloquium publications*. American Mathematical Society, Providence, 1995.

[5] P. Butkovič. Strong regularity of matrices — a survey of results. *Discrete Applied Mathematics*, 48:45–68, 1994.

[6] P. Butkovič. Simple image set of (max, +) linear mappings. *Discrete Appl. Math.*, 105(1-3):73–86, 2000.

[7] R. A. Cuninghame-Green. *Minimax Algebra*. Number 166 in Lecture notes in Economics and Mathematical Systems. Springer, 1979.

[8] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*. Jones and Barlett, Boston, MA, 1993.

[9] M. Gondran and M. Minoux. *Graphes, dioïdes et semi-anneaux*. TEC & DOC, Paris, 2001.

[10] V. Kolokoltsov. On linear, additive, and homogeneous operators. 1992. Appear in [12].

[11] V. Kolokoltsov and V. Maslov. *Idempotent analysis and applications*. Kluwer Acad. Publisher, 1997.

[12] V. Maslov and S. Samborskiĭ, editors. *Idempotent analysis*, volume 13 of *Adv. in Sov. Math.* AMS, RI, 1992.

[13] T. Neubrunn. Quasi-continuity. *Real Anal. Exchange*, 14(2):259–306, 1988/89.

[14] S. T. Rachev and L. Rüschendorf. *Mass transportation problems, volume I: theory*. Springer, 1998.

[15] R. T. Rockafellar. *Convex analysis*. Princeton University Press Princeton, N.J., 1970.

[16] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*. Springer-Verlag, Berlin, 1998.

[17] I. Singer. *Abstract convex analysis*. John Wiley & Sons Inc., New York, 1997. With a foreword by A. M. Rubinov, A Wiley-Interscience Publication.

[18] I. Singer. Further applications of the additive min-type coupling function. Privately circuled draft, 2001.

[19] K. Zimmermann. *Extremální Algebra*. Ekonomický ùstav ČSAV, Praha, 1976. (in Czech).