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Methods and Applications of (max,+) Linear Algebra

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Abstract. Exotic semirings such as the "$(\max, +)$ semiring" $(\mathbb{R} \cup \{-\infty\}, \max, +)$, or the "tropical semiring" $(\mathbb{N} \cup \{+\infty\}, \min, +)$, have been invented and reinvented many times since the late fifties, in relation with various fields: performance evaluation of manufacturing systems and discrete event system theory; graph theory (path algebra) and Markov decision processes, Hamilton-Jacobi theory; asymptotic analysis (low temperature asymptotics in statistical physics, large deviations, WKB method); language theory (automata with multiplicities).

Despite this apparent profusion, there is a small set of common, non-naive, basic results and problems, in general not known outside the $(\max, +)$ community, which seem to be useful in most applications. The aim of this short survey paper is to present what we believe to be the minimal core of $(\max, +)$ results, and to illustrate these results by typical applications, at the frontier of language theory, control, and operations research (performance evaluation of discrete event systems, analysis of Markov decision processes with average cost).

Basic techniques include: solving all kinds of systems of linear equations, sometimes with exotic symmetrization and determinant techniques; using the $(\max, +)$ Perron-Frobenius theory to study the dynamics of $(\max, +)$ linear maps. We point out some open problems and current developments.

1 Introduction: the $(\max, +)$ and tropical semirings

The "max-algebra" or "$(\max, +)$ semiring" $\mathbb{R}_{\max}$, is the set $\mathbb{R} \cup \{-\infty\}$, equipped with $\max$ as addition, and $+$ as multiplication. It is traditional to use the notation $\oplus$ for $\max$ ($2 \oplus 3 = 3$), and $\otimes$ for $+$ $(1 \otimes 1 = 2)$. We denote[1] by $\mathbb{0}$ the *zero* element for $\oplus$ (such that $\mathbb{0} \oplus a = a$, here $\mathbb{0} = -\infty$) and by $\mathbb{1}$ the *unit* element for $\otimes$ (such that $\mathbb{1} \otimes a = a \otimes \mathbb{1} = a$, here $\mathbb{1} = 0$). This structure satisfies all the semiring axioms, i.e. $\oplus$ is associative, commutative, with zero element, $\otimes$ is associative, has a unit, distributes over $\oplus$, and zero is absorbing (all the ring axioms are satisfied, except that $\oplus$ need not be

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[1] The notation for the zero and unit is one of the disputed questions of the community. The symbols ε for zero, and e for the unit, often used in the literature, are very distinctive and well suited to handwritten computations. But it is difficult to renounce to the traditional use of ε in Analysis. The notation $\mathbb{0}, \mathbb{1}$ used by the Idempotent Analysis school has the advantage of making formulæ closer to their usual analogues.

a group law). This semiring is *commutative* ($a\otimes b=b\otimes a$), *idempotent* ($a\oplus a=a$), and non zero elements have an inverse for $\otimes$ (we call *semifields* the semirings that satisfy this property). The term *dioid* is sometimes used for an *idempotent* semiring.

Using the new symbols $\oplus$ and $\otimes$ instead of the familiar $\max$ and $+$ notation is the price to pay to easily handle all the familiar algebraic constructions. For instance, we will write, in the $(\max,+)$ semiring:

$$ab = a\otimes b,\quad a^n = a\otimes\cdots\otimes a \ \ (n \text{ times}),\qquad 2^3=6\ ,\qquad\qquad \sqrt{3}=1.5\ ,$$
$$\begin{bmatrix}2 & 0\\ 4 & \mathbb{0}\end{bmatrix}\begin{bmatrix}10\\ 103\end{bmatrix} = \begin{bmatrix}2\otimes 10\oplus 0\otimes 103\\ 4\otimes 10\oplus \mathbb{0}\otimes 103\end{bmatrix} = \begin{bmatrix}103\\ 14\end{bmatrix},$$
$$(3\oplus x)^2 = (3\oplus x)(3\oplus x) = 6\oplus 3x\oplus x^2 = 6\oplus x^2 \ \ (=\max(6, 2\times x))\ .$$

We will systematically use the standard algebraic notions (matrices, vectors, linear operators, semimodules — i.e. modules over a semiring—, formal polynomials and polynomial functions, formal series) in the context of the $(\max,+)$ semiring, often without explicit mention. Essentially all the standard notions of algebra have obvious semiring analogues, provided they do not appeal to the invertibility of addition.

There are several useful variants of the $(\max,+)$ semiring, displayed in Table 1. In the sequel, we will have to consider various semirings, and will universally use the

$\mathbb{R}_{\max}$	$(\mathbb{R}\cup\{-\infty\},\max,+)$	$(\max,+)$ semiring max algebra	idempotent semifield
$\overline{\mathbb{R}}_{\max}$	$(\mathbb{R}\cup\{\pm\infty\},\max,+)$	completed $(\max,+)$ semiring	$-\infty+(+\infty)=-\infty$, for $\mathbb{0}\otimes a=\mathbb{0}$
$\mathbb{R}_{\max,\times}$	$(\mathbb{R}^+,\max,\times)$	$(\max,\times)$ semiring	isomorphic to $\mathbb{R}_{\max}$ ($x\mapsto\log x$)
$\mathbb{R}_{\min}$	$(\mathbb{R}\cup\{+\infty\},\min,+)$	$(\min,+)$ semiring	isomorphic to $\mathbb{R}_{\max}$ ($x\mapsto -x$)
$\mathbb{N}_{\min}$	$(\mathbb{N}\cup\{+\infty\},\min,+)$	tropical semiring	(famous in Language Theory)
$\mathbb{R}_{\max,\min}$	$(\mathbb{R}\cup\{\pm\infty\},\max,\min)$	bottleneck algebra	not dealt with here
$\mathbb{B}$	({false, true}, or, and)	Boolean semiring	isomorphic to $(\{\mathbb{0},\mathbb{1}\},\oplus,\otimes)$, for any of the above semirings
$\mathbb{R}_h$	$(\mathbb{R}\cup\{-\infty\},\oplus_h,+)$ $a\oplus_h b = h\log(e^{a/h}+e^{b/h})$	Maslov semirings	isomorphic to $(\mathbb{R}^+,+,\times)$ $\lim_{h\to 0^+}\mathbb{R}_h=\mathbb{R}_0=\mathbb{R}_{\max}$

Table 1. The family of $(\max,+)$ and tropical semirings . . .

notation $\oplus,\otimes,\mathbb{0},\mathbb{1}$ with a context dependent meaning (e.g. $\oplus=\max$ in $\mathbb{R}_{\max}$ but $\oplus=\min$ in $\mathbb{R}_{\min}$, $\mathbb{0}=-\infty$ in $\mathbb{R}_{\max}$ but $\mathbb{0}=+\infty$ in $\mathbb{R}_{\min}$).

The fact that $\oplus$ is idempotent instead of being invertible ($\mathbb{R}_h$ is an exception, for $h\neq 0$), is the main original feature of these "exotic" algebras, which makes them so different from the more familiar ring and field structures. In fact the idempotence and cancellativity axioms are exclusive: if for all a,b,c, $(a\oplus b=a\oplus c\Rightarrow b=c)$ and $a\oplus a=a$, we get $a=\mathbb{0}$, for all a (simplify $a\oplus a=a\oplus\mathbb{0}$).

This paper is not a survey in the usual sense. There exist several comprehensive books and excellent survey articles on the subject, each one having its own bias and motivations. Applications of $(\max,+)$ algebras are too vast (they range from asymptotic methods to decidability problems), techniques are too various (from graph theory to measure theory and large deviations) to be surveyed in a paper of this format. But there is a small common set of useful basic results, applications and problems, that we try to spotlight here. We aim neither at completeness, nor at originality. But we wish to give an honest idea of the services that one should expect from $(\max,+)$ techniques. The interested reader is referred to the books [15,44,10,2,31], to the survey papers listed in the bibliography, and to the recent collection of articles [24] for an up-to-date account of the maxplusian results and motivations. Bibliographical and historical comments are at the end of the paper.

2 Seven good reasons to use the $(\max,+)$ semiring

2.1 An Algebra for Optimal Control

A standard problem of calculus of variations, which appears in Mechanics (least action principle) and Optimal Control, is the following. Given a Lagrangian L and suitable boundary conditions (e.g. $q(0), q(T)$ fixed), compute

$$\inf_{q(\cdot)} \int_0^T L(q,\dot{q})dt \enspace . \tag{1}$$

This problem is intrinsically $(\min,+)$ linear. To see this, consider the (slightly more general) discrete variant, with $\sup$ rather than $\inf$,

$$\xi(n)=x,\ \xi(k)=f(\xi(k-1),u(k)),\ k=n+1,\ldots,N, \tag{2a}$$

$$J_n^N(x,u)=\sum_{k=n+1}^{N} c(\xi(k-1),u(k))+\Phi(\xi(N)) \enspace , \tag{2b}$$

$$V_n^N(x)=\sup_u J_n^N(x,u) \enspace , \tag{2c}$$

where the $\sup$ is taken over all sequences of *controls* $u(k), k=n+1,\cdots,N$, selected in a finite *set of controls* U, $\xi(k)$, for $k=n,\ldots,N$, belongs to a finite set X of *states*, x is a distinguished *initial state*, $f: X\times U\to X$ is the *dynamics*, $c: X\times U\to\mathbb{R}\cup\{-\infty\}$ is the *instantaneous reward*, and $\Phi: X\to\mathbb{R}\cup\{-\infty\}$ is the *final reward* (the $-\infty$ value can be used to code forbidden final states or transitions). These data form a deterministic *Markov Decision Process* (MDP) with additive reward.

The function $V_n^N(\cdot)$, which represents the optimal reward from time n to time N, as a function of the starting point, is called the *value* function. It satisfies the backward dynamic programming equation

$$V_N^N=\Phi, \qquad V_k^N(x)=\max_{u\in U}\left\{c(x,u)+V_{k+1}^N(f(x,u))\right\} \enspace . \tag{3}$$

Introducing the *transition matrix* $A\in(\mathbb{R}_{\max})^{X\times X}$,

$$A_{x,y}=\sup_{u\in U,\ f(x,u)=y} c(x,u), \tag{4}$$

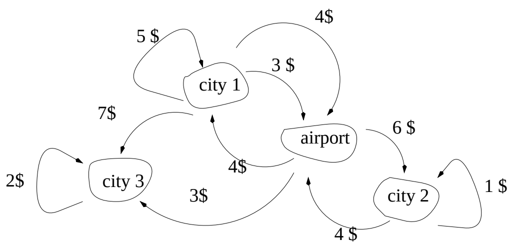
(the supremum over an empty set is $-\infty$), we obtain:

FACT 1 (DETERMINISTIC MDP $=(\max,+)$-LINEAR DYNAMICS). *The value function V_k^N of a finite deterministic Markov decision process with additive reward is given by the $(\max,+)$ linear dynamics:*

$$V_N^N = \Phi, \qquad V_k^N = AV_{k+1}^N. \tag{5}$$

The interpretation in terms of paths is elementary. If we must end at node j, we take $\Phi = \mathbb{1}_j$ (the vector with all entries $\mathbb{0}$ except the j-th equal to $\mathbb{1}$), Then, the value function $V_0^N(i) = (A^N)_{ij}$ is the maximal (additive) weight of a path of length N, from i to j, in the graph canonically associated[2] with A.

Example 1 (Taxicab). Consider a taxicab which operates between 3 cities and one airport, as shown in Fig. 1. At each state, the taxi driver has to choose his next destination, with deterministic fares shown on the graph (for simplicity, we assume that the demand is deterministic, and that the driver can choose the destination). The taxi driver considers maximizing his reward over N journeys. The $(\max,+)$ matrix associated with this MDP is displayed in Fig. 1.



$$A = \begin{array}{c} \\ c_1 \\ a \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & a & c_2 & c_3 \end{array} \\ \begin{pmatrix} 5 & 4 & \mathbb{0} & 7 \\ 4 & \mathbb{0} & 6 & 3 \\ \mathbb{0} & 4 & 1 & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 2 \end{pmatrix} \end{array}$$

Fig. 1. Taxicab Deterministic MDP and its matrix

Let us consider the optimization of the average reward:

$$\chi(x) = \sup_u \limsup_{N\to\infty} \frac{1}{N} J_0^N(x,u) \quad . \tag{6}$$

Here, the $\sup$ is taken over infinite sequences of controls $u(1), u(2), \ldots$ and the trajectory (2a) is defined for $k = 0, 1, \ldots$. We expect J_0^N to grow (or to decrease) linearly, as a function of the horizon N. Thus, $\chi(x)$ represents the optimal average reward (per time unit), starting from x. Assuming that the $\sup$ and $\limsup$ commute in (6), we get:

$$\chi(x) = \limsup_{N\to\infty} \frac{1}{N} \times (A^N \Phi)_x \tag{7}$$

(this is an hybrid formula, $A^N\Phi$ is in the $(\max,+)$ semiring , $1/N \times (\cdot)$ is in the conventional algebra). To evaluate (7), let us assume that the matrix A admits an *eigenvector* v

[2] With a $X \times X$ matrix A we associate the weighted (directed) graph, with set of nodes X, and an arc (x,y) with weight $A_{x,y}$ whenever $A_{x,y} \neq \mathbb{0}$.

in the $(\max,+)$ semiring:

$$Av = \lambda v, \text{ i.e. } \quad \max_j \{A_{ij} + v_j\} = \lambda_i + v_i \tag{8}$$

(the eigenvector v must be nonidentically $\mathbb{0}$, $\lambda \in \mathbb{R}_{\max}$ is the eigenvalue). Let us assume that v and Φ have only finite entries. Then, there exist two finite constants μ, ν such that $\nu + v \leq \Phi \leq \mu + v$. In $(\max,+)$ notation, $\nu v \leq \Phi \leq \mu v$. Then $\nu\lambda^N v = \nu A^N v \leq A^N \Phi \leq \mu A^N v = \mu \lambda^N v$, or with the conventional notation:

$$\nu + N\lambda + v \leq A^N \Phi \leq \mu + N\lambda + v. \tag{9}$$

We easily deduce from (9) the following.

FACT 2 ("EIGENELEMENTS = OPTIMAL REWARD AND POLICY"). *If the final reward Φ is* finite*, and if A has a finite eigenvector with eigenvalue λ, the optimal average reward $\chi(x)$ is a constant (independent of the starting point x), equal to the eigenvalue λ. An optimal control is obtained by playing in state i any u such that $c(i,u) = A_{ij}$ and $f(i,u) = j$, where j is in the* arg max *of (8) at state i.*

The existence of a finite eigenvector is characterized in Theorems 11 and 15 below.

We will not discuss here the extension of these results to the infinite dimensional case (e.g. (1)), which is one of the major themes of Idempotent Analysis [31]. Let us just mention that all the results presented here admit or should admit infinite dimensional generalizations, presumably up to important technical difficulties.

There is another much simpler extension, to the (discrete) semi-Markov case, which is worth being mentioned. Let us equip the above MDP with an additional map $\tau : X \times U \to \mathbb{R}^+ \setminus \{0\}$; $\tau(x(k-1), u(k))$ represents the physical time elapsed between decision k and decision $k+1$, when control $u(k)$ is chosen. This is very natural in most applications (for the taxicab example, the times of the different possible journeys in general differ). The optimal average reward per time unit now writes:

$$\chi(x) = \sup_u \limsup_{N\to\infty} \frac{\sum_{k=1}^N c(x(k-1), u(k)) + \Phi(x(N))}{\sum_{k=1}^N \tau(x(k-1), u(k))} \quad . \tag{10}$$

Of course, the specialization $\tau \equiv 1$ gives the original problem (6). Let us define $\mathcal{T}_{ij} = \{\tau(i,u) \mid f(i,u) = j\}$, and for $t \in \mathcal{T}_{ij}$,

$$A_{t,i,j} = \sup_{u \in U, f(i,u)=j, \tau(i,u)=t} c(i,u) \quad . \tag{11}$$

Arguing as in the Markov case, it is not too difficult to show the following.

FACT 3 (GENERALIZED SPECTRAL PROBLEM FOR SEMI-MARKOV PROCESSES). *If the generalized spectral problem*

$$\max_j \max_{t \in \mathcal{T}_{ij}} \{A_{t,i,j} - \lambda t + v_j\} = v_i \tag{12}$$

has a finite solution v, and if Φ is finite, then the optimal average reward is $\chi(x) = \lambda$, for all x. An optimal control is obtained by playing any u in the arg max *of (11), with j, t in the* arg max *of (12), when in state i.*

Algebraically, (12) is nothing but a generalized spectral problem. Indeed, with an obvious definition of the matrices A_t, we can write:

$$\bigoplus_{t\in\overline{\mathcal{T}}} \lambda^{-t} A_t v = v \enspace , \qquad \text{where } \overline{\mathcal{T}} = \bigcup_{i,j} \mathcal{T}_{ij} \enspace . \tag{13}$$

2.2 An Algebra for Asymptotics

In Statistical Physics, one looks at the asymptotics when the temperature h tends to zero of the spectrum of *transfer matrices*, which have the form

$$\mathcal{A}_h = (\exp(h^{-1} A_{ij}))_{1\leq i,j\leq n} \enspace .$$

The real parameters A_{ij} represent potential terms plus interaction energy terms (when two adjacent sites are in states i and j, respectively). The Perron eigenvalue[3] $\rho(\mathcal{A}_h)$ determines the free energy per site $\lambda_h = h \log \rho(\mathcal{A}_h)$. Clearly, λ_h is an eigenvalue of A in the semiring $\mathbb{R}_h$, defined in Table 1. Let $\rho_{\max}(A)$ denote the maximal $(\max,+)$ eigenvalue of A. Since $\lim_{h\to 0^+} \mathbb{R}_h = \mathbb{R}_0 = \mathbb{R}_{\max}$, the following result is natural.

FACT 4 (PERRON FROBENIUS ASYMPTOTICS). *The asymptotic growth rate of the Perron eigenvalue of $\mathcal{A}_h$ is equal to the maximal $(\max,+)$ eigenvalue of the matrix A:*

$$\lim_{h\to 0^+} h \log \rho(\mathcal{A}_h) = \rho_{\max}(A) \enspace . \tag{14}$$

This follows easily from the $(\max,+)$ spectral inequalities (24),(25) below. The normalized Perron eigenvector v_h of $\mathcal{A}_h$ also satisfies

$$\lim_{h\to 0^+} h \log(v_h)_i = u_i \enspace ,$$

where u is a special $(\max,+)$ eigenvector of A which has been characterized recently by Akian, Bapat, and Gaubert [1]. Precise asymptotic expansions of $\rho(\mathcal{A}_h)$ as sum of exponentials have been given, some of the terms having combinatorial interpretations.

More generally, $(\max,+)$ algebra arises almost everywhere in asymptotic phenomena. Often, the $(\max,+)$ algebra is involved in an elementary way (e.g. when computing exponents of Puiseux expansions using the Newton Polygon). Less elementary applications are WKB type asymptotics (see [31]), which are related to Large Deviations (see e.g. [17]).

2.3 An Algebra for Discrete Event Systems

The $(\max,+)$ algebra is popular in the Discrete Event Systems community, since $(\max,+)$ linear dynamics correspond to a well identified subclass of Discrete Event Systems, with only synchronization phenomena, called Timed Event Graphs. Indeed, consider a system with n repetitive tasks. We assume that the k-th execution of task i (firing of transition i) has to wait τ_{ij} time units for the $(k-\nu_{ij})$-th execution of task j. E.g. tasks represent the processing of parts in a manufacturing system, ν_{ij} represents an initially available stock, and τ_{ij} represents a production or transportation time.

[3] The Perron eigenvalue $\rho(B)$ of a matrix B with nonnegative entries is the maximal eigenvalue associated with a nonnegative eigenvector, which is equal to the spectral radius of B.

FACT 5 (TIMED EVENT GRAPHS ARE $(\max,+)$ LINEAR SYSTEMS). *The earliest date of occurrence of an event i in a Timed Event Graph, $x_i(k)$, satisfies*

$$x_i(k) = \max_j [\tau_{ij} + x_j(k - \nu_{ij})] \quad . \tag{15}$$

Eqn 15 coincides with the value iteration of the deterministic semi-Markov Decision Process in § 2.1, that we only wrote in the Markov version (3). Therefore, the asymptotic behavior of (15) can be dealt with as in § 2.1, using $(\max,+)$ spectral theory. In particular, if the generalized spectral problem $v_i = \max_j[\tau_{ij} - \lambda\nu_{ij} + v_j]$ has a finite solution (λ, v), then $\lambda = \lim_{k\to\infty} k^{-1} \times x_i(k)$, for all i (λ is the *cycle time*, or inverse of the *asymptotic throughput*). The study of the dynamics (15), and of its stochastic [2], and non-linear extensions [11,23] (fluid Petri Nets, minmax functions), is the major theme of $(\max,+)$ discrete event systems theory.

Another linear model is that of *heaps of pieces*. Let $\mathcal{R}$ denote a set of *positions* or *resources* (say $\mathcal{R} = \{1,\dots,n\}$). A *piece* (or *task*) a is a rigid (possibly non connected) block, represented geometrically by a set of occupied positions (or requested resources) $R(a) \subset \mathcal{R}$, a lower contour (starting time) $\ell(a) : R(a) \to \mathbb{R}$, an upper contour (release time) $h(a) : R(a) \to \mathbb{R}$, such that $\forall a \in R(a)$, $h(a) \geq \ell(a)$. The piece corresponds to the region of the $\mathcal{R}\times\mathbb{R}$ plane: $P_a = \{(r,y) \in R(a) \times \mathbb{R} \mid \ \ell(a)_r \leq y \leq h(a)_r\}$, which means that task a requires the set of resources (machines, processors, operators) $R(a)$, and that resource $r \in R(a)$ is used from time $\ell(a)_r$ to time $h(a)_r$. A piece P_a can be translated vertically of any λ, which gives the new region defined by $\ell'(a) = \lambda + \ell(a),\ h'(a) = \lambda + h(a)$. We can execute a task earlier or later, but we cannot change the differences $h(a)_r - \ell(a)_s$ which are invariants of the task. A *ground* or *initial condition* is a row vector $g \in (\mathbb{R}_{\max})^{\mathcal{R}}$. Resource r becomes initially available at time g_r. If we drop k pieces $a_1 \dots a_k$, in this order, on the ground g (letting the pieces fall down according to the gravity, forbidding horizontal translations, and rotations, as in the famous Tetris game, see Fig 2), we obtain what we call a *heap of pieces*. The upper contour $x(w)$ of the heap $w = a_1 \dots a_k$ is the row vector in $(\mathbb{R}_{\max})^{\mathcal{R}}$, whose r-th component is equal to the position of the top of the highest piece occupying resource r. The *height* of the heap is by definition $y(w) = \max_{r\in\mathcal{R}} x(w)_r$. Physically, $y(w)$ gives the *makespan* (= completion time) of the sequence of tasks w, and $x(w)_r$ is the release time of resource r.

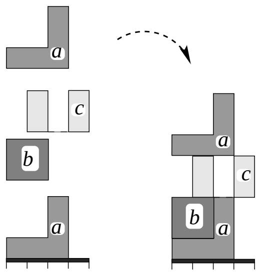
With each piece a within a set of pieces $\mathcal{T}$, we associate the matrix $M(a) \in (\mathbb{R}_{\max})^{\mathcal{R}\times\mathcal{R}}$, $M(a)_{r,s} = h(a)_s - \ell(a)_r$ if $r,s \in R(a)$, and $M(a)_{r,r} = \mathbb{1}$ for diagonal entries not in $R(a)$ (other entries are $\mathbb{0}$). The following result was found independently by Gaubert and Mairesse (in [24]), and Brilman and Vincent [6].

FACT 6 (TETRIS GAME IS $(\max,+)$ LINEAR). *The upper contour $x(w)$ and the height $y(w)$ of the heap of pieces $w = a_1 \dots a_k$, piled up on the ground g, are given by the $(\max,+)$ products:*

$$x(w) = gM(a_1)\dots M(a_k), \qquad y(w) = x(w)\mathbb{1}_{\mathcal{R}},$$

($\mathbb{1}_X$ denotes the column vector indexed by X with entries $\mathbb{1}$).

In algebraic terms, the height generating series $\bigoplus_{w\in\mathcal{T}^*} y(w)w$ is *rational* over the (max,+) semiring ($\mathcal{T}^*$ is the free monoid on $\mathcal{T}$, basic properties of rational series can be found e.g. in [38]).



$R(c) = \{2,4\}$, $\ell(c) = [\cdot, 0, \cdot, 0]$, $h(c) = [\cdot, 2, \cdot, 2]$

$R(b) = \{1,2\}$, $\ell(b) = [0, 0, \cdot, \cdot]$, $h(b) = [2, 2, \cdot, \cdot]$

$R(a) = \{1,2,3\}$, $\ell(a) = [0, 0, 0, \cdot]$, $h(a) = [1, 1, 3, \cdot]$

Fig. 2. Heap of Pieces

Let us mention an open problem. If an infinite sequence of pieces $a_1 a_2 \dots a_k \dots$ is taken at random, say in an independent identically distributed way with the uniform distribution on $\mathcal{T}$, it is known [14,2] that there exists an asymptotic growth rate $\lambda \in \mathbb{R}^+$:

$$\lambda = \lim_{k \to \infty} \frac{1}{k} y(a_1 \dots a_k) \quad \text{a.s.} \tag{16}$$

The effective computation of the constant λ (Lyapunov exponent) is one of the main open problems in (max,+) algebra. The Lyapunov exponent problem is interesting for general random matrices (not only for special matrices associated with pieces), but the heap case (even with unit height, $h(a) = 1 + \ell(a)$) is typical and difficult enough to begin with. Existing results on Lyapunov exponents can be found in [2]. See also the paper of Gaujal and Jean-Marie in [24], and [6].

2.4 An Algebra for Decision

The "tropical" semiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +)$, has been invented by Simon [39] to solve the following classical problem posed by Brzozowski: *is it decidable whether a rational language L has the Finite Power Property (FPP): $\exists m \in \mathbb{N}$, $L^* = L^0 \cup L \cup \dots \cup L^m$*. The problem was solved independently by Simon and Hashiguchi.

FACT 7 (SIMON). *The FPP problem for rational languages reduces to the finiteness problem for finitely generated semigroups of matrices with entries in $\mathbb{N}_{\min}$, which is decidable.*

Other (more difficult) decidable properties (with applications to the polynomial closure and star height problems) are the *finite section* problem, which asks, given a finitely generated semigroup of matrices S over the tropical semiring, whether the set of entries in position i, j, $\{s_{ij} \mid s \in S\}$ is finite; and the more general *limitation* problem, which asks whether the set of coefficients of a rational series in $\mathbb{N}_{\min}$, with noncommuting indeterminates, is finite. These decidability results due to Hashiguchi [25], Leung [29] and Simon [40] use structural properties of long optimal words in $\mathbb{N}_{\min}$-automata (involving multiplicative rational expressions), and combinatorial arguments. By comparison with basic Discrete Event System and Markov Decision applications, which essentially involve semigroups with a single generator ($S = \{A^k \mid k \geq 1\}$), these typically noncommutative problems represent a major jump in difficulty. We refer the reader to the survey of Pin in [24], to [40,25,29], and to the references therein. However, essential in the understanding of the noncommutative case is the one generator case, covered by the $(\max,+)$ Perron-Frobenius theory detailed below.

Let us point out an open problem. The semigroup of *linear projective maps* $\mathbb{PZ}_{\max}^{n\times n}$ is the quotient of the semigroup of matrices $\mathbb{Z}_{\max}^{n\times n}$ by the proportionality relation: $A \sim B \Leftrightarrow \exists\lambda \in \mathbb{Z},\ A = \lambda B$ (i.e. $A_{ij} = \lambda + B_{ij}$). We ask: *can we decide whether a finitely generated semigroup of linear projective maps is finite ?* The motivation is the following. If the image of a finitely generated semigroup with generators $M(a) \in \mathbb{Z}_{\max}^{n\times n}, a \in \Sigma$ by the canonical morphism $\mathbb{Z}_{\max}^{n\times n} \to \mathbb{PZ}_{\max}^{n\times n}$ is finite, then the Lyapunov exponent $\lambda =$ a.s. $\lim_{k\to\infty} k^{-1} \times \|M(a_1)\ldots M(a_k)\|$ (same probabilistic assumptions as for (16), $\|A\| = \sup_{ij} A_{ij}$, by definition) can be computed from a finite Markov Chain on the associated projective linear semigroup [19,20].

3 Solving Linear Equations in the $(\max,+)$ Semiring

3.1 A hopeless algebra?

The general system of n $(\max,+)$-linear equations with p unknowns $x_1,\ldots,x_p$ writes:

$$Ax \oplus b = Cx \oplus d, \qquad A, C \in (\mathbb{R}_{\max})^{n\times p},\ b, d \in (\mathbb{R}_{\max})^n \ . \tag{17}$$

Unlike in conventional algebra, a square linear system ($n = p$) is not generically solvable (consider $3x \oplus 2 = x \oplus 0$, which has no solution, since for all $x \in \mathbb{R}_{\max}$, $\max(3+x, 2) > \max(x, 0)$).

There are several ways to make this hard reality more bearable. One is to give general structural results. Another is to deal with natural subclasses of equations, whose solutions can be obtained by efficient methods. The *inverse* problem $Ax = b$ can be dealt with using *residuation*. The *spectral* problem $Ax = \lambda x$ (λ scalar) is solved using the $(\max,+)$ analogue of Perron-Frobenius theory. The *fixed point* problem $x = Ax \oplus b$ can be solved via rational methods familiar in language theory (introducing the "star" operation $A^* = A^0 \oplus A \oplus A^2 \oplus \cdots$). A last way, which has the seduction of forbidden things, is to say: "certainly, the solution of $3x \oplus 2 = x \oplus 0$ is $x = \ominus - 1$. For if this equation has no ordinary solution, the symmetrized equation (obtained by putting each occurrence of the unknown in the other side of the equality) $x' \oplus 2 = 3x' \oplus 0$ has the unique solution $x' = -1$. Thus, $x = \ominus - 1$ is the requested solution." Whether or not this argument is valid is the object of *symmetrization* theory.

All these approaches rely, in one way or another, on the *order* structure of idempotent semirings that we next introduce.

3.2 Natural Order Structure of Idempotent Semirings

An idempotent semiring $\mathcal{S}$ can be equipped with the following *natural* order relation

$$a \preceq b \iff a \oplus b = b. \tag{18}$$

We will write $a \prec b$ when $a \preceq b$ and $a \neq b$. The natural order endows $\mathcal{S}$ with a sup-semilattice structure, for which $a \oplus b = a \vee b = \sup\{a, b\}$ (this is the least upper bound of the set $\{a, b\}$), and $\mathbb{0} \preceq a$, $\forall a, b \in \mathcal{S}$ ($\mathbb{0}$ is the *bottom* element). The semiring laws preserve this order, i.e. $\forall a, b, c \in \mathcal{S},\ a \preceq b \implies a \oplus c \preceq b \oplus c,\ ac \preceq bc$. For the $(\max, +)$ semiring $\mathbb{R}_{\max}$, the natural order $\preceq$ coincides with the usual one. For the $(\min, +)$ semiring $\mathbb{R}_{\min}$, the natural order is the opposite of the usual one.

Since addition coincides with the sup for the natural order, there is a simple way to define infinite sums, in an idempotent semiring, setting $\bigoplus_{i \in I} x_i = \sup\{x_i \mid i \in I\}$, for any possibly infinite (even non denumerable) family $\{x_i\}_{i \in I}$ of elements of $\mathcal{S}$, when the sup exists. We say that the idempotent semiring $\mathcal{S}$ is *complete* if any family has a supremum, and if the product distributes over infinite sums. When $\mathcal{S}$ is complete, $(\mathcal{S}, \preceq)$ becomes automatically a complete lattice, the greatest lower bound being equal to $\bigwedge_{i \in I} x_i = \sup\{y \in \mathcal{S} \mid y \leq x_i,\ \forall i \in I\}$. The $(\max, +)$ semiring $\mathbb{R}_{\max}$ is not complete (a complete idempotent semiring must have a maximal element), but it can be embedded in the complete semiring $\overline{\mathbb{R}}_{\max}$.

3.3 Solving $Ax = b$ using Residuation

In general, $Ax = b$ has no solution[4], but $Ax \preceq b$ always does (take $x = \mathbb{0}$). Thus, a natural way of attacking $Ax = b$ is to relax the equality and study the set of its subsolutions. This can be formalized in terms of *residuation* [5], a notion borrowed from ordered sets theory. We say that a monotone map f from an ordered set E to an ordered set F is *residuated* if for all $y \in F$, the set $\{x \in E \mid f(x) \leq y\}$ has a maximal element, denoted $f^\sharp(y)$. The monotone map $f^\sharp$, called *residual* or *residuated map* of f, is characterized alternatively by $f \circ f^\sharp \leq \mathrm{Id}$, $f^\sharp \circ f \geq \mathrm{Id}$. An idempotent semiring $\mathcal{S}$ is *residuated* if the right and left multiplication maps $\lambda_a : x \mapsto ax$, $\rho_a : x \mapsto xa$, $\mathcal{S} \to \mathcal{S}$, are residuated, for all $a \in \mathcal{S}$. A *complete* idempotent semiring is automatically residuated. We set

$$a \backslash b \overset{\text{def}}{=} \lambda_a^\sharp(b) = \max\{x \mid ax \preceq b\} \ , \quad b/a \overset{\text{def}}{=} \rho_a^\sharp(b) = \max\{x \mid xa \preceq b\} \ .$$

In the completed $(\max, +)$ semiring $\overline{\mathbb{R}}_{\max}$, $a \backslash b = b/a$ is equal to $b - a$ when $a \neq \mathbb{0}(= -\infty)$, and is equal to $+\infty$ if $a = \mathbb{0}$. The residuated character is transfered from scalars to matrices as follows.

Proposition 2 (Matrix residuation). *Let $\mathcal{S}$ be a complete idempotent semiring. Let $A \in \mathcal{S}^{n \times p}$. The map $\lambda_A : x \mapsto Ax, \mathcal{S}^p \to \mathcal{S}^n$, is residuated. For any $y \in \mathcal{S}^n$, $A \backslash y \overset{\text{def}}{=} \lambda_A^\sharp(y)$ is given by $(A \backslash y)_i = \bigwedge_{1 \leq j \leq n} A_{ji} \backslash y_j$.*

In the case of $\overline{\mathbb{R}}_{\max}$, this reads:

$$(A \backslash y)_i = \min_{1 \leq j \leq n} (-A_{ji} + y_j) \ , \tag{19}$$

[4] It is an elementary exercise to check that the map $x \mapsto Ax$, $(\mathbb{R}_{\max})^p \to (\mathbb{R}_{\max})^n$, is surjective (resp. injective) iff the matrix A contains a monomial submatrix of size n (resp. p), a very unlikely event — recall that a square matrix B is *monomial* if there is exactly one non zero element in each row, and in each column, or (equivalently) if it is a product of a permutation matrix and a diagonal matrix with non zero diagonal elements. This implies that a matrix has a left or a right inverse iff it has a monomial submatrix of maximal size, which is the analogue of a well known result for nonnegative matrices [4, Lemma 4.3].

with the convention dual to that of $\overline{\mathbb{R}}_{\max}$, $(+\infty)+x=+\infty$, for any $x\in\mathbb{R}\cup\{\pm\infty\}$. We recognize in (19) a matrix product in the semiring $\overline{\mathbb{R}}_{\min}=(\mathbb{R}\cup\{\pm\infty\},\min,+)$, involving the transpose of the opposite of A.

Corollary 3 (Solving $Ax=y$). *Let $\mathcal{S}$ denote a complete idempotent semiring, and let $A\in\mathcal{S}^{n\times p}$, $y\in\mathcal{S}^n$. The equation $Ax=y$ has a solution iff $A(A\backslash y)=y$.*

Corollary 3 allows us to check the existence of a solution x of $Ax=y$ in time $O(np)$ (scalar operations are counted for one time unit). In the $(\max,+)$ case, a refinement (due to the total order) allows us to decide the existence of a solution by inspection of the minimizing sets in (19), see [15,44].

3.4 Basis Theorem for Finitely Generated Semimodules over $\mathbb{R}_{\max}$

A finitely generated semimodule $\mathcal{V}\subset(\mathbb{R}_{\max})^n$ is the set of linear combinations of a finite family $\{u_1,\dots,u_p\}$ of vectors of $(\mathbb{R}_{\max})^n$:

$$\mathcal{V}=\Big\{\bigoplus_{i=1}^{p}\lambda_i u_i\;\Big|\;\lambda_1,\dots,\lambda_p\in\mathbb{R}_{\max}\Big\}\enspace .$$

In matrix terms, $\mathcal{V}$ can be identified to the *column space* or *image* of the $n\times p$ matrix $A=[u_1,\dots,u_p]$, $\mathcal{V}=\operatorname{Im}A\stackrel{\text{def}}{=}\{Ax\mid\; x\in(\mathbb{R}_{\max})^p\}$. The *row space* of A is the column space of A^T (the transpose of A). The family $\{u_i\}$ is a *weak basis* of $\mathcal{V}$ if it is a generating family, minimal for inclusion. The following result, due to Moller [33] and Wagneur [42] (with variants) states that finitely generated subsemimodules of $(\mathbb{R}_{\max})^n$ have (essentially) a unique weak basis.

Theorem 4 (Basis Theorem). *A finitely generated semimodule $\mathcal{V}\subset(\mathbb{R}_{\max})^n$ has a weak basis. Any two weak bases have the same number of generators. For any two weak bases $\{u_1,\dots,u_p\}$, $\{v_1,\dots,v_p\}$, there exist invertible scalars $\lambda_1,\dots,\lambda_p$ and a permutation σ of $\{1,\dots,p\}$ such that $u_i=\lambda_i v_{\sigma(i)}$.*

The cardinality of a weak basis is called the *weak rank* of the semimodule, denoted $\mathrm{rk}_w\mathcal{V}$. The *weak column rank* (resp. weak row rank) of the matrix A is the weak rank of its column (resp. row) space. Unlike in usual algebra, the weak row rank in general differs from the weak column rank (this is already the case for Boolean matrices). Theorem 4 holds more generally in any idempotent semiring $\mathcal{S}$ satisfying the following axioms: $(a\succeq\alpha a$ and $a\neq\mathbb{0})\implies\mathbb{1}\succeq\alpha$, $(a=\alpha a\oplus b$ and $\alpha\prec\mathbb{1})\implies a=b$. The axioms needed to set up a general rank theory in idempotent semirings are not currently understood. Unlike in vector spaces, there exist finitely generated semimodules $\mathcal{V}\subset(\mathbb{R}_{\max})^n$ of arbitrarily large weak rank, if the dimension of the ambient space n is at least 3; and not all subsemimodules of $(\mathbb{R}_{\max})^n$ are finitely generated, even with $n=2$.

Example 5 (Cuninghame-Green [15],Th. 16.4). The weak column rank of the $3\times(i+1)$ matrix

$$A_i=\begin{bmatrix}0&0&\dots&0\\0&1&\dots&i\\0&-1&\dots&-i\end{bmatrix}$$

is equal to $i+1$ for all $i \in \mathbb{N}$. This can be understood geometrically using a representation due to Mairesse. We visualize the set of vectors with finite entries of a semimodule $\mathcal{V} \subset (\mathbb{R}_{\max})^3$ by the subset of $\mathbb{R}^2$, obtained by projecting $\mathcal{V}$ orthogonally, on any plane orthogonal to $(1,1,1)$. Since $\mathcal{V}$ is invariant by multiplication by any scalar λ, i.e. by the usual addition of the vector $(\lambda,\lambda,\lambda)$, the semimodule $\mathcal{V}$ is well determined by its projection. We only loose the points with $\mathbb{0}$ entries which are sent to some infinite end of the $\mathbb{R}^2$ plane. The semimodules $\mathrm{Im}\,A_1, \mathrm{Im}\,A_2, \mathrm{Im}\,A_3$ are shown on Fig 3. The generators are represented by bold points, and the semimodules by gray regions. The broken *line* between any two generators u,v represents $\mathrm{Im}\,[u,v]$. This picture should make it clear that a weak basis of a subsemimodule of $(\mathbb{R}_{\max})^3$ may have as many generators as a convex set of $\mathbb{R}^2$ may have extremal points. The notion of weak rank is therefore a very coarse one.

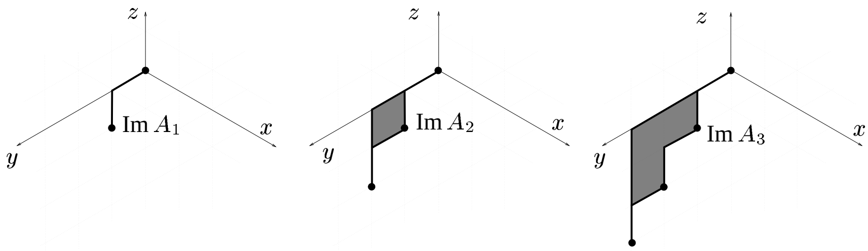


Fig. 3. An infinite ascending chain of semimodules of $(\mathbb{R}_{\max})^3$ (see Ex. 5).

Let $A \in (\mathbb{R}_{\max})^{n\times p}$. A weak basis of the semimodule $\mathrm{Im}\,A$ can be computed by a greedy algorithm. Let $A[i]$ denote the i-th column of A, and let $A(i)$ denote the $n \times (p-1)$ matrix obtained by deleting column i. We say that column i of A is *redundant* if $A[i] \in \mathrm{Im}\,A(i)$, which can be checked by Corollary 3. Replacing A by $A(i)$ when $A[i]$ is redundant, we do not change the semimodule $\mathrm{Im}\,A$. Continuing this process, we terminate in $O(np^2)$ time with a weak basis.

Application 6 (Controllability). The fact that ascending chains of semimodules need not stationnarize yields pathological features in terms of Control. Consider the controlled dynamical system:

$$x(0) = \mathbb{0},\ \ x(k) = Ax(k-1) \oplus Bu(k),\ k = 1,2,\dots \tag{20}$$

where $A \in (\mathbb{R}_{\max})^{n\times n}$, $B \in (\mathbb{R}_{\max})^{n\times q}$, and $u(k) \in (\mathbb{R}_{\max})^q, k = 1,2,\dots$ is a sequence of control vectors. Given a state $\xi \in (\mathbb{R}_{\max})^n$, the *accessibility* problem (in time N) asks whether there is a control sequence u such that $x(N) = \xi$. Clearly, ξ is accessible in time N iff it belongs to the image of the *controllability* matrix $\mathcal{C}_N = [B, AB, \dots, A^{N-1}B]$. Corollary 3 allows us to decide the accessibility of ξ. However, unlike in conventional algebra (in which $\mathrm{Im}\,\mathcal{C}_N = \mathrm{Im}\,\mathcal{C}_n$, for any $N \geq n$, thanks to Cayley-Hamilton theorem), the semimodule of accessible states $\mathrm{Im}\,\mathcal{C}_N$ may grow indefinitely as $N \to \infty$.

3.5 Solving $Ax = Bx$ by Elimination

The following theorem is due to Butkovič and Hegedüs [9]. It was rediscovered in [18, Chap. III].

Theorem 7 (Finiteness Theorem). *Let $A, B \in (\mathbb{R}_{\max})^{n\times p}$. The set $\mathcal{V}$ of solutions of the homogeneous system $Ax = Bx$ is a finitely generated semimodule.*

This is a consequence of the following universal elimination result.

Theorem 8 (Elimination of Equalities in Semirings). *Let $\mathcal{S}$ denote an arbitrary semiring. Let $A, B \in \mathcal{S}^{n\times p}$. If for any $q \geq 1$ and any row vectors $a, b \in \mathcal{S}^q$, the* hyperplane $\{x \in \mathcal{S}^q \mid \ ax = bx\}$ *is a finitely generated semimodule, then $\mathcal{V} = \{x \in \mathcal{S}^p \mid \ Ax = Bx\}$ is a finitely generated semimodule.*

The fact that hyperplanes of $(\mathbb{R}_{\max})^q$ are finitely generated can be checked by elementary means (but the number of generators can be of order q^2). Theorem 8 can be easily proved by induction on the number of equations (see [9,18]). In the $\mathbb{R}_{\max}$ case, the resulting naive algorithm has a doubly exponential complexity. But it is possible to incorporate the construction of weak bases in the algorithm, which much reduces the execution time. The making (and complexity analysis) of efficient algorithms for $Ax = Bx$ is a major open problem. When only a single solution is needed, the algorithm of Walkup and Borriello (in [24]) seems faster, in practice.

There is a more geometrical way to understand the finiteness theorem. Consider the following correspondence between semimodules of $((\mathbb{R}_{\max})^{1\times n})^2$ (couples of row vectors) and $(\mathbb{R}_{\max})^{n\times 1}$ (column vectors), respectively:

$$\begin{array}{l} \mathcal{W} \subset ((\mathbb{R}_{\max})^{1\times n})^2 \quad \longrightarrow \quad \mathcal{W}^\top = \{x \in (\mathbb{R}_{\max})^{n\times 1} \mid \ ax = bx, \ \forall (a,b) \in \mathcal{W}\} \ , \\ \mathcal{V}^\perp = \{(a,b) \in ((\mathbb{R}_{\max})^{1\times n})^2 \mid \ ax = bx, \ \forall x \in \mathcal{V}\} \quad \longleftarrow \quad \mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1} \ . \end{array} \tag{21}$$

Theorem 7 states that if $\mathcal{W}$ is a finitely generated semimodule (i.e. if all the row vectors $[a, b]$ belong to the row space of a matrix $[A, B]$) then, its orthogonal $\mathcal{W}^\top$ is finitely generated. Conversely, if $\mathcal{V}$ is finitely generated, so does $\mathcal{V}^\perp$ (since the elements (a, b) of $\mathcal{V}^\perp$ are the solutions of a finite system of linear equations). The orthogonal semimodule $\mathcal{V}^\perp$ is exactly the set of *linear equations* $(a, b) : ax = bx$ satisfied by all the $x \in \mathcal{V}$. Is a finitely generated subsemimodule $\mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1}$ defined by its equations ? The answer is positive [18, Chap. IV,1.2.2]:

Theorem 9 (Duality Theorem). *For all finitely generated semimodules $\mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1}$, $(\mathcal{V}^\perp)^\top = \mathcal{V}$.*

In general, $(\mathcal{W}^\top)^\perp \supsetneq \mathcal{W}$. The duality theorem is based on the following analogue of the Hahn-Banach theorem, stated in [18]: *if $\mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1}$ is a finitely generated semimodule, and $y \notin \mathcal{V}$, there exist $(a, b) \in ((\mathbb{R}_{\max})^{1\times n})^2$ such that $ay \neq by$ and $ax = bx, \ \forall x \in \mathcal{V}$.*

The *kernel* of a linear operator C should be defined as $\ker C = \{(x, y) \mid \ Cx = Cy\}$. When is the projector on the image of a linear operator B, parallel to $\ker C$, defined? The answer is given in [12].

3.6 Solving $x = Ax \oplus b$ using Rational Calculus

Let $\mathcal{S}$ denote a complete idempotent semiring, and let $A \in \mathcal{S}^{n\times n}, b \in \mathcal{S}^n$. The least solution of $x \succeq Ax \oplus b$ is A^*b, where the star operation is given by:

$$A^* \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{N}} A^n \quad . \tag{22}$$

Moreover, $x = A^*b$ satisfies the equation $x = Ax \oplus b$. All this is most well known (see e.g. [38]), and we will only insist on the features special to the $(\max, +)$ case. We can interpret A^*_{ij} as the *maximal weight* of a path from i to j of any length, in the graph[2] associated with A. We next characterize the convergence of A^* in $(\mathbb{R}_{\max})^{n\times n}$ (A^* is a priori defined in $(\overline{\mathbb{R}}_{\max})^{n\times n}$, but the $+\infty$ value which breaks the semifield character of $\mathbb{R}_{\max}$ is undesired in most applications). The following fact is standard (see e.g. [2, Theorem 3.20]).

Proposition 10. *Let $A \in (\mathbb{R}_{\max})^{n\times n}$. The entries of A^* belong to $\mathbb{R}_{\max}$ iff there are no circuits with positive weight in the graph[2] of A. Then, $A^* = A^0 \oplus A \oplus \cdots \oplus A^{n-1}$.*

The matrix A^* can be computed in time $O(n^3)$ using classical universal Gauss algorithms (see e.g. [21]). Special algorithms exist for the $(\max, +)$ semiring. For instance, the sequence $x(k) = Ax(k-1) \oplus b$, $x(0) = \mathbb{0}$ stationarizes before step n (with $x(n) = x(n+1) = A^*b$) iff A^*b is finite. This allows us to compute A^*b very simply. A complete account of existing algorithms can be found in [21].

3.7 The $(\max, +)$ Perron-Frobenius Theory

The most ancient, most typical, and probably most useful $(\max, +)$ results are relative to the spectral problem $Ax = \lambda x$. One might argue that 90% of current applications of $(\max, +)$ algebra are based on a complete understanding of the spectral problem. The theory is extremely similar to the well known Perron-Frobenius theory (see e.g. [4]). The $(\max, +)$ case turns out to be very appealing, and slightly more complex than the conventional one (which is not surprising, since the $(\max, +)$ spectral problem is a somehow degenerate limit of the conventional one, see §2.2). The main discrepancy is the existence of two graphs which rule the spectral elements of A, the weighted graph canonically[2] associated with a matrix A, and one of its subgraphs, called *critical* graph.

First, let us import the notion of *irreducibility* from the conventional Perron-Frobenius theory. We say that i *has access* to j if there is a path from i to j in the graph of A, and we write $i \xrightarrow{*} j$. The *classes* of A are the equivalence classes for the relation $i\mathcal{R}j \Leftrightarrow (i \xrightarrow{*} j$ and $j \xrightarrow{*} i)$. A matrix with a single class is *irreducible*. A class $\mathcal{C}$ is upstream $\mathcal{C}'$ (equivalently $\mathcal{C}'$ is downstream $\mathcal{C}$) if a node of $\mathcal{C}$ has access to a node of $\mathcal{C}'$. Classes with no other downstream classes are *final*, classes with no other upstream classes are *initial*.

The following famous $(\max, +)$ result has been proved again and again, with various degrees of generality and precision, see [37,41,15,44,22,2,31].

Theorem 11 (“$(\max,+)$ Perron-Frobenius Theorem”). *An irreducible matrix $A \in (\mathbb{R}_{\max})^{n\times n}$ has a unique eigenvalue, equal to the maximal circuit mean of A:*

$$\rho_{\max}(A) = \bigoplus_{k=1}^{n} \mathrm{tr}\,(A^k)^{\frac{1}{k}} \quad = \max_{1\leq k\leq n} \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k} \ . \tag{23}$$

We have the following refinements in terms of inequalities [18, Chap IV], [3].

Lemma 12 (“Collatz-Wielandt Properties”). *For any $A \in (\mathbb{R}_{\max})^{n\times n}$,*

$$\rho_{\max}(A) = \max\{\lambda \in \mathbb{R}_{\max} \mid \exists u \in (\mathbb{R}_{\max})^n \setminus \{\mathbb{0}\},\ \ Au \succeq \lambda u\} \ . \tag{24}$$

Moreover, if A is irreducible,

$$\rho_{\max}(A) = \min\{\lambda \in \mathbb{R}_{\max} \mid \exists u \in (\mathbb{R}_{\max})^n \setminus \{\mathbb{0}\},\ \ Au \preceq \lambda u\} \ . \tag{25}$$

The characterization (25) implies in particular that, for an irreducible matrix A, $\rho_{\max}(A)$ is the optimal value of the linear program

$$\min \lambda \text{ s.t. } \ \forall i,j \ \ A_{ij} + u_j \leq u_i + \lambda \ .$$

This was already noticed by Cuninghame-Green [15]. The standard way to compute the maximal circuit mean $\rho_{\max}(A)$ is to use Karp algorithm [27], which runs in time $O(n^3)$. The specialization of Howard algorithm (see e.g. [35]) to deterministic Markov Decision Processes with average reward, yields an algorithm whose average execution time is in practice far below that of Karp algorithm, but no polynomial bound is known for the execution time of Howard algorithm. Howard algorithm is also well adapted to the semi-Markov variants (12).

Unlike in conventional Perron-Frobenius theory, an irreducible matrix may have several (non proportional) eigenvectors. The characterization of the eigenspace uses the notion of *critical graph*. An arc (i,j) is *critical* if it belongs to a circuit $(i_1,\dots,i_k)$ whose mean weight attains the $\max$ in (23). Then, the nodes i,j are *critical*. Critical nodes and arcs form the *critical graph*. A *critical class* is a strongly connected component of the critical graph. Let $\mathcal{C}_1^c,\dots,\mathcal{C}_r^c$ denote the critical classes. Let $\tilde{A} = \rho_{\max}^{-1}(A)A$ (i.e. $\tilde{A}_{ij} = -\rho_{\max}(A) + A_{ij}$). Using Proposition 10, the existence of $\tilde{A}^*$ ($\stackrel{\text{def}}{=} (\tilde{A})^*$) is guaranteed. If i is in a critical class, we call the column $\tilde{A}^*_{\cdot,i}$ of $\tilde{A}^*$ *critical*. The following result can be found e.g. in [2,16].

Theorem 13 (Eigenspace). *Let $A \in (\mathbb{R}_{\max})^{n\times n}$ denote an irreducible matrix. The critical columns of $\tilde{A}^*$ span the eigenspace of A. If we select only one column, arbitrarily, per critical class, we obtain a weak basis of the eigenspace.*

Thus, the cardinality of a weak basis is equal to the number of critical classes. For any two i,j within the same critical class, the critical columns $\tilde{A}^*_{\cdot,i}$ and $\tilde{A}^*_{\cdot,j}$ are proportional.

We next show how the eigenvalue $\rho_{\max}(A)$ and the eigenvectors determine the asymptotic behavior of A^k as $k \to \infty$. The *cyclicity* of a critical class $\mathcal{C}_s^c$ is by definition the $\gcd$ of the lengths of its circuits. The *cyclicity* c of A is the lcm of the cyclicities of its critical classes. Let us pick arbitrarily an index i_s within each critical class $\mathcal{C}_s^c$, for $s = 1,\dots,r$, and let v_s, w_s denote the column and row of index i_s of $\tilde{A}^*$ (v_s, w_s are right and left eigenvectors of A, respectively). The following result follows from [2].

Theorem 14 (Cyclicity). *Let $A \in (\mathbb{R}_{\max})^{n\times n}$ be an irreducible matrix. There is an integer K_0 such that*

$$k \geq K_0 \implies A^{k+c} = \rho_{\max}(A)^c A^k \quad , \tag{26}$$

where c is the cyclicity of A. Moreover, if $c = 1$,

$$k \geq K_0 \implies A^k = \rho_{\max}(A)^k P, \text{ where } P = \bigoplus_{s=1}^{r} v_s w_s \quad . \tag{27}$$

The matrix P which satisfies $P^2 = P$, $AP = PA = \rho_{\max}(A)P$ is called the *spectral projector* of A. The cyclicity theorem, which writes $A_{ij}^{k+c} = \rho_{\max}(A) \times c + A_{ij}^k$ in conventional algebra, implies that $A^k x$ grows as $k \times \rho_{\max}(A)$, independently of $x \in (\mathbb{R}_{\max})^n$, and that a periodic regime is attained in finite time. The limit behavior is known a priori. Ultimately, the sequence $\rho_{\max}(A)^{-k}A^k$ visits periodically c accumulation points, which are $Q, AQ, \ldots, A^{c-1}Q$, where Q is the spectral projector of A^c. The length of the transient behavior K_0 can be arbitrarily large. In terms of Markov Decision, Theorem 14 says that optimal long trajectories stay almost all the time on the critical graph (Turnpike theorem). Theorem 14 is illustrated in Fig. 4, which shows the images of a cat (a region of the $\mathbb{R}^2$ plane) by the iterates of A (A, A^2, A^3, etc.), B and C, where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad . \tag{28}$$

We have $\rho_{\max}(A) = 2$. Since A has a unique critical circuit, the spectral projector P is rank one (its column and row spaces are lines). We find that $\tilde{A}^2 = P$: every point of the plane is sent in at most two steps to the eigenline $y = 2 \otimes x = 2 + x$, then it is translated by $(2,2)$ at each step. Similar interpretations exist for B and C.

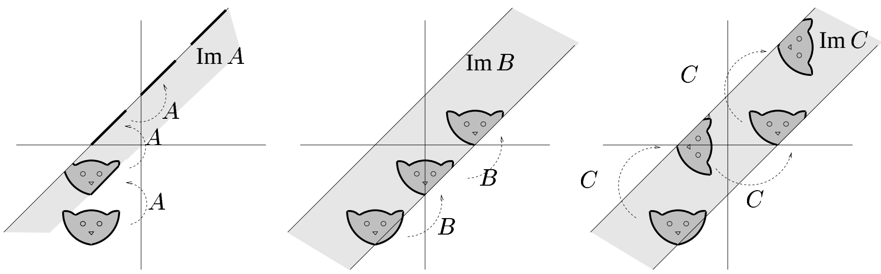


Fig. 4. A cat in a $(\max, +)$ dynamics (see (28))

Let us now consider a reducible matrix A. Given a class $\mathcal{C}$, we denote by $\rho_{\max}(\mathcal{C})$ the $(\max, +)$ eigenvalue of the restriction of the matrix A to $\mathcal{C}$. The *support* of a vector u is the set $\operatorname{supp} u = \{i \mid u_i \neq \mathbb{0}\}$. A set of nodes S is *closed* if $j \in S, i \xrightarrow{*} j$ implies $i \in S$. We say that a class $\mathcal{C} \subset S$ is *final in S* if there is no other downstream class in S.

Theorem 15 (Spectrum of reducible matrices). *A matrix $A \in (\mathbb{R}_{\max})^{n\times n}$ has an eigenvector with support $S \subset \{1,\dots,n\}$ and eigenvalue λ iff S is closed, λ is equal to $\rho_{\max}(\mathcal{C})$ for any class $\mathcal{C}$ that is final in S, and $\lambda \succeq \rho_{\max}(\mathcal{C}')$ for any other class $\mathcal{C}'$ in S.*

The proof can be found in [43,18]. See also [3]. In particular, eigenvalues of initial classes are automatically eigenvalues of A. The maximal circuit mean $\rho_{\max}(A)$ (given by (23)) is also automatically an eigenvalue of A (but the associated eigenvector need not be finite). A weak basis of the eigenspace is given in [18, Chap. IV,1.3.4].

Example 16 (Taxicab eigenproblem). The matrix of the taxicab MDP, shown in Fig 1, has 2 classes, namely $\mathcal{C}_1 = \{c_1, a, c_2\}$, $\mathcal{C}_2 = \{c_3\}$. Since $\rho_{\max}(\mathcal{C}_2) = 2 \prec \rho_{\max}(\mathcal{C}_1) = 5$, there are no finite eigenvectors (which have support $S = \mathcal{C}_1 \cup \mathcal{C}_2$). The only other closed set is $S = \mathcal{C}_1$, which is initial. Thus $\rho_{\max}(A) = \rho_{\max}(\mathcal{C}_1) = 5$ is the only eigenvalue of A. Let A' denote the restriction of A to $\mathcal{C}_1$. There are two critical circuits (c_1) and (a, c_2), and thus two critical classes $\mathcal{C}_1^c = \{c_1\}$, $\mathcal{C}_2^c = \{a, c_2\}$. A weak basis of the eigenspace of A' is given by the columns c_1 and (e.g.) c_2 of

$$(\tilde{A}')^* = \begin{array}{c} \\ c_1 \\ a \\ c_2 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & a & c_2 \end{array} \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \end{array}$$

Completing these two columns by a $\mathbb{0}$ in row 4, we obtain a basis of the eigenspace of A. The non existence of a finite eigenvector is obvious in terms of control. If such an eigenvector existed, by Fact 2, the optimal reward of the taxicab would be independent of the starting point. But, if the taxi driver starts from City 3, he remains blocked there with an income of 2 \$ per journey, whereas if he starts from any other node, he should clearly either run indefinitely in City 1, either shuttle from the airport to City 2, with an average income of 5 \$ per journey (these two policies can be obtained by applying Fact 2 to the MDP restricted to $\mathcal{C}_1$, taking the two above eigenvectors).

The following extension to the reducible case of the cyclicity theorem is worth being mentioned.

Theorem 17 (Cyclicity, reducible case). *Let $A \in (\mathbb{R}_{\max})^{n\times n}$. There exist two integers K_0 and $c \geq 1$, and a family of scalars $\lambda_{ijl} \in \mathbb{R}_{\max}$, $1 \leq i, j \leq n$, $0 \leq l \leq c-1$, such that*

$$k \geq K_0, \quad k \equiv l \mod c \implies A_{ij}^{k+c} = \lambda_{ijl}^c A_{ij}^k \ , \tag{29}$$

Characterizations exist for c and λ_{ijl}. The scalars λ_{ijl} are taken from the set of eigenvalues of the classes of A. If i, j belong to the same class $\mathcal{C}$, $\lambda_{ijl} = \rho_{\max}(\mathcal{C})$ for all l. If i, j do not belong to the same class, the theorem implies that the sequence $\frac{1}{k} \times A_{ij}^k$ may have distinct accumulation points, according to the congruence of k modulo c (see [18, Chap. VI,1.1.10]).

The cyclicity theorems for matrices are essentially equivalent to the characterization of rational series in one indeterminate with coefficient in $\mathbb{R}_{\max}$, as a merge of ultimately geometric series, see the paper of Gaubert in [13] and [28]. Transfer series and rational algebra techniques are particularly powerful for Discrete Event Systems. Timed Event Graphs can be represented by a remarkable (quotient) semiring of series with Boolean coefficients, in two commuting variables, called $\mathcal{M}_{\min}^{\max}[[\gamma,\delta]]$ (see [2, Chap. 5]). The indeterminates γ and δ have natural interpretations as *shifts* in dating and counting. The complete behavior of the system can be represented by simple —often small— commutative rational expressions [2],[18, Chap. VII–IX] (see also [28] in a more general context).

3.8 Symmetrization of the $(\max,+)$ Semiring

Let us try to imitate the familiar construction of $\mathbb{Z}$ from $\mathbb{N}$, for an arbitrary semiring $\mathcal{S}$. We build the set of couples $\mathcal{S}^2$, equipped with (componentwise) sum $(x',x'')\oplus(y',y'')=(x'\oplus y',x''\oplus y'')$, and product $(x',x'')\otimes(y',y'')=(x'y'\oplus x''y'',x'y''\oplus x''y')$. We introduce the *balance* relation

$$(x',x'')\nabla(y',y'')\iff x'\oplus y''=x''\oplus y' \enspace .$$

We have $\mathbb{Z}=\mathbb{N}^2/\nabla$, but for an idempotent semiring $\mathcal{S}$, the procedure stops, since ∇ is not transitive (e.g. $(\mathbb{1},\mathbb{0})\nabla(\mathbb{1},\mathbb{1})\nabla(\mathbb{0},\mathbb{1})$, but $(\mathbb{1},\mathbb{0})\not\nabla(\mathbb{0},\mathbb{1})$). If we renounce to quotient $\mathcal{S}^2$, we may still manipulate couples, with the $\ominus$ operation $\ominus(x',x'')=(x'',x')$. Indeed, since $\ominus$ satisfies the sign rules $\ominus\ominus x=x$, $\ominus(x\oplus y)=(\ominus x)\oplus(\ominus y)$, $\ominus(xy)=(\ominus x)y=x(\ominus y)$, and since $x\nabla y\iff x\ominus y\nabla\mathbb{0}$ (we set $x\ominus y\stackrel{\text{def}}{=}x\oplus(\ominus y)$), it is not difficult to see that *all the familiar identities valid in rings admit analogues in* $\mathcal{S}^2$*, replacing equalities by balances*. For instance, if $\mathcal{S}$ is commutative, we have for all matrices (of compatible size) with entries in $\mathcal{S}^2$ (determinants are defined as usual, with $\ominus$ instead of $-$):

$$\det(AB)\ \nabla\ \det A\ \det B, \tag{30}$$

$$P_A(A)\ \nabla\ \mathbb{0} \qquad \text{where } P_A(\lambda)=\det(A\ominus\lambda\mathrm{Id})\ \text{(Cayley Hamilton)}. \tag{31}$$

Eqn 30 can be written directly in $\mathcal{S}$, introducing the positive and negative determinants $\det^+ A=\bigoplus_{\sigma\text{ even}}\bigotimes_{1\le i\le n}A_{i\sigma(i)}$, $\det^- A=\bigoplus_{\sigma\text{ odd}}\bigotimes_{1\le i\le n}A_{i\sigma(i)}$ (the sums are taken over even and odd permutations of $\{1,\dots,n\}$, respectively). The balance (30) is equivalent to the ordinary equality $\det^+ AB\oplus\det^+ A\ \det^- B\oplus\det^- A\ \det^+ B=\det^- AB\oplus\det^+ A\ \det^+ B\oplus\det^- A\ \det^- B$, but (30) is certainly more adapted to computations. Such identities can be proved combinatorially (showing a bijection between terms on both sides), or derived automatically from their ring analogues using a simple argument due to Reutenauer and Straubing [36, Proof of Lemma 2] (see also the *transfer principle* in [18, Ch. I]).

But in the $\mathbb{R}_{\max}$ case, one can do much better. Consider the following application of the Cayley-Hamilton theorem:

$$A=\begin{bmatrix}1&3\\4&1\end{bmatrix},\quad A^2\ominus\mathrm{tr}\,(A)A\oplus\det A\nabla\mathbb{0},\ \text{i.e}\qquad A^2\oplus 2\mathrm{Id}=1A\oplus 7\mathrm{Id}\enspace .$$

Obviously, we may eliminate the 2Id term which will never saturate the identity (since $2 < 7$), and obtain $A^2 = 1A \oplus 7\mathrm{Id}$. Thus, to some extent $7 \ominus 2 = 7$. This can be formalized by introducing the congruence of semiring:

$$(x', x'')\ \mathcal{R}\ (y', y'') \Leftrightarrow (x' \neq x'', y' \neq y'' \text{ and } x' \oplus y'' = x'' \oplus y') \text{ or } (x', x'') = (y', y'') .$$

The operations $\oplus, \ominus, \otimes$ and the relation ∇ are defined canonically on the quotient semiring, $\mathbb{S}_{\max} = \mathbb{R}^2_{\max}/\mathcal{R}$, which is called the *symmetrized semiring* of $\mathbb{R}_{\max}$. This symmetrization was invented independently by G. Hegedüs [26] and M. Plus [34].

In $\mathbb{S}_{\max}$, there are three kinds of equivalence classes; classes with an element of the form $(a, \mathbb{0})$, identified to $a \in \mathbb{R}_{\max}$, and called *positive*, classes with an element of the form $(\mathbb{0}, a)$ denoted $\ominus a$, called *negative*, classes with a single element (a, a), denoted $a^\bullet$ and called *balanced*, since $a^\bullet \nabla \mathbb{0}$ (for $a = \mathbb{0}$, the three above classes coincide, we will consider $\mathbb{0}$ as both a positive, negative, and balanced element).

We have the decomposition of $\mathbb{S}_{\max}$ in sets of positive, negative, and balanced elements, respectively

$$\mathbb{S}_{\max} = \mathbb{S}^{\oplus}_{\max} \cup \mathbb{S}^{\ominus}_{\max} \cup \mathbb{S}^{\bullet}_{\max} \ .$$

This should be compared with $\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$. For instance, $3 \ominus 2 = 3$, $2 \ominus 3 = \ominus 3$, but $3 \ominus 3 = 3^\bullet$. We say that an element is *signed* if it is positive or negative.

Obviously, if a system $Ax = b$ has a solution, the balance $Ax \nabla b$ has a solution. Conversely if $Ax \nabla b$ has a positive solution x, and if A, b are positive, it is not difficult to see that $Ax = b$. It remains to solve systems of linear balances. The main difficulty is that the balance relation is not transitive. As a result, $x \nabla a$ and $cx \nabla b$ do not imply $ca \nabla b$. However, when x is signed, the implication is true. This allows us to solve linear systems of balances by elimination, when the unknowns are signed.

Theorem 18 (Cramer Formulæ). *Let $A \in (\mathbb{S}_{\max})^{n \times n}$, and $b \in (\mathbb{S}_{\max})^n$. Every signed solution of $Ax \nabla b$ satisfies the Cramer condition $Dx_i \nabla D_i$, where D is the determinant of A and D_i is the i-th Cramer determinant*[5]*. Conversely, if D_i is signed for all i, and if D is signed and nonzero, then $x = (D^{-1} D_i)_{1 \leq i \leq n}$ is the unique signed solution.*

The proof can be found in [34,2]. For the homogeneous system of n linear equations with n unknowns, $Ax \nabla \mathbb{0}$ has a signed non zero solution iff $\det A \nabla \mathbb{0}$ (see [34,18]), which extends a result of Gondran and Minoux (see [22]).

Example 19. Let us solve the taxicab eigenproblem $Ax = 5x$ by elimination in $\mathbb{S}_{\max}$ (A is the matrix shown in Fig 1). We have

$$5^\bullet x_1 \oplus 4x_2 \oplus 7x_4 \ \nabla \ \mathbb{0} \tag{32a}$$
$$4x_1 \ominus 5x_2 \oplus 6x_3 \oplus 3x_4 \ \nabla \ \mathbb{0} \tag{32b}$$
$$4x_2 \ominus 5x_3 \ \nabla \ \mathbb{0} \tag{32c}$$
$$\ominus 5x_4 \ \nabla \ \mathbb{0} \ . \tag{32d}$$

The only signed solution of (32d) is $x_4 = \mathbb{0}$. By homogeneity, let us look for the solutions such that $x_3 = 0$. Then, using (32c), we get $4x_2 \nabla 5x_3 = 5$. Since we search

[5] Obtained by replacing the i-th column of A by b.

a positive x_2, the balance can be replaced by an equality. Thus $x_2 = 1$. It remains to rewrite (32a),(32b): $5^\bullet x_1 \nabla \ominus 5$, $4x_1 \nabla 6^\bullet$, which is true for x_1 positive iff $0 \leq x_1 \leq 2$. The two extremal values give (up to a proportionality factor) the basis eigenvectors already computed in Ex. 19.

Determinants are not so easy to compute in $\mathbb{S}_{\max}$. Butkovič [8] showed that the computation of the determinant of a matrix with positive entries is polynomially equivalent (we have to solve an assignment problem) to the research of an even cycle in a (directed) graph, a problem which is not known to be polynomial. We do not know a non naive algorithm to compute the minor rank (=size of a maximal submatrix with unbalanced determinant) of a matrix in $(\mathbb{R}_{\max})^{n \times p}$. The situation is extremely strange: we have excellent polynomial iterative algorithms [34,18] to find a signed solution of the square system $Ax \nabla b$ when $\det A \neq \mathbb{0}$, but we do not have polynomial algorithms to decide whether $Ax \nabla \mathbb{0}$ has a signed non zero solution (such algorithms would allow us to compute $\det A$ in polynomial time). Moreover, the theory partly collapses if one considers rectangular systems instead of square ones. The conditions of compatibility of $Ax \nabla \mathbb{0}$ when A is rectangular cannot be expressed in terms of determinants [18, Chap. III, 4.2.6].

Historical and Bibliographical Notes

The $(\max, +)$ algebra is not classical yet, but many researchers have worked on it (we counted at least 80), and it is difficult to make a short history without forgetting important references. We will just mention here main sources of inspiration. The first use of the $(\max, +)$ semiring can be traced back at least to the late fifties, and the theory grew in the sixties, with works of Cuninghame-Green, Vorobyev, Romanovskiĭ, and more generally of the Operations Research community (on path algebra). The first enterprise of systematic study of this algebra seems to be the seminal "Minimax algebra" of Cuninghame-Green [15]. A chapter on dioids can be found in Gondran et Minoux [21]. The theory of linear independence using bideterminants, which is the ancester of symmetrization, was initiated by Gondran and Minoux (following Kuntzmann). See [22]. The last chapter of "Operatorial Methods" of Maslov [32] inaugurated the $(\max, +)$ operator and measure theory (motivated by semiclassical asymptotics). There is an "extremal algebra" tradition, mostly in East Europe, oriented towards algorithms and computational complexity. Results in this spirit can be found in the book of U. Zimmermann [44]. This tradition has been pursued, e.g. by Butkovič [7]. The *incline algebras* introduced by Cao, Kim and Roush [10] are idempotent semirings in which $a \oplus ab = a$. The tropical semiring was invented by Simon [39]. A number of language and semigroup oriented contributions are due to the tropical school (Simon, Hashiguchi, Mascle, Leung, Pin, Krob, Weber, $\ldots$). See the survey of Pin in [24], [40,25,29,28], and the references therein. Since the beginning of the eighties, Discrete Event Systems, which were previously considered by distinct communities (queuing networks, scheduling, $\ldots$), have been gathered into a common algebraic frame. "Synchronization and Linearity" by Baccelli, Cohen, Olsder, Quadrat [2] gives a comprehensive account of deterministic and stochastic (max,+) linear discrete event systems, together with recent algebraic results (such as symmetrization). Another recent text is the collection of articles edited by Maslov and Samborskiĭ [31] which is only the most visible part of the (considerable) work of the Idempotent Analysis school. A theory of probabilities in $(\max, +)$ algebra motivated by dynamic programming and large deviations, has been developed by Akian, Quadrat and Viot; and by Del Moral and Salut (see [24]). Recently, the $(\max, +)$ semiring has attracted attention from the linear algebra community (Bapat, Stanford, van den Driessche [3]). A survey with a very complete bibliography is the article of Maslov and Litvinov in [24]. Let us also mention the forthcoming book of Kolokoltsov and Maslov (an earlier version is in Russian [30]). The collection of articles edited by Gunawardena [24] will probably give the first fairly global overview of the different traditions on the subject.

References

1. M. Akian, R.B. Bapat, and S. Gaubert. Asymptotics of the Perron eigenvalue and eigenvector using max algebra. in preparation, 1996.
2. F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. Wiley, 1992.
3. R.B. Bapat, D. Stanford, and P. van den Driessche. Pattern properties and spectral inequalities in max algebra. *SIAM Journal of Matrix Analysis and Applications*, 16(3):964–976, 1995.
4. A. Berman and R.J. Plemmons. *Nonnegative matrices in the mathematical sciences*. Academic Press, 1979.
5. T.S. Blyth and M.F. Janowitz. *Residuation Theory*. Pergamon press, 1972.
6. M. Brilman and J.M. Vincent. Synchronisation by resources sharing : a performance analysis. Technical report, MAI-IMAG, Grenoble, France, 1995.
7. P. Butkovič. Strong regularity of matrices — a survey of results. *Discrete Applied Mathematics*, 48:45–68, 1994.
8. P. Butkovič. Regularity of matrices in min-algebra and its time-complexity. *Discrete Applied Mathematics*, 57:121–132, 1995.
9. P. Butkovič and G. Hegedüs. An elimination method for finding all solutions of the system of linear equations over an extremal algebra. *Ekonomicko-matematicky Obzor*, 20, 1984.
10. Z.Q. Cao, K.H. Kim, and F.W. Roush. *Incline algebra and applications*. Ellis Horwood, 1984.
11. G. Cohen, S. Gaubert, and J.P. Quadrat. Asymptotic throughput of continuous timed petri nets. In *Proceedings of the 34th Conference on Decision and Control*, New Orleans, Dec 1995.
12. G. Cohen, S. Gaubert, and J.P Quadrat. Kernels, images and projections in dioids. In *Proceedings of WODES'96*, Edinburgh, August 1996. IEE.
13. G. Cohen and J.-P. Quadrat, editors. *11th International Conference on Analysis and Optimization of Systems, Discrete Event Systems*. Number 199 in Lect. Notes. in Control and Inf. Sci. Springer Verlag, 1994.
14. J.E. Cohen. Subadditivity, generalized products of random matrices and operations research. *SIAM Review*, 30:69–86, 1988.
15. R.A. Cuninghame-Green. *Minimax Algebra*. Number 166 in Lecture notes in Economics and Mathematical Systems. Springer, 1979.
16. R.A Cuninghame-Green. Minimax algebra and applications. *Advances in Imaging and Electron Physics*, 90, 1995.

17. A. Dembo and O. Zeitouni. *Large Deviation Techniques and Applications*. Jones and Barlett, 1993.
18. S. Gaubert. *Théorie des systèmes linéaires dans les dioïdes*. Thèse, École des Mines de Paris, July 1992.
19. S. Gaubert. Performance evaluation of (max,+) automata. *IEEE Trans. on Automatic Control*, 40(12), Dec 1995.
20. S. Gaubert. On the Burnside problem for semigroups of matrices in the (max,+) algebra. *Semigroup Forum*, 52:271–292, 1996.
21. M. Gondran and M. Minoux. *Graphes et algorithmes*. Eyrolles, Paris, 1979. Engl. transl. *Graphs and Algorithms*, Wiley, 1984.
22. M. Gondran and M. Minoux. Linear algebra in dioids: a survey of recent results. *Annals of Discrete Mathematics*, 19:147–164, 1984.
23. J. Gunawardena. Min-max functions. *Discrete Event Dynamic Systems*, 4:377–406, 1994.
24. J. Gunawardena, editor. *Idempotency*. Publications of the Newton Institute. Cambridge University Press, to appear in 1996 or 1997.
25. K. Hashiguchi. Improved limitedness theorems on finite automata with distance functions. *Theoret. Comput. Sci.*, 72:27–38, 1990.
26. G. Hegedüs. Az extremális sajátvektorok meghatározása eliminációs módszerrel — az általánosított Warshall algorithmus egy javítása. *Alkalmazott Matematikai Lapok*, 11:399–408, 1985.
27. R.M. Karp. A characterization of the minimum mean-cycle in a digraph. *Discrete Maths.*, 23:309–311, 1978.
28. D. Krob and A. Bonnier-Rigny. A complete system of identities for one letter rational expressions with multiplicities in the tropical semiring. *J. Pure Appl. Algebra*, 134:27–50, 1994.
29. H. Leung. Limitedness theorem on finite automata with distance function: an algebraic proof. *Theoret. Comput. Sci*, 81:137–145, 1991.
30. V. Maslov and V. Kolokoltsov. *Idempotent analysis and its applications to optimal control theory*. Nauka, Moskow, 1994. In Russian.
31. V. Maslov and S. Samborskiĭ, editors. *Idempotent analysis*, volume 13 of *Adv. in Sov. Math.* AMS, RI, 1992.
32. V.P. Maslov. *Méthodes Operatorielles*. Mir, Moscou, 1973. French Transl. 1987.
33. P. Moller. *Théorie algébrique des Systèmes à Événements Discrets*. Thèse, École des Mines de Paris, 1988.
34. M. Plus. Linear systems in $(\max,+)$-algebra. In *Proceedings of the 29th Conference on Decision and Control*, Honolulu, Dec. 1990.
35. M.L. Puterman. Markov decision processes. *Handbook in operations research and management science*, 2:331–434, 1990.
36. C. Reutenauer and H. Straubing. Inversion of matrices over a commutative semiring. *J. Algebra*, 88(2):350–360, June 1984.
37. I.V. Romanovskiĭ. Optimization and stationary control of discrete deterministic process in dynamic programming. *Kibernetika*, 2:66–78, 1967. Engl. transl. in Cybernetics 3 (1967).
38. A. Salomaa and M. Soittola. *Automata Theoretic Aspects of Formal Powers Series*. Springer, New York, 1978.
39. I. Simon. Limited subsets of the free monoid. In *Proc. of the 19th Annual Symposium on Foundations of Computer Science*, pages 143–150. IEEE, 1978.
40. I. Simon. On semigroups of matrices over the tropical semiring. *Theor. Infor. and Appl.*, 28(3-4):277–294, 1994.
41. N.N. Vorobyev. Extremal algebra of positive matrices. *Elektron. Informationsverarbeitung und Kybernetik*, 3, 1967. in Russian.

42. E. Wagneur. Moduloids and pseudomodules. 1. dimension theory. *Discrete Math.*, 98:57–73, 1991.
43. Chen Wende, Qi Xiangdong, and Deng Shuhui. The eigen-problem and period analysis of the discrete event systems. *Systems Science and Mathematical Sciences*, 3(3), August 1990.
44. U. Zimmermann. *Linear and Combinatorial Optimization in Ordered Algebraic Structures*. North Holland, 1981.