Max-plus algebra

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Max-plus algebra has been discovered more or less independently by several schools, in relation with various mathematical fields. This chapter is limited to finite dimensional linear algebra. For more information, the reader may consult the books [CG79, Zim81, CKR84, BCOQ92, KM97, GM02]. The collections of articles [MS92, Gun98, LM05] give a good idea of current developments.

1 Preliminaries

Definitions

The **max-plus semiring** $\mathbb{R}_{\max}$ is the set $\mathbb{R}\cup\{-\infty\}$, equipped with the **addition** $(a,b)\mapsto\max(a,b)$ and the **multiplication** $(a,b)\mapsto a+b$. The identity element for the addition, **zero**, is $-\infty$, and the identity element for the multiplication, **unit**, is 0. To illuminate the linear algebraic nature of the results, the generic notations $\mathbin{+\!\!\!+}$, $\sum$, $\times$ (or concatenation), $\mathbb{0}$ and $\mathbb{1}$ are used for the addition, the sum, the multiplication, the zero and the unit of $\mathbb{R}_{\max}$ respectively, so that when a,b belong to $\mathbb{R}_{\max}$, $a \mathbin{+\!\!\!+} b$ will mean $\max(a,b)$, $a\times b$ or ab will mean the usual sum $a+b$. We use blackboard (double struck) fonts to denote the max-plus operations (compare "$\mathbin{+\!\!\!+}$" with "$+$").

The **min-plus semiring** $\mathbb{R}_{\min}$ is the set $\mathbb{R}\cup\{+\infty\}$ equipped with the addition $(a,b)\mapsto\min(a,b)$ and the multiplication $(a,b)\mapsto a+b$. The zero is $+\infty$, the unit 0. The name **tropical** is now also used essentially as a synonym of min-plus. Properly speaking, it refers to the **tropical semiring**, which is the subsemiring of $\mathbb{R}_{\min}$ consisting of the elements in $\mathbb{N}\cup\{+\infty\}$.

The **completed max-plus semiring** $\overline{\mathbb{R}}_{\max}$ is the set $\mathbb{R}\cup\{\pm\infty\}$ equipped with the addition $(a,b)\mapsto\max(a,b)$ and the multiplication $(a,b)\mapsto a+b$, with the convention that $-\infty+(+\infty)=+\infty+(-\infty)=-\infty$. The **completed min-plus semiring**, $\overline{\mathbb{R}}_{\min}$, is defined in a dual way.

Many classical algebraic definitions have max-plus analogues. For instance, $\mathbb{R}^n_{\max}$ is the set of n-dimensional **vectors** and $\mathbb{R}^{n\times p}_{\max}$ is the set of $n\times p$ **matrices** with entries in $\mathbb{R}_{\max}$. They are equipped with the vector and matrix operations, defined, and denoted, in the usual way. The $n\times p$ **zero matrix**, $\mathbf{0}_{np}$ or $\mathbf{0}$, has all its entries equal to $\mathbb{0}$. The $n\times n$ **identity matrix**, I_n or I, has diagonal entries equal $\mathbb{1}$, and non-diagonal entries equal to $\mathbb{0}$. Given a matrix $A=(A_{ij})\in\mathbb{R}^{n\times p}_{\max}$, we denote by $A_{i\cdot}$ and $A_{\cdot j}$ the i-th row and the j-th column of A. We also denote by A the linear map $\mathbb{R}^p_{\max}\to\mathbb{R}^n_{\max}$ sending a vector x to Ax. **Semimodules** and **subsemimodules** over the semiring $\mathbb{R}_{\max}$ are defined as the analogues of modules and submodules over rings. A subset F of a semimodule M over $\mathbb{R}_{\max}$ **spans** M, or is a **spanning family** of M if every element $\mathbf{x}$ of M can be expressed as a finite linear combination of the elements of F, meaning that $\mathbf{x}=\sum_{\mathbf{f}\in F}\lambda_{\mathbf{f}}.\mathbf{f}$, where $(\lambda_{\mathbf{f}})_{\mathbf{f}\in F}$ is a family of elements of $\mathbb{R}_{\max}$ such that $\lambda_{\mathbf{f}}=\mathbb{0}$ for all but finitely many $\mathbf{f}\in F$. A semimodule is **finitely generated** if it has a finite spanning family.

The sets $\mathbb{R}_{\max}$ and $\overline{\mathbb{R}}_{\max}$ are ordered by the usual order of $\mathbb{R}\cup\{\pm\infty\}$. Vectors and matrices over $\overline{\mathbb{R}}_{\max}$ are ordered with the product ordering. The supremum and the infimum operations are denoted by $\vee$ and $\wedge$, respectively. Moreover, the sum of the elements of an arbitrary set X of scalars, vectors or matrices with entries in $\overline{\mathbb{R}}_{\max}$ is by definition the supremum of X.

If $A \in \overline{\mathbb{R}}_{\max}^{n\times n}$, the **Kleene star** of A is the matrix $A^\star = I + A + A^2 + \cdots$.

The **digraph** $\Gamma(A)$ associated to a $n \times n$ matrix A with entries in $\overline{\mathbb{R}}_{\max}$ consists of the vertices $1, \ldots, n$, with an arc from vertex i to vertex j when $A_{ij} \neq \mathbb{0}$. The **weight** of a walk W given by $(i_1, i_2), \ldots, (i_{k-1}, i_k)$ is $|W|_A := A_{i_1 i_2} \cdots A_{i_{k-1} i_k}$, and its **length** is $|W| := k - 1$. The matrix A is **irreducible** if $\Gamma(A)$ is strongly connected.

Facts

1. When $A \in \overline{\mathbb{R}}_{\max}^{n\times n}$, the weight of a walk $W = ((i_1, i_2), \ldots, (i_{k-1}, i_k))$ in $\Gamma(A)$ is given by the usual sum $|W|_A = A_{i_1 i_2} + \cdots + A_{i_{k-1} i_k}$, and $A^\star_{ij}$ gives the maximal weight $|W|_A$ of a walk from vertex i to vertex j. One can also define the matrix $A^\star$ when $A \in \overline{\mathbb{R}}_{\min}^{n\times n}$. Then, $A^\star_{ij}$ is the minimal weight of a walk from vertex i to vertex j. Computing $A^\star$ is the same as the all pairs shortest path problem.

2. [CG79], [BCOQ92, Th. 3.20] If $A \in \overline{\mathbb{R}}_{\max}^{n\times n}$ and the weights of the cycles of $\Gamma(A)$ do not exceed $\mathbb{1}$, then $A^\star = I + A + \cdots + A^{n-1}$.

3. [BCOQ92, Th. 4.75 and Rk. 80] If $A \in \overline{\mathbb{R}}_{\max}^{n\times n}$ and $\mathbf{b} \in \overline{\mathbb{R}}_{\max}^{n}$, then the smallest $\mathbf{x} \in \overline{\mathbb{R}}_{\max}^{n}$ such that $\mathbf{x} = A\mathbf{x} + \mathbf{b}$ coincides with the smallest $\mathbf{x} \in \overline{\mathbb{R}}_{\max}^{n}$ such that $\mathbf{x} \geq A\mathbf{x} + \mathbf{b}$, and it is given by $A^\star\mathbf{b}$.

4. [BCOQ92, Th. 3.17] When $A \in \mathbb{R}_{\max}^{n\times n}$, $\mathbf{b} \in \mathbb{R}_{\max}^{n}$, and when all the cycles of $\Gamma(A)$ have a weight strictly less than $\mathbb{1}$, then $A^\star\mathbf{b}$ is the unique solution $\mathbf{x} \in \mathbb{R}_{\max}^{n}$ of $\mathbf{x} = A\mathbf{x} + \mathbf{b}$.

5. Let $A \in \mathbb{R}_{\max}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}_{\max}^{n}$. Construct the sequence:

$$\mathbf{x}_0 = \mathbf{b},\ \mathbf{x}_1 = A\mathbf{x}_0 + \mathbf{b},\ \mathbf{x}_2 = A\mathbf{x}_1 + \mathbf{b}, \ldots$$

The sequence $\mathbf{x}_k$ is nondecreasing. If all the cycles of $\Gamma(A)$ have a weight less than or equal to $\mathbb{1}$, then, $\mathbf{x}_{n-1} = \mathbf{x}_n = \cdots = A^\star\mathbf{b}$. Otherwise, $\mathbf{x}_{n-1} \neq \mathbf{x}_n$. Computing the sequence $\mathbf{x}_k$ to determine $A^\star\mathbf{b}$ is a special instance of label correcting shortest path algorithm [GP88].

6. [BCOQ92, Lemma 4.101] For all $a \in \overline{\mathbb{R}}_{\max}^{n\times n}, b \in \overline{\mathbb{R}}_{\max}^{n\times p}, c \in \overline{\mathbb{R}}_{\max}^{p\times n}$, and $d \in \overline{\mathbb{R}}_{\max}^{p\times p}$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\star = \begin{bmatrix} a^\star + a^\star b(ca^\star b + d)^\star ca^\star & a^\star b(ca^\star b + d)^\star \\ (ca^\star b + d)^\star ca^\star & (ca^\star b + d)^\star \end{bmatrix} .$$

This fact and the next one are special instances of well known results of language theory [Eil74], concerning unambiguous rational identities. Both are valid in more general semirings.

7. [MY60] Let $A \in \overline{\mathbb{R}}_{\max}^{n\times n}$. Construct the sequence of matrices $A^{(0)}, \ldots, A^{(n)}$ such that $A^{(0)} = A$ and

$$A^{(k)}_{ij} = A^{(k-1)}_{ij} + A^{(k-1)}_{ik}(A^{(k-1)}_{kk})^\star A^{(k-1)}_{kj} \quad ,$$

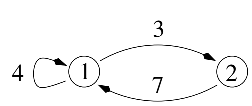
for $i, j = 1, \ldots, n$ and $k = 1, \ldots, n$. Then, $A^{(n)} = A + A^2 + \cdots$.

Example

1. Consider the matrix

$$A = \begin{bmatrix} 4 & 3 \\ 7 & -\infty \end{bmatrix} .$$

The digraph $\Gamma(A)$ is



We have

$$A^2 = \begin{bmatrix} 10 & 7 \\ 11 & 10 \end{bmatrix} \ .$$

For instance, $A^2_{11} = A_{1\cdot}A_{\cdot 1} = [4\ 3][4\ 7]^T = \max(4+4, 3+7) = 10$. This gives the maximal weight of a walk of length 2 from vertex 1 to vertex 1, which is attained by the walk $(1,2),(2,1)$. Since there is one cycle with positive weight in $\Gamma(A)$ (for instance, the cycle $(1,1)$ has weight 4), and since A is irreducible, the matrix $A^\star$ has all its entries equal to $+\infty$. To get a Kleene star with finite entries, consider the matrix

$$C = (-5)A = \begin{bmatrix} -1 & -2 \\ 2 & -\infty \end{bmatrix} \ .$$

The only cycles in $\Gamma(A)$ are $(1,1)$ and $(1,2),(2,1)$ (up to a cyclic conjugacy). They have weights -1 and 0. Applying Fact 2, we get:

$$C^\star = I \oplus C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \ .$$

Applications

1. *Dynamic programming.* Consider a deterministic Markov decision process, with a set of states $\{1,\dots,n\}$, in which one player can move from state i to state j, receiving a payoff of $A_{ij} \in \mathbb{R}\cup\{-\infty\}$. To every state i, associate an initial payoff $\mathbf{c}_i \in \mathbb{R}\cup\{-\infty\}$ and a terminal payoff, $\mathbf{b}_i \in \mathbb{R}\cup\{-\infty\}$. The value in horizon k is by definition the maximum of the sums of the payoffs (including the initial and terminal payoffs) corresponding to all the trajectories consisting exactly of k moves. It is given by $\mathbf{c}A^k\mathbf{b}$, where the product and the power are understood in the max-plus sense. The special case where the initial state is equal to some given $m \in \{1,\dots,n\}$ (and where there is no initial payoff) can be modeled by taking $\mathbf{c} := \mathbf{e}_m$, the m-th max-plus basis vector (whose entries are all equal to $\mathbb{0}$, except the m-th entry which is equal to $\mathbb{1}$). The case where the final state is fixed can be represented in a dual way. Deterministic Markov decision problems (which are the same as shortest path problems) are ubiquitous in Operations Research, Mathematical Economics and Optimal Control.

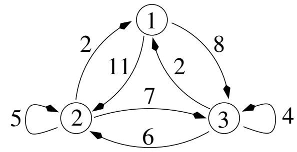
2. [BCOQ92] *Discrete event systems.* Consider a system in which certain repetitive events, denoted by $1,\dots,n$, occur. To every event i is associated a dater function $x_i : \mathbb{Z} \to \mathbb{R}$, where $x_i(k)$ represents the date of the k-th occurrence of event i. Precedence constraints between the repetitive events are given by a set of arcs $E \subset \{1,\dots,n\}^2$, equipped with two valuations $\nu : E \to \mathbb{N}$ and $\tau : E \to \mathbb{R}$: if $(i,j) \in E$, the k-th execution of event i cannot occur earlier than τ_{ij} time units before the $(k-\nu_{ij})$-th execution of event j, so that $x_i(k) \geq \max_{j:\,(i,j)\in E} \tau_{ij} + x_j(k-\nu_{ij})$. This can be rewritten, using the max-plus notation, as

$$\mathbf{x}(k) \geq A_0\mathbf{x}(k) \oplus \cdots \oplus A_{\bar\nu}\mathbf{x}(k-\bar\nu) \ ,$$

where $\bar\nu := \max_{(i,j)\in E} \nu_{ij}$ and $\mathbf{x}(k) \in \mathbb{R}^n_{\max}$ is the vector with entries $x_i(k)$. Often, the dates $x_i(k)$ are only defined for positive k, then, appropriate initial conditions must be incorporated in the model. One is particularly interested in the earliest dynamics, which, by Fact 3, is given by $\mathbf{x}(k) = A_0^\star A_1\mathbf{x}(k-1) \oplus \cdots \oplus A_0^\star A_{\bar\nu}\mathbf{x}(k-\bar\nu)$. The class of systems following dynamics of these forms is known in the Petri net literature as **timed event graphs**. It is used to model certain manufacturing systems [CDQV85], or transportation or communication networks [BCOQ92].

3. N. Bacaër [Bac03] observed that max-plus algebra appears in a familiar problem, crop rotation. Suppose n different crops can be cultivated every year. Assume for simplicity that the income of the year is a deterministic function, $(i,j) \mapsto A_{ij}$, depending only on the crop i of the preceding year, and of the crop j of the current year (a slightly more complex model in which the income of the year depends on the crops of the two preceding years is needed to explain the historical variations of crop rotations [Bac03]). The income of a sequence $i_1,\dots,i_k$ of crops can be written as $\mathbf{c}_{i_1}A_{i_1i_2}\cdots A_{i_{k-1}i_k}$, where $\mathbf{c}_{i_1}$ is the income of the first year. The maximal income in k years is given by $\mathbf{c}A^{k-1}\mathbf{b}$, where $\mathbf{b} = (\mathbb{1},\dots,\mathbb{1})$. We next show an example.

$$A = \begin{bmatrix} -\infty & 11 & 8 \\ 2 & 5 & 7 \\ 2 & 6 & 4 \end{bmatrix}$$



Here, vertices 1, 2, and 3 represent fallow (no crop), wheat, and oat, respectively. (We put no arc from 1 to 1, setting $A_{11} = -\infty$, to disallow two successive years of fallow.) The numerical values have no pretension to realism, however, the income of a year of wheat is 11 after a year of fallow, this is greater than after a year of cereal (5 or 6, depending on whether wheat or oat was cultivated). An initial vector coherent with these data may be $\mathbf{c} = [-\infty\ 11\ 8]$, meaning that the income of the first year is the same as the income after a year of fallow. We have $\mathbf{c}A\mathbf{b} = 18$, meaning that the optimal income in two years is 18. This corresponds to the optimal walk $(2, 3)$, indicating that wheat and oat should be successively cultivated during these two years.

2 The maximal cycle mean

Definitions

The **maximal cycle mean**, $\rho_{\max}(A)$, of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, is the maximum of the weight-to-length ratio over all cycles c of $\Gamma(A)$, that is:

$$\rho_{\max}(A) = \max_{c \text{ cycle of } \Gamma(A)} \frac{|c|_A}{|c|} = \max_{k\geq 1} \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k} \quad . \tag{1}$$

Denote by $\mathbb{R}_+^{n\times n}$ the set of real $n \times n$ matrices with nonnegative entries. For $A \in \mathbb{R}_+^{n\times n}$ and $p > 0$, $A^{(p)}$ is by definition the matrix such that $(A^{(p)})_{ij} = (A_{ij})^p$, and

$$\rho_p(A) := (\rho(A^{(p)}))^{1/p} \quad ,$$

where ρ denotes the (usual) spectral radius. We also define $\rho_\infty(A) = \lim_{p\to+\infty} \rho_p(A)$.

Facts

1. [CG79], [Gau92, Ch. IV], [BSvdD95] *Max-plus Collatz-Wielandt formula, I.* Let $A \in \mathbb{R}_{\max}^{n\times n}$ and $\lambda \in \mathbb{R}$. The following assertions are equivalent: (i) there exists $\mathbf{u} \in \mathbb{R}^n$ such that $A\mathbf{u} \leq \lambda\mathbf{u}$; (ii) $\rho_{\max}(A) \leq \lambda$. It follows that:

$$\rho_{\max}(A) = \inf_{\mathbf{u}\in\mathbb{R}^n} \max_{1\leq i\leq n} (A\mathbf{u})_i \mathbin{/\!\!/} \mathbf{u}_i$$

(the product $A\mathbf{u}$ and the division by $\mathbf{u}_i$ should be understood in the max-plus sense). If $\rho_{\max}(A) > \mathbb{0}$, then this infimum is attained by some $\mathbf{u} \in \mathbb{R}^n$. If in addition A is irreducible, then Assertion (i) is equivalent to the following: (i') there exists $\mathbf{u} \in \mathbb{R}_{\max}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{u} \leq \lambda\mathbf{u}$.

2. [Gau92, Ch. IV], [BSvdD95] *Max-plus Collatz-Wielandt formula, II.* Let $\lambda \in \mathbb{R}_{\max}$. The following assertions are equivalent: (i) there exists $\mathbf{u} \in \mathbb{R}_{\max}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{u} \geq \lambda\mathbf{u}$; (ii) $\rho_{\max}(A) \geq \lambda$. It follows that:

$$\rho_{\max}(A) = \max_{\mathbf{u}\in\mathbb{R}_{\max}^n\setminus\{\mathbf{0}\}} \min_{\substack{1\leq i\leq n \\ \mathbf{u}_i\neq \mathbb{0}}} (A\mathbf{u})_i \mathbin{/\!\!/} \mathbf{u}_i \quad .$$

3. [Fri86] For $A \in \mathbb{R}_+^{n\times n}$, we have $\rho_\infty(A) = \exp(\rho_{\max}(\log(A)))$, where log is interpreted entrywise.

4. [KO85] For all $A \in \mathbb{R}_+^{n\times n}$, and $1 \leq q \leq p \leq \infty$, we have $\rho_p(A) \leq \rho_q(A)$.

5. For all $A, B \in \mathbb{R}_+^{n\times n}$, we have

$$\rho(A\circ B) \leq \rho_p(A)\rho_q(B) \text{ for all } p,q\in[1,\infty] \text{ such that } \frac{1}{p}+\frac{1}{q}=1 \ .$$

This follows from the classical Kingman's inequality [Kin61] which states that the map $\log\circ\rho\circ\exp$ is convex ($\exp$ is interpreted entrywise). We have in particular $\rho(A\circ B)\leq \rho_\infty(A)\rho(B)$.

6. [Fri86] For all $A\in\mathbb{R}_+^{n\times n}$, we have

$$\rho_\infty(A)\leq\rho(A)\leq\rho_\infty(A)\rho(\hat{A})\leq\rho_\infty(A)n \ ,$$

where $\hat{A}$ is the pattern matrix of A, that is, $\hat{A}_{ij}=1$ if $A_{ij}\neq 0$ and $\hat{A}_{ij}=0$ if $A_{ij}=0$.

7. [Bap98, EvdD99] For all $A\in\mathbb{R}_+^{n\times n}$, we have $\lim_{k\to\infty}(\rho_\infty(A^k))^{1/k}=\rho(A)$.

8. [CG79] *Computing $\rho_{\max}(A)$ by linear programming.* For $A\in\mathbb{R}_{\max}^{n\times n}$, $\rho_{\max}(A)$ is the value of the linear program

$$\inf\lambda \text{ s.t. } \exists \mathbf{u}\in\mathbb{R}^n, \forall(i,j)\in E,\ A_{ij}+\mathbf{u}_j\leq\lambda+\mathbf{u}_i$$

where $E=\{(i,j)\mid 1\leq i,j\leq n,\ A_{ij}\neq\mathbb{0}\}$ is the set of arcs of $\Gamma(A)$.

9. *Dual linear program to compute $\rho_{\max}(A)$.* Let $\mathscr{C}$ denote the set of nonnegative vectors $x=(x_{ij})_{(i,j)\in E}$ such that

$$\forall 1\leq i\leq n,\quad \sum_{1\leq k\leq n,\ (k,i)\in E} x_{ki} = \sum_{1\leq j\leq n, (i,j)\in E} x_{ij}, \quad \text{and} \quad \sum_{(i,j)\in E} x_{ij}=1 \ .$$

To every cycle c of $\Gamma(A)$ corresponds bijectively the extreme point of the polytope $\mathscr{C}$ which is given by $x_{ij}=1/|c|$ if (i,j) belongs to c, and $x_{ij}=0$ otherwise. Moreover, $\rho_{\max}(A)=\sup\{\sum_{(i,j)\in E}A_{ij}x_{ij}\mid x\in\mathscr{C}\}$.

10. [Kar78] *Karp's formula.* If $A\in\mathbb{R}_{\max}^{n\times n}$ is irreducible, then, for all $1\leq i\leq n$,

$$\rho_{\max}(A)=\max_{\substack{1\leq j\leq n\\ A^n_{ij}\neq\mathbb{0}}}\ \min_{1\leq k\leq n}\frac{(A^n)_{ij}-(A^{n-k})_{ij}}{k} \ . \tag{2}$$

To evaluate the right hand side expression, compute the sequence $\mathbf{u}^0=\mathbf{e}_i$, $\mathbf{u}^1=\mathbf{u}^0A$, $\mathbf{u}^n=\mathbf{u}^{n-1}A$, so that $\mathbf{u}^k=A^k_{i\cdot}$ for all $0\leq k\leq n$. This takes a time $O(nm)$, where m is the number of arcs of $\Gamma(A)$. One can avoid storing the vectors $\mathbf{u}^0,\dots,\mathbf{u}^n$, at the price of recomputing the sequence $\mathbf{u}^0,\dots,\mathbf{u}^{n-1}$ once $\mathbf{u}^n$ is known. The time and space complexity of Karp's algorithm are $O(nm)$ and $O(n)$, respectively. The policy iteration algorithm of [CTCG$^+$98] seems experimentally more efficient than Karp's algorithm. Other algorithms are given in particular in [CGL96], [BO93], [EvdD99]. A comparison of maximal cycle mean algorithms appears in [DGI98]. When the entries of A take only two finite values, the maximal cycle mean of A can be computed in linear time [CGB95]. The Karp and policy iteration algorithms, as well as the general max-plus operations (full and sparse matrix products, matrix residuation, etc.) are implemented in the **Maxplus toolbox** of **Scilab**, freely available in the contributed section of the web site www.scilab.org.

Example

1. For the matrix A in Application 3 of Section 1, we have $\rho_{\max}(A)=\max(5,4,(2+11)/2,(2+8)/2,(7+6)/2,(11+7+2)/3,(8+6+2)/3)=20/3$, which gives the maximal reward per year. This is attained by the cycle $(1,2),(2,3),(3,1)$, corresponding to the rotation of crops: fallow, wheat, oat.

3 The max-plus eigenproblem

The results of this section and of the next one constitute max-plus spectral theory. Early and fundamental contributions are due to Cuninghame-Green (see [CG79]), Vorobyev [Vor67], Romanovskiĭ [Rom67], Gondran and Minoux [GM77], and Cohen, Dubois, Quadrat, and Viot [CDQV83]. General presentations are included in [CG79, BCOQ92, GM02]. The infinite dimensional max-plus spectral theory (which is not covered here) has been developed particularly after Maslov, in relation with Hamilton-Jacobi partial differential equations, see [MS92, KM97]. See also [MPN02, AGW05, Fat05] for recent developments.

In this section and in the two next ones, A denotes a matrix in $\mathbb{R}_{\max}^{n\times n}$.

Definitions

An **eigenvector** of A is a vector $\mathbf{u} \in \mathbb{R}_{\max}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{u} = \lambda\mathbf{u}$, for some scalar $\lambda \in \mathbb{R}_{\max}$, which is called the **(geometric) eigenvalue** corresponding to $\mathbf{u}$. With the notation of classical algebra, the equation $A\mathbf{u} = \lambda\mathbf{u}$ can be rewritten as

$$\max_{1\leq j\leq n} A_{ij} + \mathbf{u}_j = \lambda + \mathbf{u}_i,\ \forall 1 \leq i \leq n\ .$$

If λ is an eigenvalue of A, the set of vectors $\mathbf{u} \in \mathbb{R}_{\max}^n$ such that $A\mathbf{u} = \lambda\mathbf{u}$ is the **eigenspace** of A for the eigenvalue λ.

The **saturation digraph** with respect to $\mathbf{u} \in \mathbb{R}_{\max}^n$, $\mathrm{Sat}(A, \mathbf{u})$, is the digraph with vertices $1, \ldots, n$ and an arc from vertex i to vertex j when $A_{ij}\mathbf{u}_j = (A\mathbf{u})_i$.

A cycle $c = ((i_1, i_2), \ldots, (i_k, i_1))$ that attains the maximum in (1) is called **critical**. The **critical digraph** is the union of the critical cycles. The **critical vertices** are the vertices of the critical digraph.

The **normalized matrix** is $\tilde{A} = \rho_{\max}(A)^{-1}A$ (when $\rho_{\max}(A) \neq \mathbb{0}$).

For a digraph Γ, vertex i **has access** to a vertex j, if there is a walk from i to j in Γ. The **(access equivalent) classes** of Γ are the equivalence classes of the set of its vertices for the relation "i has access to j and j has access to i". A class C **has access** to a class C' if some vertex of C has access to some vertex of C'. A class is **final** if it has access only to itself.

The **classes** of a matrix A are the classes of $\Gamma(A)$, and the **critical classes** of A are the classes of the critical digraph of A. A class C of A is **basic** if $\rho_{\max}(A[C, C]) = \rho_{\max}(A)$.

Facts The proof of most of the following facts can be found in particular in [CG79] or [BCOQ92, Section 3.7], we give specific references when needed.

1. For any matrix A, $\rho_{\max}(A)$ is an eigenvalue of A, and any eigenvalue of A is less than or equal to $\rho_{\max}(A)$.

2. An eigenvalue of A associated with an eigenvector in $\mathbb{R}^n$ must be equal to $\rho_{\max}(A)$.

3. [ES75] *Max-plus diagonal scaling.* Assume that $\mathbf{u} \in \mathbb{R}^n$ is an eigenvector of A. Then the matrix B such that $B_{ij} = \mathbf{u}_i^{-1}A_{ij}\mathbf{u}_j$ has all its entries less than or equal to $\rho_{\max}(A)$, and the maximum of every of its rows is equal to $\rho_{\max}(A)$.

4. If A is irreducible, then $\rho_{\max}(A) > \mathbb{0}$ and it is the only eigenvalue of A.

From now on, we assume that $\Gamma(A)$ *has at least one cycle, so that* $\rho_{\max}(A) > \mathbb{0}$.

5. For all critical vertices i of A, the column $\tilde{A}^{\star}_{\cdot i}$ is an eigenvector of A for the eigenvalue $\rho_{\max}(A)$. Moreover, if i and j belong to the same critical class of A, then $\tilde{A}^{\star}_{\cdot i} = \tilde{A}^{\star}_{\cdot j}\tilde{A}^{\star}_{ji}$.

6. *Eigenspace for the eigenvalue* $\rho_{\max}(A)$. Let $C_1,\dots,C_s$ denote the critical classes of A, and let us choose arbitrarily one vertex $i_t \in C_t$, for every $t=1,\dots,s$. Then, the columns $\tilde{A}^\star_{\cdot,i_t}$, $t=1,\dots,s$ span the eigenspace of A for the eigenvalue $\rho_{\max}(A)$. Moreover, any spanning family of this eigenspace contains some scalar multiple of every column $\tilde{A}^\star_{\cdot,i_t}$, $t=1,\dots,s$.

7. Let C denote the set of critical vertices, and let $T=\{1,\dots,n\}\setminus C$. The following facts are proved in a more general setting in [AG03, Th. 3.4], with the exception of (ii), which follows from Fact 4 of Section 1.

(i) The restriction $\mathbf{v}\mapsto \mathbf{v}[C]$ is an isomorphism from the eigenspace of A for the eigenvalue $\rho_{\max}(A)$ to the eigenspace of $A[C,C]$ for the same eigenvalue.

(ii) An eigenvector $\mathbf{u}$ for the eigenvalue $\rho_{\max}(A)$ is determined from its restriction $\mathbf{u}[C]$ by $\mathbf{u}[T] = (\tilde{A}[T,T])^\star \tilde{A}[T,C]\mathbf{u}[C]$.

(iii) Moreover, $\rho_{\max}(A)$ is the only eigenvalue of $A[C,C]$ and the eigenspace of $A[C,C]$ is stable by infimum and by convex combination in the usual sense.

8. *Complementary slackness.* If $\mathbf{u}\in\mathbb{R}^n_{\max}$ is such that $A\mathbf{u}\le \rho_{\max}(A)\mathbf{u}$, then $(A\mathbf{u})_i = \rho_{\max}(A)\mathbf{u}_i$, for all critical vertices i.

9. *Critical digraph vs saturation digraph.* Let $\mathbf{u}\in\mathbb{R}^n$ be such that $A\mathbf{u}\le\rho_{\max}(A)\mathbf{u}$. Then, the union of the cycles of $\mathrm{Sat}(A,\mathbf{u})$ is equal to the critical digraph of A.

10. [CQD90], [Gau92, Ch. IV], [BSvdD95] *Spectrum of reducible matrices.* A scalar $\lambda \neq \mathbb{0}$ is an eigenvalue of A if and only if there is at least one class C of A such that $\rho_{\max}(A[C,C])=\lambda$ and $\rho_{\max}(A[C,C])\ge\rho_{\max}(A[C',C'])$ for all classes C' that have access to C.

11. [CQD90], [BSvdD95] The matrix A has an eigenvector in $\mathbb{R}^n$ if and only if all its final classes are basic.

12. [Gau92, Ch. IV] *Eigenspace for an eigenvalue* λ. Let $C^1,\dots,C^m$ denote all the classes C of A such that $\rho_{\max}(A[C,C])=\lambda$ and $\rho_{\max}(A[C',C'])\le\lambda$ for all classes C' that have access to C. For every $1\le k\le m$, let $C^k_1,\dots,C^k_{s_k}$ denote the critical classes of the matrix $A[C^k,C^k]$. For all $1\le k\le m$ and $1\le t\le s_k$, let us choose arbitrarily an element $j_{k,t}$ in C^k_t. Then, the family of columns $(\lambda^{-1}A)^\star_{\cdot,j_{k,t}}$, indexed by all these k and t, spans the eigenspace of A for the eigenvalue λ, and any spanning family of this eigenspace contains a scalar multiple of every $(\lambda^{-1}A)^\star_{\cdot,j_{k,t}}$.

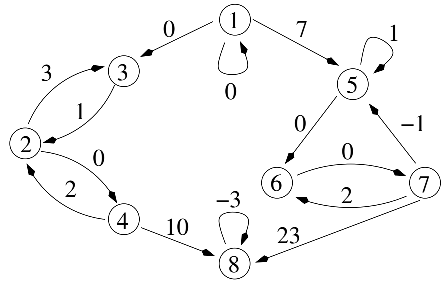
13. *Computing the eigenvectors.* Observe first that any vertex j which attains the maximum in Karp's formula (2) is critical. To compute one eigenvector for the eigenvalue $\rho_{\max}(A)$, it suffices to compute $\tilde{A}^\star_{\cdot j}$ for some critical vertex j. This is equivalent to a single source shortest path problem, which can be solved in $O(nm)$ time and $O(n)$ space. Alternatively, one may use the *policy iteration algorithm* of [CTCG$^+$98] or the improvement in [EvdD99] of the *power algorithm* [BO93]. Once a particular eigenvector is known, the critical digraph can be computed from Fact 9 in $O(m)$ additional time.

Examples

1. For the matrix A in Application 3 of Section 1, the only critical cycle is $(1,2),(2,3),(3,1)$ (up to a circular permutation of vertices). The critical digraph consists of the vertices and arcs of this cycle. By Fact 6, any eigenvector $\mathbf{u}$ of A is proportional to $\tilde{A}^\star_{\cdot 1} = [0\ {-13/3}\ {-14/3}]^T$ (or equivalently, to $\tilde{A}^\star_{\cdot 2}$ or $\tilde{A}^\star_{\cdot 3}$). Observe that an eigenvector yields a relative price information between the different states.

2. Consider the matrix and its associated digraph:

$$A = \left[\begin{array}{c|ccc|ccc|c} 0 & \cdot & 0 & \cdot & 7 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & 3 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 10 \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & 2 & \cdot & 23 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -3 \end{array}\right]$$



(We use $\cdot$ to represent the element $-\infty$.) The classes of A are $C^1 = \{1\}$, $C^2 = \{2,3,4\}$, $C^3 = \{5,6,7\}$ and $C^4 = \{8\}$. We have $\rho_{\max}(A) = \rho_{\max}(A[C^2,C^2]) = 2$, $\rho_{\max}(A[C^1,C^1]) = 0$, $\rho_{\max}(A[C^3,C^3]) = 1$, and $\rho_{\max}(A[C^4,C^4]) = -3$. The critical digraph is reduced to the critical cycle $(2,3)(3,2)$. By Fact 6, any eigenvector for the eigenvalue $\rho_{\max}(A)$ is proportional to $\tilde{A}^\star_{\cdot 2} = [-3\ 0\ -1\ 0\ -\infty\ -\infty\ -\infty\ -\infty]^T$. By Fact 10, the other eigenvalues of A are 0 and 1. By Fact 12, any eigenvector for the eigenvalue 0 is proportional to $A^\star_{\cdot 1} = \mathbf{e}_1$. Observe that the critical classes of $A[C^3,C^3]$ are $C^3_1 = \{5\}$ and $C^3_2 = \{6,7\}$. Therefore, by Fact 12, any eigenvector for the eigenvalue 1 is a max-plus linear combination of $(1^{-1}A)^\star_{\cdot 5} = [6\ -\infty\ -\infty\ -\infty\ 0\ -3\ -2\ -\infty]^T$ and $(1^{-1}A)^\star_{\cdot 6} = [5\ -\infty\ -\infty\ -\infty\ -1\ 0\ 1\ -\infty]^T$. The eigenvalues of A^T are $2, 1$ and -3. So A and A^T have only two eigenvalues in common.

4 Asymptotics of matrix powers

Definitions

A sequence $s_0, s_1, \ldots$ of elements of $\mathbb{R}_{\max}$ is **recognizable** if there exists a positive integer p, vectors $\mathbf{b} \in \mathbb{R}_{\max}^{p\times 1}$ and $\mathbf{c} \in \mathbb{R}_{\max}^{1\times p}$, and a matrix $M \in \mathbb{R}_{\max}^{p\times p}$ such that $s_k = \mathbf{c}M^k\mathbf{b}$, for all nonnegative integers k.

A sequence $s_0, s_1, \ldots$ of elements of $\mathbb{R}_{\max}$ is **ultimately geometric** with **rate** $\lambda \in \mathbb{R}_{\max}$ if $s_{k+1} = \lambda s_k$ for k large enough.

The **merge** of q sequences $s^1, \ldots, s^q$ is the sequence s such that $s_{kq+i-1} = s^i_k$, for all $k \geq 0$ and $1 \leq i \leq q$.

Facts

1. [Gun94, CTGG99] If every row of the matrix A has at least one entry different from $\mathbb{0}$, then, for all $1 \leq i \leq n$ and $\mathbf{u} \in \mathbb{R}^n$, the limit

$$\chi_i(A) = \lim_{k\to\infty} (A^k\mathbf{u})_i^{1/k} \quad ,$$

exists and is independent of the choice of $\mathbf{u}$. The vector $\chi(A) = (\chi_i(A))_{1\leq i\leq n} \in \mathbb{R}^n$ is called the **cycle-time** of A. It is given by

$$\chi_i(A) = \max\{\rho_{\max}(A[C,C]) \mid C \text{ is a class of } A \text{ to which } i \text{ has access}\} \ .$$

In particular, if A is irreducible, then $\chi_i(A) = \rho_{\max}(A)$ for all $i = 1, \ldots, n$.

2. The following constitutes the cyclicity theorem, due to Cohen, Dubois, Quadrat, and Viot [CDQV83]. See [BCOQ92] and [AGW05] for more accessible accounts.

(i) If A is irreducible, there exists a positive integer γ such that $A^{k+\gamma} = \rho_{\max}(A)^\gamma A^k$ for k large enough. The minimal value of γ is called the **cyclicity** of A.

(ii) Assume again that A is irreducible. Let $C_1, \ldots, C_s$ be the critical classes of A and for $i = 1, \ldots, s$, let γ_i denote the g.c.d. of the lengths of the critical cycles of A belonging to C_i. Then, the cyclicity γ of A is the l.c.m. of $\gamma_1, \ldots, \gamma_s$.

(iii) Assume that $\rho_{\max}(A) \neq \mathbb{0}$. The **spectral projector** of A is the matrix $P := \lim_{k\to\infty} \tilde{A}^k \tilde{A}^\star = \lim_{k\to\infty} \tilde{A}^k \oplus \tilde{A}^{k+1} \oplus \cdots$. It is given by $P = \sum_{i\in C} \tilde{A}^\star_{\cdot i}\tilde{A}^\star_{i\cdot}$, where C denotes the set of critical vertices of A. When A is irreducible, the limit is attained in finite time. If in addition A has cyclicity one, then $A^k = \rho_{\max}(A)^k P$ for k large enough.

3. Assume that A is irreducible, and let m denote the number of arcs of its critical digraph. Then, the cyclicity of A can be computed in $O(m)$ time from the critical digraph of A, using the algorithm of Denardo [Den77].

4. The smallest integer k such that $A^{k+\gamma} = \rho_{\max}(A)^\gamma A^k$ is called the **coupling time**. It is estimated in [HA99, BG01, AGW05] (assuming again that A is irreducible).

5. [AGW05, Th. 7.5] *Turnpike theorem.* Define a walk of $\Gamma(A)$ to be optimal if it has a maximal weight amongst all walks with the same ends and length. If A is irreducible, then the number of non-critical vertices of an optimal walk (counted with multiplicities) is bounded by a constant depending only on A.

6. [Mol88, Gau94, KB94, DS00] A sequence of elements of $\mathbb{R}_{\max}$ is recognizable if and only if it is a merge of ultimately geometric sequences. In particular, for all $1 \leq i,j \leq n$, the sequence $(A^k)_{ij}$ is a merge of ultimately geometric sequences.

7. [Sim78, Has90, Sim94, Gau96] One can decide whether a finitely generated semigroup S of matrices with effective entries in $\mathbb{R}_{\max}$ is finite. One can also decide whether the set of entries in a given position of the matrices of S is finite (limitedness problem). However [Kro94], whether this set contains a given entry is undecidable (even when the entries of the matrices belong to $\mathbb{Z} \cup \{-\infty\}$).

Examples

1. For the matrix A in Application 3 of Section 1, the cyclicity is 3, and the spectral projector is

$$P = \tilde{A}^\star_{\cdot 1}\tilde{A}^\star_{1\cdot} = \begin{bmatrix} 0 \\ -13/3 \\ -14/3 \end{bmatrix} \begin{bmatrix} 0 & 13/3 & 14/3 \end{bmatrix}^T = \begin{bmatrix} 0 & 13/3 & 14/3 \\ -13/3 & 0 & 1/3 \\ -14/3 & -1/3 & 0 \end{bmatrix} .$$

2. For the matrix A in Example 2 of Section 3, the cycle-time is $\chi(A) = [2\ 2\ 2\ 2\ 1\ 1\ 1\ -3]^T$. The cyclicity of $A[C^2, C^2]$ is 2, because there is only one critical cycle, which has length 2. Let $B := A[C^3, C^3]$. The critical digraph of B has two strongly connected components, consisting respectively of the cycles $(5,5)$ and $(6,7),(7,6)$. So B has cyclicity $\mathrm{lcm}(1,2) = 2$. The sequence $s_k := (A^k)_{18}$ is such that $s_{k+2} = s_k + 4$, for $k \geq 24$, with $s_{24} = s_{25} = 51$. Hence, s_k is the merge of two ultimately geometric sequences, both with rate 4. To get an example where different rates appear, replace the entries A_{11} and A_{88} of A, by $-\infty$. Then, the same sequence s_k is such that $s_{k+2} = s_k + 4$, for all even $k \geq 24$, and $s_{k+2} = s_k + 2$, for all odd $k \geq 5$, with $s_5 = 31$ and $s_{24} = 51$.

5 The max-plus permanent

Definitions

The (max-plus) **permanent** of A is $\operatorname{per} A = \sum_{\sigma\in S_n} A_{1\sigma(1)} \cdots A_{n\sigma(n)}$, or with the usual notation of classical algebra, $\operatorname{per} A = \max_{\sigma\in S_n} A_{1\sigma(1)} + \cdots + A_{n\sigma(n)}$, which is the value of the optimal assignment problem with weights A_{ij}.

A **max-plus polynomial function** P is a map $\mathbb{R}_{\max} \to \mathbb{R}_{\max}$ of the form $P(x) = \sum_{i=0}^{n} p_i x^i$ with $p_i \in \mathbb{R}_{\max}$, $i = 0, \ldots, n$. If $p_n \neq \mathbb{0}$, P is of **degree** n.

The **roots** of a non-zero max-plus polynomial function P are the points of non-differentiability of P, together with the point $\mathbb{0}$ when the derivative of P near $-\infty$ is positive. The **multiplicity** of a root α of P is defined as the variation of the derivative of P at the point α, $P'(\alpha^+) - P'(\alpha^-)$, when $\alpha \neq \mathbb{0}$, and as its derivative near $-\infty$, $P'(\mathbb{0}^+)$, when $\alpha = \mathbb{0}$.

The (max-plus) **characteristic polynomial function** of A is the polynomial function P_A given by $P_A(x) = \mathrm{per}(A + xI)$ for $x \in \mathbb{R}_{\max}$. The **algebraic eigenvalues** of A are the roots of P_A.

Facts

1. [CGM80] Any non-zero max-plus polynomial function P can be factored uniquely as $P(x) = a(x + \alpha_1)\cdots(x + \alpha_n)$, where $a \in \mathbb{R}$, n is the degree of P and the α_i are the roots of P, counted with multiplicities.

2. [CG83], [ABG04, Th. 4.6 and 4.7]. The greatest algebraic eigenvalue of A is equal to $\rho_{\max}(A)$. Its multiplicity is less than or equal to the number of critical vertices of A, with equality if and only if the critical vertices can be covered by disjoint critical cycles.

3. Any geometric eigenvalue of A is an algebraic eigenvalue of A (this can be deduced from Fact 2 of this section, and Fact 10 of Section 3).

4. [Yoe61] If $A \geq I$ and $\mathrm{per}\, A = \mathbb{1}$, then $A^\star_{ij} = \mathrm{per}\, A(j,i)$, for all $1 \leq i, j \leq n$.

5. [But00] Assume that all the entries of A are different from $\mathbb{0}$. The following are equivalent: (i) there is a vector $b \in \mathbb{R}^n$ that has a unique preimage by A; (ii) there is only one permutation σ such that $|\sigma|_A := A_{1\sigma(1)} \cdots A_{n\sigma(n)} = \mathrm{per}\, A$. Further characterizations can be found in [But00, DSS05].

6. [Bap95] *Alexandroff inequality over* $\mathbb{R}_{\max}$. Construct the matrix B with columns $A_{\cdot 1}$, $A_{\cdot 1}$, $A_{\cdot 3}$,..., $A_{\cdot n}$ and the matrix C with columns $A_{\cdot 2}$, $A_{\cdot 2}$, $A_{\cdot 3}$,..., $A_{\cdot n}$. Then $(\mathrm{per}\, A)^2 \geq (\mathrm{per}\, B)(\mathrm{per}\, C)$, or with the notation of classical algebra, $2 \times \mathrm{per}\, A \geq \mathrm{per}\, B + \mathrm{per}\, C$.

7. [BB03] The max-plus characteristic polynomial function of A can be computed by solving $O(n)$ optimal assignment problems.

Example

1. For the matrix A in Example 2 of Section 3, the characteristic polynomial of A is the product of the characteristic polynomials of the matrices $A[C^i, C^i]$, for $i = 1, \ldots, 4$. Thus, $P_A(x) = (x + 0)(x + 2)^2 x(x + 1)^3(x + (-3))$, and so, the algebraic eigenvalues of A are $-\infty, -3, 0, 1$ and 2, with respective multiplicities 1, 1, 1, 3 and 2.

6 Linear inequalities and projections

Definitions

If $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$, the **range** of A, denoted $\mathrm{range}\, A$, is $\{A\mathbf{x} \mid \mathbf{x} \in \overline{\mathbb{R}}_{\max}^p\} \subset \overline{\mathbb{R}}_{\max}^n$. The **kernel** of A, denoted $\ker A$, is the set of **equivalence classes modulo** A, which are the classes for the equivalence relation "$\mathbf{x} \sim \mathbf{y}$ if $A\mathbf{x} = A\mathbf{y}$".

The **support** of a vector $\mathbf{b} \in \overline{\mathbb{R}}_{\max}^n$ is $\mathrm{supp}\, \mathbf{b} := \{i \in \{1, \ldots, n\} \mid \mathbf{b}_i \neq \mathbb{0}\}$.

The **orthogonal congruence** of a subset U of $\overline{\mathbb{R}}_{\max}^n$ is $U^\perp := \{(\mathbf{x}, \mathbf{y}) \in \overline{\mathbb{R}}_{\max}^n \times \overline{\mathbb{R}}_{\max}^n \mid \mathbf{u}\cdot\mathbf{x} = \mathbf{u}\cdot\mathbf{y}\ \forall \mathbf{u} \in U\}$, where "$\cdot$" denotes the max-plus scalar product. The **orthogonal space** of a subset C of $\overline{\mathbb{R}}_{\max}^n \times \overline{\mathbb{R}}_{\max}^n$ is $C^\top := \{\mathbf{u} \in \overline{\mathbb{R}}_{\max}^n \mid \mathbf{u}\cdot\mathbf{x} = \mathbf{u}\cdot\mathbf{y}\ \forall(\mathbf{x}, \mathbf{y}) \in C\}$.

Facts

1. For all a, $b \in \overline{\mathbb{R}}_{\max}$, the maximal $c \in \overline{\mathbb{R}}_{\max}$ such that $ac \leq b$, denoted by $a \backslash b$ (or b / a), is given by $a \backslash b = b - a$ if $(a, b) \notin \{(-\infty, -\infty), (+\infty, +\infty)\}$, and $a \backslash b = +\infty$ otherwise.

2. [BCOQ92, Eqn 4.82] If $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$ and $B \in \overline{\mathbb{R}}_{\max}^{n\times q}$, then the inequation $AX \leq B$ has a maximal solution $X \in \overline{\mathbb{R}}_{\max}^{p\times q}$ given by the matrix $A \setminus B$ defined by $(A \setminus B)_{ij} = \bigwedge_{1\leq k\leq n} A_{ki} \setminus B_{kj}$. Similarly, for $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$ and $C \in \overline{\mathbb{R}}_{\max}^{r\times p}$, the maximal solution $C /\!\!/ A \in \overline{\mathbb{R}}_{\max}^{r\times n}$ of $XA \leq C$ exists and is given by $(C /\!\!/ A)_{ij} = \bigwedge_{1\leq k\leq p} C_{ik} /\!\!/ A_{jk}$.

3. The equation $AX = B$ has a solution if and only if $A(A \setminus B) = B$.

4. For $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$, the map $A^\sharp : \mathbf{y} \in \overline{\mathbb{R}}_{\min}^{n} \to A \setminus \mathbf{y} \in \overline{\mathbb{R}}_{\min}^{p}$ is linear. It is represented by the matrix $-A^T$.

5. [BCOQ92, Table 4.1] For matrices A, B, C with entries in $\overline{\mathbb{R}}_{\max}$ and with appropriate dimensions, we have:

$$A(A \setminus (AB)) = AB, \qquad A \setminus (A(A \setminus B)) = A \setminus B,$$
$$(A + B) \setminus C = (A \setminus C) \wedge (B \setminus C), \qquad A \setminus (B \wedge C) = (A \setminus B) \wedge (A \setminus C),$$
$$(AB) \setminus C = B \setminus (A \setminus C), \qquad A \setminus (B /\!\!/ C) = (A \setminus B) /\!\!/ C.$$

The first five identities have dual versions, with $/\!\!/$ instead of $\setminus$. Due to the last identity, we shall write $A \setminus B /\!\!/ C$ instead of $A \setminus (B /\!\!/ C)$.

6. [CGQ97] Let $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$, $B \in \overline{\mathbb{R}}_{\max}^{n\times q}$ and $C \in \overline{\mathbb{R}}_{\max}^{r\times p}$. We have $\operatorname{range} A \subset \operatorname{range} B \iff A = B(B \setminus A)$, and $\ker A \subset \ker C \iff C = (C /\!\!/ A)A$.

7. [CGQ96] Let $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$. The map $\Pi_A := A \circ A^\sharp$ is a projector on the range of A, meaning that $(\Pi_A)^2 = \Pi_A$ and $\operatorname{range} \Pi_A = \operatorname{range} A$. Moreover, $\Pi_A(x)$ is the greatest element of the range of A which is less than or equal to x. Similarly, the map $\Pi^A := A^\sharp \circ A$ is a projector on the range of $A^\sharp$, and $\Pi^A(x)$ is the smallest element of the range of $A^\sharp$ which is greater than or equal to x. Finally, every equivalence class modulo A meets the range of $A^\sharp$ at a unique point.

8. [CGQ04, DS04] For any $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$, the map $x \mapsto A(-x)$ is a bijection from $\operatorname{range}(A^T)$ to $\operatorname{range}(A)$, with inverse map $x \mapsto A^T(-x)$.

9. [CGQ96, CGQ97] *Projection onto a range parallel to a kernel.* Let $B \in \overline{\mathbb{R}}_{\max}^{n\times p}$ and $C \in \overline{\mathbb{R}}_{\max}^{q\times n}$. For all $x \in \overline{\mathbb{R}}_{\max}^{n}$, there is a greatest ξ on the range of B such that $C\xi \leq Cx$. It is given by $\Pi_B^C(x)$, where $\Pi_B^C := \Pi_B \circ \Pi^C$. We have $(\Pi_B^C)^2 = \Pi_B^C$. Assume now that every equivalence class modulo C meets the range of B at a unique point. This is the case if, and only if, $\operatorname{range}(CB) = \operatorname{range} C$ and $\ker(CB) = \ker B$. Then $\Pi_B^C(x)$ is the unique element of the range of B which is equivalent to x modulo C, the map Π_B^C is a linear projector on the range of B, and it is represented by the matrix $(B /\!\!/ (CB))C$ which is equal to $B((CB) \setminus C)$.

10. [CGQ97] *Regular matrices.* Let $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$. The following assertions are equivalent: (i) there is a linear projector from $\overline{\mathbb{R}}_{\max}^{n}$ to $\operatorname{range} A$; (ii) $A = AXA$ for some $X \in \overline{\mathbb{R}}_{\max}^{p\times n}$; (iii) $A = A(A \setminus A /\!\!/ A)A$.

11. [Vor67], [Zim76, Ch. 3] (see also [But94, AGK05]). *Vorobyev-Zimmermann covering theorem.* Assume that $A \in \mathbb{R}_{\max}^{n\times p}$ and $\mathbf{b} \in \overline{\mathbb{R}}_{\max}^{n}$. For $j \in \{1, \ldots, p\}$, let

$$S_j = \{i \in \{1, \ldots, n\} \mid A_{ij} \neq \mathbb{0} \text{ and } A_{ij} \setminus \mathbf{b}_i = (A \setminus \mathbf{b})_j\} \enspace .$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\cup_{1\leq j\leq p} S_j \supset \operatorname{supp} \mathbf{b}$ or equivalently $\cup_{j\in \operatorname{supp}(A \setminus \mathbf{b})} S_j \supset \operatorname{supp} \mathbf{b}$. It has a unique solution if, and only if, $\cup_{j\in \operatorname{supp}(A \setminus \mathbf{b})} S_j \supset \operatorname{supp} \mathbf{b}$ and $\cup_{j\in J} S_j \not\supset \operatorname{supp} \mathbf{b}$ for all strict subsets J of $\operatorname{supp}(A \setminus \mathbf{b})$.

12. [Zim77, SS92, CGQ04, CGQS05, DS04] *Separation theorem.* Let $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$ and $\mathbf{b} \in \overline{\mathbb{R}}_{\max}^{n}$. If $\mathbf{b} \notin \operatorname{range} A$, then there exists $\mathbf{c}, \mathbf{d} \in \overline{\mathbb{R}}_{\max}^{n}$ such that the **halfspace** $H := \{\mathbf{x} \in \overline{\mathbb{R}}_{\max}^{n} \mid \mathbf{c} \cdot \mathbf{x} \geq \mathbf{d} \cdot \mathbf{x}\}$ contains $\operatorname{range} A$ but not $\mathbf{b}$. We can take $\mathbf{c} = -\mathbf{b}$ and $\mathbf{d} = -\Pi_A(\mathbf{b})$. Moreover, when A and $\mathbf{b}$ have entries in $\mathbb{R}_{\max}$, $\mathbf{c}, \mathbf{d}$ can be chosen with entries in $\mathbb{R}_{\max}$.

13. [GP97] For any $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$, we have $((\operatorname{range} A)^{\perp})^{\top} = \operatorname{range} A$.

14. [LMS01, CGQ04] A linear form defined on a finitely generated subsemimodule of $\overline{\mathbb{R}}_{\max}^{n}$ can be extended to $\overline{\mathbb{R}}_{\max}^{n}$. This is a special case of a max-plus analogue of the Riesz representation theorem.

15. [BH84, GP97] Let $A, B \in \overline{\mathbb{R}}_{\max}^{n\times p}$. The set of solutions $\mathbf{x} \in \overline{\mathbb{R}}_{\max}^{p}$ of $A\mathbf{x} = B\mathbf{x}$ is a finitely generated subsemimodule of $\overline{\mathbb{R}}_{\max}^{p}$.

16. [GP97, Gau98] Let X, Y be finitely generated subsemimodules of $\overline{\mathbb{R}}_{\max}^{n}$, $A \in \overline{\mathbb{R}}_{\max}^{n\times p}$ and $B \in \overline{\mathbb{R}}_{\max}^{r\times n}$. Then $X \cap Y$, $X + Y := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in X,\ \mathbf{y} \in Y\}$, and $X - Y := \{\mathbf{z} \in \overline{\mathbb{R}}_{\max}^{n} \mid \exists \mathbf{x} \in X, \mathbf{y} \in Y,\ \mathbf{x} = \mathbf{y} + \mathbf{z}\}$ are finitely generated subsemimodules of $\overline{\mathbb{R}}_{\max}^{n}$. Also, $A^{-1}(X)$, $B(X)$, and $X^{\perp}$ are finitely generated subsemimodules of $\overline{\mathbb{R}}_{\max}^{p}$, $\overline{\mathbb{R}}_{\max}^{r}$, and $\overline{\mathbb{R}}_{\max}^{n} \times \overline{\mathbb{R}}_{\max}^{n}$, respectively. Similarly, if Z is a finitely generated subsemimodule of $\overline{\mathbb{R}}_{\max}^{n} \times \overline{\mathbb{R}}_{\max}^{n}$, then $Z^{\top}$ is a finitely generated subsemimodule of $\overline{\mathbb{R}}_{\max}^{n}$.

17. Facts 13–16 still hold if $\overline{\mathbb{R}}_{\max}$ is replaced by $\mathbb{R}_{\max}$.

18. When $A, B \in \mathbb{R}_{\max}^{n\times p}$, algorithms to find one solution of $A\mathbf{x} = B\mathbf{x}$ are given in [WB98] or [CGB03]. One can also use the general algorithm of [GG98] to compute a finite fixed point of a min-max function, together with the observation that $\mathbf{x}$ satisfies $A\mathbf{x} = B\mathbf{x}$ if and only if $\mathbf{x} = f(\mathbf{x})$, where $f(\mathbf{x}) = \mathbf{x} \wedge (A \setminus\!\!\setminus (B\mathbf{x})) \wedge (B \setminus\!\!\setminus (A\mathbf{x}))$.

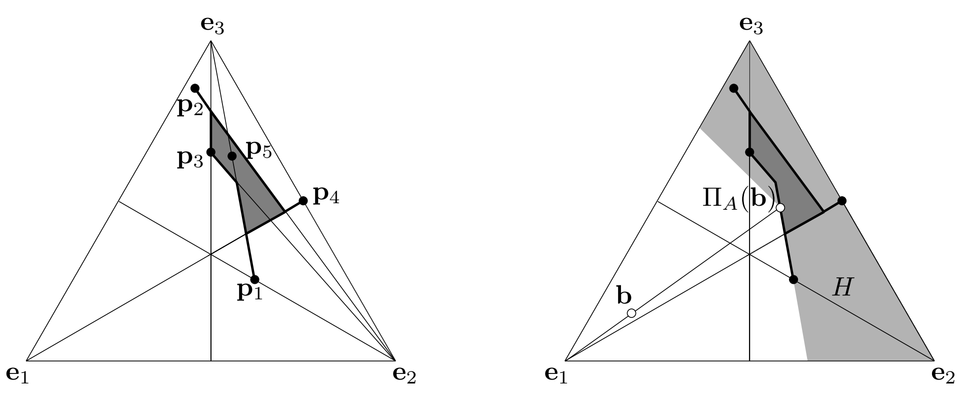
Examples

1. In order to illustrate Fact 11, consider

$$A = \begin{bmatrix} 0 & 0 & 0 & -\infty & 0.5 \\ 1 & -2 & 0 & 0 & 1.5 \\ 0 & 3 & 2 & 0 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 0.5 \end{bmatrix}. \tag{3}$$

Let $\bar{\mathbf{x}} := A \setminus\!\!\setminus \mathbf{b}$. We have $\bar{\mathbf{x}}_1 = \min(-0 + 3, -1 + 0, -0 + 0.5) = -1$, and so, $S_1 = \{2\}$, because the minimum is attained only by the second term. Similarly, $\bar{\mathbf{x}}_2 = -2.5$, $S_2 = \{3\}$, $\bar{\mathbf{x}}_3 = -1.5$, $S_3 = \{3\}$, $\bar{\mathbf{x}}_4 = 0$, $S_4 = \{2\}$, $\bar{\mathbf{x}}_5 = -2.5$, $S_5 = \{3\}$. Since $\cup_{1\le j\le 5} S_j = \{2, 3\} \not\supset \operatorname{supp} \mathbf{b} = \{1, 2, 3\}$, Fact 11 shows that the equation $Ax = \mathbf{b}$ has no solution. This also follows from the fact that $\Pi_A(\mathbf{b}) = A(A \setminus\!\!\setminus \mathbf{b}) = [-1\ 0\ 0.5]^T < \mathbf{b}$.

2. The range of the previous matrix A is represented on the following picture (left).



A non-zero vector $\mathbf{x} \in \mathbb{R}_{\max}^{3}$ is represented by the point that is the barycenter with weights $(\exp(\beta\mathbf{x}_i))_{1\le i\le 3}$ of the vertices of the simplex, where $\beta > 0$ is a fixed scaling parameter. Every vertex of the simplex represents one basis vector $\mathbf{e}_i$. Proportional vectors are represented by the same point. The i-th column of A, $A_{\cdot i}$, is represented by the point $\mathbf{p}_i$ on the figure. Observe that the broken segment from $\mathbf{p}_1$ to $\mathbf{p}_2$, which represents the semimodule generated by $A_{\cdot 1}$ and $A_{\cdot 2}$, contains $\mathbf{p}_5$. Indeed, $A_{\cdot 5} = 0.5A_{\cdot 1} + A_{\cdot 2}$. The range of A is represented by the closed region in dark grey and by the bold segments joining the points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4$ to it.

We next compute a half-space separating the point b defined in (3) from range A. Recall that $\Pi_A(\mathbf{b}) = [-1\ 0\ 0.5]^T$. So, by Fact 12, a half-space containing range A and not $\mathbf{b}$ is $H := \{\mathbf{x} \in \overline{\mathbb{R}}^3_{\max} \mid (-3)\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus (-0.5)\mathbf{x}_3 \geq 1\mathbf{x}_1 \oplus \mathbf{x}_2 \oplus (-0.5)\mathbf{x}_3\}$. We also have $H \cap \mathbb{R}^3_{\max} = \{\mathbf{x} \in \mathbb{R}^3_{\max} \mid \mathbf{x}_2 \oplus (-0.5)\mathbf{x}_3 \geq 1\mathbf{x}_1\}$. The set of non-zero points of $H \cap \mathbb{R}^3_{\max}$ are represented by the light gray region on the picture, at right.

7 Max-plus linear independence and rank

Definitions

If M is a subsemimodule of $\mathbb{R}^n_{\max}$, $\mathbf{u} \in M$ is an **extremal generator** of M, or $\mathbb{R}_{\max}\mathbf{u} := \{\lambda.\mathbf{u} \mid \lambda \in \mathbb{R}_{\max}\}$ is an **extreme ray** of M, if $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} = \mathbf{v} \oplus \mathbf{w}$ with $\mathbf{v}, \mathbf{w} \in M$ imply that $\mathbf{u} = \mathbf{v}$ or $\mathbf{u} = \mathbf{w}$.

A family $\mathbf{u}_1, \ldots, \mathbf{u}_r$ of vectors of $\mathbb{R}^n_{\max}$ is **linearly independent in the Gondran-Minoux sense** if for all disjoints subsets I and J of $\{1, \ldots, r\}$, and all $\lambda_i \in \mathbb{R}_{\max}$, $i \in I \cup J$, we have $\sum_{i \in I} \lambda_i.\mathbf{u}_i \neq \sum_{j \in J} \lambda_j.\mathbf{u}_j$, unless $\lambda_i = \mathbb{0}$ for all $i \in I \cup J$.

For $A \in \mathbb{R}^{n \times n}_{\max}$, we define

$$\det{}^+ A := \sum_{\sigma \in S_n^+} A_{1\sigma(1)} \cdots A_{n\sigma(n)}, \qquad \det{}^- A := \sum_{\sigma \in S_n^-} A_{1\sigma(1)} \cdots A_{n\sigma(n)} \ ,$$

where S_n^+ and S_n^- are respectively the sets of even and odd permutations of $\{1, \ldots, n\}$. The **bideterminant** [GM84] of A is $(\det^+ A, \det^- A)$.

For $A \in \mathbb{R}^{n \times p}_{\max} \setminus \{\mathbf{0}\}$, we define

- the **row rank** (resp. the **column rank**) of A, denoted $\mathrm{rk}_{\mathrm{row}}(A)$ (resp. $\mathrm{rk}_{\mathrm{col}}(A)$), as the number of extreme rays of range A^T (resp. range A);
- the **Schein** rank of A as $\mathrm{rk}_{\mathrm{Sch}}(A) := \min\{r \geq 1 \mid A = BC, \text{ with } B \in \mathbb{R}^{n \times r}_{\max}, C \in \mathbb{R}^{r \times p}_{\max}\}$.
- the **strong rank** of A, denoted $\mathrm{rk}_{\mathrm{st}}(A)$, as the maximal $r \geq 1$ such that there exists a $r \times r$ submatrix B of A for which there is only one permutation σ such that $|\sigma|_B = \mathrm{per}\, B$;
- the row (resp. column) **Gondran-Minoux rank** of A, denoted $\mathrm{rk}_{\mathrm{GMr}}(A)$ (resp. $\mathrm{rk}_{\mathrm{GMc}}$), as the maximal $r \geq 1$ such that A has r linearly independent rows (resp. columns) in the Gondran-Minoux sense;
- the **symmetrized rank** of A, denoted $\mathrm{rk}_{\mathrm{sym}}(A)$, as the maximal $r \geq 1$ such that A has a $r \times r$ submatrix B such that $\det^+ B \neq \det^- B$.

(A new rank notion, **Kapranov rank**, which is not discussed here, has been recently studied [DSS05]. We also note that the Schein rank is called in this reference Barvinok rank.)

Facts

1. [Hel88, Mol88, Wag91, Gau98, DS04, CGQ05] Let M be a finitely generated subsemimodule of $\mathbb{R}^n_{\max}$. A subset of vectors of M spans M if, and only if, it contains at least one non-zero element of every extreme ray of M.

2. [GM02] The columns of $A \in \mathbb{R}^{n \times n}_{\max}$ are linearly independent in the Gondran-Minoux sense if and only if $\det^+ A \neq \det^- A$.

3. [Plu90], [BCOQ92, Th. 3.78]. *Max-plus Cramer's formula.* Let $A \in \mathbb{R}^{n \times n}_{\max}$, let $\mathbf{b}^-, \mathbf{b}^+ \in \mathbb{R}^n_{\max}$. Define the i-th positive Cramer's determinant by

$$D_i^+ := \det{}^+(A_{\cdot 1} \ldots A_{\cdot, i-1} \mathbf{b}^+ A_{\cdot, i+1} \ldots A_{\cdot n}) \oplus \det{}^-(A_{\cdot 1} \ldots A_{\cdot, i-1} \mathbf{b}^- A_{\cdot, i+1} \ldots A_{\cdot n}) \ ,$$

and the i-th negative Cramer's determinant, D_i^-, by exchanging $\mathbf{b}^+$ and $\mathbf{b}^-$ in the definition of D_i^+. Assume that $\mathbf{x}^+, \mathbf{x}^- \in \mathbb{R}^n_{\max}$ have disjoint supports. Then, $A\mathbf{x}^+ \oplus \mathbf{b}^- = A\mathbf{x}^- \oplus \mathbf{b}^+$ implies that

$$(\det{}^+ A)\mathbf{x}_i^+ \oplus (\det{}^- A)\mathbf{x}_i^- \oplus D_i^- = (\det{}^- A)\mathbf{x}_i^+ \oplus (\det{}^+ A)\mathbf{x}_i^- \oplus D_i^+ \;\; \forall 1 \leq i \leq n \ . \tag{4}$$

The converse implication holds, and the vectors $\mathbf{x}^+$ and $\mathbf{x}^-$ are uniquely determined by (4), if $\det^+ A \neq \det^- A$, and if $D_i^+ \neq D_i^-$ or $D_i^+ = D_i^- = \mathbb{0}$, for all $1 \leq i \leq n$. This result is formulated in a simpler way in [Plu90, BCOQ92] using the **symmetrization** of the max-plus semiring, which leads to more general results. We note that the converse implication relies on the following semiring analogue of the classical adjugate identity: $A \operatorname{adj}^+ A \oplus \det^- A\, I = A \operatorname{adj}^- A \oplus \det^+ A\, I$, where $\operatorname{adj}^\pm A := (\det^\pm A(j,i))_{1 \leq i,j \leq n}$. This identity, as well as analogues of many other determinantal identities, can be obtained using the general method of [RS84]. See for instance [GBCG98], where the derivation of the Binet-Cauchy identity is detailed.

4. For $A \in \mathbb{R}^{n \times p}_{\max}$, we have

$$\mathrm{rk}_{\mathrm{st}}(A) \leq \mathrm{rk}_{\mathrm{sym}}(A) \leq \left\{ \begin{array}{l} \mathrm{rk}_{\mathrm{GMr}}(A) \\ \mathrm{rk}_{\mathrm{GMc}}(A) \end{array} \right\} \leq \mathrm{rk}_{\mathrm{Sch}}(A) \leq \left\{ \begin{array}{l} \mathrm{rk}_{\mathrm{row}}(A) \\ \mathrm{rk}_{\mathrm{col}}(A) \end{array} \right. \ .$$

The second inequality follows from Fact 2, the third one follows from Facts 2 and 3. The other inequalities are immediate. Moreover, all these inequalities become equalities if A is regular [CGQ05].

Example

1. The matrix A in Example 1 of Section 6 has column rank 4: the extremal rays of range A are generated by the first four columns of A. All the other ranks of A are equal to 3.

References

[ABG04] M. Akian, R. Bapat, and S. Gaubert. Min-plus methods in eigenvalue perturbation theory and generalised Lidskiĭ-Višik-Ljusternik theorem. arXiv:math.SP/0402090, 04.

[AG03] M. Akian and S. Gaubert. Spectral theorem for convex monotone homogeneous maps, and ergodic control. *Nonlinear Anal.*, 52(2):637–679, 2003.

[AGK05] M. Akian, S. Gaubert, and V. Kolokoltsov. Set coverings and invertibility of functional Galois connections. In *Idempotent Mathematics and Mathematical Physics*, Contemp. Math., pages 19–51. Amer. Math. Soc., 2005.

[AGW05] M. Akian, S. Gaubert, and C. Walsh. Discrete max-plus spectral theory. In *Idempotent Mathematics and Mathematical Physics*, Contemp. Math., pages 19–51. Amer. Math. Soc., 2005.

[Bac03] N. Bacaër. Modèles mathématiques pour l'optimisation des rotations. *Comptes Rendus de l'Académie d'Agriculture de France*, 89(3):52, 2003. Electronic version available on www.academie-agriculture.fr.

[Bap95] R. B. Bapat. Permanents, max algebra and optimal assignment. *Linear Algebra Appl.*, 226/228:73–86, 1995.

[Bap98] R. B. Bapat. A max version of the Perron-Frobenius theorem. In *Proceedings of the Sixth Conference of the International Linear Algebra Society (Chemnitz, 1996)*, volume 275/276, pages 3–18, 1998.

[BB03] R. E. Burkard and P. Butkovič. Finding all essential terms of a characteristic maxpolynomial. *Discrete Appl. Math.*, 130(3):367–380, 2003.

[BCOQ92] F. Baccelli, G. Cohen, G.-J. Olsder, and J.-P. Quadrat. *Synchronization and Linearity.* Wiley, 1992.

[BG01] A. Bouillard and B. Gaujal. Coupling time of a (max,plus) matrix. In *Proceedings of the Workshop on Max-Plus Algebras, a satellite event of the first IFAC Symposium on System, Structure and Control (Praha, 2001).* Elsevier, 2001.

[BH84] P. Butkovič and G. Hegedűs. An elimination method for finding all solutions of the system of linear equations over an extremal algebra. *Ekonom.-Mat. Obzor*, 20(2):203–215, 1984.

[BO93] J. G. Braker and G. J. Olsder. The power algorithm in max algebra. *Linear Algebra Appl.*, 182:67–89, 1993.

[BSvdD95] R.B. Bapat, D. Stanford, and P. van den Driessche. Pattern properties and spectral inequalities in max algebra. *SIAM Journal of Matrix Analysis and Applications*, 16(3):964–976, 1995.

[But94] P. Butkovič. Strong regularity of matrices—a survey of results. *Discrete Appl. Math.*, 48(1):45–68, 1994.

[But00] P. Butkovic. Simple image set of (max, +) linear mappings. *Discrete Appl. Math.*, 105(1-3):73–86, 2000.

[CDQV83] G. Cohen, D. Dubois, J.-P. Quadrat, and M. Viot. Analyse du comportement périodique des systèmes de production par la théorie des dioïdes. Rapport de recherche 191, INRIA, Le Chesnay, France, 1983.

[CDQV85] G. Cohen, D. Dubois, J.-P. Quadrat, and M. Viot. A linear system theoretic view of discrete event processes and its use for performance evaluation in manufacturing. *IEEE Trans. on Automatic Control*, AC–30:210–220, 1985.

[CG79] R. A. Cuninghame-Green. *Minimax Algebra*, volume 166 of *Lect. Notes in Econom. and Math. Systems.* Springer-Verlag, Berlin, 1979.

[CG83] R. A. Cuninghame-Green. The characteristic maxpolynomial of a matrix. *J. of Math. Analysis and Appl.*, 95:110–116, 1983.

[CGB95] R. A. Cuninghame-Green and P. Butkovič. Extremal eigenproblem for bivalent matrices. *Linear Algebra Appl.*, 222:77–89, 1995.

[CGB03] R. A. Cuninghame-Green and P. Butkovic. The equation $A \otimes x = B \otimes y$ over $(\max, +)$. *Theoret. Comput. Sci.*, 293(1):3–12, 2003.

[CGL96] R. A. Cuninghame-Green and Y. Lin. Maximum cycle-means of weighted digraphs. *Appl. Math. JCU*, 11B:225–234, 1996.

[CGM80] R. A. Cuninghame-Green and P. F. J. Meijer. An algebra for piecewise-linear minimax problems. *Discrete Appl. Math*, 2:267–294, 1980.

[CGQ96] G. Cohen, S. Gaubert, and J.-P. Quadrat. Kernels, images and projections in dioids. In *Proceedings of WODES'96*, pages 151–158, Edinburgh, August 1996. IEE.

[CGQ97] G. Cohen, S. Gaubert, and J.-P. Quadrat. Linear projectors in the max-plus algebra. In *Proceedings of the IEEE Mediterranean Conference*, Cyprus, 1997. IEEE.

[CGQ04] G. Cohen, S. Gaubert, and J.-P. Quadrat. Duality and separation theorems in idempotent semimodules. *Linear Algebra and Appl.*, 379:395–422, 2004.

[CGQ05] G. Cohen, S. Gaubert, and J.-P. Quadrat. Regular matrices in max-plus algebra. preprint, 2005.

[CGQS05] G. Cohen, S. Gaubert, J.-P. Quadrat, and I. Singer. Max-plus convex sets and functions. In *Idempotent Mathematics and Mathematical Physics*, Contemp. Math., pages 105–129. Amer. Math. Soc., 2005.

[CKR84] Z.Q. Cao, K.H. Kim, and F.W. Roush. *Incline algebra and applications.* Ellis Horwood, 1984.

[CQD90] W. Chen, X. Qi, and S. Deng. The eigen-problem and period analysis of the discrete event systems. *Systems Science and Mathematical Sciences*, 3(3), August 1990.

[CTCG+98] J. Cochet-Terrasson, G. Cohen, S. Gaubert, M. Mc Gettrick, and J.-P. Quadrat. Numerical computation of spectral elements in max-plus algebra. In *Proc. of the IFAC Conference on System Structure and Control*, Nantes, July 1998.

[CTGG99] J. Cochet-Terrasson, S. Gaubert, and J. Gunawardena. A constructive fixed point theorem for min-max functions. *Dynamics and Stability of Systems*, 14(4):407–433, 1999.

[Den77] E. V. Denardo. Periods of connected networks and powers of nonnegative matrices. *Math. Oper. Res.*, 2(1):20–24, 1977.

[DGI98] A. Dasdan, R. K. Gupta, and S. Irani. An experimental study of minimum mean cycle algorithms. Technical Report 32, UCI-ICS, 1998.

[DS00] B. De Schutter. On the ultimate behavior of the sequence of consecutive powers of a matrix in the max-plus algebra. *Linear Algebra Appl.*, 307(1-3):103–117, 2000.

[DS04] M. Develin and B. Sturmfels. Tropical convexity. *Doc. Math.*, 9:1–27, 2004. (Erratum pp. 205–206).

[DSS05] M. Develin, F. Santos, and B. Sturmfels. On the rank of a tropical matrix. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 213–242. Cambridge Univ. Press, Cambridge, 2005.

[Eil74] S. Eilenberg. *Automata, languages, and machines. Vol. A.* Academic Press, New York, 1974. Pure and Applied Mathematics, Vol. 58.

[ES75] G. M. Engel and H. Schneider. Diagonal similarity and equivalence for matrices over groups with 0. *Czechoslovak Math. J.*, 25(100)(3):389–403, 1975.

[EvdD99] L. Elsner and P. van den Driessche. On the power method in max algebra. *Linear Algebra Appl.*, 302/303:17–32, 1999.

[Fat05] A. Fathi. Weak KAM theorem in Lagrangian dynamics. Lecture notes, to be published by Cambridge University Press, 2005.

[Fri86] S. Friedland. Limit eigenvalues of nonnegative matrices. *Linear Algebra Appl.*, 74:173–178, 1986.

[Gau92] S. Gaubert. *Théorie des systèmes linéaires dans les dioïdes.* Thèse, École des Mines de Paris, July 1992.

[Gau94] S. Gaubert. Rational series over dioids and discrete event systems. In *Proc. of the 11th Conf. on Anal. and Opt. of Systems: Discrete Event Systems*, volume 199 of *Lect. Notes. in Control and Inf. Sci*, Sophia Antipolis, June 1994. Springer.

[Gau96] S. Gaubert. On the Burnside problem for semigroups of matrices in the (max,+) algebra. *Semigroup Forum*, 52:271–292, 1996.

[Gau98] S. Gaubert. Exotic semirings: Examples and general results. Support de cours de la 26$^{\text{ième}}$ École de Printemps d'Informatique Théorique, Noirmoutier, 1998.

[GBCG98] S. Gaubert, P. Butkovič, and R. Cuninghame-Green. Minimal (max,+) realization of convex sequences. *SIAM Journal on Control and Optimization*, 36(1):137–147, January 1998.

[GG98] S. Gaubert and J. Gunawardena. The duality theorem for min-max functions. *C. R. Acad. Sci. Paris.*, 326, Série I:43–48, 1998.

[GM77] M. Gondran and M. Minoux. Valeurs propres et vecteurs propres dans les dioïdes et leur interprétation en théorie des graphes. *E.D.F., Bulletin de la Direction des Études et Recherches, Série C, Mathématiques Informatique*, 2:25–41, 1977.

[GM84] M. Gondran and M. Minoux. Linear algebra in dioids: a survey of recent results. *Annals of Discrete Mathematics*, 19:147–164, 1984.

[GM02] M. Gondran and M. Minoux. *Graphes, dioïdes et semi-anneaux.* Éditions TEC & DOC, Paris, 2002.

[GP88] G. Gallo and S. Pallotino. Shortest path algorithms. *Annals of Operations Research*, 13:3–79, 1988.

[GP97] S. Gaubert and M. Plus. Methods and applications of (max,+) linear algebra. In *STACS'97*, volume 1200 of *Lect. Notes Comput. Sci.*, pages 261–282, Lübeck, March 1997. Springer.

[Gun94] J. Gunawardena. Cycle times and fixed points of min-max functions. In *Proceedings of the 11th International Conference on Analysis and Optimization of Systems*, volume 199 of *Lect. Notes. in Control and Inf. Sci*, pages 266–272. Springer, 1994.

[Gun98] J. Gunawardena, editor. *Idempotency*, volume 11 of *Publications of the Newton Institute.* Cambridge University Press, Cambridge, 1998.

[HA99] M. Hartmann and C. Arguelles. Transience bounds for long walks. *Math. Oper. Res.*, 24(2):414–439, 1999.

[Has90] K. Hashiguchi. Improved limitedness theorems on finite automata with distance functions. *Theoret. Comput. Sci.*, 72:27–38, 1990.

[Hel88] S. Helbig. On Carathéodory's and Kreĭn-Milman's theorems in fully ordered groups. *Comment. Math. Univ. Carolin.*, 29(1):157–167, 1988.

[Kar78] R. M. Karp. A characterization of the minimum mean-cycle in a digraph. *Discrete Maths.*, 23:309–311, 1978.

[KB94] D. Krob and A. Bonnier Rigny. A complete system of identities for one letter rational expressions with multiplicities in the tropical semiring. *J. Pure Appl. Algebra*, 134:27–50, 1994.

[Kin61] J. F. C. Kingman. A convexity property of positive matrices. *Quart. J. Math. Oxford Ser. (2)*, 12:283–284, 1961.

[KM97] V. N. Kolokoltsov and V. P. Maslov. *Idempotent analysis and its applications*, volume 401 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1997.

[KO85] S. Karlin and F. Ost. Some monotonicity properties of Schur powers of matrices and related inequalities. *Linear Algebra Appl.*, 68:47–65, 1985.

[Kro94] D. Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. *Internat. J. Algebra Comput.*, 4(3):405–425, 1994.

[LM05] G. L. Litvinov and V. P. Maslov, editors. *Idempotent Mathematics and Mathematical Physics*. Number 377 in Contemp. Math. Amer. Math. Soc., 2005.

[LMS01] G. L. Litvinov, V. P. Maslov, and G. B. Shpiz. Idempotent functional analysis: an algebraic approach. *Math. Notes*, 69(5):696–729, 2001.

[Mol88] P. Moller. *Théorie algébrique des Systèmes à Événements Discrets*. Thèse, École des Mines de Paris, 1988.

[MPN02] J. Mallet-Paret and R. Nussbaum. Eigenvalues for a class of homogeneous cone maps arising from max-plus operators. *Discrete and Continuous Dynamical Systems*, 8(3):519–562, July 2002.

[MS92] V. P. Maslov and S. N. Samborskiĭ, editors. *Idempotent analysis*, volume 13 of *Advances in Soviet Mathematics*. Amer. Math. Soc., Providence, RI, 1992.

[MY60] R. McNaughton and H. Yamada. Regular expressions and state graphs for automata. *IRE trans on Electronic Computers*, 9:39–47, 1960.

[Plu90] M. Plus. Linear systems in $(\max, +)$-algebra. In *Proceedings of the 29th Conference on Decision and Control*, Honolulu, Dec. 1990.

[Rom67] I. V. Romanovskiĭ. Optimization of stationary control of discrete deterministic process in dynamic programming. *Kibernetika*, 3(2):66–78, 1967.

[RS84] C. Reutenauer and H. Straubing. Inversion of matrices over a commutative semiring. *J. Algebra*, 88(2):350–360, June 1984.

[Sim78] I. Simon. Limited subsets of the free monoid. In *Proc. of the 19th Annual Symposium on Foundations of Computer Science*, pages 143–150. IEEE, 1978.

[Sim94] I. Simon. On semigroups of matrices over the tropical semiring. *Theor. Infor. and Appl.*, 28(3-4):277–294, 1994.

[SS92] S. N. Samborskiĭ and G. B. Shpiz. Convex sets in the semimodule of bounded functions. In *Idempotent analysis*, pages 135–137. Amer. Math. Soc., Providence, RI, 1992.

[Vor67] N. N. Vorob′ev. Extremal algebra of positive matrices. *Elektron. Informationsverarbeit. Kybernetik*, 3:39–71, 1967. (In Russian).

[Wag91] E. Wagneur. Moduloïds and pseudomodules. I. Dimension theory. *Discrete Math.*, 98(1):57–73, 1991.

[WB98] E. A. Walkup and G. Borriello. A general linear max-plus solution technique. In *Idempotency*, volume 11 of *Publ. Newton Inst.*, pages 406–415. Cambridge Univ. Press, Cambridge, 1998.

[Yoe61] M. Yoeli. A note on a generalization of boolean matrix theory. *Amer. Math. Monthly*, 68:552–557, 1961.

[Zim76] K. Zimmermann. *Extremální Algebra*. Ekonomický ùstav ČSAV, Praha, 1976. (in Czech).

[Zim77] K. Zimmermann. A general separation theorem in extremal algebras. *Ekonom.-Mat. Obzor*, 13(2):179–201, 1977.

[Zim81] U. Zimmermann. *Linear and Combinatorial Optimization in Ordered Algebraic Structures*. North Holland, 1981.