EXISTENCE OF EIGENVECTORS FOR MONOTONE HOMOGENEOUS FUNCTIONS

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Abstract. We consider functions \( f : \mathbb{R}^n \to \mathbb{R}^n \) which are additively homogeneous and monotone in the product ordering on \( \mathbb{R}^n \) (topical functions). We show that if some non-empty sub-eigen-space of \( f \) is bounded in the Hilbert semi-norm then \( f \) has an additive eigenvector and we give a Collatz-Wielandt characterisation of the corresponding eigenvalue. The boundedness condition is satisfied if a certain directed graph associated to \( f \) is strongly connected.

The Perron-Frobenius theorem for non-negative matrices, its analogue for the max-plus semiring, a version of the mean ergodic theorem for Markov chains and theorems of Bather and Zijn all follow as immediate corollaries.

1. Introduction

1.1. Notation. The partial order on \( \mathbb{R} \) will be extended pointwise to functions \( f, g : X \to \mathbb{R} \) so that \( f \leq g \) if and only if \( f(x) \leq g(x) \) for all \( x \in X \). The least upper bound and greatest lower bound with respect to this ordering, will be denoted in infix form by \( \vee \) and \( \wedge \), respectively: \( (f \vee g)(x) = \max(f(x), g(x)) \) and \( (f \wedge g)(x) = \min(f(x), g(x)) \). In particular, taking \( X = \{1, \ldots, n\} \), this gives the product ordering on \( \mathbb{R}^n \) with its usual structure as a distributive lattice.

It will also be convenient to use the following “vector-scalar” convention: if, in an operation or a relation, a vector and a scalar appear together, then the operation is applied to, or the relation is taken to hold for, each component of the vector. For example, if \( \lambda \in \mathbb{R} \) and \( x \in \mathbb{R}^n \), then \( \lambda + x = (\lambda + x_1, \ldots, \lambda + x_n) \) and \( x \leq \lambda \) if and only if \( x_i \leq \lambda \) for \( 1 \leq i \leq n \).

1.2. Topical functions. A function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is (additively) homogeneous if

\[
\forall \lambda \in \mathbb{R} \text{ and } \forall x \in \mathbb{R}^n, \quad f(\lambda + x) = \lambda + f(x),
\]

and monotone if

\[
\forall x, y \in \mathbb{R}^n, \quad x \leq y \implies f(x) \leq f(y).
\]

Functions which are monotone and homogeneous have been called “topical functions” in [12] and we adopt this terminology here. If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a topical function, we say that \( x \in \mathbb{R}^n \) is an (additive) eigenvector for the eigenvalue \( \lambda \in \mathbb{R} \) if \( f(x) = \lambda x \).

The main results of this paper are existence theorems for eigenvectors, Theorems 1 and 2, and a Collatz-Wielandt characterisation of the eigenvalue. Proposition 1. Our methods are elementary.


Key words and phrases. Collatz-Wielandt property, Hilbert projective metric, nonexpansive function, nonlinear eigenvalue, Perron-Frobenius theorem, strongly connected graph, sub-eigenspace.

Topical functions include many examples that have been extensively studied: non-negative matrices (see below); max-plus matrices, (see §4 and [2]), and other models of discrete event systems, [3, 9, 10, 22]; operators arising in Markov decision theory and the theory of stochastic games, [6, 13]; problems in fixed point theory, [17, 19]; matrix scaling problems and related problems of entropy minimisation, [15, 20]. This paper shows the emergence of elementary general results of wide applicability: we recover some well-known theorems as immediate corollaries of our main results.

1.3. The multiplicative context. Non-negative matrices are familiar in a multiplicative form so it will be helpful to note first that the additive and multiplicative contexts are interchangeable.

Let \( \mathbb{R}^+ \) denote the positive reals; \( \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} \). The whole space \( \mathbb{R}^n \), can be placed in bijective correspondence with the positive cone, \((\mathbb{R}^+)^n\), via the mutually inverse functions \( \exp: \mathbb{R}^n \to (\mathbb{R}^+)^n \) and \( \log: (\mathbb{R}^+)^n \to \mathbb{R}^n \), where \( \exp(x)_i = \exp(x_i) \), for \( x \in \mathbb{R}^n \), and \( \log(x) = \log(x) \), for \( x \in (\mathbb{R}^+)^n \). If \( A : (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n \) is any self-map of the positive cone then \( \mathcal{E}(A) : \mathbb{R}^n \to \mathbb{R}^n \) will denote the function defined by \( \mathcal{E}(A)(x) = \log(A(\exp(x))). \) This induces a bijective functional between self-maps of \((\mathbb{R}^+)^n\) and self-maps of \( \mathbb{R}^n \). Clearly, \( \mathcal{E}(AB) = \mathcal{E}(A)\mathcal{E}(B) \), so that the dynamics of \( A \) on \((\mathbb{R}^+)^n\) and \( \mathcal{E}(A) \) on \( \mathbb{R}^n \) are equivalent.

If \( A : (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n \) is represented by a non-negative matrix in the standard basis (for which the same notation, \( A \), will be used) then it is easy to see that \( \mathcal{E}(A) \) is a topical function. Furthermore, \( x \in \mathbb{R}^n \) is an (additive) eigenvector of \( \mathcal{E}(A) \), with eigenvalue \( \lambda \in \mathbb{R} \), if and only if, \( \exp(x) \in (\mathbb{R}^+)^n \) is an eigenvector of \( A \) in the usual sense, with eigenvalue \( \exp(\lambda) : A \exp(x) = \exp(\lambda) \exp(x) \).

Note that (additive) eigenvectors of \( \mathcal{E}(A) \) correspond bijectively to the (multiplicative) eigenvectors of \( A \) all of whose components are positive. The word eigenvector will be used in both contexts; the reader should have no difficulty inferring the right meaning. Note further that a non-negative matrix \( A \) corresponds to a topical function under \( \mathcal{E} \) if, and only if, no row of \( A \) is the zero vector:

\[
\forall i, \exists j \text{ such that } A_{ij} \neq 0.
\]

1.4. Nonexpansiveness. A key property of topical functions is their nonexpansiveness with respect to certain norms. Let \( t, b : \mathbb{R}^n \to \mathbb{R} \) be defined as ("top") \( t(x) = x_1 \lor \cdots \lor x_n \), and ("bottom") \( b(x) = -t(-x) = x_1 \land \cdots \land x_n \), both of which are topical functions. The supremum, or \( \ell^\infty \), norm on \( \mathbb{R}^n \) can then be defined as \( ||x||_{\infty} = t(x)\lor b(x) \). We shall also need the Hilbert semi-norm, \( ||x||_{H} = t(x) - b(x) \), which defines a metric on the space of lines parallel to the main diagonal in \( \mathbb{R}^n \).

This metric is the additive version of the Hilbert projective metric while \( ||x||_{\infty} \) gives rise to the additive version of Thompson’s “part” metric on \((\mathbb{R}^+)^n\), [17].

An elementary application of (1) and (2), [12, Proposition 1.1], shows that a function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is topical if, and only if,

\[
\forall x, y \in \mathbb{R}^n, t(f(x) - f(y)) \leq t(x - y) \; .
\]

(This provides some justification for the term topical.) We see immediately that a topical function is nonexpansive with respect to both the supremum norm and the Hilbert semi-norm: \( \forall x, y \in \mathbb{R}^n \),

\[
\|f(x) - f(y)\|_{\infty} \leq \|x - y\|_{\infty} \; ,
\]

\[
\|f(x) - f(y)\|_{H} \leq \|x - y\|_{H} \; .
\]

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In fact, as first observed by Crandall and Tartar [7], if \( f \) is homogeneous, then it is monotone if, and only if, it is nonexpansive in the supremum norm, [12, Proposition 1.1].

2. Sub-eigenspaces and the Collatz-Wielandt property

The results of the present paper originate in the study of sub-eigenspaces. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \). For any \( \lambda \in \mathbb{R} \), define the sub-eigenspace of \( f \) associated to \( \lambda \),

\[ S_\lambda(f) = \{ x \in \mathbb{R}^n \mid f(x) \leq \lambda + x \} . \]

If \( S_\lambda(f) \neq \emptyset \) then \( \lambda \) is said to be a sub-eigenvalue, and any \( x \in S_\lambda(f) \) is a sub-eigenvector. Let \( \Lambda(f) \subseteq \mathbb{R} \) denote the set of sub-eigenvalues: \( \Lambda(f) = \{ \lambda \in \mathbb{R} \mid S_\lambda(f) \neq \emptyset \} \). For any functions \( f, g : \mathbb{R}^n \to \mathbb{R}^n \) and any \( \lambda, \mu \in \mathbb{R} \), the following are easily seen to hold.

\[
\begin{align*}
(6a) & \quad f \leq g \quad \Rightarrow \quad S_\lambda(f) \supseteq S_\lambda(g) , \\
(6b) & \quad \lambda \leq \mu \quad \Rightarrow \quad S_\lambda(f) \subseteq S_\mu(f) , \\
(6c) & \quad S_{\lambda+\mu}(f) = S_\lambda(f - \mu) .
\end{align*}
\]

It follows immediately from (6b) that for any function \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( \Lambda(f) \) must be an interval of the form \((-\infty, \infty)\), \((\alpha, \infty)\) or \([\alpha, \infty)\) and it is easy to see that all three forms can appear. For a topical function the first form can be ruled out. To see this, it is helpful to recall first some well-understood facts about the asymptotic dynamics of a topical function. \( f : \mathbb{R}^n \to \mathbb{R}^n \).

First, (4) implies that all trajectories of \( f \) are asymptotically the same:

\[ f^k(x) = f^k(y) + O(1) \quad \text{as } k \to \infty . \]

Second, an elementary argument using (1) and (2) shows that the sequence \( t(f^k(0)) \) is sub-additive,

\[ t(f^{k+l}(0)) \leq t(f^k(0)) + t(f^l(0)) . \]

It follows from (7) that the sequence \( t(f^k(x)/k) \) converges as \( k \to \infty \) and that the limit is independent of \( x \), [12, 22]. The upper cycle-time of \( f \), \( \overline{\lambda}(f) \in \mathbb{R} \) is defined as

\[ \overline{\lambda}(f) = \lim_{k \to \infty} t(f^k(x)/k) . \]

Dually, the lower cycle-time is \( \underline{\lambda}(f) = \lim_{k \to \infty} b(f^k(x)/k) \). The existence of the cycle-time vector of \( f \), \( \lambda(f) = \lim_{k \to \infty} f^k(x)/k \), is another matter altogether. It does not always exist. [12, Theorem 3.1], and one of the central problems in the field is to characterise those topical functions for which it does.

Now suppose that \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a topical function and that \( f(x) \leq \lambda + x \) for some \( x \in \mathbb{R}^n \) and some \( \lambda \in \mathbb{R} \). Using (1) and (2), \( f^k(x) \leq k\lambda + x \). Hence,

\[ t(f^k(x)/k) \leq \lambda + t(x/k) , \]

from which the following lemma immediately follows.

**Lemma 1.** If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a topical function then either \( \Lambda(f) = (a, \infty) \) or \( \Lambda(f) = [a, \infty) \), where \( \overline{\lambda}(f) \leq a \).
Both possibilities can occur. It follows from Proposition 1 and (10a) below that if \( f \) has an eigenvector, \( f(x) = \lambda + x \), then \( \Lambda(f) = [\lambda, \infty) \). If \( f = \mathcal{E}(A) \) where \( A \) is the non-negative matrix below

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

then it is easy to see that \( \Lambda(f) = (0, \infty) \).

We shall now show that \( \overline{\Lambda}(f) = a \). This requires the following simple but crucial observation.

**Lemma 2.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a topical function and let \( k \) be any positive integer. If \( S_{\lambda}(f^k) \neq \emptyset \), then \( S_{\overline{\Lambda}(f)}(f^k) \neq \emptyset \).

**Proof.** If \( S_{\lambda}(f^k) \neq \emptyset \), then \( f^k(x) \leq \lambda + x \) for some \( x \in \mathbb{R}^n \). Let

\[
y = x \land (f(x) - \lambda/k) \land \cdots \land (f^{k-1}(x) - (k-1)\lambda/k).
\]

Using (1) and (2) we see that

\[
f(y) \leq f(x) \land (f^2(x) - \lambda/k) \land \cdots \land (f^k(x) - (k-1)\lambda/k) \\
\leq f(x) \land (f^2(x) - \lambda/k) \land \cdots \land (x + \lambda/k) \\
= y + \lambda/k.
\]

Thus, \( y \in S_{\overline{\Lambda}(f)}(f^k) \neq \emptyset \). \( \square \)

Lemma 2 allows us to give the following characterisation of \( \overline{\Lambda}(f) \) which may be thought of as a generalised Collatz-Wielandt formula. [16, §1.3].

**Proposition 1.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a topical function. Then,

\[
\inf \Lambda(f) = \inf_{x \in \mathbb{R}^n} t(f(x) - x) = \overline{\Lambda}(f).
\]

**Proof.** Let \( a = \inf \Lambda(f) \). Since \( f(x) \leq x + \lambda \) if, and only if, \( t(f(x) - x) \leq \lambda \) the first equality in (8) follows easily. Lemma 1 has already shown that \( \overline{\Lambda}(f) \leq a \). Now choose \( \epsilon > 0 \). For sufficiently large \( k \), \( f^k(0) \leq (\overline{\Lambda}(f) + \epsilon)k \). Hence, \( S_{\overline{\Lambda}(f) + \epsilon}(f^k) \neq \emptyset \). By Lemma 2, \( S_{\overline{\Lambda}(f) + \epsilon}(f^k) \neq \emptyset \). Hence, \( a \leq \overline{\Lambda}(f) + \epsilon \). Since \( \epsilon \) was chosen arbitrarily, \( a \leq \overline{\Lambda}(f) \) and so \( a = \overline{\Lambda}(f) \). \( \square \)

A result on topical functions can be dualised by applying it to the topical function \(-f(-x)\). Using this method on the Collatz-Wielandt formula, we deduce that

\[
\underline{\Lambda}(f) = \sup_{x \in \mathbb{R}^n} b(f(x) - x).
\]

If \( f \) has an eigenvector, so that \( f(x) = \lambda + x \) then it follows from (1) that

\[
(10a) \quad \underline{\Lambda}(f) = \lambda = \overline{\Lambda}(f)
\]

and

\[
(10b) \quad \Lambda(f) = (\lambda, \ldots, \lambda).
\]

3. Existence of eigenvectors

3.1. The main result. It is convenient for the proofs that follow to make use of the normalised sub-eigenspace, \( S'_{\lambda}(f) \subseteq \mathbb{R}^n \), defined by

\[
S'_{\lambda}(f) = \{ x \in \mathbb{R}^n \mid f(x) \leq \lambda + x \text{ and } b(x) = 0 \}.
\]
If $b(x) = 0$ then $\|x\|_H = \|x\|_\infty$. It follows that if $f$ is homogeneous then $S_\lambda(f)$ is non-empty and bounded in the Hilbert semi-norm if, and only if, $S'_\lambda(f)$ is non-empty and bounded in the supremum norm.

**Theorem 1.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a topical function for which some sub-eigenspace is non-empty and bounded in the Hilbert semi-norm. Then $f$ has an eigenvector.

**Proof.** Assume that $S_\mu(f)$ is non-empty and bounded in the Hilbert semi-norm. Let $a = \inf \Lambda(f)$. Evidently $a \leq \mu$. We may assume, without loss of generality, that $a = \mu$. To see why, suppose that $a < \mu$. Since $S_\mu(f)$ is bounded in the Hilbert semi-norm, the sets $S'_\mu(f)$, for $a < b \leq \mu$ are compact in the supremum norm. It follows easily from (6b) that $S'_\mu(f) = \bigcap_{a < b \leq \mu} S'_\lambda(f)$. The right hand side is a decreasing intersection of non-empty compact sets and so $S'_\lambda(f)$ is also non-empty and compact. Hence $S_\lambda(f)$ is non-empty and bounded in the Hilbert semi-norm as claimed.

Let $g = -a + f$. By (6c), $\Lambda(g) = [0, \infty)$, so that we can find $x \in \mathbb{R}^n$ such that $g(x) \leq x$. Hence $g^{k+1}(x) \leq g^k(x)$ and $g^k(x) \in S_\lambda(g) = S_\lambda(f)$ for all $k \in \mathbb{N}$. If $\lim_{k \to \infty} t(g^k(x)) = -\infty$, then $g^k(x) \leq -1 + x$, for some sufficiently large $k$ and Lemma 2 shows that $S_{-1}\lambda(g) \neq \emptyset$, contradicting $\Lambda(g) = [0, \infty)$. Hence $t(g^k(x))$ is bounded from below as $k \to \infty$. By hypothesis, $||g^k(x)||_H$ remains bounded and this can only happen if $g^k(x)$ itself remains bounded. Let $y = \lim_{k \to \infty} g^k(x)$. Then by continuity of $g$, $g(y) = y$, so that $f(y) = a + y$. \qed

The following examples are instructive in the light of this result. Consider the topical functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f_1(x) = ((x_1 - 1) \lor x_2) \land (x_1 + 1) \quad \text{and} \quad g_1(x) = x_1 \land x_2$$

$$f_2(x) = x_1 \lor x_2 \quad \text{and} \quad g_2(x) = x_1 \lor x_2 .$$

We leave it to the reader to show that $\Lambda(f) = [0, \infty)$ and

$$S_\lambda(f) = \begin{cases} 
\{ x \in \mathbb{R}^2 | -\lambda + x_1 \leq x_2 \leq \lambda + x_1 \} & \text{for } 0 \leq \lambda < 1 \\
\{ x \in \mathbb{R}^2 | -\lambda + x_1 \leq x_2 \leq \lambda + x_1 \} & \text{for } \lambda \geq 1 .
\end{cases}$$

It follows that $S_1(f)$ is bounded for $0 \leq \lambda < 1$ and unbounded for $1 \leq \lambda$. As for $g$, it has the eigenvector $(0, 0)$ and $\Lambda(g) = [0, \infty)$ but $S_\lambda(g) = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \leq \lambda + x_2 \}$ is unbounded for all $\lambda \geq 0$. (The dual super-eigenspaces are also unbounded.)

### 3.2. Graphs associated to topical functions.

If $A$ is a $n \times n$ non-negative matrix, its associated graph, $G(A)$, is the directed graph with vertices $\{1, \ldots, n\}$ and an edge from $i$ to $j$ if, and only if, $A_{ij} \neq 0$. [5, Chapter 2]. The matrix $A$ is irreducible if, and only if, $G(A)$ is strongly connected if there is a directed path between any two vertices. The Perron-Frobenius theorem (see Corollary 1 below) asserts that an irreducible non-negative matrix has an eigenvector all of whose components are positive. We now generalise this to topical functions.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$. Define the associated graph of $f$, $G(f)$, to be the directed graph with vertices $\{1, \ldots, n\}$ and an edge from $i$ to $j$ if, and only if, $\lim_{\nu \to \infty} f_i(\nu e_j) = \infty$ where $e_j$ is the $j$-th vector of the canonical basis of $\mathbb{R}^n$.

**Theorem 2.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a topical function whose associated graph is strongly connected. Then all non-empty sub-eigenspaces of $f$ are bounded. In particular, $f$ has an eigenvector.
Proof. For each edge from \( i \) to \( j \) of \( G(f) \) define \( h_{ij} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \) by
\[
h_{ij}(x) = \sup \{ \nu \in \mathbb{R} \mid f_i(\nu e_j) \leq x \},
\]
where we use the convention that \( \sup \emptyset = -\infty \). For any \( \lambda \in \mathbb{R} \), let \( h_{ij}^\lambda(x) = h_{ij}(\lambda + x) \). Let \( S_\lambda(f) \) be any non-empty sub-eigenspace of \( f \) and choose \( x \in S_\lambda(f) \), which we may assume to satisfy \( b(x) = 0 \). Let \( i \in \{1, \cdots, n\} \) be the component for which \( x_i = 0 \). Choose any other component \( j \in \{1, \cdots, n\} \). By hypothesis there exists a directed path from \( i \) to \( j \) in \( G(f) \). Suppose that the nodes on this are \( i = i_1, \cdots, i_k = j \), where there is an edge from \( i_{p-1} \) to \( i_p \) for \( 1 \leq p \leq k \). Since \( b(x) = 0 \), we must have \( x_{i_p} e_{i_p} \leq x \). Hence
\[
f_{i_{p-1}}(x_{i_p} e_{i_p}) \leq f_{i_{p-1}}(x) \leq \lambda + x_{i_{p-1}},
\]
and so \( x_{i_p} \leq h_{i_{p-1}i_p}^\lambda(x_{i_{p-1}}) \). Putting these together we find that
\[
x_j \leq h_{i_n i_{n-1}}^\lambda \circ \cdots \circ h_{i_1 i_2}^\lambda(0).
\]
It follows that \( S_\lambda(f) \) is bounded in the Hilbert semi-norm. By Theorem 1, \( f \) has an eigenvector.

Amghibech and Dellacherie state a similar but weaker result in [1]. They use a different graph which is, in general, not strongly connected for the examples studied in the next section. With the exception of that in Corollary 3. However, the proof technique of [1] based on an approximation procedure, could be used to obtain an independent proof of Theorem 2.

Consider the topical function \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by
\[
f_1(x) = x_1 \vee (x_2 \wedge x_3),
\]
\[
f_2(x) = x_1 \vee x_2 \vee x_3,
\]
\[
f_3(x) = x_1 \vee x_2 \vee x_3.
\]
\( G(f) \) is not strongly connected since there are no edges from 1 to 2 and from 1 to 3. Nevertheless it is easy to check that \( f \) has bounded sub-eigenspaces. Is there a combinatorial object associated to a topical function which determines when the function has bounded sub-eigenspaces? This is an interesting problem which we hope to address elsewhere.

For convex topical functions, Theorem 2 has a converse. Recall that a function \( h : \mathbb{R}^n \to \mathbb{R} \) is convex if for all \( x, y \in \mathbb{R}^n \),
\[
h(\lambda x + \mu y) \leq \lambda h(x) + \mu h(y),
\]
where \( 0 \leq \lambda, \mu \leq 1 \) and \( \lambda + \mu = 1 \). A function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is convex if each component function \( f_i : \mathbb{R}^n \to \mathbb{R} \) is convex. A simple deduction from (11), which is left to the reader, captures the intuition that the derivative of \( h \) is increasing. With the same notation as above, let \( x' = \lambda x + \mu y = x + \mu(y - x) = y - \lambda(y - x) \). Then,
\[
\frac{h(x') - h(x)}{\mu} \leq \frac{h(y) - h(x')}{\lambda}.
\]

For any function \( f : \mathbb{R}^n \to \mathbb{R}^n \) define its syntactic graph, \( G^s(f) \), to be the directed graph with vertices \( 1, \cdots, n \) and an edge from \( i \) to \( j \) if and only if \( f_i \) depends on \( x_j \) in the following sense: there is no map \( h : \mathbb{R}^{n-1} \to \mathbb{R} \) such that
\[
f_i(x) = h(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n).
Proposition 2. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a convex topical function. Then \( G(f) = G^s(f) \). Moreover, \( G^s(f) \) is strongly connected if, and only if, all sub-eigenspaces of \( f \) are bounded in the Hilbert semi-norm.

Proof. Clearly, an edge of \( G(f) \) is an edge of \( G^s(f) \). Conversely, if there is an edge from \( i \) to \( j \) in \( G^s(f) \), then we can find \( x, x' \in \mathbb{R}^n \) such that \( x_k = x'_k \) for all \( k \neq j \), \( x_j \neq x'_j \), and \( f_i(x) \neq f_i(x') \). Without loss of generality, assume that \( x'_j > x_j \). Choose \( \nu > 0 \) and let \( y = x' + \nu e_j \). Let \( \alpha = x'_j - x_j + \nu \), \( \lambda = \nu/\alpha \) and \( \mu = (x'_j - x_j)/\alpha \). Evidently, \( 0 \leq \lambda, \mu \leq 1 \) and \( \lambda + \mu = 1 \) and it is easy to check that \( x' = \lambda x + \mu y \). In accordance with the notation used in (12). Using this inequality we see that

\[
\frac{f_i(y) - f_i(x')}{\lambda} \geq \frac{f_i(x') - f_i(x)}{\mu}
\]

which can be rewritten as

\[
f_i(x' + \nu e_j) \geq \frac{\nu}{x'_j - x_j}(f_i(x') - f_i(x)) + f_i(x').
\]

Since this holds for any \( \nu > 0 \), it follows that \( \lim_{\nu \to \infty} f_i(x' + \nu e_j) = \infty \). But, \( x' + \nu e_j \leq t(x') + \nu e_j \). Using (2), we see that \( f_i(\nu e_j) \leq f_i(x' + \nu e_j) - t(x') \) and so \( \lim_{\nu \to \infty} f_i(\nu e_j) = \infty \). It follows that there is an edge from \( i \) to \( j \) in \( G(f) \) and so \( G^s(f) \) is identical to \( G(f) \).

If \( G^s(f) = G(f) \) is strongly connected then Theorem 2 shows that all the sub-eigenspaces of \( f \) are bounded. Conversely, suppose that \( G^s(f) \) is not strongly connected. Then, by standard arguments, \([5, \text{Chapter 2}]\), we can, after possibly reordering the variables so that \( x = (y, z) \), where \( y \in \mathbb{R}^p \), \( z \in \mathbb{R}^q \), \( n = p + q \) and \( f(x) = (g(y, z), h(z)) \), for some topical functions \( g : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \) and \( h : \mathbb{R}^q \to \mathbb{R}^q \). Suppose that \( S_{\lambda}(f) \) is non-empty and choose \( x \in S_{\lambda}(f) \). Since \( f(x) \leq x \), we must have \( g(y, z) \leq \lambda + y \) and \( h(z) \leq \lambda + z \). Now choose \( \mu \geq 0 \). Using (1) and (2) it follows that \( g(\lambda + y, z) \leq g(\mu + y, \mu + z) \leq \lambda + \mu + y \). Hence \( (\mu + y, z) \in S_{\lambda}(f) \) for all \( \mu > 0 \) and so all non-empty sub-eigenspaces of \( f \) are unbounded in the Hilbert semi-norm. \(\square\)

4. Applications

We now show that several well-known theorems are immediate corollaries of the elementary results above.

Corollary 1. (Perron-Frobenius theorem, [5]) Let \( A \) be a \( n \times n \) non-negative matrix. If \( A \) is irreducible then its spectral radius is an eigenvalue, for which \( A \) has an eigenvector all of whose components are positive.

Proof. Since \( A \) is irreducible the nondegeneracy condition (3) holds. Let \( f = \mathcal{E}(A) \). It is easy to see that \( G(f) \), the graph associated to \( f \), is identical to \( G(A) \), the graph associated to \( A \). Since \( A \) is irreducible, \( G(A) \) is strongly connected and so, by Theorem 2, \( f \) has an eigenvector: \( f(x) = r + x \). Evidently, \( A \exp(x) = \exp(r) \exp(x) \), where \( \exp(x) \) has all its components positive. It remains to show that \( \exp(r) \) is the spectral radius of \( A \). For completeness, we reproduce the standard argument using the Collatz-Wielandt property. Suppose that \( z \in \mathbb{C}^n \) is a (multiplicative) eigenvector of \( A \) with eigenvalue \( \lambda \in \mathbb{C} \). \( Az = \lambda z \). Let \( |z| \in \mathbb{R}^n \) be the vector of absolute values: \(|z| = (|z_1|, \cdots, |z_n|) \) and let \( x = \log(|z|) \). A simple application
of the triangle inequality shows that $A|z| \geq |\lambda||z|$. Hence $f(x) \geq \log(|\lambda|) + x$. It follows, using (9) and (10a), that
\[
\log(|\lambda|) \leq b(f(x) - x) \leq \Delta(f) = r
\]
and so $\exp(r)$ is the spectral radius of $A$. \qed

The function $\mathbb{R}^2 \to \mathbb{R}$ which takes $x \mapsto \log(\exp(x_1) + \exp(x_2))$ is convex, from which it follows that $\mathcal{E}(A)$ is a convex topical function. Hence we could have used Proposition 2 together with Theorem 1 in the proof of Corollary 1. This applies also to the topical functions in the corollaries below, which are all convex.

Property (1) applied to the function $t : \mathbb{R}^2 \to \mathbb{R}$ illustrates that addition distributes over maximum. It follows that the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of maximum (as addition) and addition (as multiplication) forms an idempotent semiring (a semiring whose addition satisfies $a + a = a$), called the \textit{max-plus semiring} and denoted $\mathbb{R}_{\max}$. [11]. Suppose that $A$ is a $n \times n$ matrix over $\mathbb{R}_{\max}$ which satisfies a similar nondegeneracy condition to (3):
\[
\forall i, \exists j \text{ such that } A_{ij} \neq -\infty.
\]
If $x \in \mathbb{R}^n$ then it is easy to see that $x \mapsto Ax$ defines a topical function. For instance, the matrix on the left below gives rise to the function on the right.
\[
\begin{pmatrix}
2 & -1 \\
-\infty & 4
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
f_1(x) \\
f_2(x)
\end{array}
\end{pmatrix}
= \begin{pmatrix}
(x_1 + 2) \vee (x_2 - 1) \\
x_2 + 4
\end{pmatrix}.
\]

If $A$ is a $n \times n$ matrix over $\mathbb{R}_{\max}$, its associated graph, $G(A)$, is the directed graph with vertices $\{1, \ldots, n\}$ and an edge from $i$ to $j$ if, and only if, $A_{ij} \neq -\infty$. It is customary, in max-plus theory, to adjoin labels ("weights") to the edges in $G(A)$. [2]. This unlabelled version will be sufficient for our purposes. A is said to be irreducible if $G(A)$ is strongly connected. It is easy to see that if $f$ is the topical function corresponding to $A$ then the graphs $G(A)$ and $G(f)$ coincide. The following result follows immediately. The cited reference is to a standard source but the result has been proved independently many times.

**Corollary 2.** (Perron-Frobenius for max-plus. [2, Theorem 3.28]) An irreducible max-plus matrix has an eigenvector.

In max-plus theory, the eigenvectors of a matrix lie in $(\mathbb{R} \cup \{-\infty\})^n$. The point of Corollary 2 is that such an eigenvector can be found in $\mathbb{R}^n$. The formula for the eigenvalue, based on the structure of the circuits of $G(A)$, lies outside the scope of the present paper. [2].

For the next result, assume that $P$ is the transition matrix of a Markov chain (so that $P$ is row-stochastic) and let $f(x) = c + Px$, for some $c \in \mathbb{R}^n$. Evidently, $f$ is a topical function. By (10b), if $f$ has an eigenvector with eigenvalue $\lambda$, then
\[
\frac{(1 + P + \cdots + P^{k-1})c}{k} \leadsto \frac{f^k(0)}{k}
\]
converges to $(\lambda, \ldots, \lambda)$. The next result can hence be thought of as a version of the mean ergodic theorem for Markov chains, [23, Chapter XIII, §1, Theorem 2].

**Corollary 3.** Let $c \in \mathbb{R}^n$ and let $P$ denote a $n \times n$ irreducible row-stochastic matrix. The function $f(x) = c + Px$ has an eigenvector.

\[Proof.\] $f$ is a topical function and $G(f)$ is strongly connected so the result follows from Theorem 2. \qed
A family of non-negative matrices \( \{ P^u \}_{u \in U} \) is said to be communicating if the matrix \( \sup_{u \in U} P^u \), obtained by taking entrywise suprema, has finite entries and is irreducible. The following result is due to Bather.

**Corollary 4.** (Bather’s theorem, [4, Theorem 2.4]) Let \( \{ P^u \}_{u \in U} \) be a communicating family of row-stochastic matrices, and let \( \{ e^u \}_{u \in U} \) be a family of vectors \( e^u \in \mathbb{R}^n \) that is bounded above. Then the function \( f(x) = \sup_{u \in U}(e^u + P^u x) \) has an eigenvector.

**Proof.** It is easy to check that \( G(f) = G(\sup_{u \in U} P^u) \). Since the latter is strongly connected by hypothesis, the result follows immediately from Theorem 2. \( \Box \)

The next result was proved by Zijm in the special case of a finite communicating family. It follows by combining the argument of Corollary 4 with that of Corollary 1.

**Corollary 5.** (Zijm’s theorem, [24, Theorem 3.4]) Let \( \{ A^u \}_{u \in U} \) be a communicating family of non-negative matrices. Then the function \( f(x) = \sup_{u \in U} A^u x \) has a (multiplicative) eigenvector.

As a last illustration of the ideas developed here, consider the topological function \( \mathcal{E}(f) \) where \( f : (\mathbb{R}^+)^3 \to (\mathbb{R}^+)^3 \) is defined by:

\[
\begin{align*}
f_1(x) &= 2x_1 \lor 3x_2 \\
f_2(x) &= \sqrt{x_1(4x_2 + 15x_3)} \\
f_3(x) &= x_2.
\end{align*}
\]

None of the above corollaries can be applied to \( f \). However, \( G(\mathcal{E}(f)) \) is strongly connected. By Theorem 2, \( f \) has a (multiplicative) eigenvector. In fact, \( f(3, 3, 1) = 3(3, 3, 1) \).

5. Conclusions

An alternative approach to the eigenvector problem stems from the observation in (7) that all trajectories of a topological function \( f \) are asymptotically equivalent. This suggests that the asymptotics of \( f^k(x) \) contain information on the existence of fixed points, an idea confirmed in recent work, [8, 10].

Topological functions can be defined and studied on cones in Banach spaces, as Krein and Rutman have done for Perron-Frobenius theory. Some attractive examples have emerged here, [21], but with the exception of Nussbaum’s work, [17, 18], little general progress has been made.

**References**


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