EXISTENCE OF EIGENVECTORS FOR MONOTONE HOMOGENEOUS FUNCTIONS

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Abstract. We consider functions $f : \mathbb{R}^n \to \mathbb{R}^n$ which are additively homogeneous and monotone in the product ordering on $\mathbb{R}^n$ (topical functions). We show that if some non-empty sub-eigenspace of f is bounded in the Hilbert semi-norm then f has an additive eigenvector and we give a Collatz-Wielandt characterisation of the corresponding eigenvalue. The boundedness condition is satisfied if a certain directed graph associated to f is strongly connected. The Perron-Frobenius theorem for non-negative matrices, its analogue for the max-plus semiring, a version of the mean ergodic theorem for Markov chains and theorems of Bather and Zijm all follow as immediate corollaries.

1. Introduction

1.1. **Notation.** The partial order on $\mathbb{R}$ will be extended pointwise to functions $f, g : X \to \mathbb{R}$ so that $f \leq g$ if, and only if, $f(x) \leq g(x)$ for all $x \in X$. The least upper bound and greatest lower bound with respect to this ordering, will be denoted in infix form by $\vee$ and $\wedge$, respectively: $(f \vee g)(x) = \max(f(x), g(x))$ and $(f \wedge g)(x) = \min(f(x), g(x))$. In particular, taking $X = \{1, \cdots, n\}$, this gives the product ordering on $\mathbb{R}^n$ with its usual structure as a distributive lattice.

It will also be convenient to use the following *vector-scalar* convention: if, in an operation or a relation, a vector and a scalar appear together, then the operation is applied to, or the relation is taken to hold for, each component of the vector. For example, if $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then $\lambda + x = (\lambda + x_1, \cdots, \lambda + x_n)$ and $x \leq \lambda$ if, and only if, $x_i \leq \lambda$ for $1 \leq i \leq n$.

1.2. **Topical functions.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is (additively) *homogeneous* if

$$\forall \lambda \in \mathbb{R} \text{ and } \forall x \in \mathbb{R}^n, \quad f(\lambda + x) = \lambda + f(x) \ , \tag{1}$$

and *monotone* if

$$\forall x, y \in \mathbb{R}^n, \quad x \leq y \implies f(x) \leq f(y) \ . \tag{2}$$

Functions which are monotone and homogeneous have been called *topical functions* in [12] and we adopt this terminology here. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a topical function, we say that $x \in \mathbb{R}^n$ is an (additive) *eigenvector* for the *eigenvalue* $\lambda \in \mathbb{R}$ if $f(x) = \lambda + x$. The main results of this paper are existence theorems for eigenvectors, Theorems 1 and 2, and a Collatz-Wielandt characterisation of the eigenvalue, Proposition 1. Our methods are elementary.

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Topical functions include many examples that have been extensively studied: non-negative matrices (see below); max-plus matrices, (see §4 and [2]), and other models of discrete event systems, [3, 9, 10, 22]; operators arising in Markov decision theory and the theory of stochastic games, [6, 13]; problems in fixed point theory, [17, 19]; matrix scaling problems and related problems of entropy minimisation, [15, 20]. This paper shows the emergence of elementary general results of wide applicability: we recover some well-known theorems as immediate corollaries of our main results.

1.3. **The multiplicative context.** Non-negative matrices are familiar in a multiplicative form so it will be helpful to note first that the additive and multiplicative contexts are interchangeable.

Let $\mathbb{R}^+$ denote the positive reals: $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. The whole space, $\mathbb{R}^n$, can be placed in bijective correspondence with the positive cone, $(\mathbb{R}^+)^n$, via the mutually inverse functions $\exp : \mathbb{R}^n \to (\mathbb{R}^+)^n$ and $\log : (\mathbb{R}^+)^n \to \mathbb{R}^n$, where $\exp(x)_i = \exp(x_i)$, for $x \in \mathbb{R}^n$, and $\log(x)_i = \log(x_i)$, for $x \in (\mathbb{R}^+)^n$. If $A : (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n$ is any self-map of the positive cone then $\mathcal{E}(A) : \mathbb{R}^n \to \mathbb{R}^n$ will denote the function defined by $\mathcal{E}(A)(x) = \log(A(\exp(x)))$. This induces a bijective functional between self-maps of $(\mathbb{R}^+)^n$ and self-maps of $\mathbb{R}^n$. Clearly, $\mathcal{E}(AB) = \mathcal{E}(A)\mathcal{E}(B)$, so that the dynamics of A on $(\mathbb{R}^+)^n$ and $\mathcal{E}(A)$ on $\mathbb{R}^n$ are equivalent.

If $A : (\mathbb{R}^+)^n \to (\mathbb{R}^+)^n$ is represented by a non-negative matrix in the standard basis (for which the same notation, A, will be used) then it is easy to see that $\mathcal{E}(A)$ is a topical function. Furthermore, $x \in \mathbb{R}^n$ is an (additive) eigenvector of $\mathcal{E}(A)$, with eigenvalue $\lambda \in \mathbb{R}$, if, and only if, $\exp(x) \in (\mathbb{R}^+)^n$ is an eigenvector of A in the usual sense, with eigenvalue $\exp(\lambda)$: $A\exp(x) = \exp(\lambda)\exp(x)$.

Note that (additive) eigenvectors of $\mathcal{E}(A)$ correspond bijectively to the (multiplicative) eigenvectors of A *all of whose components are positive.* The word eigenvector will be used in both contexts; the reader should have no difficulty inferring the right meaning. Note further that a non-negative matrix A corresponds to a topical function under $\mathcal{E}$ if, and only if, no row of A is the zero vector:

$$\forall i,\ \exists j \ \text{ such that } \ A_{ij} \neq 0 \ . \tag{3}$$

1.4. **Nonexpansiveness.** A key property of topical functions is their nonexpansiveness with respect to certain norms. Let $\mathsf{t}, \mathsf{b} : \mathbb{R}^n \to \mathbb{R}$ be defined as ("top") $\mathsf{t}(x) = x_1 \vee \cdots \vee x_n$, and ("bottom") $\mathsf{b}(x) = -\mathsf{t}(-x) = x_1 \wedge \cdots \wedge x_n$, both of which are topical functions. The supremum, or ℓ^∞, norm on $\mathbb{R}^n$ can then be defined as $\|x\|_\infty = \mathsf{t}(x) \vee -\mathsf{b}(x)$. We shall also need the Hilbert semi-norm, $\|x\|_{\mathsf{H}} = \mathsf{t}(x) - \mathsf{b}(x)$, which defines a metric on the space of lines parallel to the main diagonal in $\mathbb{R}^n$. This metric is the additive version of the Hilbert projective metric while $\|x\|_\infty$ gives rise to the additive version of Thompson's "part" metric on $(\mathbb{R}^+)^n$, [17].

An elementary application of (1) and (2), [12, Proposition 1.1], shows that a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is topical if, and only if,

$$\forall x, y \in \mathbb{R}^n, \mathsf{t}(f(x) - f(y)) \leq \mathsf{t}(x - y) \ .$$

(This provides some justification for the term *topical.*) We see immediately that a topical function is nonexpansive with respect to both the supremum norm and the Hilbert semi-norm: $\forall x, y \in \mathbb{R}^n$,

$$\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty \tag{4}$$

$$\|f(x) - f(y)\|_{\mathsf{H}} \leq \|x - y\|_{\mathsf{H}} \ . \tag{5}$$

In fact, as first observed by Crandall and Tartar [7], if f is homogeneous, then it is monotone if, and only if, it is nonexpansive in the supremum norm, [12, Proposition 1.1].

2. Sub-eigenspaces and the Collatz-Wielandt property

The results of the present paper originate in the study of sub-eigenspaces. Let $f : \mathbb{R}^n \to \mathbb{R}^n$. For any $\lambda \in \mathbb{R}$, define the sub-eigenspace of f associated to λ, $S_\lambda(f) \subseteq \mathbb{R}^n$, by

$$S_\lambda(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \lambda + x\} \ .$$

If $S_\lambda(f) \neq \emptyset$ then λ is said to be a *sub-eigenvalue*, and any $x \in S_\lambda(f)$ is a *sub-eigenvector*. Let $\Lambda(f) \subseteq \mathbb{R}$ denote the set of sub-eigenvalues: $\Lambda(f) = \{\lambda \in \mathbb{R} \mid S_\lambda(f) \neq \emptyset\}$. For any functions $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and any $\lambda, \mu \in \mathbb{R}$, the following are easily seen to hold.

$$f \leq g \quad \Rightarrow \quad S_\lambda(f) \supset S_\lambda(g) \ , \tag{6a}$$

$$\lambda \leq \mu \quad \Rightarrow \quad S_\lambda(f) \subset S_\mu(f) \ , \tag{6b}$$

$$S_{(\lambda+\mu)}(f) \quad = \quad S_\lambda(f - \mu) \ . \tag{6c}$$

It follows immediately from (6b) that for any function $f : \mathbb{R}^n \to \mathbb{R}^n$, $\Lambda(f)$ must be an interval of the form $(-\infty, \infty)$, (a, ∞) or $[a, \infty)$ and it is easy to see that all three forms can appear. For a topical function the first form can be ruled out. To see this, it is helpful to recall first some well-understood facts about the asymptotic dynamics of a topical function, $f : \mathbb{R}^n \to \mathbb{R}^n$.

First, (4) implies that all trajectories of f are asymptotically the same:

$$f^k(x) = f^k(y) + O(1) \ \text{ as } k \to \infty \ . \tag{7}$$

Second, an elementary argument using (1) and (2) shows that the sequence $\mathsf{t}(f^k(0))$ is sub-additive,

$$\mathsf{t}(f^{k+l}(0)) \leq \mathsf{t}(f^k(0)) + \mathsf{t}(f^l(0)) \ .$$

It follows from (7) that the sequence $\mathsf{t}(f^k(x)/k)$ converges as $k \to \infty$ and that the limit is independent of x, [12, 22]. The *upper cycle-time* of f, $\overline{\chi}(f) \in \mathbb{R}$, is defined as

$$\overline{\chi}(f) = \lim_{k\to\infty} \mathsf{t}(f^k(x)/k) \ .$$

Dually, the *lower cycle-time* is $\underline{\chi}(f) = \lim_{k\to\infty} \mathsf{b}(f^k(x)/k)$. The existence of the *cycle-time vector* of f, $\chi(f) = \lim_{k\to\infty} f^k(x)/k$, is another matter altogether. It does not always exist, [12, Theorem 3.1], and one of the central problems in the field is to characterise those topical functions for which it does.

Now suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a topical function and that $f(x) \leq \lambda + x$ for some $x \in \mathbb{R}^n$ and some $\lambda \in \mathbb{R}$. Using (1) and (2), $f^k(x) \leq k\lambda + x$. Hence,

$$\mathsf{t}(f^k(x)/k) \leq \lambda + \mathsf{t}(x/k) \ ,$$

from which the following lemma immediately follows.

Lemma 1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a topical function then either $\Lambda(f) = (a, \infty)$ or $\Lambda(f) = [a, \infty)$, where $\overline{\chi}(f) \leq a$.*

Both possibilities can occur. It follows from Proposition 1 and (10a) below that if f has an eigenvector, $f(x) = \lambda + x$, then $\Lambda(f) = [\lambda, \infty]$. If $f = \mathcal{E}(A)$ where A is the non-negative matrix below

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then it is easy to see that $\Lambda(f) = (0, \infty)$.

We shall now show that $\overline{\chi}(f) = a$. This requires the following simple but crucial observation.

Lemma 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a topical function and let k be any positive integer. If $S_\lambda(f^k) \neq \emptyset$, then $S_{\lambda/k}(f) \neq \emptyset$.*

Proof. If $S_\lambda(f^k) \neq \emptyset$, then $f^k(x) \leq \lambda + x$ for some $x \in \mathbb{R}^n$. Let

$$y = x \wedge (f(x) - \lambda/k) \wedge \cdots \wedge (f^{k-1}(x) - (k-1)\lambda/k) \ .$$

Using (1) and (2) we see that

$$\begin{aligned} f(y) &\leq f(x) \wedge (f^2(x) - \lambda/k) \wedge \cdots \wedge (f^k(x) - (k-1)\lambda/k) \\ &\leq f(x) \wedge (f^2(x) - \lambda/k) \wedge \cdots \wedge (x + \lambda/k) \\ &= y + \lambda/k \ . \end{aligned}$$

Thus, $y \in S_{\lambda/k}(f) \neq \emptyset$. □

Lemma 2 allows us to give the following characterisation of $\overline{\chi}(f)$ which may be thought of as a generalised Collatz-Wielandt formula, [16, §1.3].

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a topical function. Then,*

$$\inf \Lambda(f) = \inf_{x \in \mathbb{R}^n} \mathsf{t}(f(x) - x) = \overline{\chi}(f) \ . \tag{8}$$

Proof. Let $a = \inf \Lambda(f)$. Since $f(x) \leq x + \lambda$ if, and only if, $\mathsf{t}(f(x) - x) \leq \lambda$ the first equality in (8) follows easily. Lemma 1 has already shown that $\overline{\chi}(f) \leq a$. Now choose $\epsilon > 0$. For sufficiently large k, $f^k(0) \leq (\overline{\chi}(f) + \epsilon)k$. Hence, $S_{(\overline{\chi}(f)+\epsilon)k}(f^k) \neq \emptyset$. By Lemma 2, $S_{\overline{\chi}(f)+\epsilon}(f) \neq \emptyset$. Hence, $a \leq \overline{\chi}(f) + \epsilon$. Since ϵ was chosen arbitrarily, $a \leq \overline{\chi}(f)$ and so $a = \overline{\chi}(f)$. □

A result on topical functions can be dualised by applying it to the topical function $-f(-x)$. Using this method on the Collatz-Wielandt formula, we deduce that

$$\underline{\chi}(f) = \sup_{x \in \mathbb{R}^n} \mathsf{b}(f(x) - x) \ . \tag{9}$$

If f has an eigenvector, so that $f(x) = \lambda + x$ then it follows from (1) that

$$\underline{\chi}(f) = \lambda = \overline{\chi}(f) \text{ and} \tag{10a}$$

$$\chi(f) = (\lambda, \cdots, \lambda) \ . \tag{10b}$$

3. Existence of eigenvectors

3.1. **The main result.** It is convenient for the proofs that follow to make use of the normalised sub-eigenspace, $S'_\lambda(f) \subseteq \mathbb{R}^n$, defined by

$$S'_\lambda(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \lambda + x \text{ and } \mathsf{b}(x) = 0\} \ .$$

If $\mathsf{b}(x) = 0$ then $\|x\|_{\mathsf{H}} = \|x\|_\infty$. It follows that if f is homogeneous then $S_\lambda(f)$ is non-empty and bounded in the Hilbert semi-norm if, and only if, $S'_\lambda(f)$ is non-empty and bounded in the supremum norm.

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a topical function for which some sub-eigenspace is non-empty and bounded in the Hilbert semi-norm. Then f has an eigenvector.*

Proof. Assume that $S_\mu(f)$ is non-empty and bounded in the Hilbert semi-norm. Let $a = \inf \Lambda(f)$. Evidently $a \le \mu$. We may assume, without loss of generality, that $a = \mu$. To see why, suppose that $a < \mu$. Since $S_\mu(f)$ is bounded in the Hilbert semi-norm, the sets $S'_b(f)$, for $a < b \le \mu$ are compact in the supremum norm. It follows easily from (6b) that $S'_a(f) = \bigcap_{a<b\le\mu} S'_b(f)$. The right hand side is a decreasing intersection of non-empty compact sets and so $S'_a(f)$ is also non-empty and compact. Hence $S_a(f)$ is non-empty and bounded in the Hilbert semi-norm, as claimed.

Let $g = -a + f$. By (6c), $\Lambda(g) = [0, \infty)$, so that we can find $x \in \mathbb{R}^n$ such that $g(x) \le x$. Hence $g^{k+1}(x) \le g^k(x)$ and $g^k(x) \in S_0(g) = S_a(f)$ for all $k \in \mathbb{N}$. If $\lim_{k\to\infty} \mathsf{t}(g^k(x)) = -\infty$, then $g^k(x) \le -1 + x$, for some sufficiently large k and Lemma 2 shows that $S_{-1/k}(g) \ne \emptyset$, contradicting $\Lambda(g) = [0, \infty)$. Hence $\mathsf{t}(g^k(x))$ is bounded from below as $k \to \infty$. By hypothesis, $\|g^k(x)\|_{\mathsf{H}}$ remains bounded and this can only happen if $g^k(x)$ itself remains bounded. Let $y = \lim_{k\to\infty} g^k(x)$. Then by continuity of g, $g(y) = y$, so that $f(y) = a + y$. $\square$

The following examples are instructive in the light of this result. Consider the topical functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{array}{rcl} f_1(x) & = & ((x_1 - 1) \vee x_2) \wedge (x_1 + 1) \\ f_2(x) & = & x_1 \vee x_2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g_1(x) & = & x_1 \wedge x_2 \\ g_2(x) & = & x_1 \vee x_2 \end{array} .$$

We leave it to the reader to show that $\Lambda(f) = [0, \infty)$ and

$$S_\lambda(f) = \begin{cases} \{x \in \mathbb{R}^2 \mid -\lambda + x_1 \le x_2 \le \lambda + x_1\} & \text{for } 0 \le \lambda < 1 \\ \{x \in \mathbb{R}^2 \mid -\lambda + x_1 \le x_2\} & \text{for } \lambda \ge 1 \end{cases} .$$

It follows that $S_\lambda(f)$ is bounded for $0 \le \lambda < 1$ and unbounded for $1 \le \lambda$. As for g, it has the eigenvector $(0, 0)$ and $\Lambda(g) = [0, \infty)$ but $S_\lambda(g) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le \lambda + x_2\}$ is unbounded for all $\lambda \ge 0$. (The dual super-eigenspaces are also unbounded.)

3.2. **Graphs associated to topical functions.** If A is a $n \times n$ non-negative matrix, its associated graph, $G(A)$, is the directed graph with vertices $\{1, \cdots, n\}$ and an edge from i to j if, and only if, $A_{ij} \ne 0$, [5, Chapter 2]. The matrix A is irreducible if, and only if, $G(A)$ is *strongly connected*: if there is a directed path between any two vertices. The Perron-Frobenius theorem (see Corollary 1 below) asserts that an irreducible non-negative matrix has an eigenvector all of whose components are positive. We now generalise this to topical functions.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$. Define the associated graph of f, $G(f)$, to be the directed graph with vertices $\{1, \cdots, n\}$ and an edge from i to j if, and only if, $\lim_{\nu\to\infty} f_i(\nu e_j) = \infty$, where e_j is the j-th vector of the canonical basis of $\mathbb{R}^n$.

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a topical function whose associated graph is strongly connected. Then all non-empty sub-eigenspaces of f are bounded. In particular, f has an eigenvector.*

Proof. For each edge from i to j of $G(f)$ define $h_{ji} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$h_{ji}(x) = \sup\{\nu \in \mathbb{R} \mid f_i(\nu e_j) \leq x\} \ ,$$

where we use the convention that $\sup \emptyset = -\infty$. For any $\lambda \in \mathbb{R}$, let $h^{\lambda}_{ji}(x) = h_{ji}(\lambda + x)$. Let $S_\lambda(f)$ be any non-empty sub-eigenspace of f and choose $x \in S_\lambda(f)$, which we may assume to satisfy $\mathsf{b}(x) = 0$. Let $i \in \{1, \cdots, n\}$ be the component for which $x_i = 0$. Choose any other component $j \in \{1, \cdots, n\}$. By hypothesis there exists a directed path from i to j in $G(f)$. Suppose that the nodes on this are $i = i_1, \cdots, i_k = j$, where there is an edge from i_{p-1} to i_p for $1 < p \leq k$. Since $\mathsf{b}(x) = 0$, we must have $x_{i_p} e_{i_p} \leq x$. Hence

$$f_{i_{p-1}}(x_{i_p} e_{i_p}) \leq f_{i_{p-1}}(x) \leq \lambda + x_{i_{p-1}}$$

and so $x_{i_p} \leq h^{\lambda}_{i_p i_{p-1}}(x_{i_{p-1}})$. Putting these together we find that

$$x_j \leq h^{\lambda}_{i_k i_{k-1}} \circ \cdots \circ h^{\lambda}_{i_2 i_1}(0) \ .$$

It follows that $S_\lambda(f)$ is bounded in the Hilbert semi-norm. By Theorem 1, f has an eigenvector. □

Amghibech and Dellacherie state a similar but weaker result in [1]. They use a different graph which is, in general, not strongly connected for the examples studied in the next section, with the exception of that in Corollary 3. However, the proof technique of [1], based on an approximation procedure, could be used to obtain an independent proof of Theorem 2.

Consider the topical function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{array}{rcl} f_1(x) & = & x_1 \vee (x_2 \wedge x_3) \\ f_2(x) & = & x_1 \vee x_2 \vee x_3 \\ f_3(x) & = & x_1 \vee x_2 \vee x_3 \ . \end{array}$$

$G(f)$ is not strongly connected since there are no edges from 1 to 2 and from 1 to 3. Nevertheless it is easy to check that f has bounded sub-eigenspaces. Is there a combinatorial object associated to a topical function which determines when the function has bounded sub-eigenspaces? This is an interesting problem which we hope to address elsewhere.

For convex topical functions, Theorem 2 has a converse. Recall that a function $h : \mathbb{R}^n \to \mathbb{R}$ is convex if, for all $x, y \in \mathbb{R}^n$,

$$h(\lambda x + \mu y) \leq \lambda h(x) + \mu h(y) \ , \tag{11}$$

where $0 \leq \lambda, \mu \leq 1$ and $\lambda + \mu = 1$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is convex if each component function $f_i : \mathbb{R}^n \to \mathbb{R}$ is convex. A simple deduction from (11), which is left to the reader, captures the intuition that the derivative of h is increasing. With the same notation as above, let $x' = \lambda x + \mu y = x + \mu(y - x) = y - \lambda(y - x)$. Then,

$$\frac{h(x') - h(x)}{\mu} \leq \frac{h(y) - h(x')}{\lambda} \ . \tag{12}$$

For any function $f : \mathbb{R}^n \to \mathbb{R}^n$ define its *syntactic* graph, $G^s(f)$, to be the directed graph with vertices $1, \cdots, n$ and an edge from i to j if, and only if, f_i depends on x_j in the following sense: there is no map $h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f_i(x) = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

Proposition 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a convex topical function. Then $G(f) = G^s(f)$. Moreover, $G^s(f)$ is strongly connected if, and only if, all sub-eigenspaces of f are bounded in the Hilbert semi-norm.*

Proof. Clearly, an edge of $G(f)$ is an edge of $G^s(f)$. Conversely, if there is an edge from i to j in $G^s(f)$, then we can find $x, x' \in \mathbb{R}^n$ such that $x_k = x'_k$ for all $k \neq j$, $x_j \neq x'_j$, and $f_i(x) \neq f_i(x')$. Without loss of generality, assume that $x'_j > x_j$. Choose $\nu > 0$ and let $y = x' + \nu e_j$. Let $\alpha = x'_j - x_j + \nu$, $\lambda = \nu/\alpha$ and $\mu = (x'_j - x_j)/\alpha$. Evidently, $0 \leq \lambda, \mu \leq 1$ and $\lambda + \mu = 1$ and it is easy to check that $x' = \lambda x + \mu y$, in accordance with the notation used in (12). Using this inequality we see that

$$\frac{f_i(y) - f_i(x')}{\lambda} \geq \frac{f_i(x') - f_i(x)}{\mu} \ ,$$

which can be rewritten as

$$f_i(x' + \nu e_j) \geq \frac{\nu}{x'_j - x_j}(f_i(x') - f_i(x)) + f_i(x') \ .$$

Since this holds for any $\nu > 0$, it follows that $\lim_{\nu \to \infty} f_i(x' + \nu e_j) = \infty$. But, $x' + \nu e_j \leq \mathsf{t}(x') + \nu e_j$. Using (2), we see that $f_i(\nu e_j) \geq f_i(x' + \nu e_j) - \mathsf{t}(x')$ and so $\lim_{\nu \to \infty} f_i(\nu e_j) = \infty$. It follows that there is an edge from i to j in $G(f)$ and so $G^s(f)$ is identical to $G(f)$.

If $G^s(f) = G(f)$ is strongly connected then Theorem 2 shows that all the sub-eigenspaces of f are bounded. Conversely, suppose that $G^s(f)$ is not strongly connected. Then, by standard arguments, [5, Chapter 2], we can, after possibly reordering, partition the variables so that $x = (y, z)$, where $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, $n = p + q$ and $f(x) = (g(y, z), h(z))$, for some topical functions $g : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, and $h : \mathbb{R}^q \to \mathbb{R}^q$. Suppose that $S_\lambda(f)$ is non-empty and choose $x \in S_\lambda(f)$. Since $f(x) \leq \lambda + x$, we must have $g(y, z) \leq \lambda + y$ and $h(z) \leq \lambda + z$. Now choose $\mu \geq 0$. Using (1) and (2) it follows that $g(\mu + y, z) \leq g(\mu + y, \mu + z) \leq \lambda + \mu + y$. Hence $(\mu + y, z) \in S_\lambda(f)$ for all $\mu > 0$ and so all non-empty sub-eigenspaces of f are unbounded in the Hilbert semi-norm. □

4. Applications

We now show that several well-known theorems are immediate corollaries of the elementary results above.

Corollary 1. (Perron-Frobenius theorem, [5]) *Let A be a $n \times n$ non-negative matrix. If A is irreducible then its spectral radius is an eigenvalue, for which A has an eigenvector all of whose components are positive.*

Proof. Since A is irreducible the nondegeneracy condition (3) holds. Let $f = \mathcal{E}(A)$. It is easy to see that $G(f)$, the graph associated to f, is identical to $G(A)$, the graph associated to A. Since A is irreducible, $G(A)$ is strongly connected and so, by Theorem 2, f has an eigenvector: $f(x) = r + x$. Evidently, $A\exp(x) = \exp(r)\exp(x)$, where $\exp(x)$ has all its components positive. It remains to show that $\exp(r)$ is the spectral radius of A. For completeness, we reproduce the standard argument using the Collatz-Wielandt property. Suppose that $z \in \mathbb{C}^n$ is a (multiplicative) eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$: $Az = \lambda z$. Let $|z| \in \mathbb{R}^n$ be the vector of absolute values: $|z| = (|z_1|, \cdots, |z_n|)$ and let $x = \log(|z|)$. A simple application of the triangle inequality shows that $A|z| \geq |\lambda||z|$. Hence $f(x) \geq \log(|\lambda|) + x$. It follows, using (9) and (10a), that

$$\log(|\lambda|) \leq \mathsf{b}(f(x) - x) \leq \underline{\chi}(f) = r$$

and so $\exp(r)$ is the spectral radius of A. $\square$

The function $\mathbb{R}^2 \to \mathbb{R}$ which takes $x \mapsto \log(\exp(x_1) + \exp(x_2))$ is convex, from which it follows that $\mathcal{E}(A)$ is a convex topical function. Hence we could have used Proposition 2, together with Theorem 1 in the proof of Corollary 1. This applies also to the topical functions in the corollaries below, which are all convex.

Property (1) applied to the function $\mathsf{t} : \mathbb{R}^2 \to \mathbb{R}$ illustrates that addition distributes over maximum. It follows that the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of maximum (as addition) and addition (as multiplication) forms an idempotent semiring (a semiring whose addition satisfies $a + a = a$), called the *max-plus semiring* and denoted $\mathbb{R}_{\max}$, [11]. Suppose that A is a $n \times n$ matrix over $\mathbb{R}_{\max}$ which satisfies a similar nondegeneracy condition to (3):

$$\forall i,\ \exists j \ \text{ such that } \ A_{ij} \neq -\infty \ .$$

If $x \in \mathbb{R}^n$ then it is easy to see that $x \mapsto Ax$ defines a topical function. For instance, the matrix on the left below gives rise to the function on the right.

$$\begin{pmatrix} 2 & -1 \\ -\infty & 4 \end{pmatrix} \qquad \begin{array}{rcl} f_1(x) & = & (x_1 + 2) \vee (x_2 - 1) \\ f_2(x) & = & x_2 + 4 \ . \end{array}$$

If A is a $n \times n$ matrix over $\mathbb{R}_{\max}$, its associated graph, $G(A)$, is the directed graph with vertices $\{1, \cdots, n\}$ and an edge from i to j if, and only if, $A_{ij} \neq -\infty$. It is customary, in max-plus theory, to adjoin labels ("weights") to the edges in $G(A)$, [2]. This unlabelled version will be sufficient for our purposes. A is said to be irreducible if $G(A)$ is strongly connected. It is easy to see that if f is the topical function corresponding to A then the graphs $G(A)$ and $G(f)$ coincide. The following result follows immediately. The cited reference is to a standard source but the result has been proved independently many times.

Corollary 2. (Perron-Frobenius for max-plus, [2, Theorem 3.28]) *An irreducible max-plus matrix has an eigenvector.*

In max-plus theory, the eigenvectors of a matrix lie in $(\mathbb{R} \cup \{-\infty\})^n$. The point of Corollary 2 is that such an eigenvector can be found in $\mathbb{R}^n$. The formula for the eigenvalue, based on the structure of the circuits of $G(A)$, lies outside the scope of the present paper, [2].

For the next result, assume that P is the transition matrix of a Markov chain (so that P is row-stochastic) and let $f(x) = c + Px$, for some $c \in \mathbb{R}^n$. Evidently, f is a topical function. By (10b), if f has an eigenvector with eigenvalue λ, then

$$\frac{(1 + P + \cdots + P^{k-1})c}{k} = \frac{f^k(0)}{k}$$

converges to $(\lambda, \cdots, \lambda)$. The next result can hence be thought of as a version of the mean ergodic theorem for Markov chains, [23, Chapter XIII, §1, Theorem 2].

Corollary 3. *Let $c \in \mathbb{R}^n$ and let P denote a $n \times n$ irreducible row-stochastic matrix. The function $f(x) = c + Px$ has an eigenvector.*

Proof. f is a topical function and $G(f)$ is strongly connected so the result follows from Theorem 2. $\square$

A family of non-negative matrices $\{P^u\}_{u\in U}$ is said to be *communicating* if the matrix $\sup_{u\in U} P^u$, obtained by taking entrywise suprema, has finite entries and is irreducible. The following result is due to Bather.

Corollary 4. (Bather's theorem, [4, Theorem 2.4]) *Let $\{P^u\}_{u\in U}$ be a communicating family of row-stochastic matrices, and let $\{c^u\}_{u\in U}$ be a family of vectors $c^u \in \mathbb{R}^n$ that is bounded above. Then the function $f(x) = \sup_{u\in U}(c^u + P^u x)$ has an eigenvector.*

Proof. It is easy to check that $G(f) = G(\sup_{u\in U} P^u)$. Since the latter is strongly connected by hypothesis, the result follows immediately from Theorem 2. □

The next result was proved by Zijm in the special case of a finite communicating family. It follows by combining the argument of Corollary 4 with that of Corollary 1.

Corollary 5. (Zijm's theorem, [24, Theorem 3.4]) *Let $\{A^u\}_{u\in U}$ be a communicating family of non-negative matrices. Then the function $f(x) = \sup_{u\in U} A^u x$ has a (multiplicative) eigenvector.*

As a last illustration of the ideas developed here, consider the topical function $\mathcal{E}(f)$ where $f : (\mathbb{R}^+)^3 \to (\mathbb{R}^+)^3$ is defined by:

$$\begin{aligned} f_1(x) &= 2x_1 \vee 3x_2 \\ f_2(x) &= \sqrt{x_1(4x_2 + 15x_3)} \\ f_3(x) &= x_2 \ . \end{aligned}$$

None of the above corollaries can be applied to f. However, $G(\mathcal{E}(f))$ is strongly connected. By Theorem 2, f has a (multiplicative) eigenvector. In fact, $f(3,3,1) = 3(3,3,1)$.

5. Conclusions

An alternative approach to the eigenvector problem stems from the observation in (7) that all trajectories of a topical function f are asymptotically equivalent. This suggests that the asymptotics of $f^k(x)$ contain information on the existence of fixed points, an idea confirmed in recent work, [8, 10].

Topical functions can be defined and studied on cones in Banach spaces, as Krein and Rutman have done for Perron-Frobenius theory. Some attractive examples have emerged here, [21], but with the exception of Nussbaum's work, [17, 18], little general progress has been made.

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