Neuvième réunion du groupe de travail

ALGÈBRES TROPICALES ...

GdR/PRC ALP et GdR/PRC Automatique
sur le thème

Systèmes linéaires positifs

Mercredi 15 Dec 1999.

salle U-V, (niveau -2, passage rouge), Departement de Mathematiques et d'Informatique, ENS, 45 rue d'Ulm, 75005 Paris

Nonnegative Realizations

Lorenzo Farina
Dipartimento di Informatica e Sistemistica
Universita' di Roma ''La Sapienza''
Via Eudossiana, 18
00184 Roma, Italy
e-mail: farina@dis.uniroma1.it

Positive System: *Definition*

A positive system is a system in which the state variables are always positive (or at least nonnegative) in value.

D.G. Luenberger, Positive linear systems, Chapter 6 in: *Introduction to dynamic systems*, J. Wiley & Sons, New York, 1979

- Basically, the interest for such systems is motivated by the fact that state variables may have a physical meaning provided that they are nonnegative.

- With respect to the nonnegativity constraint, two approaches are possible :
  - only trajectories in the positive orthant are meaningful
  - the model is consistent, it is a **positive system**

- For **discrete-time** LTI homogeneous systems:

$$x(t+1) = Ax(t) \Leftrightarrow A \geq \mathbf{0}$$

- For **continuous-time** LTI homogeneous systems:

$$\dot{x}(t) = Ax(t) \Leftrightarrow A \geq_{\mathrm{e}} \mathbf{0}$$

(i.e. all off-diagonal entries of A are nonnegative)

- In view the above definition of positive systems, the analysis reduce to that of nonnegative (or essentially nonnegative) matrices.

- Nonnegative matrices is the subject of many books and a huge number of results are available in the literature.

- In most cases, however, also the **input** and the **output** take nonnegative values

- For example, in population dynamics, the input may represent immigration and the output the total population

- Consequently, we shall consider the following definition

POSITIVE SYSTEMS

A positive system is a system in which the state, input and output variables are always positive (or at least nonnegative) in value.

- Questions arising when considering inputs and/or outputs (such as reachability, observability, realizability...) are the **subject of research** in positive systems theory

Examples of Positive Systems:

Age-Structured Population (Leslie Model)

- The time t is discrete and denotes the reproduction season

- The state variables x_i represent the number of females, at the beginning of year t, of age 1,2,...,n

- The ageing process of the population may be described by

$$x_{i+1}(t+1) = s_i x_i(t) \qquad i = 1, \ldots, n-1$$

where s_i is the survival rate at age i

- The reproduction process may be described by

$$x_1(t+1) = s_0\big(f_1 x_1(t) + \ldots + f_n x_n(t)\big)$$

where f_i is the fertility rate at age i

- The dynamic matrix, known as **Leslie matrix**, is of the form

$$A = \begin{pmatrix} s_0 f_1 & s_0 f_2 & \dots & s_0 f_{n-1} & s_0 f_n \\ s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & s_{n-1} & 0 \end{pmatrix}$$

- The input to the Leslie system may represent immigration and the output the total population

- Since the s_i's and f_i's are positive then the Leslie systems are positive systems

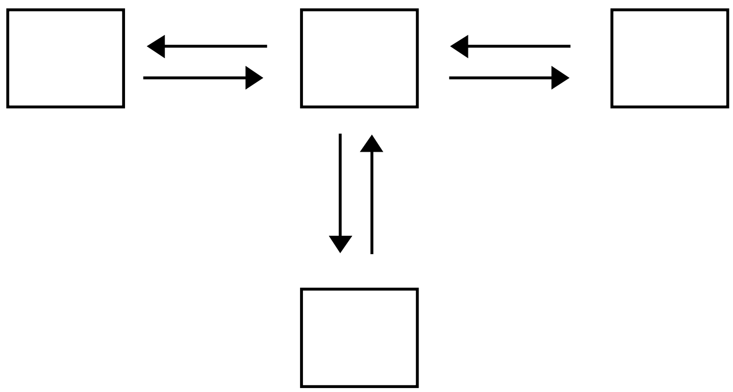
- The estimated survival and fertility rates for the first ten age groups of four animals are reported next

	s_0 / f_1	s_1 / f_2	s_2 / f_3	s_3 / f_4	s_4 / f_5	s_5 / f_6	s_6 / f_7	s_7 / f_8	s_8 / f_9	s_9 / f_{10}
fish	$6 \cdot 10^{-5}$	0.45	0.27	0.26	0.26	0.25	0.25	0.25	0.25	0.25
	---	5000	11000	18000	24000	31000	34000	41000	45000	46000
bird	0.50	0.80	0.36	0.37	0.38	0.39	0.39	0.38	0.38	0.37
	---	0.40	0.45	0.50	0.50	0.50	0.50	0.50	0.50	0.50
deer	0.70	0.92	0.48	0.49	0.48	0.42	0.28	0.25	0.22	0.20
	---	---	0.10	0.40	0.50	0.50	0.50	0.50	0.45	0.40
squirrel	0.40	0.24	0.30	0.33	0.34	0.33	0.30	0.28	0.24	0.27
	0.6	1.2	1.9	1.9	1.9	1.9	1.9	1.9	1.8	1.6

Examples of Positive Systems:
Compartmental Models

- The time t is continuous

- There is a set of interconnected regions



  the arrows indicate flow of material between regions (compartments)

- Each region is taken to contain a quantity of some material which passes from one compartment to another over time

- The compartments may correspond to actual entities (such as the bloodstream or gastrointestinal tract) or may represent only convenient mathematical fictions

- The major field of use of compartmental system analysis is pharmacokinetics and in tracer experiments

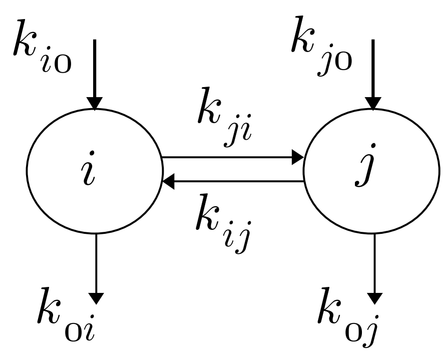
- Let x_i denote the amount of material in the i-th compartment. The mass balance equation is

$$\dot{x}_i(t) = \text{inflow rate} \ - \ \text{outflow rate}$$

which may be described by a first order dynamic system

$$\dot{x}_i(t) = f_{i0} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(k_{ij} x_j(t) - k_{ji} x_i(t) \right) - k_{0i} x_i(t)$$

where k_{ij} is the mass flow rate to compartment i from compartment j, with subscript 0 denoting the environment



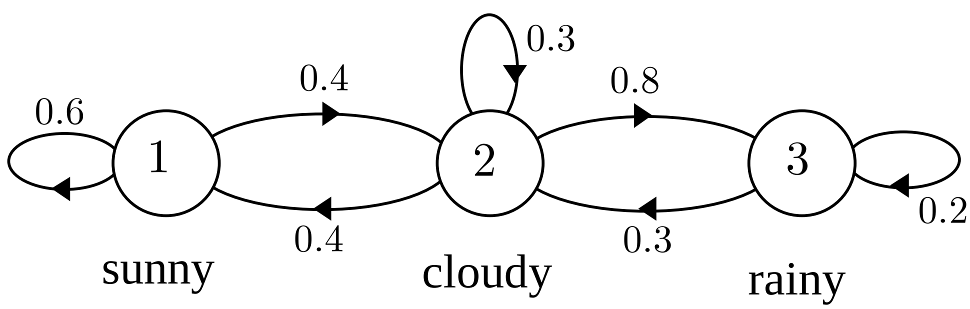
- Since the k_{ij}'s are positive, then compartmental systems are positive systems

Examples of Positive Systems:
Hidden Markov Models

- Suppose a system can be described at any time t as being in one of a set of N distinct states $X_1,...,X_N$.

- At regularly spaced discrete times, the system undergoes a change of state according to a set of probabilities associated with the state which may be described by a transition probability matrix A

$$\Pi_{k+1} = A\Pi_k$$
$$a_{ij} = \Pr\left(X_{k+1} = i \mid X_k = j\right)$$

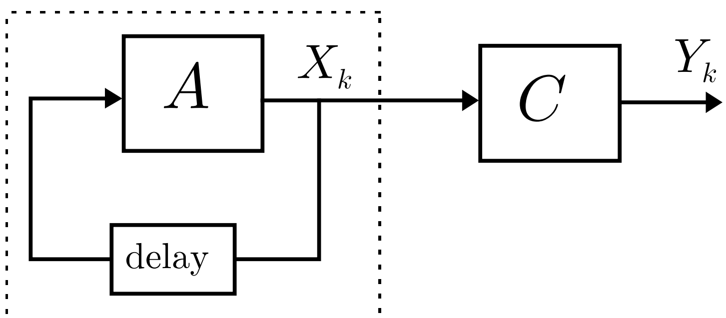


$$A = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.3 & 0.3 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

- Suppose the obsvervation is a probabilistic function of the state, *i.e.* the underlying stochastic process is not directly observable (it is hidden)

- The measurement process may be described by a transition probability matrix C

$$\boldsymbol{\sigma}_k = C\,\Pi_k$$
$$c_{mi} = \Pr\left(Y_k = m \mid X_k = i\right)$$



- Since the transition probabilities are positive, then the hidden Markov models are positive systems

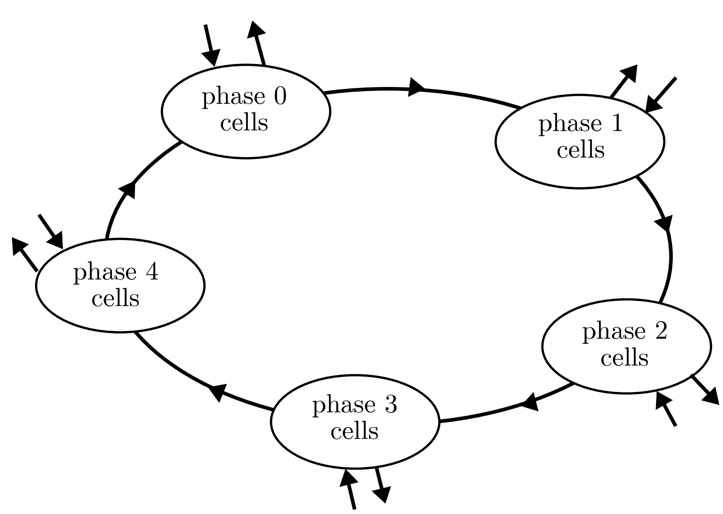
Examples of Positive Systems: *Charge-Routing Networks*

- A charge-routing network is a MOS integrated-circuit chip for discrete-time signal processing

- The admissible charge cells operations: *storage*, *injection*, *tranfer*, *splitting*, *addition*, *extraction*

- It is possible to produce a current proportional to a weighted sum of the packet size using nondestructive sensing

- Charge transfer is controlled by a clock whose period is divided into p equal phases



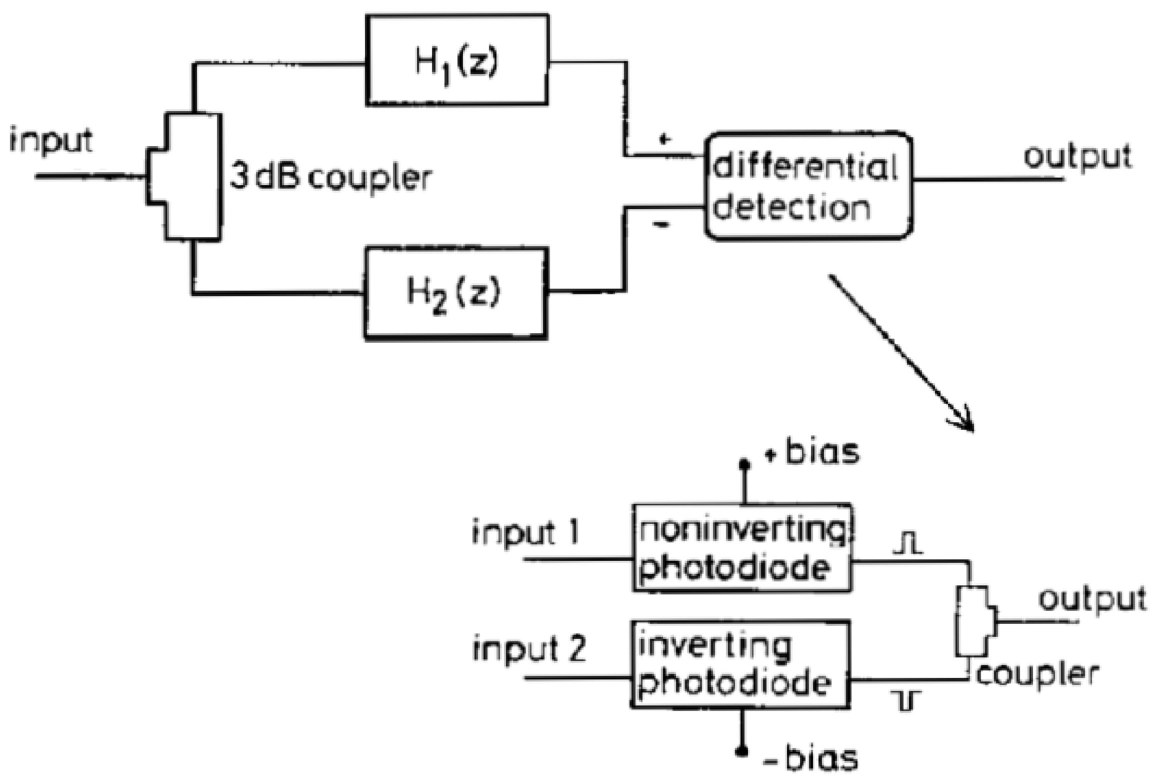
- Every cell in the network can be uniquely classified as
  - a *source cell* which receives a new charge input to the network
  - a *sink cell* whose charge is released from the network
  - an *internal cell* that is neither source nor sink

- A charge-routing network may be described by a linear discrete-time dynamic routing scheme

$$\begin{aligned} x_{i+1}(t+1) &= A_i x_i(t) + B_i y_i(t) \\ w_{i+1}(t+1) &= C_i x_i(t) + D_i y_i(t) \end{aligned} \quad t = i \pmod{p}$$

  where x_i, y_i and w_i represent the the size of the charge packets contained in internal, source and sink cells

- Since the charge transfer coefficients are nonnegative, a charge-routing network is a positive system



input
3 dB coupler
$H_1(z)$
$H_2(z)$
+
-
differential detection
output
+ bias
input 1
noninverting photodiode
input 2
inverting photodiode
output
coupler
- bias

Basic Results on Nonnegative Matrices

- A nonnegative matrix A is reducible if it can be written, by some reordering of the state variable, as

$$A = \begin{pmatrix} B & \mathbf{0} \\ C & D \end{pmatrix}$$

where B and D are square matrices. Otherwise A is irreducible.

- Any nonnegative matrix A can be reduced, by a suitable reordering or the state variables, to a triangular block form

$$A = \begin{pmatrix} A_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ A_{21} & A_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ A_{s1} & A_{s2} & & A_{ss} \end{pmatrix}$$

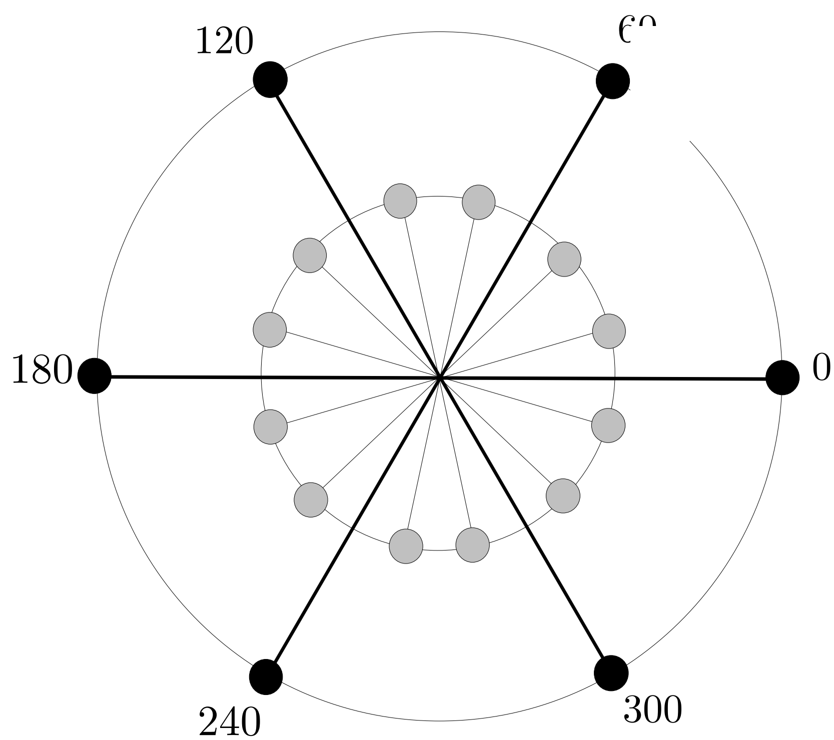
where each block A_{ii} is square and irreducible.

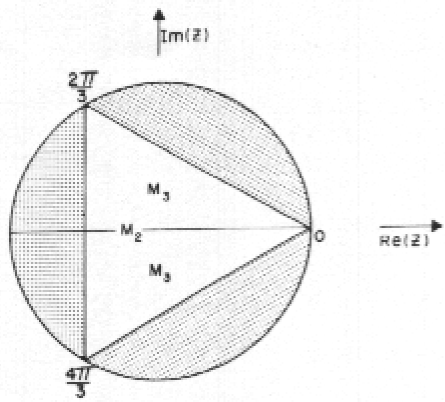
- If A is nonnegative irreducible then:

  1. $\rho(A)$ is a simple positive eigenvalue and the corresponding eigenvector is strictly positive. Moreover, no other eigenvector is nonnegative

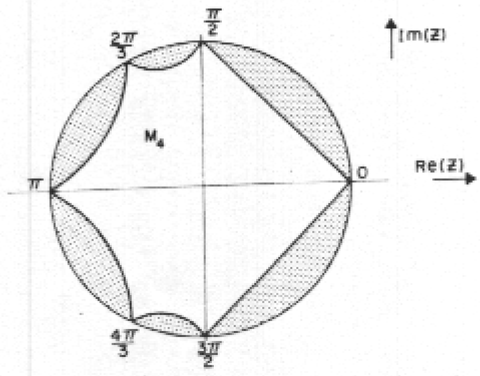
  2. if A has h eigenvalues of maximum modulus, then these numbers are distinct roots of $\lambda^h - \rho^h = 0$

  3. the whole spectrum of A goes over into itself under a rotation of the complex plane by $2\pi/h$

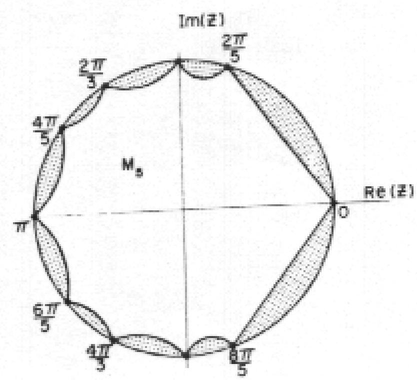



Im(z)
2π/3
M3
M2
O
Re(z)
M3
4π/3

(a)


π/2
Im(z)
2π/3
M4
Re(z)
π
O
4π/3
3π/2

(b)

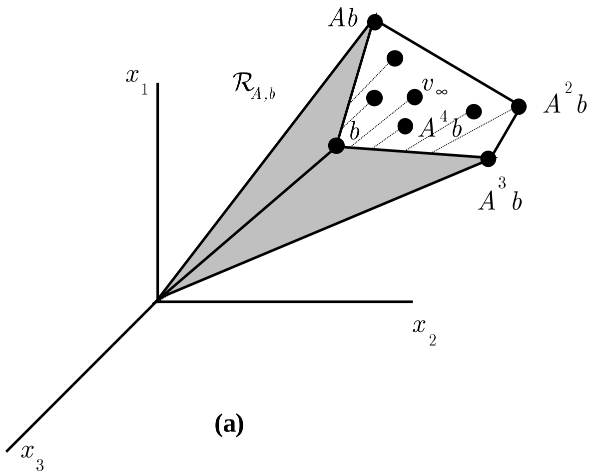

Im(z)
2π/5
2π/3
4π/5
M5
Re(z)
π
O
6π/5
4π/3
8π/5

(c)

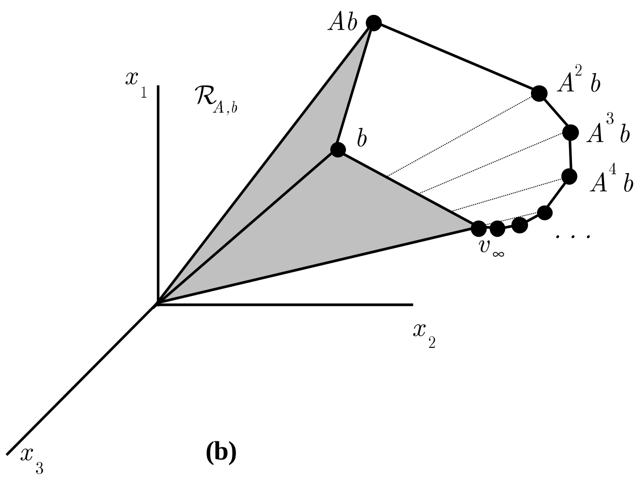
Existence of a Positive Realization of LTI Discrete-time Systems: *Geometric Conditions*

- The *reachability cone*:

$$\mathcal{R}_{A,b} = \text{cone}\big(b, Ab, A^2b, \ldots\big)$$



(a)



(b)

- The *observability cone* :

$$\mathcal{O}_{A,c^T} = \left\{ x : c^T A^{k-1} x \geq 0,\, k = 1,2,\ldots \right\}$$

- A duality property holds

$$\mathcal{R}^*_{A,b} = \mathcal{O}_{A^T,b^T}$$
$$\mathcal{O}^*_{A,c^T} = \mathcal{R}_{A^T,c}$$

where $\mathcal{K}^* \doteq \left\{ y : x^T y \geq 0,\, \forall x \in \mathcal{K} \right\}$

Theorem. $H(z) = c^T (zI - A)^{-1} b$ *is positive realizable if and only if there exists a polyhedral convex cone* $\mathcal{P}$ *such that*

$$(i)\quad A\mathcal{P} \subset \mathcal{P}$$
$$(ii)\quad \mathcal{R} \subset \mathcal{P} \subset \mathcal{O}$$

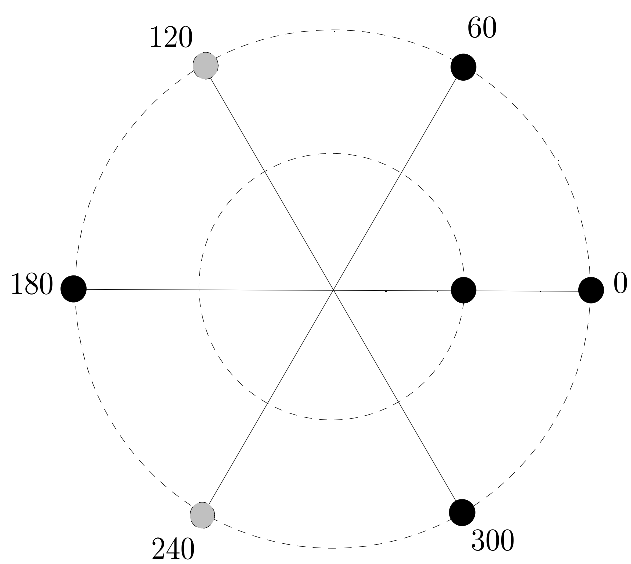
H. Maeda and S. Kodama, Reachability, observability and realizability of linear systems with positive constraints, *IECE Trans.* **63-A** (1980) 688-694 (in Japanese)

- $AP = PA_+ \quad b = Pb_+ \quad c_+^T = c^T P$

with $\mathcal{P} = \mathrm{cone}(P)$, yields a positive realization

Existence of a Positive Realization of LTI Discrete-time Systems: *Input-Output Conditions*

Definition. A transfer function with nonnegative impulse response is *cyclic of index r* if its maximum modulus poles are a subset of those which are allowed eigenvalues of maximum modulus of a nonnegative matrix of size r, with r minimal. If $r = 1$, then it will be said to be *primitive*.



$$r = l.c.m.(2,6) = 6$$

- $h^{(0)}(k) = c^{(0)T} A^{(0)k-1} b^{(0)}$ cyclic of index $r^{(0)}$:

$$
\begin{aligned}
&h^{(0,1)}(k) = h^{(0)}\left(1 + (k-1)r^{(0)}\right) \\
&h^{(0,2)}(k) = h^{(0)}\left(2 + (k-1)r^{(0)}\right) \\
&\vdots \\
&h^{\left(0,r^{(0)}\right)}(k) = h^{(0)}\left(r^{(0)} + (k-1)r^{(0)}\right) = h^{(0)}\left(kr^{(0)}\right)
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{array}{lll}
A^{(0,1)} = \left[A^{(0)}\right]^{r^{(0)}} & b^{(0,1)} = b^{(0)} & c^{(0,1)^T} = c^{(0)T} \\
A^{(0,2)} = \left[A^{(0)}\right]^{r^{(0)}} & b^{(0,2)} = A^{(0)} b^{(0)} & c^{(0,2)^T} = c^{(0)T} \\
\vdots & \vdots & \vdots \\
A^{\left(0,r^{(0)}\right)} = \left[A^{(0)}\right]^{r^{(0)}} & b^{\left(0,r^{(0)}\right)} = \left[A^{(0)}\right]^{r^{(0)}-1} b^{(0)} & c^{(0,1)^T} = c^{(0)T}
\end{array}
$$

- $n^{(0,i_0)} < n^{(0)} \Rightarrow$ finite number of levels

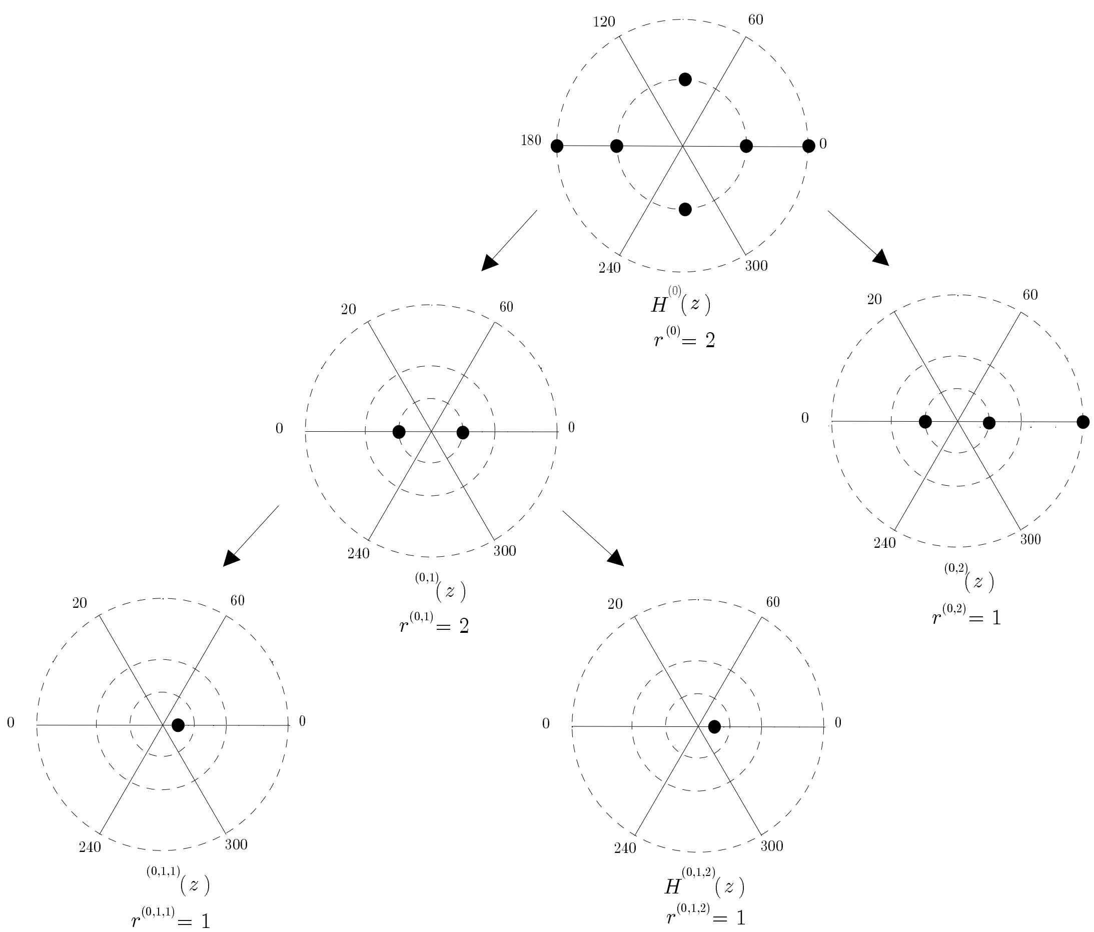
- It's a tree-like structure ...

$$
\begin{array}{ccccc}
 & & H^{(0)}(z) & & \\
 & \swarrow & \downarrow & \searrow & \\
H^{(0,1)}(z) & & H^{(0,2)}(z) & & H^{(0,3)}(z)
\end{array}
$$

... having primitive leaves

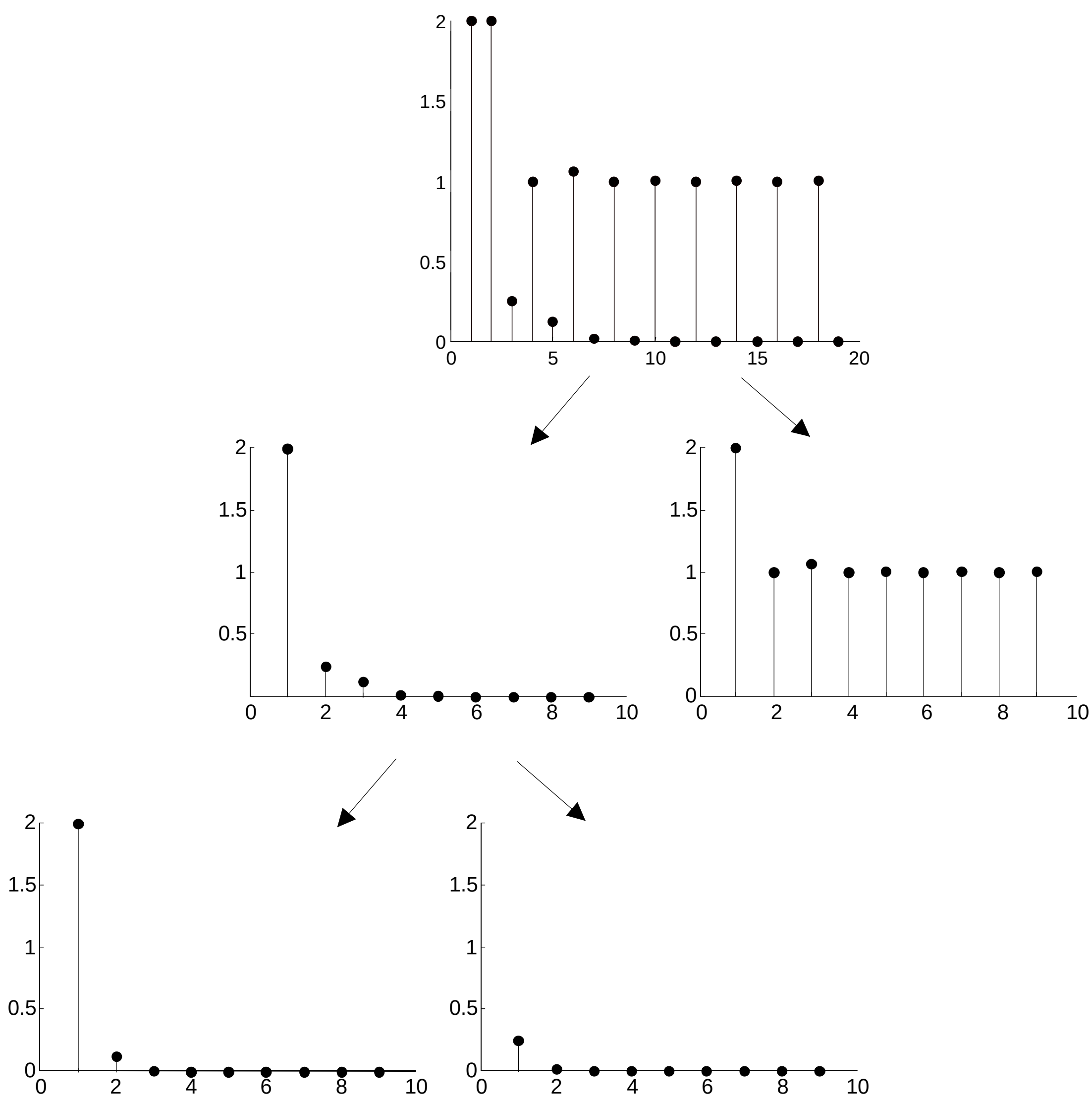
Example.

$$H^{(0)}(z) = \frac{2z^5 + 2z^4 - 1.75z^3 - z^2 - 0.25z - 0.0625}{z^6 - z^4 - 0.0625z^2 + 0.0625}$$



$$H^{(0,1)}(z) = \frac{2z + 0.25}{z^2 - 0.0625} \qquad H^{(0,2)}(z) = \frac{2z^2 - z - 0.0625}{z^3 - z^2 - 0.0625z + 0.0625}$$

$$H^{(0,1,1)}(z) = \frac{2}{z - 0.0625} \qquad H^{(0,1,2)}(z) = \frac{0.25}{z - 0.0625}$$



- $H(z)$ primitive $\Leftrightarrow$ $\rho_{H(z)}$ unique (possibly multiple)

Theorem. *If*

$$(i)\ \ h(k) \geq 0$$
$$(ii)\ \ H(z) \text{ is primitive}$$

then $H(z)$ *is positively realizable.*

B.D.O. Anderson, M. Deistler, L. Farina and L. Benvenuti, Nonnegative realization of linear systems with nonnegative impulse response, *IEEE Trans. CAS-I* 43 (1996) 134-142

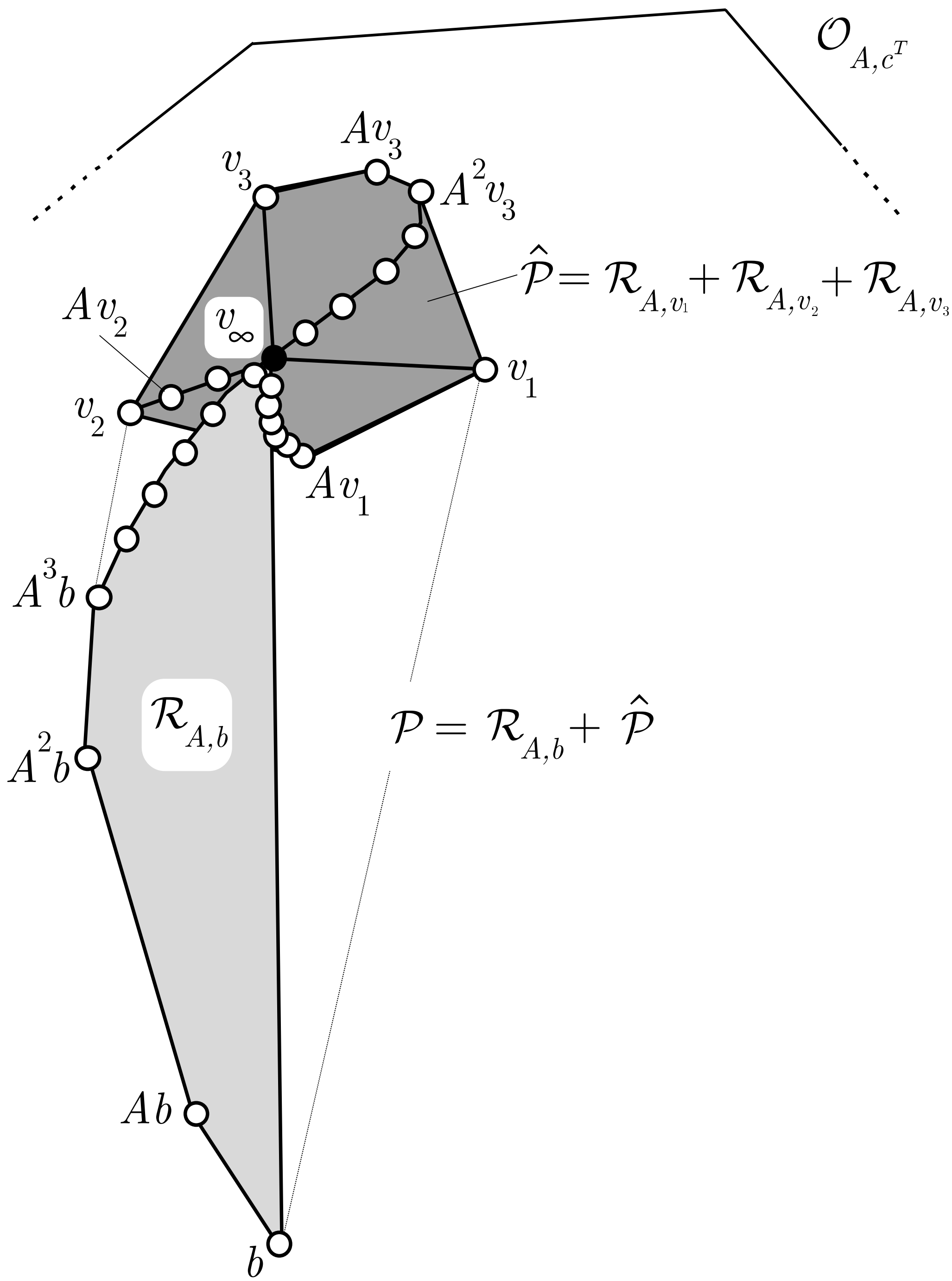
- The theorem has been also extended in [Anderson *et al.*, 1996] to the case in which

$$\liminf_{k\to\infty} \rho_{H^{(0)}(z)}{}^{-k} h^{(0)}(k) > 0 \underset{\not\Leftarrow}{\Rightarrow} \text{primitivity of } H^{(0,i_0)}(z)$$

$$\Updownarrow$$

$$\lim_{k\to\infty} \frac{c^{(0)T}\left[A^{(0)}\right]^{i+(k-1)r^{(0)}} b^{(0)}}{\left\| c^{(0)T}\left[A^{(0)}\right]^{i+(k-1)r^{(0)}} b^{(0)} \right\|} > 0, \quad \forall i$$

- **The proof is constructive!**



$\mathcal{O}_{A,c^T}$
Av_3
v_3
A^2v_3
$\hat{\mathcal{P}} = \mathcal{R}_{A,v_1} + \mathcal{R}_{A,v_2} + \mathcal{R}_{A,v_3}$
Av_2
v_∞
v_1
v_2
Av_1
A^3b
$\mathcal{R}_{A,b}$
$\mathcal{P} = \mathcal{R}_{A,b} + \hat{\mathcal{P}}$
A^2b
Ab
b

Theorem. *$H(z)$ is positively realizable if and only if*

$$(i)\ h^{(0)}(k) \geq 0$$
$$(ii)\ H^{\left(0,i_0,i_1,\ldots,i_q\right)}(z) \text{ are cyclic}$$

L. Farina, On the existence of a positive realization, *Systems & Control* Letters 28 (1996) 219-226

- Condition (ii) is equivalent to primitivity of the leaves

Corollary. *If*

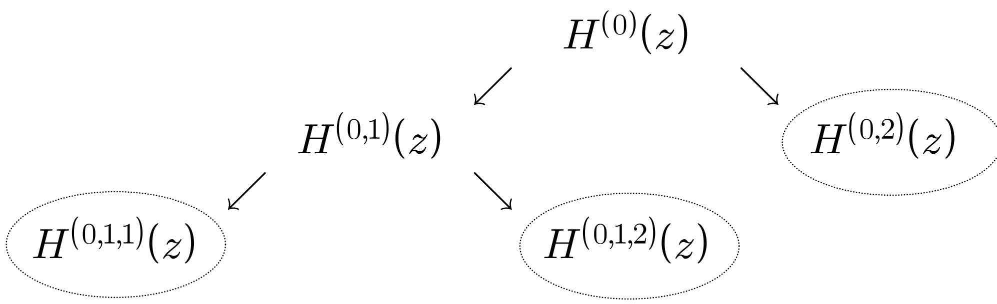
(i) $h(k) \geq 0$

(ii) *each pole has a polar angle which a rational multiple of* π.

then $H(z)$ is positively realizable

Example. (reprise)

$$H^{(0)}(z) = \frac{2z^5 + 2z^4 - 1.75z^3 - z^2 - 0.25z - 0.0625}{z^6 - z^4 - 0.0625z^2 + 0.0625}$$



$$H^{(0,2)}(z) = \frac{2z^2 - z - 0.0625}{z^3 - z^2 - 0.0625z + 0.0625}$$

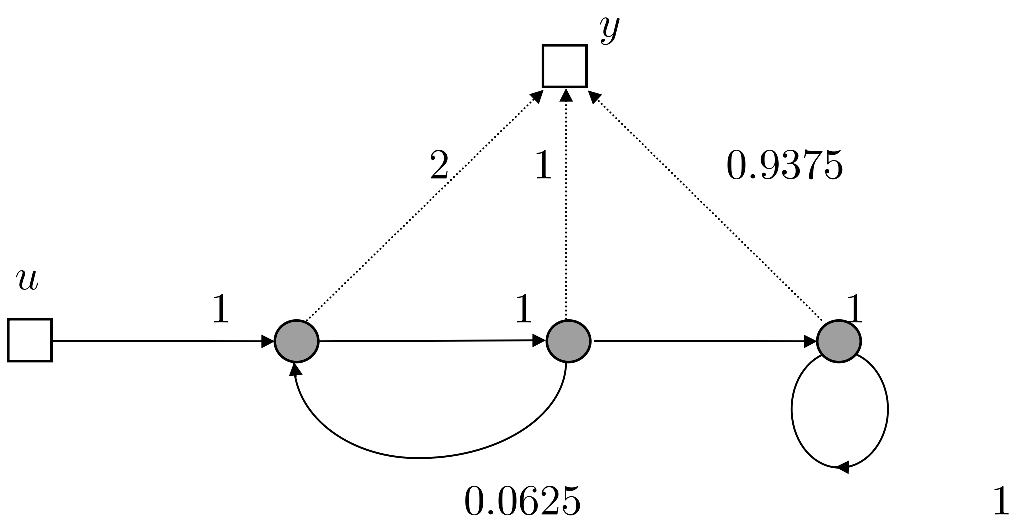
$$H^{(0,1,1)}(z) = \frac{2}{z - 0.0625}$$

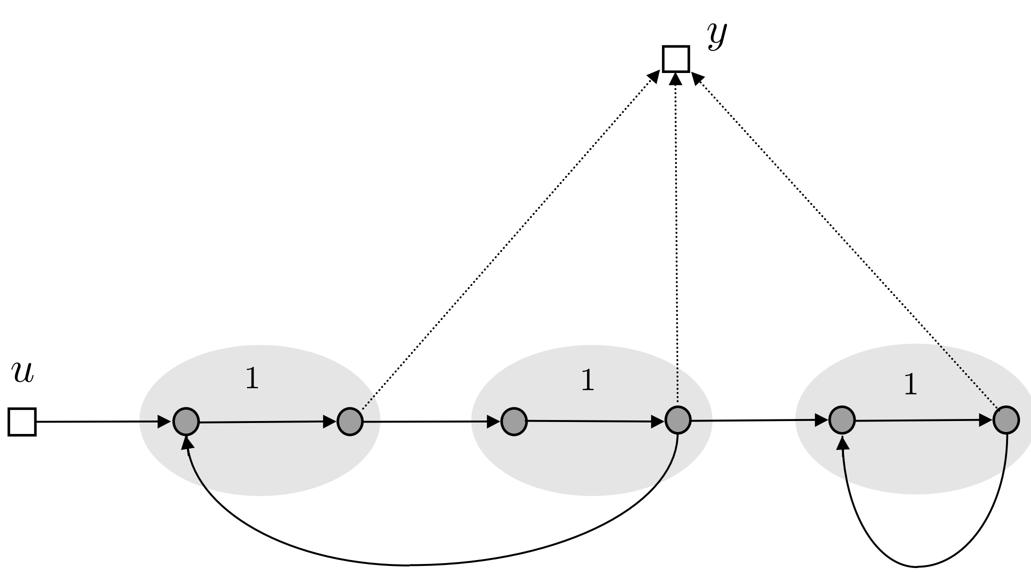
$$H^{(0,1,2)}(z) = \frac{0.25}{z - 0.0625}$$

$$F_+^{(0,2)} = \begin{pmatrix} 0 & 0.0625 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad g_+^{(0,2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad h_+^{(0,2)} = \begin{pmatrix} 2 \\ 1 \\ 0.9375 \end{pmatrix}$$

$$F_+^{(0,1,1)} = 0.0625 \qquad g_+^{(0,1,1)} = 1 \quad h_+^{(0,1,1)} = 2$$

$$F_+^{(0,1,2)} = 0.0625 \qquad g_+^{(0,1,2)} = 1 \quad h_+^{(0,1,2)} = 0.25$$





$$A_+^{(0,2)} = \begin{pmatrix} 0 & 0 & 0 & 0.0625 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad b_+^{(0,2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad c_+^{(0,2)} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0.9375 \end{pmatrix}$$

$$A_+^{(0,1,1)} = \begin{pmatrix} 0 & 0 & 0 & 0.0625 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad b_+^{(0,1,1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad c_+^{(0,1,1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$A_+^{(0,1,2)} = \begin{pmatrix} 0 & 0 & 0 & 0.0625 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad b_+^{(0,1,2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad c_+^{(0,1,2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.25 \end{pmatrix}$$

$$A = \begin{pmatrix} A_+^{(0,2)} & 0 & 0 \\ 0 & A_+^{(0,1,1)} & 0 \\ 0 & 0 & A_+^{(0,1,2)} \end{pmatrix} \quad b = \begin{pmatrix} b_+^{(0,2)} \\ b_+^{(0,1,1)} \\ b_+^{(0,1,2)} \end{pmatrix} \quad c = \begin{pmatrix} c_+^{(0,2)} \\ c_+^{(0,1,1)} \\ b_+^{(0,1,2)} \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.0625 & 0 & 0 & 0 & 0.9375 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} 2 \\ 2 \\ 0.25 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad h = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The Positive Realization Problem

(LTI SISO discrete-time systems)

• Let $\left\{F, g, h^T\right\}$ be any minimal realization of a prescribed n-th order transfer function

$$W(z) = h^T (zI - F)^{-1} g$$

(*i*) Is there a positive realization A_+, b_+, c_+^T (*i.e.* $A_+ \in \mathbb{R}_+^{N \times N}$, b_+, $c_+ \in \mathbb{R}_+^N$) of some finite dimension N?

(*ii*) If so, how may it be found?

(*iii*) What is the minimal value for N over all realizations?

(*iv*) Is there a set of realizations, and how are members of the set related, especially those of minimal dimension?

• The system $\left\{A_+, b_+, c_+^T\right\}$ is a *positive system*, in fact

$$x(k) \geq 0 \text{ and } y(k) \geq 0 \text{ for any } u(k) \geq 0,\ k \geq 0$$

$$\Updownarrow$$

$$A_+ \in \mathbb{R}_+^{N \times N},\ b_+, c_+ \in \mathbb{R}_+^N$$

and $w(k) = h^T F^{k-1} g,\ k = 1, 2, \ldots$ is nonnegative for all $k \geq 0$.

• From nonnegativity of the impulse response one can immediately derive the following:

(1) One of the dominant poles of $W(z)$, say λ_1, is positive

(2) The residue r_1 associated to λ_1 is positive

• A general sistematic finite procedure to check non-negativity of $w(k)$ is not known.

- *Theorem.* (*B.D.O Anderson et al*, *1996*)

If

(1) The impulse response function $w(k)$ is nonnegative for all $k \geq 0$

(2) The dominant real pole of $W(z)$ is unique (possibly multiple)

then $W(z)$ has a positive realization.

• In this talk we shall give a partial answer to the question (*iii*), *i.e.* to the minimality problem.

• We shall give necessary and sufficient conditions for a given third order transfer function $W(z)$ with distinct positive real poles, to be realizable as a positive system of the same order.

• Such conditions are easily testable and the proof also provides a tool for constructing a positive realization when existing.

Preliminary Results

- A set $\mathcal{K}$ is said to be a *cone* provided that $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$.

- If $\mathcal{K}$ contains an open ball of $\mathbb{R}^n$ then $\mathcal{K}$ is said to be *solid*.

- If $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$ then $\mathcal{K}$ is said to be *pointed*.

- A cone $\mathcal{K}$ which is closed, convex, solid and pointed is a *proper cone*

- A cone $\mathcal{K}$ is said to be *polyhedral* if it is expressible as the intersection of a finite family of closed half-spaces.

- The notation cone$(v_1, \dots, v_M)$ indicates the polyhedral closed convex cone consisting of all finite non-negative linear combinations of vectors $v_1, \dots, v_M$, the vectors v_i will be called the *generators* of the cone.

• Theorem. *(Maeda and Kodama, 1980)*

Let $\{F, g, h^T\}$ be any minimal realization of $W(z)$. Then, $W(z)$ has a positive realization if and only if there exists a polyhedral proper cone $\mathcal{K}$ such that

(1) $F\mathcal{K} \subset \mathcal{K}$, *i.e.* $\mathcal{K}$ is F-invariant;

$\Leftrightarrow FK = KA_+, \quad A_+ \geq 0$ with $\mathcal{K} = \text{cone}\,(K)$.

(2) $\mathcal{K} \subset \mathcal{O}$

(3) $g \in \mathcal{K}$

where

$$\mathcal{O} = \{x \mid h^T F^k x \geq 0, k = 0, 1, \dots\}$$

is called the *observability cone*.

• Conditions (1-3) will be called the *MK* conditions

• A positive realization $\{A_+, b_+, c_+^T\}$ with $A_+ \in \mathbb{R}_+^{N \times N}$, b_+, $c_+ \in \mathbb{R}_+^N$ is obtained by solving

$$FK = KA_+, \qquad g = Kb_+, \qquad c_+^T = h^T K$$

where $\mathcal{K} = \text{cone}\,(K)$ has N generators, $K \in \mathbb{R}_+^{3 \times N}$

• Positive realizations of minimal order correspond to cones $\mathcal{K}$ with minimal number of generators satisfying the *MK* conditions.

• Consider the case of a third order transfer function with positive real poles $1 = \lambda_1 > \lambda_2 > \lambda_3$

$$W(z) = \frac{1}{z-1} + \frac{r_2}{z-\lambda_2} + \frac{r_3}{z-\lambda_3}$$

and its Jordan canonical realization

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad g = \begin{pmatrix} 1 \\ r_2 \\ r_3 \end{pmatrix} \quad h = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$W(z) = h^T (zI - F)^{-1} g$$

• One can prove that w.l.o.g.

(1) $\lambda_1 = 1$

(2) $r_1 = 1$

• It is possible to reformulate the *MK* conditions on a plane as follows

(1) $F^*\mathcal{P} \subset \mathcal{P}$ where

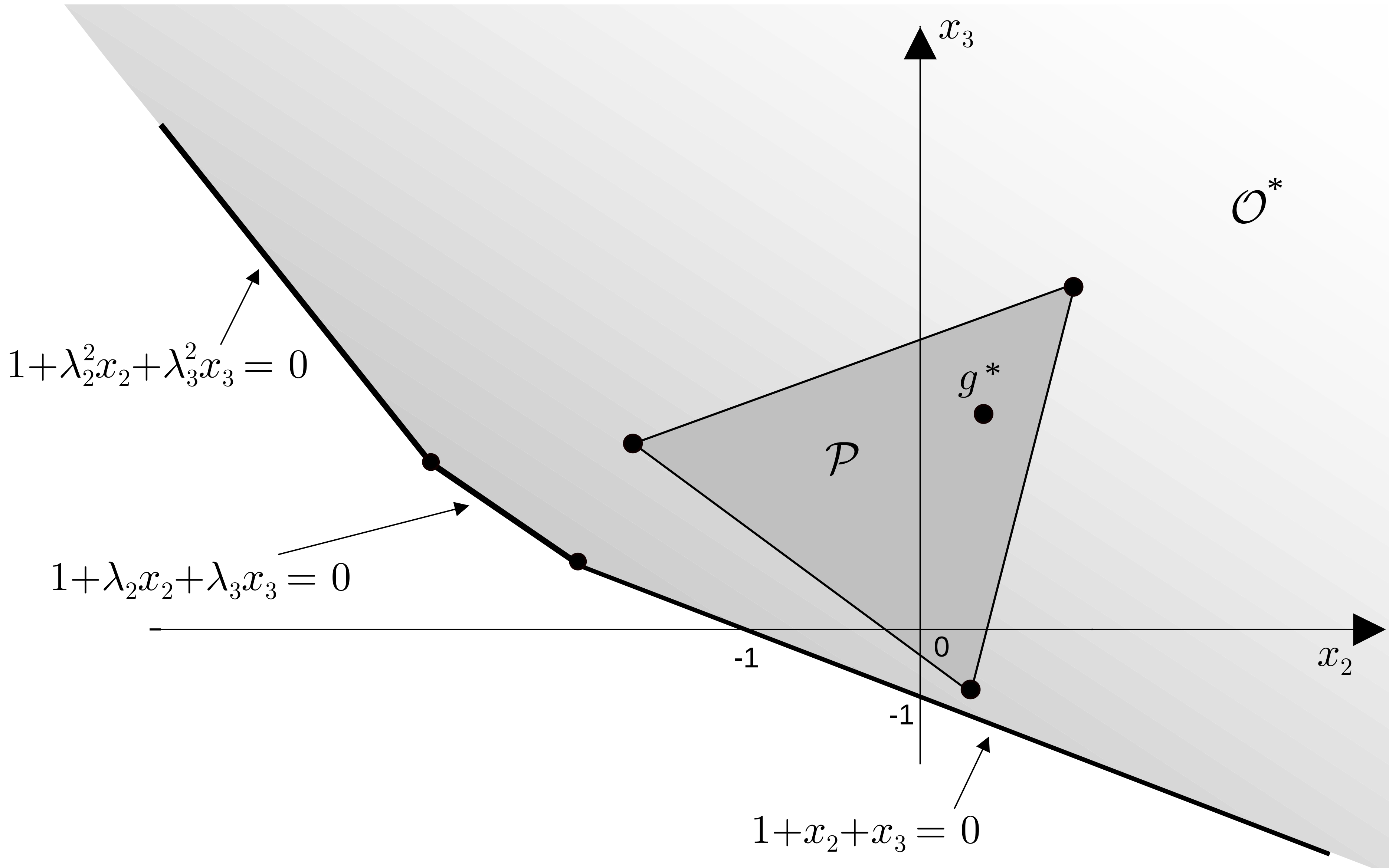
$$F^* = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix}$$

(2) $\mathcal{P} \subset \{x_2, x_3 : 1 + \lambda_2^k x_2 + \lambda_3^k x_3\} := \mathcal{O}^*$

(3) $g^* := \begin{pmatrix} r_2 \\ r_3 \end{pmatrix} \in \mathcal{P}$

where $\mathcal{P} = \text{conv}\,(P)$ is an F^*-invariant polytope in the $\{x_2, x_3\}$ plane

• Conditions (1-3) will be called the *MK** conditions



x_3
$\mathcal{O}^*$
$1+\lambda_2^2 x_2+\lambda_3^2 x_3 = 0$
g^*
$\mathcal{P}$
$1+\lambda_2 x_2+\lambda_3 x_3 = 0$
0
-1
x_2
-1
$1+x_2+x_3 = 0$

Our problem is:

• Given a third order transfer function with positive real poles $1 = \lambda_1 > \lambda_2 > \lambda_3$

$$W(z) = \frac{1}{z-1} + \frac{r_2}{z-\lambda_2} + \frac{r_3}{z-\lambda_3}$$

find a positive realization with a state space of dimension 3.

$$\Updownarrow$$

• Given the set $\mathcal{O}^*$ and the vector g^*, find an F^*-invariant polytope $\mathcal{P}$ contained in $\mathcal{O}^*$ and containing g^* in the $\{x_2, x_3\}$ plane

• We define a one parameter family of F^*-invariant politopes $P_M(\alpha)$ as follows

$$P_M(\alpha) = D \begin{pmatrix} -\dfrac{1-\alpha}{\lambda_2-\alpha} & -1 & -\dfrac{\lambda_2-\beta(\alpha)}{1-\beta(\alpha)} \\ \dfrac{1-\alpha}{\lambda_3-\alpha} & 1 & \dfrac{\lambda_3-\beta(\alpha)}{1-\beta(\alpha)} \end{pmatrix}$$

$$:= (v_1(\alpha), v_2, v_3(\beta(\alpha)))$$

where

$$D = \begin{pmatrix} \dfrac{1-\lambda_3}{\lambda_2-\lambda_3} & 0 \\ 0 & \dfrac{1-\lambda_2}{\lambda_2-\lambda_3} \end{pmatrix}$$

with

$$\bar{\alpha} \leq \alpha < \lambda_3$$

$$\bar{\alpha} = \max\left\{\frac{1+\lambda_2+\lambda_3-2\sqrt{(\lambda_2-\lambda_3)^2+(1-\lambda_2)(1-\lambda_3)}}{3}, 0\right\}$$

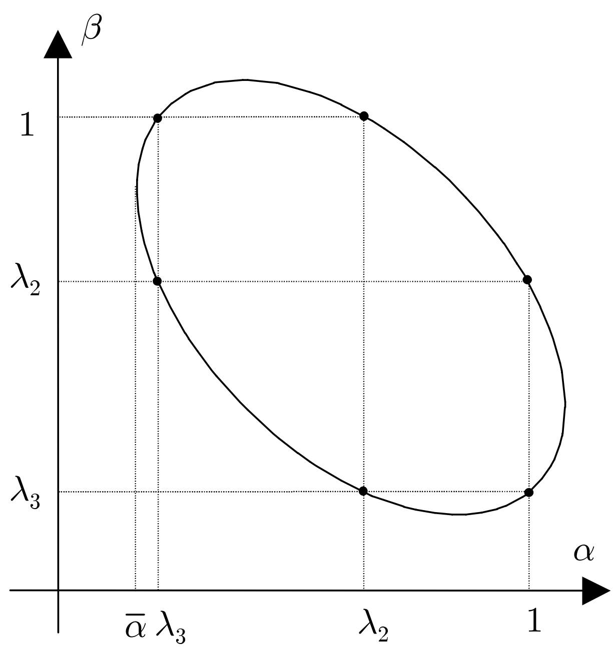
and $\beta(\alpha)$ such that

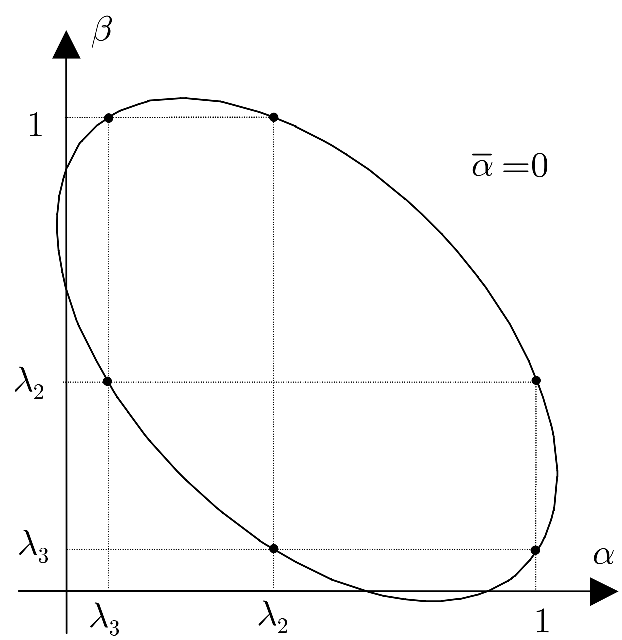
$$\alpha^2 + \beta(\alpha)^2 + \alpha\beta(\alpha) +$$

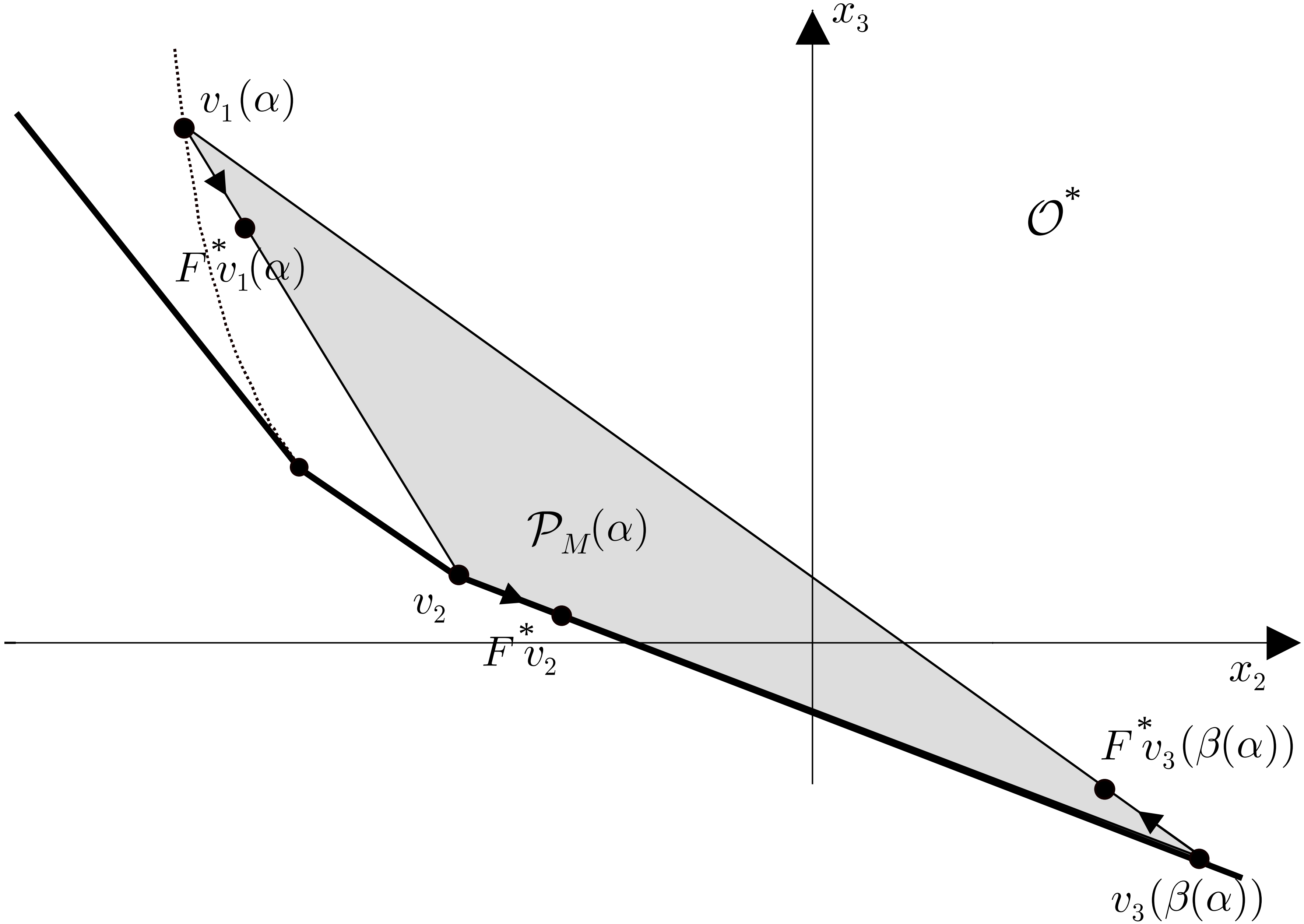
$$+ (\alpha + \beta(\alpha))(1 + \lambda_2 + \lambda_3) - \lambda_2 - \lambda_3 - \lambda_2\lambda_3 = 0$$

with

$$\lambda_2 \leq \beta(\alpha) < 1$$







x_3
$v_1(\alpha)$
$\mathcal{O}^*$
$F^* v_1(\alpha)$
$\mathcal{P}_M(\alpha)$
v_2
$F^* v_2$
x_2
$F^* v_3(\beta(\alpha))$
$v_3(\beta(\alpha))$

• $\mathcal{P}_M(\alpha) := \operatorname{conv} P_M(\alpha)$ is a *maximal* one parameter family of F^*-invariant politopes (triangles). In fact:

• If $\mathcal{Q} = \operatorname{conv} Q$ is a triangle such that

$$\mathcal{Q} \supset \mathcal{P}_M\left(\alpha'\right)$$

for some α' and

(1) $\mathcal{Q}$ is F^*-invariant
(2) $\mathcal{Q} \subset \mathcal{O}^*$

then

$$\mathcal{Q} \equiv \mathcal{P}_M\left(\alpha'\right)$$

• The family $\mathcal{P}_M(\alpha)$ describe a region $\mathcal{P}_M$ as α varies in $\bar{\alpha} \leq \alpha < \lambda_3$

• If $g^* \in \mathcal{P}_M$ then, by construction, there exists an α' which defines a politope $\mathcal{P}_M(\alpha')$ satisfying the *MK** conditions

• If $g^* \in \mathcal{P}_M$, a third order positive realization $\{A_+, b_+, c_+^T\}$ is given by

$$A_+ = K^{-1}FK, \qquad g = Kb_+, \qquad c_+^T = h^TK$$

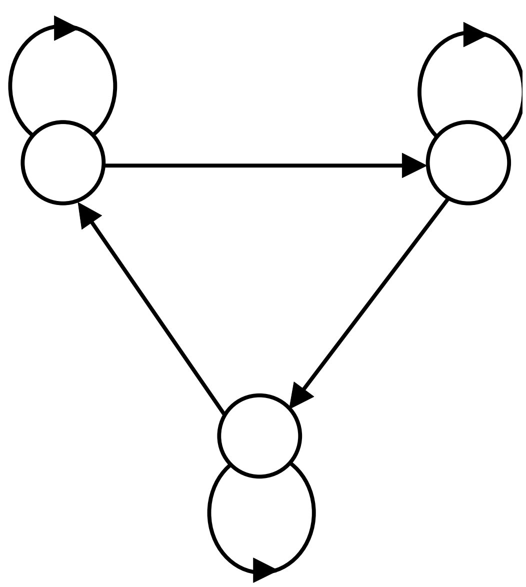
with

$$K = \begin{pmatrix} 1 \quad 1 \quad 1 \\ P_M(\alpha') \end{pmatrix}$$

- The zero pattern of $\{A_+, b_+, c_+^T\}$ is

$$A_+ = \begin{pmatrix} * & 0 & * \\ * & * & 0 \\ 0 & * & * \end{pmatrix}, \quad b_+ = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

$$c_+ = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$$



- The region $\mathcal{P}_M$ is described by the set of inequalities

$$1 + x_2 + x_3 \geq 0$$

$$(1 - \bar{\alpha}) + (\lambda_2 - \bar{\alpha})\, x_2 + (\lambda_3 - \bar{\alpha})\, x_3 \geq 0$$

$$(1 - \alpha)^2 + (\lambda_2 - \alpha)^2\, x_2 + (\lambda_3 - \alpha)^2\, x_3 \geq 0$$

for all α such that $\bar{\alpha} \leq \alpha < \lambda_3$

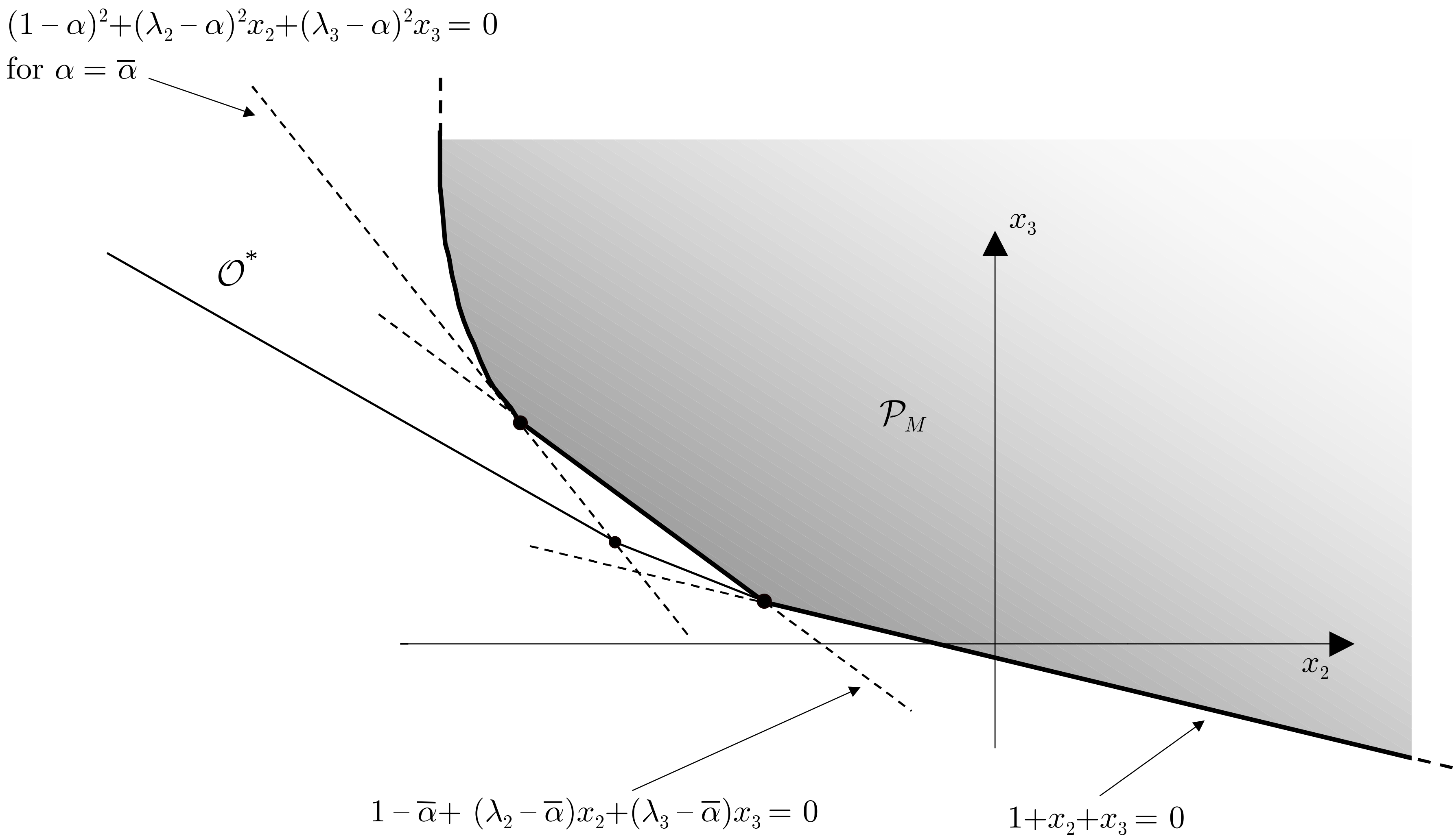
- Consequently, $g^* := \begin{pmatrix} r_2 \\ r_3 \end{pmatrix} \in \mathcal{P}_M$ iff

$$1 + r_2 + r_3 \geq 0$$

$$(1 - \bar{\alpha}) + (\lambda_2 - \bar{\alpha})\, r_2 + (\lambda_3 - \bar{\alpha})\, r_3 \geq 0$$

$$(1 - \alpha)^2 + (\lambda_2 - \alpha)^2\, r_2 + (\lambda_3 - \alpha)^2\, r_3 \geq 0$$

for all α such that $\bar{\alpha} \leq \alpha < \lambda_3$


$(1-\alpha)^2+(\lambda_2-\alpha)^2x_2+(\lambda_3-\alpha)^2x_3=0$
for $\alpha=\overline{\alpha}$
$\mathcal{O}^*$
x_3
$\mathcal{P}_M$
x_2
$1-\overline{\alpha}+(\lambda_2-\overline{\alpha})x_2+(\lambda_3-\overline{\alpha})x_3=0$
$1+x_2+x_3=0$

• *Theorem*. Let

$$W(z) = \frac{r_1}{z-\lambda_1} + \frac{r_2}{z-\lambda_2} + \frac{r_3}{z-\lambda_3}$$

be a third order transfer function with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$. Then, $W(z)$ has a third order positive realization if and only if

(1) $r_1 > 0$

(2) $r_1 + r_2 + r_3 \geq 0$

(3) $(1-\bar{\alpha})\, r_1 + (\lambda_2 - \bar{\alpha})\, r_2 + (\lambda_3 - \bar{\alpha})\, r_3 \geq 0$

(4) $(1-\alpha)^2\, r_1 + (\lambda_2 - \alpha)^2\, r_2 + (\lambda_3 - \alpha)^2\, r_3 \geq 0$
for all α such that $\bar{\alpha} \leq \alpha \leq \lambda_3$

where

$$\bar{\alpha} = \max\left\{ \frac{1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (1-\lambda_2)(1-\lambda_3)}}{3}, 0 \right\}$$

- *Theorem*. Let

$$W(z) = \sum_{k=1}^{\infty} w_k z^{-k}$$

be a third order transfer function with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$. Then, $W(z)$ has a third order positive realization if and only if the following conditions hold:

(1) $w_3 - (\lambda_2 + \lambda_3)\, w_2 + \lambda_2 \lambda_3 w_1 > 0$

(2) $w_1 \geq 0$

(3) $w_2 - \bar{\alpha} w_1 \geq 0$

(4) $w_3 - 2w_2\alpha + w_1^2\alpha \geq 0$
for all α such that $\bar{\alpha} \leq \alpha \leq \lambda_3$

where

$$\bar{\alpha} = \max\left\{\frac{1+\lambda_2+\lambda_3-2\sqrt{(\lambda_2-\lambda_3)^2+(1-\lambda_2)(1-\lambda_3)}}{3}, 0\right\}$$

- *Theorem*. Let

$$W(z) = \frac{a_2 z^2 + a_1 z + a_0}{(z-1)(z-\lambda_2)(z-\lambda_3)}$$

be a third order transfer function with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$. Then, $W(z)$ has a third order positive realization if and only if the following conditions hold:

(1) $a_0 + a_1 + a_2 > 0$

(2) $a_2 \geq 0$

(3) $a_1 + (1 + \lambda_2 + \lambda_3 - \bar{\alpha})\, a_2 \geq 0$

(4) $\left(\alpha^2 - 2\gamma_1\alpha + \gamma_0\right) a_2 + (1 + \lambda_2 + \lambda_3 - 2\alpha)\, a_1 + a_0 \geq 0$
for all α such that $\bar{\alpha} \leq \alpha \leq \lambda_3$ where

$$\begin{aligned}\gamma_1 &= 1 + \lambda_2 + \lambda_3 \\ \gamma_0 &= 1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2\end{aligned}$$

and

$$\bar{\alpha} = \max\left\{\frac{1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (1 - \lambda_2)(1 - \lambda_3)}}{3}, 0\right\}$$

• *Theorem*. Let $W(z)$ be a third order transfer function with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$ and let $\{F, g, h^T\}$ be any minimal realization of $W(z)$. Then, $W(z)$ has a third order positive realization if and only if the following conditions hold:

(1) $\lim_{k\to\infty} h^T F^k g > 0$

(2) $h^T g \geq 0$

(3) $h^T (F - \bar{\alpha} I)\, g \geq 0$

(4) $h^T (F - \alpha I)^2\, g \geq 0$
for all α such that $\bar{\alpha} \leq \alpha \leq \lambda_3$

where

$$\bar{\alpha} = \max\left\{\frac{1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (1 - \lambda_2)(1 - \lambda_3)}}{3}, 0\right\}$$

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