Tropicalizing discrete convex geometry

...some unexpected results

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Materials mostly from: arXiv:math/0904.3436, with X. Allamigeon and É. Goubault; arXiv:math/0906.3492, with X. Allamigeon and R. Katz

Introduction: a classical question

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Naive answer: cannot be more than $\binom{p}{d}$, since d independent inequalities among p must be saturated.

Theorem (McMullen upper bound 1970, confirmation of a conjecture made by Motzkin in 1953)

Among the polytopes of dimension d with p vertices, the cyclic polytope maximizes the number of faces of each dimension.

The cyclic polytope $C(p,d)$ is the convex hull of p points of the moment curve $t \mapsto z(t) := (t, t^2, \ldots, t^d)$.

Definition

A $0/1$ sequence satisfies Gale's evenness condition if the number of 1 between any two 0 is even.

Eg., 0110111100011001111110000111

Let $C(p, d) := \mathrm{co}(z(t_1), \ldots, z(t_p))$ with $t_1 < t_2 < \cdots < t_p$.

Fact

The points $z(t_{i_1}), \ldots, z(t_{i_{d+1}})$ define a facet iff the associated word satisfies Gale's evenness condition.

Eg., $i_1 = 2, i_2 = 3,\ i_3 = 5,\ p = 5, d = 2 \to 01101$

Corollary

The number of facets of a polytope of dimension d with p vertices is at most

$$U(p,d) := \binom{p-d/2}{d/2} + \binom{p-d/2-1}{d/2-1} \qquad \textit{for } d \textit{ even}$$

$$U(p,d) := 2\binom{p-(d+1)/2}{(d-1)/2} \qquad \textit{for } d \textit{ odd.}$$

This is $\Theta(p^{d/2})$ has $p \to \infty$, keeping d fixed, so much smaller than the naive bound $\binom{p}{d} = \Theta(p^d)$.

The same questions can be raised for max-plus or tropical convex sets/ cones

- $\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +)$ the max-plus semiring.
- A subset $\mathcal{V}$ of $\mathbb{R}^d_{\max}$ is a (convex) cone if
  $u, v \in \mathcal{V}, \lambda, \mu \in \mathbb{R}_{\max} \implies \sup(\lambda + u, \mu + v) \in \mathcal{V}$.

Considered by: Zimmermann; Litvinov, Maslov, Samborski, Shpiz; Cohen, Moller, Quadrat, Viot; Butkovič, Hegedus, Helbig; Gaubert; Wagneur ; Singer, Nitica; Briec, Horvath; Develin, Sturmfels, Joswig, Yu; Sergeev, Schneider, Meunier ...

Definition

A vector $u \in \mathcal{V}$ is an extreme generator of $\mathcal{V}$ if $u = \sup(v, w), v, w \in \mathcal{V} \implies u = v$ or $u = w$.

A closed tropical cone of $\mathbb{R}^d_{\max}$ is known to be generated by its extreme generators (SG+Katz 2006,2007; Butkovič, Sergeev, Schneider, 2007; finitely generated case: Moller 1988, Wagneur 1991).

The notion of face is problematic in the tropical setting.

Theorem (Allamigeon, SG, Katz, arXiv:math/0906.3492)

The number of extreme rays of a tropical cone $\mathcal{V}$ defined by p inequalities in dimension d cannot exceed $U(p+d, d-1)$.

$$\mathcal{V} := \{x \in \mathbb{R}^d_{\max} \mid \max_{j\in[d]} a_{ij} + x_j \leq \max_{j\in[d]} b_{ij} + x_j,\ i \in [p]\} \ .$$

The bound is $\Theta(p^{\lfloor (d-1)/2 \rfloor})$ for d fixed and $p \to \infty$.

Proof (by dequantization)

For $\beta > 0$, consider the classical convex cone $\mathcal{V}(\beta)$ defined by the $p + d$ inequalities

$$y_j \geq 0 \ , \qquad j \in [d] \ ,$$
$$\frac{1}{d} \sum_{j \in [d]} \exp(\beta a_{ij}) y_j \leq \sum_{j \in [d]} \exp(\beta b_{ij}) y_j \ , i \in [p] \ .$$

By the McMullen upper bound theorem, $\mathcal{V}(\beta)$ has a generating family $(u_k(\beta))_{k \in [K]}$ with $K \leq U(p + d, d - 1)$.

If $x \in \mathcal{V}$, then $E_\beta(x) := (\exp(\beta x_j)) \in \mathcal{V}(\beta)$. WLOG, normalize $u_k(\beta)$ (entries sum to one). Let $v_k(\beta) := E_\beta^{-1}(u_k)$.

$$\max_{j\in[d]} v_k(\beta)_j \leq 0 \leq \beta^{-1}\log d + \max_{j\in[d]} v_k(\beta)_j \enspace ,$$

$$-\beta^{-1}\log d + \max_{j\in[d]} a_{ij} + v_k(\beta)_j \leq \beta^{-1}\log d + \max_{j\in[d]} b_{ij} + v_k(\beta)_j \enspace .$$

Then, it can be checked that any accumulation point of the family $(v_k(\beta))_{k\in[K]}$ yields a generating family of $\mathcal{V}$ (use $\mathcal{V}(\beta) \supset \mathcal{V}$ thanks to the $1/d$ trick).

$\square$

Is the tropical upper bound attained?

The usual bound of $\sharp$ vertices for a dim d polytope with p facets is attained by the polar of the cyclic polytope

$$C(p,d)^\circ := \{y \mid z(t_i)\cdot(y-w) \leq 1, i \in [p]\},\ w \in \mathrm{int}(C(p,d)) \ .$$

In the tropical case

$$z(t) := \text{“}(1, t, \ldots, t^{d-1})\text{”} = (1, t, \ldots, (d-1)t) \in \mathbb{R}^d_{\max} \ ,$$

and homogeneizing naively $C(p,d)^\circ$ yields

$$\{y \mid \text{“}z(t_i)\cdot y \leq 0\text{”}, i \in [p]\}$$

which is tropical nonsense.

- Introduce a sign pattern $\epsilon_{ij} \in \{\pm 1\}$
- Set formally "$z(t_i)_j = \epsilon_{ij} t_i^{j-1}$" in the symmetrized maxplus semiring $\mathbb{S}_{\max}$, so $z(t_i) =$ "$z^+(t_i) - z^-(t_i)$" where

$$z^\pm(t_i) \in \mathbb{R}^d_{\max}, \qquad z^\pm(t_i)_j = \begin{cases} \text{"}t_i^{j-1}\text{"} & \text{if } \epsilon_{ij} = \pm 1 \\ \text{"}0\text{"} & \text{otherwise} \end{cases}$$

Definition

The signed cyclic polyhedral cone $C(p,d;\epsilon)$, is generated by p pairs of vectors $(z^-(t_i), z^+(t_i)) \in (\mathbb{R}^d_{\max})^2$, $i \in [p]$. Its polar $\mathcal{K}(p,d;\epsilon)$ is the set of vectors $x \in \mathbb{R}^d_{\max}$ such that

$$\text{"}z^-(t_i) \cdot x \leq z^-(t_i) \cdot x\text{"}, \quad i \in [p], \text{ i.e.}$$

$$\max_{j \in [d], \epsilon_{ij} = -1} (j-1)t_i + x_j \leq \max_{j \in [d], \epsilon_{ij} = +1} (j-1)t_i + x_j, \qquad i \in [p] \ .$$

See SG and Katz, LAA 09 for more information on tropical polars.

The analogy with the classical case . . .

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may suggest that there should be some choice of sign ϵ such that the polar $\mathcal{K}(p, d; \epsilon)$ of the signed cyclic polyhedral cone has exactly $U(p + d, d - 1)$ extreme rays. . .

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may suggest that there should be some choice of sign ϵ such that the polar $\mathcal{K}(p, d; \epsilon)$ of the signed cyclic polyhedral cone has exactly $U(p+d, d-1)$ extreme rays...

we shall see that this is not true.

Some lattice paths

southward / eastward paths in the sign pattern ϵ_{ij}

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccccc} & j_1 & & & j_2 & j_3 & & & & j_4 & & j_5 & j_6 & & \end{array} \\
\begin{array}{c} \\ \\ i_1 \\ \\ \\ i_2 \\ \\ i_3 \\ \\ \\ \\ i_4 \\ \\ \\ \\ \\ i_5 \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
\cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & + & \star & \star & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & - & \star & \star & \star & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot
\end{pmatrix}
\end{array}
$$

A lattice path for the sign pattern ϵ_{ij} is tropically allowed if

(i) every sign occurring on the initial vertical segment, except possibly the sign at the bottom of the segment, is positive;

(ii) every sign occurring on the final vertical segment, except possibly the sign at the top of the segment, is positive;

(iii) every sign occurring in some other vertical segment, except possibly the signs at the top and bottom of this segment, is positive;

(iv) for every horizontal segment, the pair of signs consisting of the signs of the leftmost and rightmost positions of the segment is of the form $(+,-)$ or $(-,+)$;

(v) as soon as a pair $(-,+)$ occurs as the extreme signs of an horizontal segment, the pairs of the next horizontals segments must also be equal to $(-,+)$.

If only (i)–(iv) hold, we say that the path is classically allowed.

Theorem (Allamigeon, SG, Katz, arXiv:math/0906.3492)

There are fewer extreme points in the tropical case.

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- *The extreme rays of the polar of the tropical signed cyclic polyhedral cone correspond bijectively to the tropically allowed lattice paths.*

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Theorem (Allamigeon, SG, Katz, arXiv:math/0906.3492)

- *The extreme rays of the polar of the tropical signed cyclic polyhedral cone correspond bijectively to the tropically allowed lattice paths.*
- *For $t_1 \ll t_2 \ll \cdots \ll t_p$, the extreme rays of the classical analogue of this polar correspond bijectively to the classically allowed lattice paths.*

There are fewer extreme points in the tropical case.

This apparently mysterious result relies on a characterization of the extreme points of a tropical polyhedron in terms of the inequalities which define it: Allamigeon, SG, Goubault, arXiv:math/0904.3436
Recall first.

Fact (See Butkovič, Sergeev, Schneider; SG, Katz, both LAA 07)

A vector g of a tropical cone $\mathcal{C} \in \mathbb{R}^d_{\max}$ is extreme iff $\exists t \in [d]$ such that g is a minimal element of the set $\{ x \in \mathcal{C} \mid x_t = g_t \}$. In that case, g is said to be extreme of type t.

Definition

The tangent cone of $\mathcal{C} := \{x \mid \text{“}Ax \leq Bx\text{”}\}$ at g is defined as the tropical cone $\mathcal{T}(g, \mathcal{C})$ of $\mathbb{R}^d_{\max}$ given by the system of inequalities

$$\max_{i \in \arg\max(A_k g)} x_i \leq \max_{j \in \arg\max(B_k g)} x_j$$

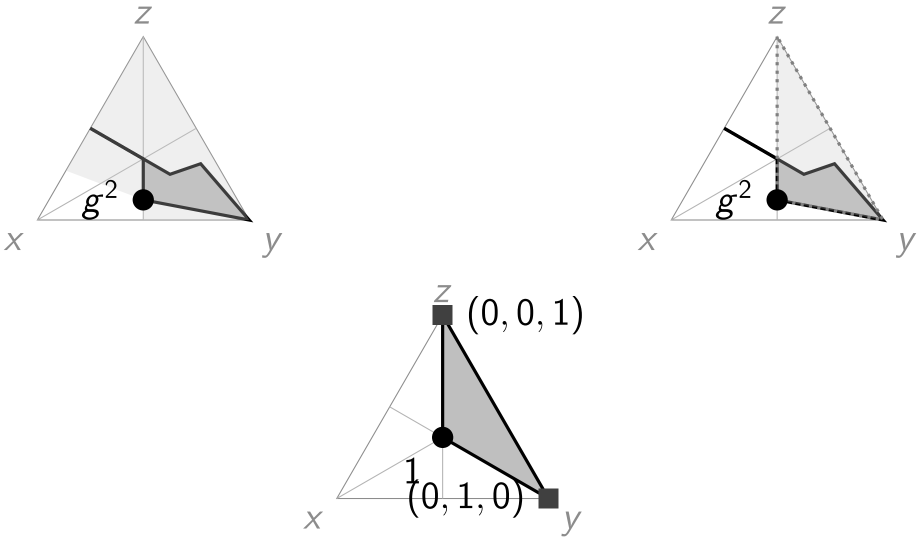
for all k such that $A_k g = B_k g$.

Fact (Allamigeon, SG, Goubault)

There exists a neighborhood N of g such that for all $x \in N$, x belongs to $\mathcal{C}$ if and only if it is an element of $g + \mathcal{T}(g, \mathcal{C})$.

Fact (ibid.)

The element g is extreme in $\mathcal{C}$ if and only if the vector 1 is extreme in $\mathcal{T}(g, \mathcal{C})$.


z
x
y
g^2
z
x
y
g^2
z
(0, 0, 1)
1
(0, 1, 0)
x
y

Theorem (Allamigeon, SG, Goubault, ibid.)

A vector $y \in \mathbb{R}^d_{\max}$ belongs to an extreme ray of a tropical polyhedral cone $\mathcal{C}$ if, and only if, there exists $s \in \{1, \dots, d\}$ such that

$$(x \in \mathcal{T}(\mathcal{C}, y) \cap \{1, 0\}^d \text{ and } x_s = 1) \Rightarrow (x_r = 1 \text{ or } y_r = 0)$$

for all $r \in \{1, \dots, d\}$.

Corollary

If t entries of y are zero, then y must saturate at least $d - t - 1$ inequalities among $A_r x \leq B_r x$, $1 \leq r \leq p$.

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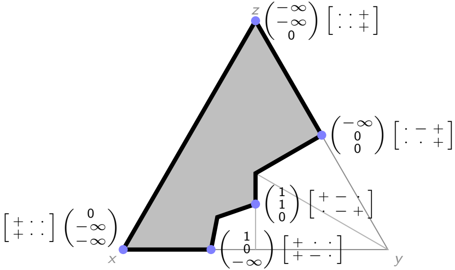
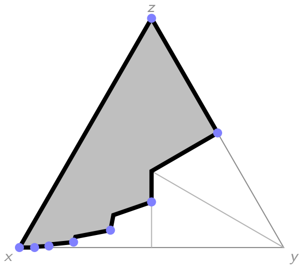
- the proof uses also the Cramer theory of M. Plus (1990);
- the tangent cones turn out to be described by line directed graphs, which must have a unique terminal node. This explains the mysterious condition (v)





$$\begin{pmatrix} 0 & -\infty & 0 \\ 0 & -\infty & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} -\infty & 0 & -\infty \\ -\infty & 1 & -\infty \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

Usually, a point y in $\{x \mid Ax \leq b\}$ is extreme iff the family of rows A_k arising from active constraint is of full rank. The same is not true in the tropical case.

- $N^{\text{tpath}}(\epsilon)$ (resp. $N^{\text{path}}(\epsilon)$) := ♯ tropically (resp. non-tropically) allowed lattice paths for the sign pattern ϵ.
- $N^{\text{trop}}(p,d)$:= maximal ♯ extreme rays of a tropical cone in dimension d defined as the intersection of p half-spaces.

$$\max_{\epsilon \in \{\pm 1\}^{p\times d}} N^{\text{tpath}}(\epsilon) \leq N^{\text{trop}}(p,d) \leq U(p+d, d-1) = \max_{\epsilon \in \{\pm 1\}^{p\times d}} N^{\text{path}}(\epsilon) \ .$$

Conjecture (ibid.)

The maximum of the ♯ of extreme points is attained among the polars of signed cyclic polyhedra:

$$\max_{\epsilon \in \{\pm 1\}^{p\times d}} N^{tpath}(\epsilon) = N^{trop}(p,d)$$

The signed cyclic polyhedron is the simplest candidate: we know that a maximizing object is "in general position".

Fact (ibid.)

For every p, d,

$$N^{tpath}(p,d) \leq (p(d-1)+1)2^{d-1} \enspace .$$

Hence, if the conjecture was true, the curse of dimensionality would be reduced in the tropical world. For a fixed dimension d, the number of extreme points would grow only linearly in the number of constraints p (to be compared with the classical $p^{\lfloor (d-1)/2 \rfloor}$ growth).

However. . .

Fact

For $d \geq 2p+1$, we have

$$N^{tpath}(p,d) \geq U(d, d-p-1) \ . \tag{1}$$

It follows that the tropical upper bound is asymptotically tight for a fixed number of constraints p, as the dimension tends to infinity (curse of dimensionality is still here):

$$N^{\text{trop}}(p,d) \sim U(p+d, d-1) \qquad \text{as } d \to \infty \ .$$

Lower and upper bounds for $N^{\text{trop}}(p, d)$, the maximal number of extreme rays of a tropical polyhedral cone defined by p inequalities in dimension d.

$d \setminus p$	1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	11	12
4	6	8	10	12	14	16	18	20	22
5	9	14	20	[26, 27]	[32, 35]	[38, 44]	[44, 54]	[50, 65]	[56, 77]
6	12	20	30	42	[55, 56]	[68, 72]	[82, 90]	[96, 110]	[110, 132]
7	16	30	50	[71, 77]	[96, 112]	[124, 156]	[152, 210]	[180, 275]	[208, 352]
8	20	40	70	112	[159, 168]	[216, 240]	[280, 330]	[340, 440]	[401, 572]
9	25	55	105	[172, 182]	[250, 294]	[321, 450]	[436, 660]	[613, 935]	[751, 1287]
10	30	70	140	252	[370, 420]	[538, 660]	[668, 990]	[898, 1430]	[1320, 2002]
11	36	91	196	[363, 378]	[584, 672]	[805, 1122]	[1122, 1782]	[1357, 2717]	[1799, 4004]

Smallest unsettled instance of the conjecture: $N^{\text{trop}}(4, 5) = 26$ or 27?

A glimpse at the sign patterns maximizing $N^{\text{tpath}}(p, d)$.

$N^{\text{tpath}}(p, d)$ can be computed in $O(pd)$ time by an automata trick. We experimentally got patterns like

$$\begin{pmatrix}
+ & + & - & + & - & + & - & + & - & + \\
+ & + & + & - & + & - & + & - & + & + \\
+ & - & + & + & - & + & - & + & - & + \\
+ & - & + & + & + & - & + & - & + & + \\
+ & - & + & + & + & + & - & + & - & + \\
+ & + & - & + & + & + & + & - & + & + \\
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+ & + & - & + & - & + & + & + & - & + \\
+ & - & + & - & + & - & + & + & - & + \\
+ & + & - & + & - & + & - & + & + & + \\
+ & - & + & - & + & - & + & - & + & +
\end{pmatrix}.$$

The shape of optimal patterns critically depends on (p, d) (no simple rule).

So, the first good news is that tropical polyhedra seem to have fewer extreme rays than their classical counter parts (at least, the polars of cyclic polytopes do).

Can we compute more efficiently the extreme points in the tropical case ?

So, the first good news is that tropical polyhedra seem to have fewer extreme rays than their classical counter parts (at least, the polars of cyclic polytopes do).

Can we compute more efficiently the extreme points in the tropical case ?

...second good news!

Tropical double description method

Allamigeon, SG, Goubault, arXiv:math/0904.3436

Theorem (Elementary step)

Let $\mathcal{K}$ be a tropical polyhedral cone generated by a set $G \subset \mathbb{R}^d_{\max}$, and $\mathcal{H} = \{ x \mid ax \leq bx \}$ a halfspace ($a, b \in \mathbb{R}^{1\times d}_{\max}$). The cone $\mathcal{K} \cap \mathcal{H}$ is generated by the vectors:

$$g \text{ s.t. } g \in G,\ ag \leq bg$$
$$(ah)g + (bg)h \text{ s.t. } g, h \in G,\ ag \leq bg, \text{ and } ah > bh \}.$$

The tropical double description algorithm

1: **procedure** $\textsc{ComputeExtreme}(A, B, n)$ $\quad \triangleright$ $A, B \in \mathbb{R}_{\max}^{n\times d}$
2: $\quad$ **if** $n = 0$ **then** $\quad \triangleright$ *Base case*
3: $\quad\quad$ **return** $(\epsilon_i)_{1\leq i\leq d}$
4: $\quad$ **else** $\quad \triangleright$ *Inductive case*
5: $\quad\quad$ split $Ax \leq Bx$ into $Cx \leq Dx$ and $ax \leq bx$, with $C, D \in \mathbb{R}_{\max}^{(n-1)\times d}$ and $a, b \in \mathbb{R}_{\max}^{1\times d}$
6: $\quad\quad$ $G := \textsc{ComputeExtreme}(C, D, n-1)$
7: $\quad\quad$ $G^{\leq} := \{ g^i \in G \mid ag^i \leq bg^i \}$, $G^{>} := \{ g^j \in G \mid ag^j > bg^j \}$, $H := G^{\leq}$
8: $\quad\quad$ **for all** $g^i \in G^{\leq}$ and $g^j \in G^{>}$ **do**
9: $\quad\quad\quad$ $h := (ag^j)g^i + (bg^i)g^j$
10: $\quad\quad\quad$ **if** h is extreme in $\{ x \mid Ax \leq Bx \}$ **then**
11: $\quad\quad\quad\quad$ append κh to H, where κ is the opposite of the first non-0 coefficient of h
12: $\quad\quad\quad$ **end**
13: $\quad\quad$ **done**
14: $\quad$ **end**
15: $\quad$ **return** H
16: **end**

depends critically on the is extreme test in line 10.

Recall our characterization: a vector $y \in \mathbb{R}^d_{\max}$ belongs to an extreme ray of a tropical polyhedral cone $\mathcal{C}$ if, and only if, there exists $s \in \{1, \dots, d\}$ such that

$$(x \in \mathcal{T}(\mathcal{C}, y) \cap \{1, 0\}^d \text{ and } x_s = 1) \Rightarrow (x_r = 1 \text{ or } y_r = 0)$$

for all $r \in \{1, \dots, d\}$.

Eg, when y is finite, does there exists s such that, for $x \in \{0, 1\}^d$,

$$x_s = 1 \text{ and } \max_{i \in \arg\max(A_k y)} x_i \leq \max_{j \in \arg\max(B_k y)} x_j \implies x \equiv 1?$$

This is expressed as a problem concerning Horn clauses (in arXiv:math/0904.3436) or as an hypergraph reachability problem (current work).

In arXiv:math/0904.3436, we develop a propagation (fixed point type) algorithm, which allows us to compute the least model of a compact Horn formula, and so to decide the extremality of y in a time $O(pd^2)$ (d iterations taking each a linear time in the size of the hypergraph).

Recall the double description algorithm:

procedure $\textsc{ComputeExtreme}(A, B, n)$ $\quad \triangleright\ A, B \in \mathbb{R}_{\max}^{n\times d}$
    if $n = 0$ **then** $\quad \triangleright$ *Base case*
        return $(\epsilon_i)_{1\leq i\leq d}$
    else $\quad \triangleright$ *Inductive case*
        split $Ax \leq Bx$ into $Cx \leq Dx$ and $ax \leq bx$, with $C, D \in \mathbb{R}_{\max}^{(n-1)\times d}$ and $a, b \in \mathbb{R}_{\max}^{1\times d}$
        $G := \textsc{ComputeExtreme}(C, D, n-1)$
        $G^{\leq} := \{\, g^i \in G \mid ag^i \leq bg^i \,\}$, $G^{>} := \{\, g^j \in G \mid ag^j > bg^j \,\}$, $H := G^{\leq}$
        for all $g^i \in G^{\leq}$ and $g^j \in G^{>}$ **do**
            $h := (ag^j)g^i + (bg^i)g^j$
            if h is extreme in $\{\, x \mid Ax \leq Bx \,\}$ **then**
                append κh to H, where κ is the opposite of the first non-0 coefficient of h
            end
        done
    end
    return H
end

Corollary

In the tropical double description method, each whole step of elimination of nonextreme elements via the Horn formula approach, for a current generating family G, takes a time $O(pd^2|G|^2)$.

Compare with the naive approach consisting in generating the candidates, and eliminating them by residuation (check whether one is a combination of the others). This is more or less what is done by the `mpsolve` function (SG) of the maxplus toolbox of `Scilab`, now in `ScicosLab`. This requires a time of $O(d|G|^4)$, which is in general much worse, because $|G|$ is expected to be huge.

This has been implemented in OCaml by Allamigeon (time T) that he compared with a much improved version, also in OCaml (time T') of the residuation based method of the maxplus toolbox `mpsolve` function.

d	n	final nb. of gen.		max. nb. of gen.		T (s)		T' (s) (alternative)		ratio T/T'
		fastest	slowest	fastest	slowest	fastest	slowest	fastest	slowest	(slowest)
10	15	7	20	56	148	0.24	0.53	0.79	5.55	0.095
11	15	53	640	128	890	0.49	79.87	3.13	2188.24	0.036
12	15	24	608	451	631	5.00	17.7	93.17	448.34	0.039
14	15	146	608	350	971	9.49	85.39	121.83	2328.77	0.037
15	15	36	182	477	1600	8.21	342.71	126.30	6 h 35 min	0.014
15	12	31	2226	192	3244	1.71	973.39	19.97	2 days 10 h	0.005
17	12	486	2130	886	3934	17.31	909.11	835.66	1 day 20 h	0.005
20	12	—	7821	—	6565	—	3256.51	—	$>$ 10 days	$<$ 0.004

In a current work, we are even decreasing the $O(pd^2)$ time of a single elimination step by almost a factor d, thanks to a generalization of Tarjan algorithm, to determine the minimal SCC of an hypergraph, but this is another story.

Thank you!