Tropical modules, zero-sum games and non-archimedean optimization

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Noncommutative geometry - Workshop #3: number theory
A celebration of Alain Connes' 70th birthday
Shanghai
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Based on work with Cohen, Quadrat (tropical modules) Akian and Guterman (tropical geometry and games) and Allamigeon, Benchimol and Joswig (tropical linear programming).

- Recent work by Connes and Consani, on the arithmetic site and the scaling site, led them to develop homological algebra in characteristic one
- Modules, and congruences over the tropical semifield are the basic objects
- Cohen, Quadrat, SG: modules / congruences over the tropical semifield, $\mathbb{R}_{\max}$, used in the 90'-00', with motivations from Control and Optimization
- This talk: link between tropical modules and zero-sum games.
- guided tour of "pre-NCG era" results on tropical modules, with an application to computational complexity

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Talk scheduled at 10:40, break ends at time x, orator ready at time y, when do we start?

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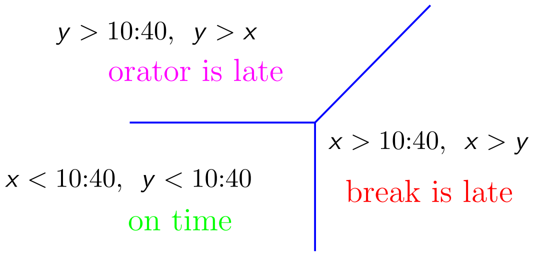

tropical modules arose as spaces of reachable dates

tropical congruences arose as observability relations

Initial motivation 2: Hamilton-Jacobi PDE

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}), \qquad H(x, \cdot) \text{ convex}$$

If u, v are two solutions, $\alpha \in \mathbb{R}$, then $\max(u, v)$ and $\alpha + v$ are also solutions (space of solutions is a tropical module)

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$$v(t,x) = \sup_{X(0)=x,\, \dot{X}(\cdot)} -\int_0^t L(X(s), \dot{X}(s))ds + v(0, X(t))$$

$$L(x,u) := \sup_{p\in\mathbb{R}^n} \langle p, x\rangle - H(x,p) \ .$$

"One can also interpret [these magical properties] in terms of the so-called exotic algebra $(\mathbb{R}, \max, +)$", Crandall, Ishii, Lions User's guide of viscosity solutions. Also: Maslov's "dequantization"

Part I.

Two open problems concerning zero-sum games and linear programming

The mean payoff problem

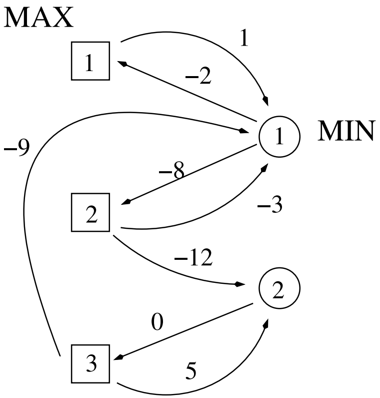
Mean payoff games

$G = (V, E)$ bipartite graph. $r_{ij} \in \mathbb{Z}$ price of the arc $(i, j) \in E$.

MAX and MIN move a token, alternatively (square states: MAX plays; circle states: MIN plays). n MIN nodes, m MAX nodes.

MIN always pays to MAX the price of the arc (having a negative fortune is allowed)

v_i^k value of MAX, initial state (i, MIN).



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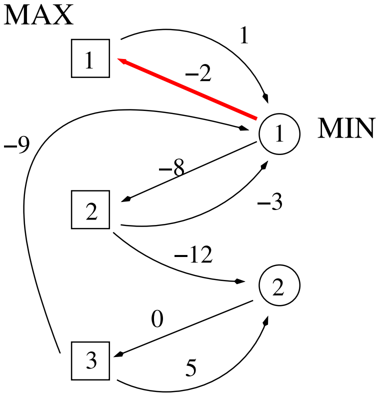

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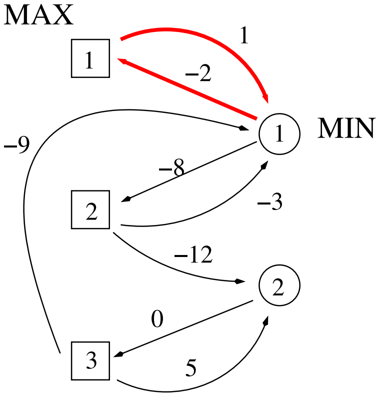

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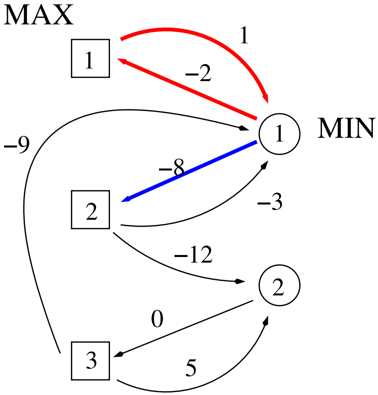

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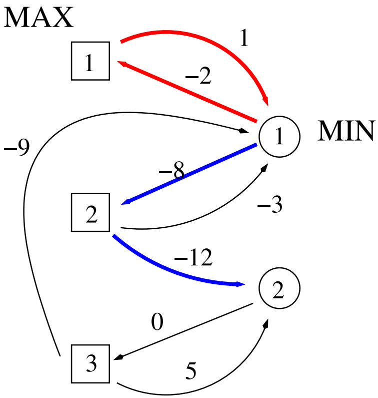

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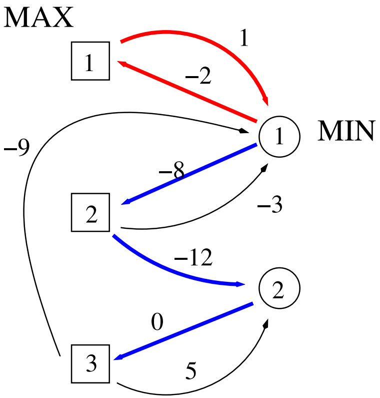

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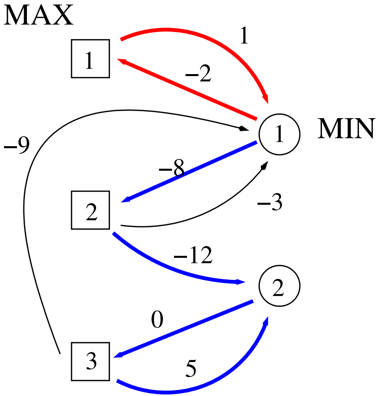

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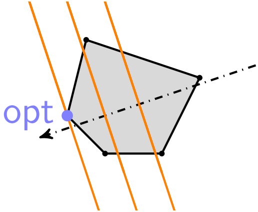
Zwick and Paterson [1996] showed that value iteration solves MPG in pseudo polynomial time $O((n+m)^5 W)$ where $W = \max_{ij} |r_{ij}| = O(2^L)$.

Complexity issues in linear programming

A linear program is an optimization problem:

$$\min c \cdot x;\ Ax \leqslant b,\ x \in \mathbb{R}^n \ ,$$

where $c \in \mathbb{Q}^n$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.



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Can linear programming be solved in strongly polynomial time?

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$\neq$ strongly polynomial (arithmetic model): number of arithmetic operations bounded by poly(m, n), **and** the size of operands of arithmetic operations is bounded by poly(L).

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Classical approaches to solve LP problem

- pivoting algorithms, simplex Dantzig (1947)

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- ellipsoid, Khachyan (1979), polynomial time
- interior points, Karmarkar (1984) ..., polynomial time.

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The simplex method (Dantzig, 1947)

Iterate over adjacent vertices (basic points) of the polyhedron while improving the objective function

$$c^\top v^1 \geqslant c^\top v^2 \geqslant \ldots \geqslant c^\top v^N$$

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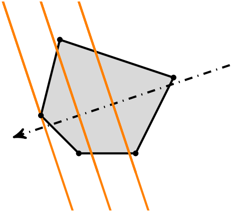

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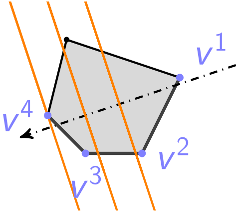

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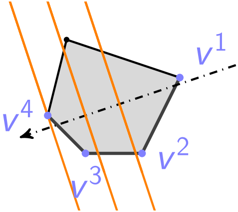

the algorithm is parametrized by a *pivoting rule*, which selects the next edge to be followed.

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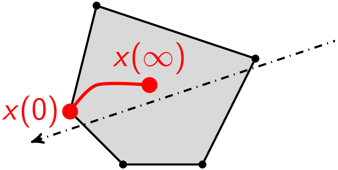
- Every iteration (pivoting from a basic point to the next one) can be done with a strongly polynomial complexity (linear system over $\mathbb{Q}$).
- is there a pivoting rule ensuring that the number of iterations in the worst case is polynomially bounded?
- It is not even known that the graph of the polyhedron has polynomial diameter (polynomial Hirsch conjecture), ie that the perfectly lucid pivoting rule makes a polynomial number of steps.

Interior points

For all $\mu > 0$, consider the *barrier problem*

$$\min c \cdot x - \mu\Big(\sum_{i=1}^{m} \log(b_i - A_i x)\Big), \quad b_i - A_i x > 0 \; i \in [m]$$

$\mu \mapsto x(\mu)$ optimal solution, is the central path. branch of an algebraic curve. $x(0)$ is the solution of the LP.



"*the good convergence properties of Karmarkar's algorithm arise from good geometric properties of the set of trajectories*", Bayer, Lagarias 89.

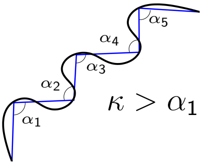
Total curvature

The total curvature of a path γ, parametrized by arc length, so that $\|\gamma'(s)\| = 1$, is given by

$$\kappa := \int_0^L \|\gamma''(s)\| ds$$

or

$$\kappa = \sup_{q \geqslant 2} \sup_{0 \leqslant \lambda_0 < \cdots < \lambda_q \leqslant L} \angle \gamma(\lambda_{k-1})\gamma(\lambda_k)\gamma(\lambda_{k+1})$$



$$\kappa > \alpha_1 + \cdots + \alpha_5$$

Continuous analogue of Hirsch's conjecture

Dedieu and Shub (2005) initially conjectured that the total curvature of the central path is $O(n)$ (n number of variables).

This was motivated by a theorem of Dedieu-Malajovich-Shub (2005): total curvature is $O(n)$, averaged over all 2^{n+m} LP's (cells of the arrangement of hyperplanes), $\epsilon_i A_i x \leqslant b_i$, $\eta_j x_j \geqslant 0$, $\epsilon_i, \eta_j = \pm 1$.

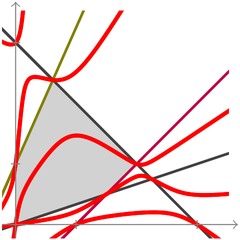


Illustration from Benchimol's Phd

Deza, Terlaky and Zinchenko (2008) constructed a redundant Klee-Minty cube, showing that a total curvature exponential in n is possible, and revised the conjecture of Dedieu and Shub:

Conjecture (Continuous analogue of Hirsch conjecture, [Deza, Terlaky, and Zinchenko, 2008])

The total curvature of the central path is $O(m)$, where m is the number of constraints.

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Theorem (Allamigeon, Benchimol, SG, Joswig, arXiv:1405.4161+followup)

- *There is a LP with* $2r+2$ *variables and* $3r+4$ *inequalities such that the central path has a total curvature in* $\Omega(2^r)$. *(Disproves DTZ).*

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- *There is a LP with $2r+2$ variables and $3r+4$ inequalities such that the central path has a total curvature in $\Omega(2^r)$. (Disproves DTZ).*
- *Moreover, every log-barrier interior point method which iterates within the neighborhood $\mathcal{N}_\theta^{-\infty}$ of the primal-dual central path of "width" $\theta < 1$ independent of the instance makes $\Omega(2^r)$ iterations on the same LP.*

Although the word “tropical” appears in none of these statements, the proofs rely on tropical geometry in an essential way, through linear programming over non-archimedean fields, and tropical modules.

Part II.
Operator approach to mean payoff games

Value of a zero-sum game

k horizon, i initial state. σ, π strategies of both players determine a run:

$$(i, \mathrm{MIN}), (j_1, \mathrm{MAX}), (i_1, \mathrm{MIN}), (j_2, \mathrm{MAX}), \dots, (j_k, \mathrm{MAX}), (i_k, \mathrm{MIN})$$

Max receives

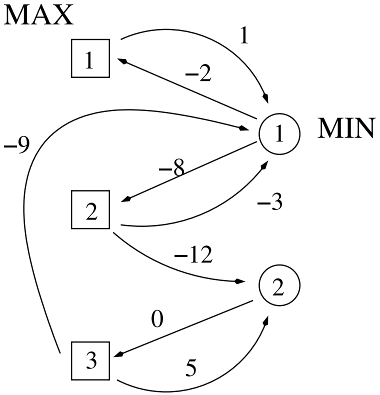
$$R_i^k(\sigma, \pi) = r_{ij_1} + r_{j_1 i_1} + r_{i_1 j_2} + \cdots + r_{j_k i_k} \quad .$$

The value of the game is v_i^k and σ^*, π^* are optimal strategies if

$$R_i^k(\sigma^*, \pi) \leqslant v_i^k = R_i^k(\sigma^*, \pi^*) \leqslant R_i^k(\sigma, \pi^*), \qquad \forall \sigma, \pi$$

v_i^k value of MAX, initial state (i, MIN).

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



$$v^1 = (0, 0)$$
$$v^2 = (-11, 5)$$
$$v^3 = (-15, 10)$$
$$v^4 = (-16, 15)$$
$$\lim_k v^k/k = (-1, 5)$$

Theorem (Shapley)

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$$[T(x)]_j = \min_{i\in[m],\ j\to i} \left(r_{ji} + \max_{k\in[n],\ i\to k} (r_{ik} + x_k) \right)$$

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$$[T^k(u)]_i$$

is the value of a modified game in horizon k with initial state i, in which MAX receives an additional payment of u_j in the terminal state j

Shapley operators of games are
monotone (or order preserving)

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Known axioms in non-linear potential theory / PDE viscosity solutions theory, eg Crandall and Tartar, PAMS 80, Dellacherie.

Theorem (Bewley, Kohlerg 76, Neyman 03)

The mean payoff vector

$$\lim_{k \to \infty} T^k(0)/k$$

does exist if $T : \mathbb{R}^n \to \mathbb{R}^n$ is semi-algebraic and nonexpansive in any norm.

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Finite action space and perfect information implies T piecewise linear.

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$T : \mathbb{R}^n \to \mathbb{R}^n$ extends continuously $(\mathbb{R} \cup \{-\infty\})^n \to (\mathbb{R} \cup \{-\infty\})^n$. (Burbanks, Nussbaum, Sparrow).

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The converse follows from a fixed point theorem of Kohlberg (a nonexpansive piecewise linear map has an invariant half-line).

Part III.
Tropical modules

Tropical semifield $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, equipped with

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For any totally ordered abelian group $(G, +, \leqslant)$, one can define $G_{\max}$.
$G = (\mathbb{R}^N, +, \leqslant_{\text{lex}})$ specially useful.

Tropical modules

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V is a tropical convex set if the same is true conditionnally to "$\lambda + \mu = 1$", i.e., $\max(\lambda, \mu) = 0$.

By the subharmonic certificates theorem, the game is winning for MAX, i.e., $\exists j, \lim_{k\to\infty}[T^k(0)]_j/k \geqslant 0$, iff

$$V = \{v \in \mathbb{R}^n_{\max} \mid T(v) \geqslant v\} \not\equiv \text{“}0\text{”} = (-\infty, \ldots, -\infty) \ .$$

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All closed tropical submodules of $\mathbb{R}^n_{\max}$ arise from a Shapley operator T (infinite number of actions on one side allowed).

Tropical adjoints

Let $A \in \mathbb{R}_{\max}^{m \times n}, x \in \mathbb{R}_{\max}^{n},\ y \in \mathbb{R}_{\max}^{n}$

$$(Ax)_i = \max_{j \in [n]}(A_{ij} + x_j), \qquad i \in [m]$$

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More on adjoints: Cohen, SG, Quadrat, LAA 04

The Shapley operator of a MPG can be written as

$$[T(v)]_j = \min_{i\in[m],\ j\to i} \left(-A_{ij} + \max_{k\in[n],\ i\to k} (B_{ik} + v_k) \right)$$

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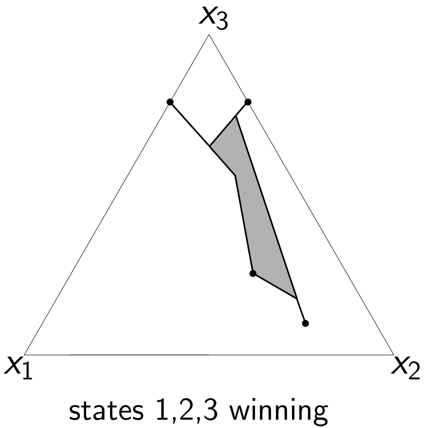
The set of subharmonic certificates $\{v \mid Av \leqslant Bv\}$ is a tropical convex polyhedral cone.

Modules of subharmonic vectors



Part IV.
The tropical analogues of Hilbert spaces

Scott's topology and duality

L complete lattice. $U \subset L$ is Scott open if it is a *upper set* (i.e., $u \in U, v \geqslant u \implies v \in U$) and for all directed sets $D \subset L$, $\sup D \in U$ implies $D \cap U \neq \emptyset$.

A map $f : L \to L'$ is Scott continuous iff it commutes with directed sups.

Complete tropical module V: modules over $\overline{\mathbb{R}}_{\max}$, stable by arbitrary suprema, structure laws Scott continuous. V' topological dual.

Theorem (Cohen, SG, Quadrat, LAA 04)

Suppose that V is a complete tropical module. Then, V and V' are anti-isomorphic lattices. Moreover, $V \simeq (V')'$.

See Gierz et al., book: Continuous lattices and domains for background on Scott topology.

Tropical sesquilinear form and Hilbert's metric

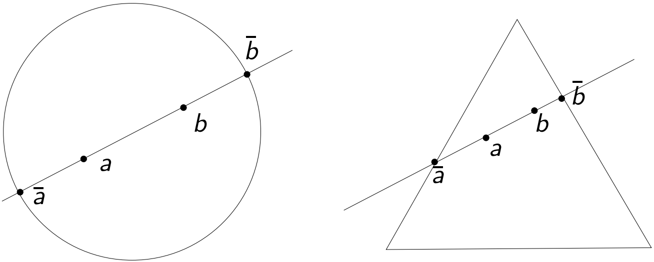
$$
\begin{aligned}
x/v &:= \max\{\lambda \mid \text{“}\lambda v\text{”} \leqslant x\} \\
&= \min_i (x_i - v_i) \quad \text{if } x, v \in \mathbb{R}^n \ .
\end{aligned}
$$

$$
\delta(x,y) = \text{“}(x/y)(y/x)\text{”} = \min_i (x_i - y_i) + \min_j (y_j - x_j)
$$

$d = -\delta$ is the (additive) Hilbert's projective metric

$$
d(x,y) = \|x - y\|_H, \ \ \|z\|_H := \max_{1 \leqslant i \leqslant d} z_i - \min_{1 \leqslant i \leqslant d} z_i \ .
$$

Hilbert's metric on an open convex set



$$d_H(a,b) = \log \frac{|b-\bar a||a-\bar b|}{|a-\bar a|||b-\bar b|} \quad .$$

disc: Klein model of the hyperbolic space;

simplex: d_H conjugate to the additive (tropical) Hilbert metric (take $\exp : \mathbb{R}^n \to \mathbb{R}^n_+$ and a cross section of $\mathbb{R}^n_+$).

Projection on a tropical cone

If $V \subset \overline{\mathbb{R}}_{\max}^n$ is the complete tropical module generated by U,

$$\begin{aligned} P_V(x) &= \max\{v \in V \mid v \leqslant x\} \\ &= \max_{u \in U} (x/u) + u \ . \end{aligned}$$

Similar to $\qquad P_V(x) = \sum_{u \in U} \langle x, u \rangle u$

$$V = \mathrm{Col}(A), \quad [P_V(x)]_i = \max_{k \in [p]} \min_{j \in [n]} (A_{ik} - A_{jk} + x_j), \quad i \in [n]$$

Cuninghame-Green; Gondran, Minoux; Cohen, SG, Quadrat

Separation

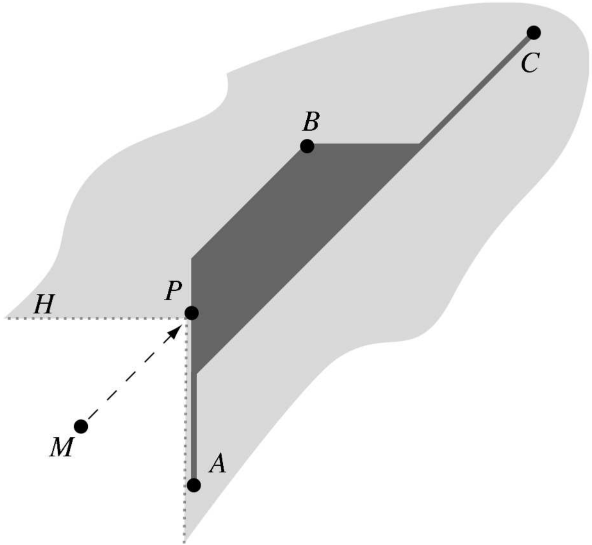
V complete tropical submodule of $\overline{\mathbb{R}}^d_{\max}$
If $y \notin V$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leqslant P_V(y)/v\}$$

contains V and not y.

Compare with the optimality condition for the projection on a space V: $\langle y - P_V(y), v\rangle = 0, \forall v \in V$

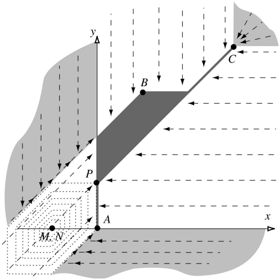
First separation theorem in Zimmermann 77. Hilbert's construction in Cohen, SG, Quadrat in Bensoussan01, LAA04.


B
C
P
H
M
A

Best approximation in Hilbert's projective metric

Prop.(Cohen, SG, Quadrat, in Bensoussan Festschrift 01)

$$d(x, P_{\mathcal{V}}(x)) = \min_{y \in \mathcal{V}} d(x, y) \enspace .$$



Tropical direct summands

$V \subset X$ tropical modules, $E \subset V^2$ congruence. $X = V \oplus E$ (direct sum) if

$$\forall x \in X, \qquad \exists! v \in V, \qquad (x, v) \in E \enspace .$$

Suppose $U \xrightarrow{B} X \xrightarrow{C} Y$, $V = \operatorname{Range} B$,
$E = \ker C = \{(x, z) \mid Cx = Cz\}$, projection $X \mapsto V$?

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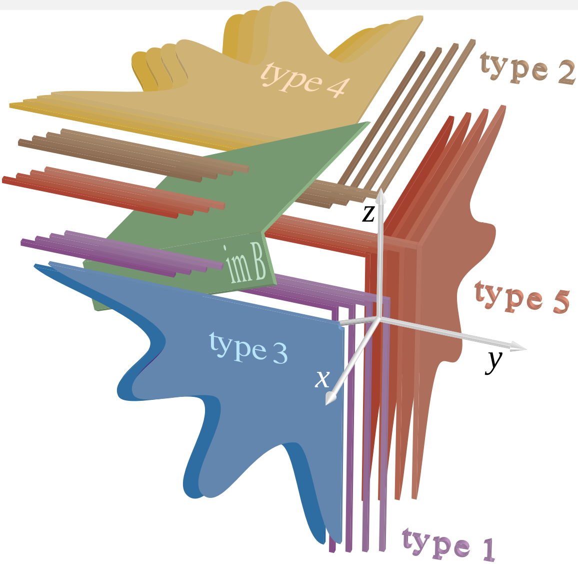
$$\begin{aligned} \Pi_B^C(x) &:= \max\{z \in \mathrm{Range}\, B \mid Cz \leqslant Cx\} \\ &= B \circ B^\sharp \circ C^\sharp \circ C(x) = B(B\backslash(C\backslash(Cx))) \end{aligned}$$

Theorem (Cohen, SG, Quadrat, Wodes'96, IEEEMed'97)

TFAE

- $X = \mathrm{Range}(B) \oplus \ker C$
- $C \circ \Pi_B^C = C, \qquad \Pi_B^C \circ B = B$
- $B = (B/(CB))CB \qquad C = CB((CB)\backslash C)$.

Tropical direct summands illustrated



$$B = \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \\ 2 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

Part V.
Tropical polyhedra

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leqslant bx\text{”}\}$$

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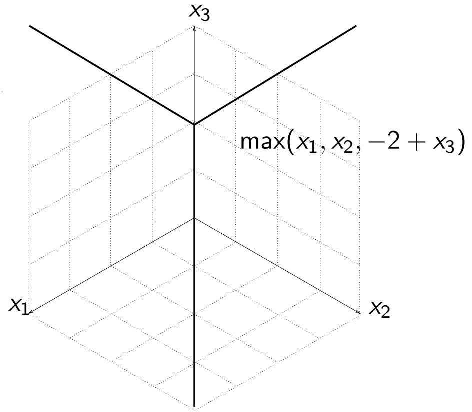

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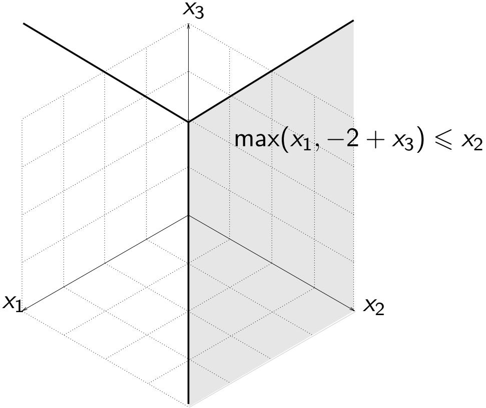

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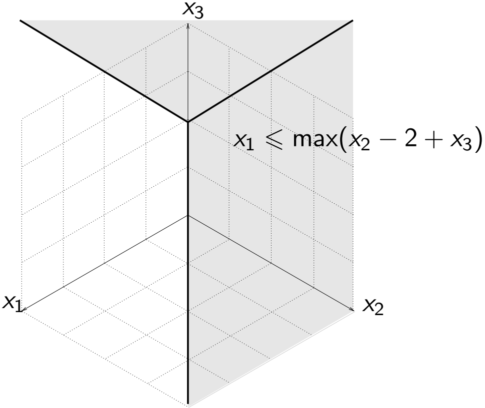

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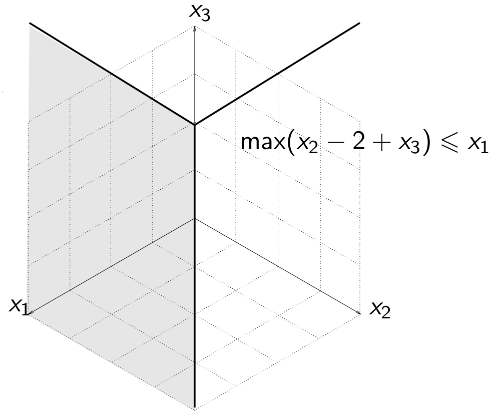
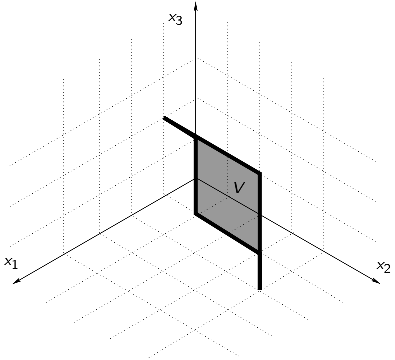


Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



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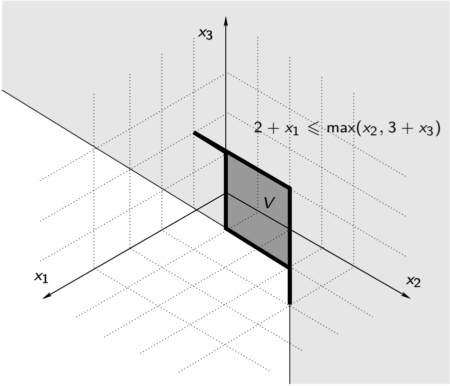

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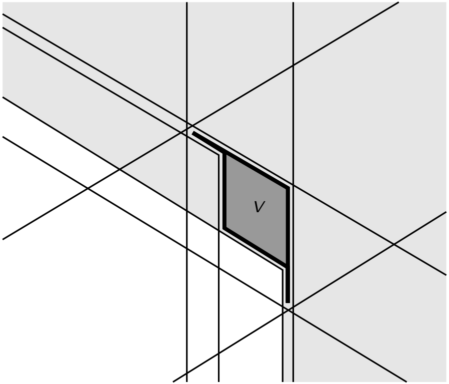
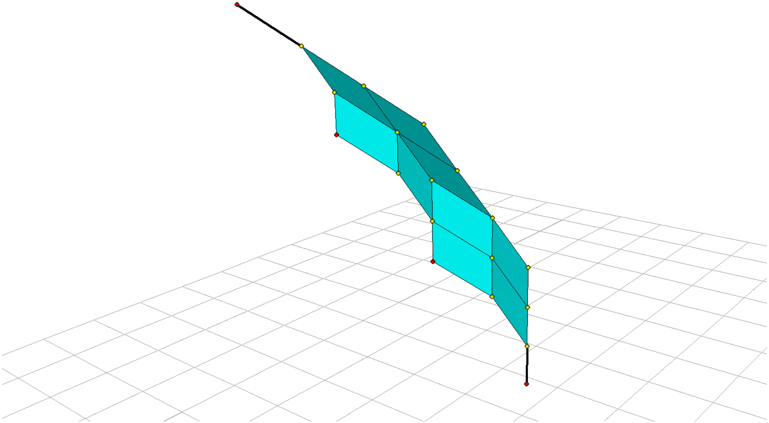


A tropical polytope with four vertices



Structure of the polyhedral complex: Develin, Sturmfels

Tropical objects arise by considering non-archimedean valuations.

There is a convenient choice of non-archimedean field in tropical geometry . . .

Puiseux series with real exponents $\mathbb{C}\{\{t^{-\mathbb{R}}\}\}$,

$$f(t) = \sum_{k \in \mathbb{N}} a_k t^{b_k}$$

were $a_k \in \mathbb{C}$, and the sequence $b_k \in \mathbb{R}$ tends to $-\infty$.

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$\mathbb{K} := \mathbb{R}\{\{t^{-\mathbb{R}}\}\}_{\mathrm{cvg}}$, absolutely convergent real generalized Puiseux series constitute a real closed field, van den Dries and Speissegger (TAMS) (definable functions in a variant of the o-minimal structure $\mathbb{R}_{an,*}$).

Non-archimedean valuation of

$$f(t)=\sum_{k\in\mathbb{N}} a_k t^{b_k}$$

$$\operatorname{val} f = \max_{k, a_k\neq 0} b_k = \lim_{t\to\infty} \frac{|\log f(t)|}{\log t}$$

$$\operatorname{val}(f+g) \leqslant \max(\operatorname{val} f, \operatorname{val} g), \qquad \text{and} \;=\; \text{holds if } f,g\geqslant 0$$
$$\operatorname{val}(fg) = \operatorname{val} f + \operatorname{val} g \;.$$

Theorem (Combines Develin and Yu and Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

1. *Every tropical polyhedron P can be written as $P = \operatorname{val} \mathcal{P}$ where $\mathcal{P}$ is a polyhedron in $\mathbb{K}_+^n$, here $\mathbb{K} = \mathbb{R}\{\{t^{-\mathbb{R}}\}\}_{cvg}$.*
2. *Moreover, P is the uniform (Hausdorff) limit of*

$$\log_t \mathcal{P} := \{\frac{\log z}{\log t} \mid z \in \mathcal{P}\}$$

as $t \to \infty$.

Related deformation in Briec and Horvath.

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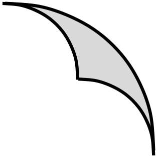

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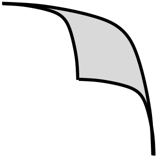

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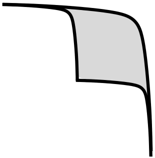


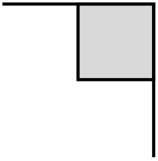
Related deformation in Briec and Horvath.

Theorem (Combines Develin and Yu and Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

1. *Every tropical polyhedron P can be written as $P = \mathrm{val}\,\mathcal{P}$ where $\mathcal{P}$ is a polyhedron in $\mathbb{K}_+^n$, here $\mathbb{K} = \mathbb{R}\{\{t^{-\mathbb{R}}\}\}_{cvg}$.*
2. *Moreover, P is the uniform (Hausdorff) limit of*

$$\log_t \mathcal{P} := \{\frac{\log z}{\log t} \mid z \in \mathcal{P}\}$$

as $t \to \infty$.



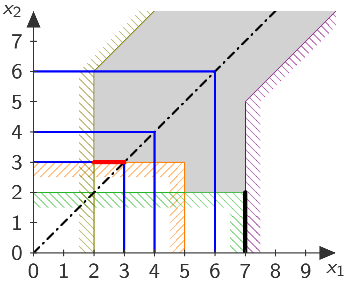
Related deformation in Briec and Horvath.

Tropical linear program

$$\min \text{“}c^\top x\text{”}; \qquad \text{“}A^+x + b^+ \geqslant A^-x + b^-\text{”}$$

$$\min \max_j c_j + x_j$$

$$\max(\max_j(A^+_{ij} + x_j), b^+_i) \geqslant \max(\max_j(A^-_{ij} + x_j), b^-_i) \quad .$$



Correspondence classical $\leftrightarrow$ tropical LP

Theorem (Allamigeon, Benchimol, SG, Joswig, SIAM J. Disc. Math)

Suppose that $\mathcal{P} = \{x \in \mathbb{K}^n \mid \mathbf{A}\mathbf{x} + \mathbf{b} \geqslant 0\}$ is included in the positive orthant of $\mathbb{K}^n$ and that the tropicalization of $(\mathbf{A}, \mathbf{b})$ is sign generic (to be defined soon). Then,

$$\mathrm{val}(\mathcal{P}) = \{x \in \mathbb{R}^n_{\max} \mid \text{“}A^+x + b^+ \geqslant A^-x + b^-\text{”}\} ,$$

where $(A^+ \; b^+) = \mathrm{val}(\mathbf{A}^+\mathbf{b}^+)$ and $(A^- \; b^-) = \mathrm{val}(\mathbf{A}^- \; \mathbf{b}^-)$. Moreover the classical and tropical polyhedron have the same combinatorics: valuation sends basic points to basic points, edges to edges, etc.

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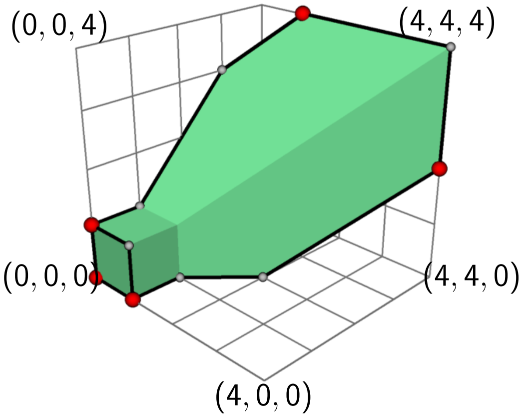
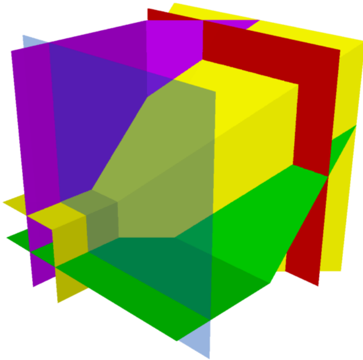
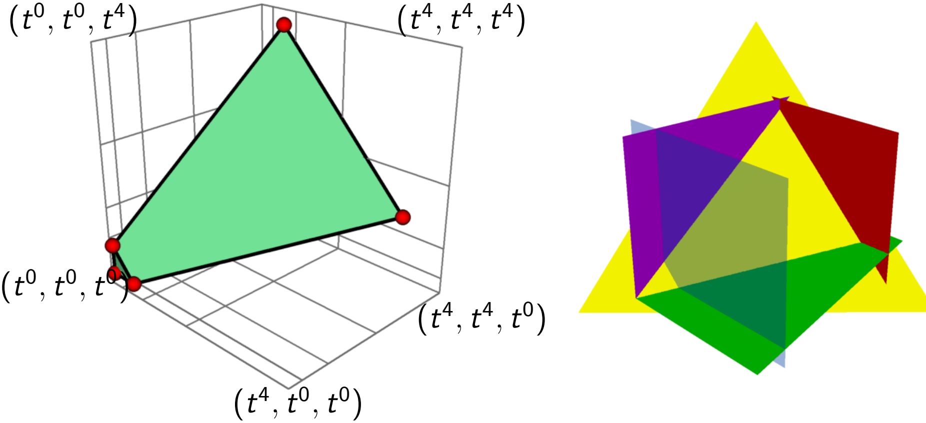
A point of a tropical polyhedron is basic if it saturates n inequalities. A tropically extreme point (member of a minimal generating family) is basic, but not vice versa.





Picture from [Allamigeon, Benchimol, Gaubert, and Joswig, 2015a].



Picture from [Allamigeon, Benchimol, Gaubert, and Joswig, 2015a].

Symmetrized tropical semiring

$s^2 = 1$ ($s =$ "-1"), $\mathbb{S}_{\max} := \mathbb{R}_{\max}[s]/\sim$, $a + sb \sim a$ for $b < a$, $a + sb \sim sb$ for $a < b$.

Three kind of equivalence classes: a, sa, $a + sa$ (a and sa are said to be signed).

$\mathbb{S}_{\max} \simeq \mathbb{K}/\mathbb{K}_0^+$, $\mathbb{K}_0^+ := \{f \in \mathbb{K} \mid \operatorname{val} f = 0,\ f > 0\}$, i.e., $\mathbb{S}_{\max} =$ powerset semiring of Krasner hyperfield of signed tropical numbers.

$\mathbb{S}_{\max} \mapsto \mathbb{R}_{\max}$, $a + sb \mapsto |a + sb| := a + b$, morphism of semirings.

$\mathbb{S}_{\max}$ used in Maxplus 90, showing a tropical Cramer theorem.

Tropical determinant

$A \in \mathbb{S}_{\max}^{n\times n}$, $M_{ij} := |A_{ij}|$

$$\det A = \sum_{\sigma} \epsilon(\sigma) \prod_i A_{i\sigma(i)} \quad \in \mathbb{S}_{\max}$$

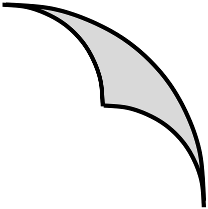
$|\det A|$ is the value of the optimal assignment problem:

$$\max_{\sigma} \sum_i M_{i\sigma(i)}$$

A is sign generic if $\det A \neq 0$ is signed (ie., if all the monomials in the determinant expansion have the same sign).

Sign-genericity is related to even cycle problem and Polya's permanent problem. Checkable in polynomial time Robertson, Seymour, Thomas

sign generic condition not satisfied, valuation does not commute with the external representation.



$$x_1 + x_2 \leqslant 1, \qquad t^1 x_1 + x_2 \geqslant 1, \qquad x_1 + t^1 x_2 \geqslant 1$$

$$X_i = \log x_i / \log t, \qquad t \to 0.$$

$$\max(X_1, X_2) \leqslant 0,\ \max(1 + X_1, X_2) \geqslant 0,\ \max(X_1, 1 + X_2) \geqslant 0 \ .$$

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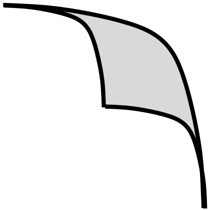

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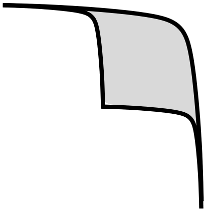

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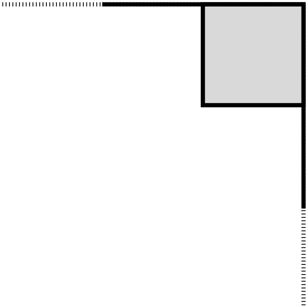

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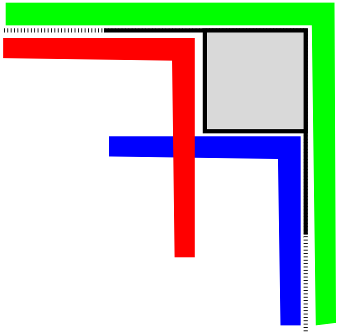

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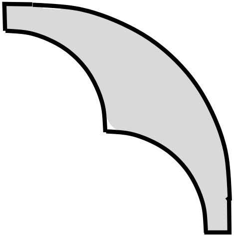

$$x_1 + x_2 \leqslant t^{\epsilon},\ t^1 x_1 + x_2 \geqslant 1,\ x_1 + t^1 x_2 \geqslant 1,\ t^2 x_1 \geqslant 1,\ t^2 x_2 \geqslant 1$$

$X_i = \log x_i / \log t, \qquad t \to 0.$

$$\max(X_1, X_2) \leqslant \epsilon,\ \max(1+X_1, X_2) \geqslant 0,\ \max(X_1, 1+X_2) \geqslant 0,\ X_1 \geqslant 2, X_2 \geqslant 2\ .$$

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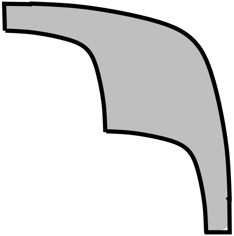

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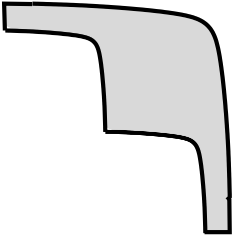

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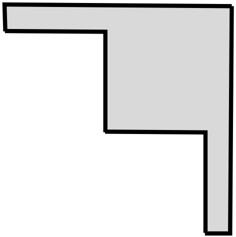

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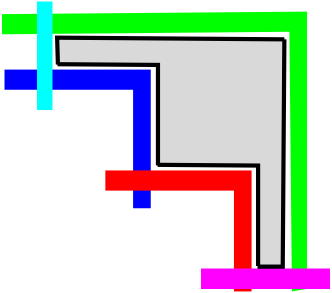

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Assume that the data are tropically in general position.

Theorem (Allamigeon, Benchimol, SG, Joswig SIAM J. Disc. Math)

The valuation of the path of the simplex algorithm over $\mathbb{K}$ can be computed tropically (with a compatible pivoting rule). One iteration takes $O(n(m+n))$ time.

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Tropical Cramer determinants = opt. assignment used to compute reduce costs.

Pivoting is more subttle tropically. Tropical general position assumption (only one optimal permutation) stronger than sign general position (all optimal permutations yield monomials with the same sign). The $O(n(n+m))$ bound arises by tracking the deformations of a hypergraph along a tropical edge. We can still pivot tropically if the data are only in sign general position, but then we (currently) loose a factor n in time.

Example of compatible pivoting rule. A rule is **combinatorial** if any entering/leaving inequalities are functions of the history (sequence of bases) and of the signs of the minors of the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^\top & 0 \end{pmatrix} \enspace .$$

(eg signs of reduced costs).
Most known pivoting rules are combinatorial.

Mean payoff games reduce to tropical LP

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1309.5925, SIAM J. Opt, + refinement in Benchimol's PhD)

If any combinatorial (or even "semialgebraic") rule in classical linear programming would run in strongly polynomial time, then, mean payoff games could be solved in strongly polynomial time.

The "semialgebraic" rule must satisfy a mild technical assumption (polynomial time solvability of LP's over Newton polytopes).

Sketch of Proof

1. Mean payoff games are equivalent to feasibility problems in tropical linear programming (Akian, SG, Guterman)
2. Tropical linear programs can be lifted to a subclass of classical linear programs over $\mathbb{K}$.
3. The set of runs (sequences of bases) of the classical simplex algorithm equipped with a combinatorial (or even semialgebraic) pivoting rule is independent of the real closed field. Being a run is a first order property, apply Tarski's theorem. So, number of iterations of classical simplex over $\mathbb{K}$ is the same as over $\mathbb{R}$.
4. Can simulate the classical simplex on $\mathbb{K}$ tropically, every pivot being strongly polynomial.

Technicalities were hidden here.
Instead of $\mathbb{K}$, we eventually use a field of formal Hahn series

$$\mathbb{R}[[t^{\mathbb{R}^N}]], \quad \text{the value group } \mathbb{R}^N \text{ is equipped with lex order}$$

to encode a symbolic perturbation scheme (needs to encode a MPG by a LP in general position).

Part VI.
Tropicalization of the central path

Primal-dual central path

$$\text{minimize} \quad \frac{c^\top x}{\mu} - \sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i) \tag{1}$$
$$\text{subject to} \quad Ax + w = b,\ x > 0, w > 0.$$

$$\begin{aligned} Ax + w &= b \\ -A^\top y + s &= c \\ w_i y_i &= \mu \quad \text{for all } i \in [m] \\ x_j s_j &= \mu \quad \text{for all } j \in [n] \\ x, w, y, s &> 0\ . \end{aligned} \tag{2}$$

For any $\mu > 0$, $\exists!$ $(x^\mu, w^\mu, y^\mu, s^\mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$. The central path is the image of the map $\mathcal{C} : \mathbb{R}_{>0} \to \mathbb{R}^{2m+2n}$ which sends $\mu > 0$ to the vector $(x^\mu, w^\mu, y^\mu, s^\mu)$.

The tropical central path

Assume now that $\mathbf{A}(t), \mathbf{b}(t), \mathbf{c}(t)$ have entries in $\mathbb{K}$ (absolutely converging Puiseux series with real exponents, $t \to \infty$).

The tropical central path is the log-limit:

$$\mathcal{C}^{\mathrm{trop}} : \lambda \mapsto \lim_{t \to \infty} \frac{\log \mathcal{C}(t^{\lambda})}{\log t} . \tag{3}$$

The pointwise limit does exist since $\mathcal{C}(\cdot)$ is definable in a polynomially bounded o-minimal structure.

$\mathcal{C}^{\mathrm{trop}}$ can be computed by combinatorial means.

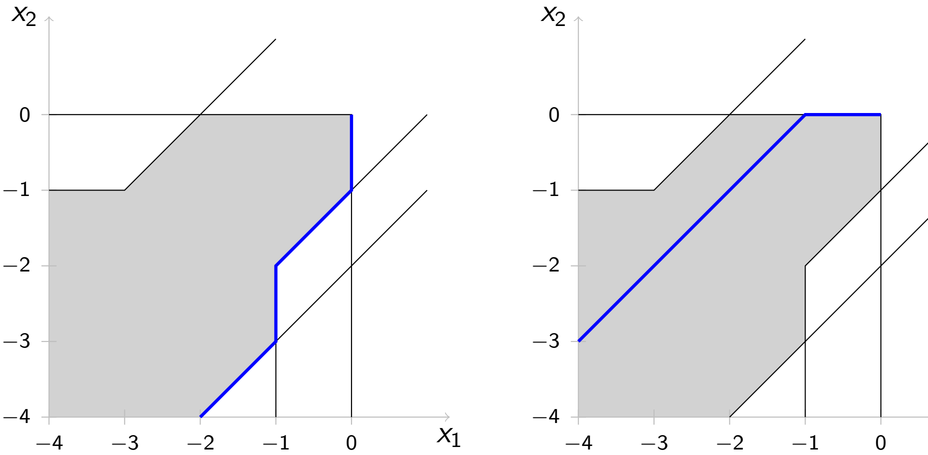


Figure : Tropical central paths for the objective function $\min \mathbf{x}_1$ (left) and $\min t^1\mathbf{x}_1 + \mathbf{x}_2$ (right).

The counter example . . .

$$
\begin{aligned}
\min \quad & \mathbf{v}_0 \\
\text{s.t.} \quad & \mathbf{u}_0 \leqslant t^1 \\
& \mathbf{v}_0 \leqslant t^2 \\
& \mathbf{v}_i \leqslant t^{(1-\frac{1}{2^i})}(\mathbf{u}_{i-1} + \mathbf{v}_{i-1}) && \text{for } 1 \leqslant i \leqslant r \qquad\qquad \mathbf{LP}_r \\
& \mathbf{u}_i \leqslant t^1 \mathbf{u}_{i-1} && \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_i \leqslant t^1 \mathbf{v}_{i-1} && \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_r \geqslant 0, \ \mathbf{v}_r \geqslant 0
\end{aligned}
$$

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

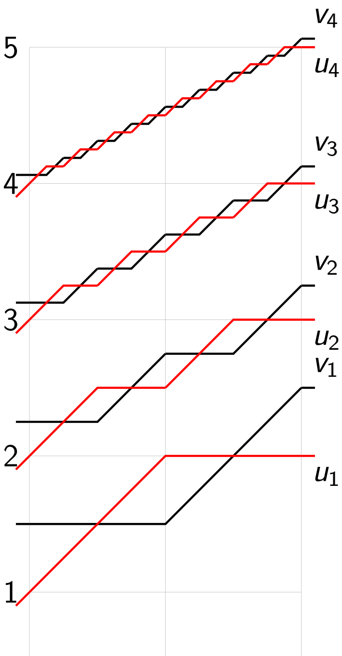
For t large enough, the total curvature of the central path is $\geqslant (2^{r-1} - 1)\pi/2$.

Large enough: $\log_2 t = \Omega(2^r)$. Need an exponential number of bits.

$$
\begin{aligned}
\mathbf{u}_0 &\leqslant t^1 & u_0 &\leqslant 1\\
\mathbf{v}_0 &\leqslant t^2 & v_0 &\leqslant 2\\
\mathbf{v}_i &\leqslant t^{(1-\frac{1}{2^i})}(u_{i-1}+v_{i-1}) & v_i &\leqslant 1-\frac{1}{2^i}+\max(u_{i-1},v_{i-1})\\
\mathbf{u}_i &\leqslant t^1\mathbf{u}_{i-1} & u_i &\leqslant 1+u_{i-1}\\
\mathbf{u}_i &\leqslant t^1\mathbf{v}_{i-1} & u_i &\leqslant 1+v_{i-1}\\
\mathbf{u}_r &\geqslant 0,\ \mathbf{v}_r \geqslant 0 & c^\top x &= v_0 \leqslant \lambda
\end{aligned}
$$

The tropical central path is given by

$$
\begin{aligned}
u_0 &= 1\\
v_0 &= \min(2,\lambda)\\
v_i &= 1-\frac{1}{2^i}+\max(u_{i-1},v_{i-1})\\
u_i &= 1+\min(u_{i-1},v_{i-1})
\end{aligned}
$$


v_4
5
u_4
v_3
4
u_3
v_2
3
u_2
v_1
2
u_1
1

Concluding remarks

This talk: relation between zero-sum games and tropical modules

- linear maps, their adjoints, linear projectors $\rightarrow$ Shapley operators

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- complete tropical modules, similar to Hilbert spaces, Hilbert's projective metric plays the role of the Euclidean metric

Concluding remarks

This talk: relation between zero-sum games and tropical modules

- linear maps, their adjoints, linear projectors $\to$ Shapley operators
- $\{$winning certificates$\} = \{$subharmonic vectors$\}$ is a tropical module
- complete tropical modules, similar to Hilbert spaces, Hilbert's projective metric plays the role of the Euclidean metric
- gave an application to computational complexity, MPG $\subset$ nonarchimedean LP

Thank you !

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