Tropical methods for ergodic control and zero-sum games

Minilecture, Part I

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INRIA and CMAP, École Polytechnique

Dynamical Optimization in PDE and Geometry
Applications to Hamilton-Jacobi
Ergodic Optimization, Weak KAM
Université Bordeaux 1, December 12-21 2011

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $=$

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- "2×3" $= 5$

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- $\text{“}2^3\text{''} = \text{“}2\times 2\times 2\text{''} = 6$
- $\text{“}\sqrt{-1}\text{''} =$

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- “$\sqrt{-1}$” $=-0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} =$$

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So

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- “2×3” $=5$
- “$5/2$” $=3$
- “2^3” $=$ “$2\times 2\times 2$” $= 6$
- “$\sqrt{-1}$” $=-0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

The notation $a \oplus b := \max(a, b)$, $a \odot b := a + b$, $\mathbb{0} := -\infty$, $\mathbb{1} := 0$ is also used in the tropical/max-plus litterature

The sister algebra: min-plus

$$\text{“}a + b\text{”} = \min(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

- “$2 + 3$” $= 2$
- “2×3” $= 5$

In the late 80's in France, the term "algèbres exotiques" was used

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SÉMINAIRE SUR LES ALGÈBRES EXOTIQUES ET LES SYSTÈMES A ÉVÉNEMENTS DISCRETS

3 et 4 juin 1987

Organisé par :

le Centre National de la Recherche Scientifique
le Centre National d'Etudes des Télécommunications
l'Institut National de la Recherche en Informatique et Automatique au :

Centre National d'Etudes des Télécommunications
38-40, rue du Général Leclerc
92131 Issy-les-Moulineaux

Pour la modélisation des processus continus, on dispose aujourd'hui de théories ayant atteint une certaine maturité. Il n'en va pas de même pour ce qu'il est désormais convenu d'appeler « systèmes à événements discrets » et que l'on rencontre dans l'étude des ateliers flexibles, des réseaux d'ordinateurs ou de télécommunications, des circuits VLSI spécialisés en traitement du signal, pour ne citer que quelques exemples. Diverses approches et théories de ces systèmes s'appuyant sur des outils mathématiques variés ont néanmoins émergé.

Ce séminaire à caractère didactique, organisé dans le cadre de l'ATP-CNRS « Méthodologie de l'Automatique et de l'Analyse des Systèmes », avec le concours du CNET et de l'INRIA, a pour objectifs d'une part d'initier les participants à certaines de ces théories et aux outils correspondants, et d'autre part de constituer un lieu de rencontre et de confrontation de ces approches.

Conférenciers invités (liste provisoire) : P. Caspi, IMAG Grenoble ; P. Chretienne, Univ. de Paris VI ; R. A. Cuninghame Green, Univ. de Birmingham, UK ; G. Cohen, Ecole des Mines Fontainebleau ; N. Halbwachs, IMAG Grenoble ; M. Minoux, STEI Issy-les-Moulineaux ; P. Moller, IIASA Vienne, AUT ; G. J. Olsder, Univ. Delft, Pays-Bas ; J. P. Quadrat, INRIA Rocquencourt ; Ch. Reutenauer, Univ. Paris VI ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Comité d'organisation : P. Chemouil, CNET Issy-les-Moulineaux ; G. Cohen, Ecole des Mines Fontainebleau ; J. P. Quadrat, INRIA Rocquencourt ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Toutes les personnes intéressées sont invitées à contacter le plus vite possible :

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Tél. (1) 64.22.48.21

The term “exotic” appeared also in the User’s guide of viscosity solutions of Crandall, Ishii, Lions (Bull. AMS, 92)

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of its properties are given there. See also [104]. Its "magical properties" can be seen as related to the Lax formula for the solution of

$$\frac{\partial w}{\partial t} - \frac{1}{2}|\nabla w|^2 = 0 \quad \text{for } x \in \mathbb{R}^N,\ t \geq 0,\ w|_{t=0} = v \text{ on } \mathbb{R}^N,$$

which is

$$w(x,t) = \sup_y \left\{ v(y) - \frac{1}{2t}|x-y|^2 \right\}.$$

Indeed, the coincidence of this solution formula and solutions produced by the method of charteristics leads to the properties used. Of course, this is a heuristic connection, since characteristic methods require too much regularity to be rigorous here.

The inf convolution can also be seen as a nonlinear analogue of the standard mollification when replacing the "linear structure of L^2 and its duality" by the "nonlinear structure of L^∞ or C." One can also interpret this analogy in terms of the so-called exotic algebra $(\mathbb{R}, \max, +)$.

The term "tropical" is in the honor of Imre Simon, 1943 - 2009

who lived in Sao Paulo (south tropic).

These algebras were invented by various schools in the world

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev ~65 ... Zimmerman, Butkovic; Optimization
- Maslov ~ 80'- ... Kolokoltsov, Litvinov, Samborskii, Shpiz... Quasi-classic analysis, variations calculus
- Simon ~ 78- ... Hashiguchi, Leung, Pin, Krob, ... Automata theory
- Gondran, Minoux ~ 77 Operations research
- Cohen, Quadrat, Viot ~ 83- ... Olsder, Baccelli, S.G., Akian initially discrete event systems, then optimal control, idempotent probabilities, combinatorial linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney ~00- max-plus approximation of HJB
- Del Moral ~95 Puhalskii ~99, idempotent probabilities.

now in **tropical geometry**, after Viro, Mikhalkin, Passare, Sturmfels and many.

Menu: connections between...

- tropical convexity
- dynamic programming / zero-sum games
- Perron-Frobenius theory
- metric geometry

Tropical convex sets and cones

Definition

A set C of functions $X \to \mathbb{R}_{\max}$ is a tropical convex set if $u, v \in C$, $\lambda, \mu \in \mathbb{R}_{\max}$, $\max(\lambda, \mu) = 0$ implies $\sup(\lambda + u, \mu + v) \in C$.

A tropical convex cone or semimodule is defined similarly, ommiting the requirement that $\max(\lambda, \mu) = 0$.

Semimodules are analogous both to classical convex cones and to linear spaces

They can also be defined and studied abstractly

Korbut 65, Vorobyev 65, Zimmermann 77-, Cuninghame-Green 79-, Butkovič, Hegedus 84, Helbig 88; idempotent functional analysis by Litvinov, Maslov, Samborski, Shpiz 92-; max-plus / abstract convexity Cohen, Gaubert, Quadrat 96-, Briec and Horvath 04-, Singer 04-, . . .

Tropical point of view in Develin and Sturmfels, 04-. Since that time, many works by some of the above authors and others Joswig, Santos, Yu, Ardila, Nitica, Sergeev, Schneider, Meunier, Werner. . .

the term tropical linear space is ambiguous, may refer to elements of the tropical Grassmanian of Speyer and Sturmfels which are special tropical convex cones.

Several examples of tropical convex sets

Motivations from optimal control

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S^t)_{t\geq 0}$, $S^t\phi := v(t,\cdot)$.

Superposition principle: $\forall \lambda \in \mathbb{R}$, $\forall \phi, \psi$,

$$\begin{aligned} S^t(\sup(\phi,\psi)) &= \sup(S^t\phi, S^t\psi) \\ S^t(\lambda+\phi) &= \lambda + S^t\phi \end{aligned}$$

So S^t is max-plus linear.

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$$\begin{aligned} S^t(\text{“}\phi+\psi\text{”}) &= \text{“}S^t\phi + S^t\psi\text{”} \\ S^t(\text{“}\lambda\phi\text{”}) &= \text{“}\lambda S^t\phi\text{”} \end{aligned}$$

So S^t is max-plus linear.

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0, \cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \sup_{y \in \mathbb{R}^n} tL(\frac{x - y}{t}) + \phi(y) \ .$$

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Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \text{``} \int G(x - y)\phi(y)dy\text{''} \quad .$$

$$\mathcal{V}_\lambda := \{\phi \mid S^t\phi = \lambda t + \phi, \forall t > 0\}$$

is a max-plus or tropical cone in infinite dimension. The functions ϕ are the weak-KAM solutions of Fathi.

S^t is an instance of Moreau conjugacy:

$$S^t\phi(x) = \sup_y a(x,y) + \phi(y) \enspace .$$

Metric geometry

(X, d) metric space.
1-Lip:= $\{u \mid u(x) - u(y) \leq d(x, y)\}$ is a tropical convex cone.
TFAE

$$u \text{ is } 1 - \text{Lip}$$

$$u(y) = \max_{x \in X} -d(x, y) + u(x)$$

$$u(y) = \max_{x \in X} -d(x, y) + v(x), \quad \exists v$$

$$u \in \text{Span}\{-d(x, \cdot) \mid x \in X\} \qquad \text{make picture!}$$

Question. What are the tropical extreme rays ?

We shall see they are precisely the maps $-d(x, \cdot)$, $x \in X$, together with the horofunctions associated with Busemann points (limits of infinite geodesics).

Gromov's horoboundary compactification of X.
Analogous to the probabilistic Martin boundary.
Related to results of Fathi and Maderna, Contreras, Ishii and Mitake for optimal control problems with noncompact state space.

Spaces of semiconvex functions

Fleming, McEneaney

$$C_\alpha := \{u : \mathbb{R}^n \to \mathbb{R} \mid u + \alpha\|x\|^2/2 \text{ is convex}\}$$

is a tropical convex cone.

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

- $[n] := \{1, \dots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$

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T is order preserving and additively homogeneous:

$$x \le y \implies T(x) \le T(y)$$
$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$

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Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic)
Gunawardena, Sparrow; Singer, Rubinov,

$$T_i(x) = \sup_{y \in \mathbb{R}} \Big(T_i(y) + \min_{1 \le i \le n} (x_i - y_i) \Big)$$

Variant. T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

Order-preserving + additively (sub)homogeneous $\implies$ sup-norm nonexpansive

$$\|T(x) - T(y)\|_\infty \leq \|x - y\|_\infty \ .$$

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Order-preserving + additively homogeneous $\iff$ top nonexpansive

$$\mathrm{t}(T(x) - T(y)) \leq \mathrm{t}(x - y), \qquad \mathrm{t}(z) := \max_i z_i \ .$$

Variant. T is additively subhomogeneous if

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Order-preserving + additively subhomogeneous $\iff$ top$^+$ nonexpansive

$$\mathrm{t}^+(T(x) - T(y)) \leq \mathrm{t}^+(x - y), \quad \mathrm{t}^+(z) := \max(\max_i z_i, 0) \ .$$

If T is order preserving and additively homogeneous, then the set of subsolutions

$$C = \{u \mid T(u) \geq u\}$$

(showing the game is superfair) is a tropical (max-plus) convex cone.

Similarly, if S^t is the semigroup of the Isaacs equation

$$v_t - H(x, Dv, D^2 v) = 0, \qquad H(x, p, \cdot) \text{ order preserving}$$

S^t is order preserving and additively homogeneous

$$C = \{u \mid S^t u \geq u, \ \forall t \geq 0\}$$

is a tropical convex cone.

Supersolutions constitute a min-plus convex cone.

Discounted case = tropical convex sets

If T is only order preserving and additively subhomogeneous

$$C = \{u \mid T(u) \geq u\}$$

is a tropical (max-plus) convex set.
Proof. If $u, v \in C, \beta \in \mathbb{R}_+$

$$\begin{aligned} T(\sup(u, -\beta + v)) &\geq \sup(T(u), T(-\beta + v)) \\ &\geq \sup(T(u), -\beta + T(v)) \\ &\geq \sup(u, -\beta + v) \end{aligned}$$

Population dynamics = games with exponential glasses

K closed convex pointed cone in a Banach space, say $K = \mathbb{R}^n_+$.

$$x \leq y \implies F(x) \leq F(y)$$

$$F(\lambda x) = \lambda F(x), \qquad \lambda > 0$$

If $K = \mathbb{R}^n_+$, or $\mathcal{C}(X)$, then:

$$T(x) = \log \circ F \circ \exp$$

is a Shapley operator.

Example: Perron-Frobenius $\subset$ stochastic control

$$F(X) = MX, \qquad M_{ij} \geq 0$$

$$x = \log X$$

$$T(x) = \sup_P \left(Px - S(P; M)\right)$$

where the sup is taken over the set of stochastic matrices, and S is the relative entropy

$$S_i(P; M) = \sum_j P_{ij} \log(P_{ij}/M_{ij})$$

The Perron eigenvector

$$F(U) = \mu U, \qquad U \in \operatorname{int} \mathbb{R}_+^n$$

corresponds to the additive eigenvector $u = \log U$,

$$\begin{aligned} T(u) &= \lambda + u, \qquad \lambda = \log \mu \\ &= \sup_P \left(Pu - S(P; M) \right) \ . \end{aligned}$$

So, the log of the Perron root μ is

$$\log \rho(M) = \sup_{P,m} -m \cdot S(P; M)$$

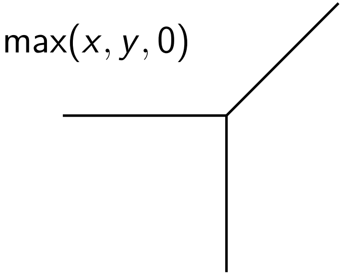
where the sup is over all invariant measures m of P.

Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$\text{``}ax + by + c\text{''}$$

is attained at least twice.



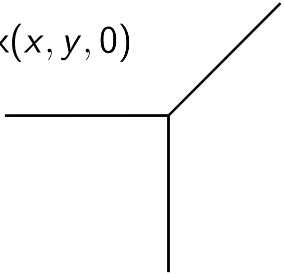
Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

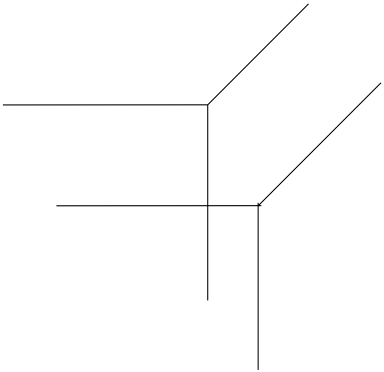
$$\max(a + x, b + y, c)$$

is attained at least twice.

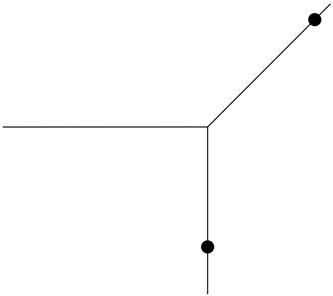
$$\max(x, y, 0)$$



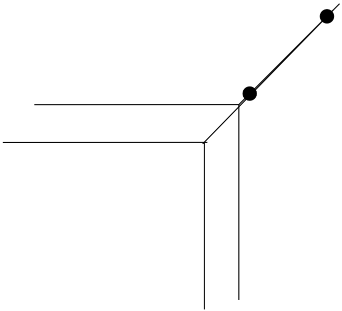
Two generic tropical lines meet at a unique point



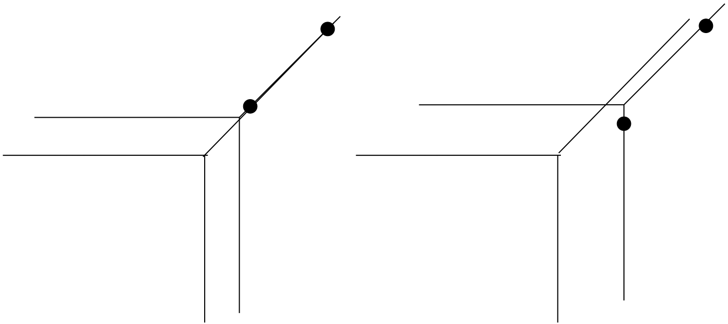
By two generic points passes a unique tropical line



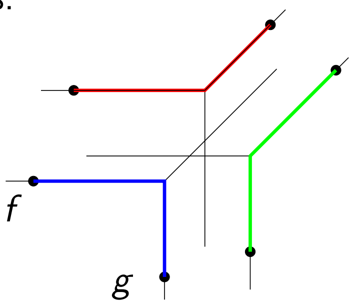
non generic case



non generic case resolved by perturbation



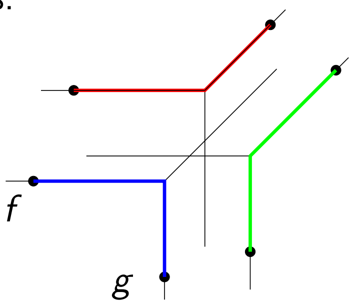
Tropical segments:



$$[f,g] := \{\text{“}\lambda f + \mu g\text{”} \mid \lambda,\mu \in \mathbb{R}\cup\{-\infty\},\ \text{“}\lambda+\mu = 1\text{”}\}.$$

(The condition “$\lambda, \mu \geq 0$” is automatic.)

Tropical segments:

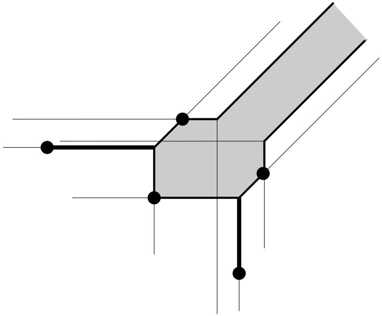


$[f, g] := \{ \sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}, \max(\lambda, \mu) = 0\}$.

(The condition $\lambda, \mu \geq -\infty$ is automatic.)

Exercise: draw a convex set.

Tropical convex set: $f, g \in C \implies [f, g] \in C$



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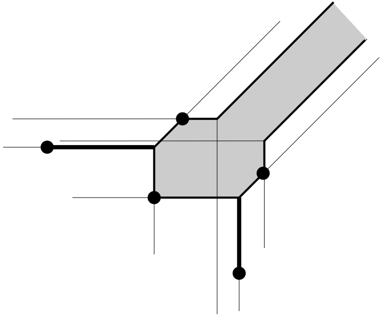


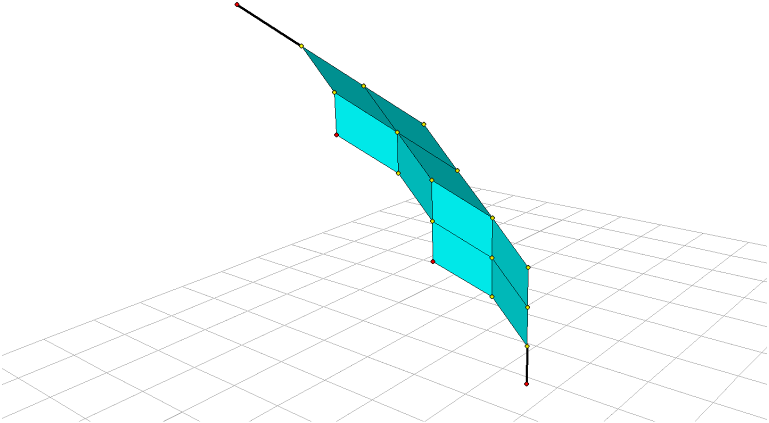
Tropical convex cone: ommit "$\lambda + \mu = 1$", i.e., replace $[f, g]$ by $\{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}\}$

Homogeneization

A convex set C in $\mathbb{R}^n_{\max}$ corresponds to a convex cone $\hat{C}$ in $\mathbb{R}^{n+1}_{\max}$,

$$\hat{C} := \{(u, \lambda + u) \mid u \in C, \lambda \in \mathbb{R}_{\max}\}$$

A max-plus "tetrahedron"?

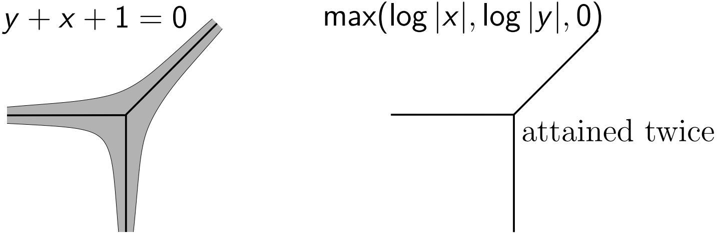


The previous drawing was generated by POLYMAKE of Gawrilow and Joswig, in which an extension allows one to handle easily tropical polyhedra. They were drawn with JAVAVIEW. See Joswig arXiv:0809.4694 for more information.

Why?

Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the “log-log plot”

$$A(V) := \{(\log|z_1|, \ldots, \log|z_n|) \mid (z_1, \ldots, z_n) \in V\} \ .$$



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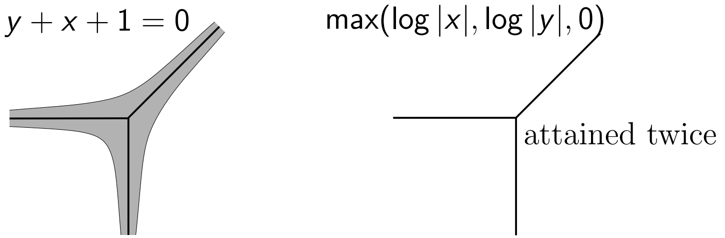

$|y| \leq |x| + 1,\ |x| \leq |y| + 1,\ 1 \leq |x| + |y|$

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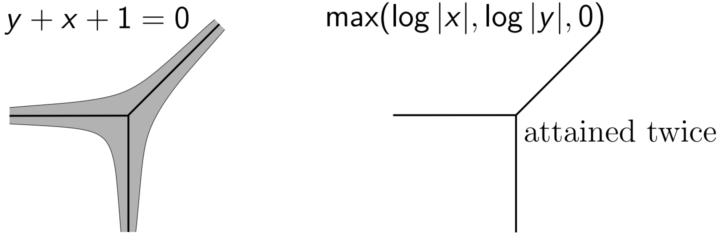
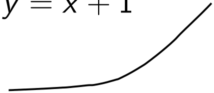


$X := \log|x|, \ Y := \log|y|$

$Y \leq \log(e^X + 1), \ X \leq \log(e^Y + 1), \ 1 \leq e^X + e^Y$

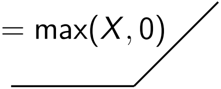
real tropical lines

$y = x + 1$



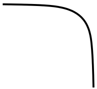
$X = \log(e^X + 1)$

$Y = \max(X, 0)$



real tropical lines

$x + y = 1$

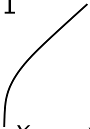


$\log(e^X + e^Y) = 1$

$\max(X, Y) = 0$

real tropical lines

$x = y + 1$



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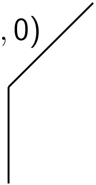
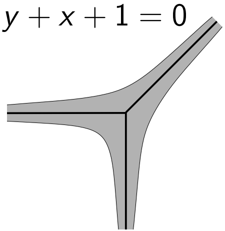
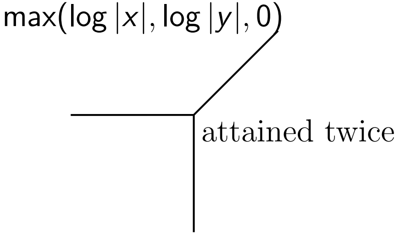


Viro's log-glasses, related to Maslov's dequantization

$$a +_h b := h \log(e^{a/h} + e^{b/h}), \qquad h \to 0^+$$

With h-log glasses, the amoeba of the line retracts to the tropical line as $h \to 0^+$





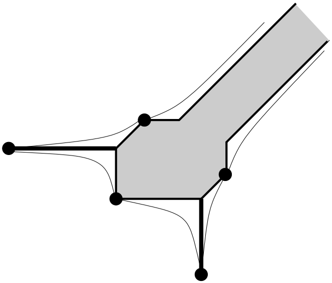
$$\max(a, b) \leq a +_h b \leq h \log 2 + \max(a, b)$$

Similar to convergence of p-norm to sup-norm

$$[a, b] := \{\lambda a +_p \mu b \mid \lambda, \mu \geq 0, \lambda +_p \mu = 1\}$$

$$a +_p b = (a^p + b^p)^{1/p}$$

The convex hull in the $+_h$ / $+_p$ sense converges to the tropical convex hull as $h \to 0$ / $p \to \infty$ (Briec and Horvath).



See Passare & Rullgard, Duke Math. 04 for more information on amoebas

Introduction to amoebas: lecture notes by Yger.

All the results of classical convexity have tropical analogues, sometimes more degenerate...

- generation by extreme points Helbig; SG, Katz 07; Butkovič, Sergeev, Schneider 07; Choquet Akian, SG, Walsh 09, Poncet 11 infinite dim.
- projection / best-approximation : Cohen, SG, Quadrat 01,04; Singer
- Hahn-Banach analytic Litvinov, Maslov, Shpiz 00; Cohen, SG, Quadrat 04; geometric Zimmermann 77, Cohen, SG, Quadrat 01,05; Develin, Sturmfels 04, Joswig 05
- cyclic projections Butkovic, Cuninghame-Green TCS03; SG, Sergeev 06
- Radon, Helly, Carathéodory, Colorful Carathéodory, Tverberg: SG, Meunier DCG09

This lecture

Tropical convexity is equivalent to dynamic programming (zero-sum games).

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- finite dimensional convex sets (cones) $\sim$ stochastic games with finite state spaces

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Tropical convexity is equivalent to dynamic programming (zero-sum games).

- finite dimensional convex sets (cones) $\sim$ stochastic games with finite state spaces
- leads to: equivalence (computational complexity) results, algorithms, approximation methods, ...

Some results and techniques . . .

Residuation / Galois correspondences in lattices

Let $A \in M_{dp}(\mathbb{R}_{\max})$. Then, for $x \in \mathbb{R}_{\max}^p$ and $b \in \mathbb{R}_{\max}^d$,

$$Ax \leq b \iff x \leq A^\sharp b$$

where

$$(A^\sharp b)_j = \min_{1\leq i\leq d} -A_{ij} + b_j, \qquad 1 \leq j \leq p$$

$$AA^\sharp A = A \qquad A^\sharp A A^\sharp = A^\sharp$$

The row and column spaces of A are anti-isomorphic semi-lattices, $x \mapsto (A(-x))^T$, $y \mapsto ((-y)A)^T$, general residuation result (infinite dim OK, Cohen, SG, Quadrat 01,04), different proof by Develin and Sturmfels, 04.

The tropical sesquilinear form

$$x/v := \max\{\lambda \mid \text{“}\lambda v\text{”} \leq x\}$$
$$= \min_i (x_i - v_i) \quad \text{if } x, v \in \mathbb{R}^n \ .$$

$$\delta(x,y) = \text{“}(x/y)(y/x)\text{”} = \min_i (x_i - y_i) + \min_j (y_j - x_i)$$

$d = -\delta$ is the (additive) Hilbert's projective metric

$$d(x,y) = \|x - y\|_H, \ \ \|z\|_H := \max_{1 \leq i \leq d} z_i - \min_{1 \leq i \leq d} z_i \ .$$

Projection on a tropical cone

If $C \subset \mathbb{R}^d_{\max}$ is a tropical convex cone stable by sups (closed in Scott topology -non-Haussdorf-):

$$P_C(x) = \max\{v \in C \mid v \leq x\}$$
$$= \max_{u \in U} (x/u) + u \quad .$$

for any generating set U of C.

Compare with

$$P_C(x) = \sum_{u \in U} \langle x, u \rangle u$$

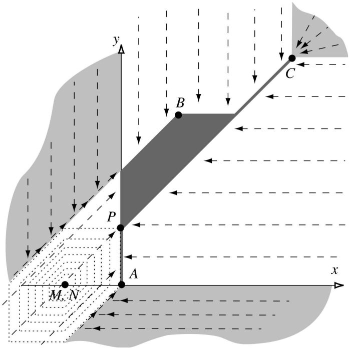
if U is a Hilbert base of a Hilbert space.

When $C = \mathrm{Col}(A)$, $P_C(x) = AA^\sharp x$.

Best approximation in Hilbert's projective metric

Prop.(Cohen, SG, Quadrat, in Bensoussan Festschrift 01)

$$d(x, P_{\mathcal{V}}(x)) = \min_{y \in \mathcal{V}} d(x, y) \quad .$$


y
B
C
P
A
x
M, N

Separation

Goes back to Zimmermann 77, simple geometric construction in Cohen, SG, Quadrat in Ben01, LAA04.
C closed linear cone of $\mathbb{R}^d_{\max}$, or complete semimodule
If $y \notin C$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leq P_C(y)/v\}$$

contains C and not y.
Compare with the optimality condition for the projection on a convex cone C: $\langle y - P_C(y), v\rangle \leq 0, \forall v \in C$

Separation

Goes back to Zimmermann 77, simple geometric construction in Cohen, SG, Quadrat in Ben01, LAA04.
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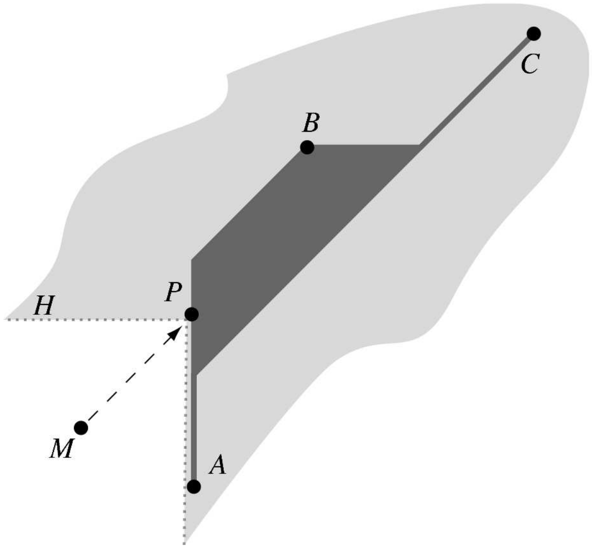
If $y \notin C$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leq P_C(y)/v\}$$

contains C and not y.
Let $\bar{y} := P_C(y)$ and $I := \{i \mid y_i = \bar{y}_i\}$. Then,

$$\mathcal{H} = \{v \mid \max_{i \in I^c} v_i - \bar{y}_i \leq \max_{i \in I} v_i - \bar{y}_i\}$$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Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{``}ax \leq bx\text{''}\}$$

Tropical half-spaces

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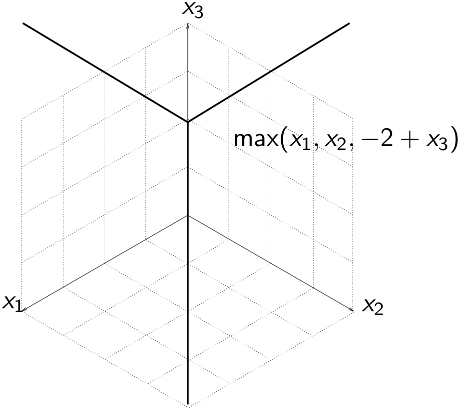

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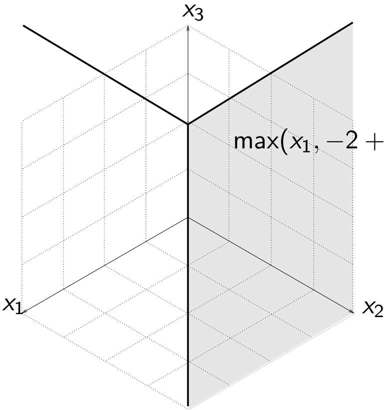

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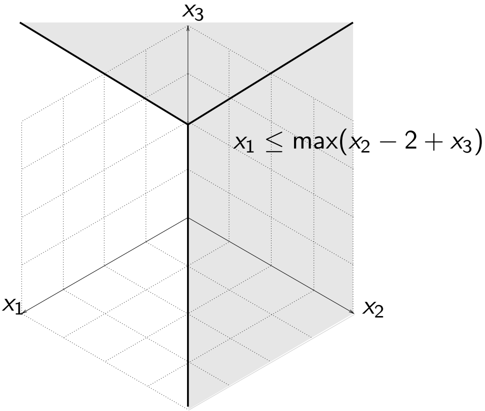

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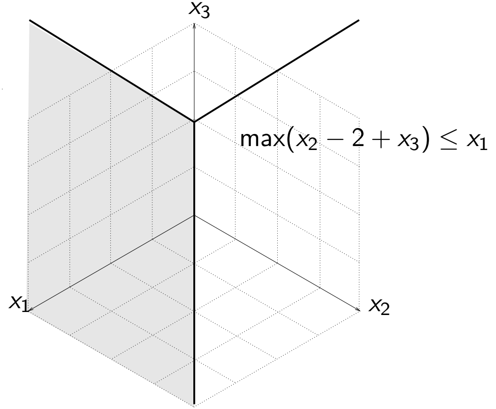


A halfspace can always be written as:

$$\max_{i\in I} a_i + x_i \leq \max_{j\in J} b_j + x_j, \qquad I \cap J = \emptyset \enspace .$$

Apex: $v_i := -\max(a_i, b_i)$.

If $v \in \mathbb{R}^n$, H is the union of sectors of the tropical hyperplane with apex v:

$$\max_{1\leq i\leq n} x_i - v_i \qquad \text{attained twice}$$

Halfspaces appeared in: Joswig 04; Cohen, Quadrat SG 00; Zimmermann 77, ...

Corollary (Zimmermann; Samborski, Shpiz; Cohen, SG, Quadrat, Singer; Develin, Sturmfels; Joswig. . .)

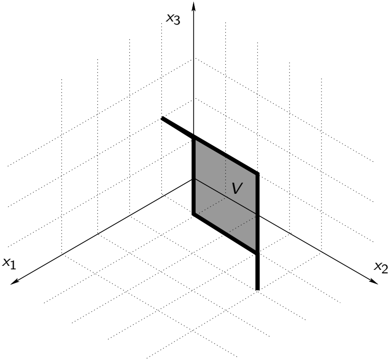
A tropical convex cone closed (in the Euclidean topology) is the intersection of tropical half-spaces.

$\mathbb{R}_{\max}$ is equipped with the topology of the metric $(x, y) \mapsto \max_i |e^{x_i} - e^{y_i}|$ inherited from the Euclidean topology by log-glasses.

The apex $-P_C(y)$ of the algebraic separating half-space $\mathcal{H}$ above may have some $+\infty$ coordinates, and therefore may not be closed in the Euclidean topology (always Scott closed). The proof needs a perturbation argument, this is where the assumption that C is closed (and not only stable by arbitrary sups = Scott closed) is needed.

Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



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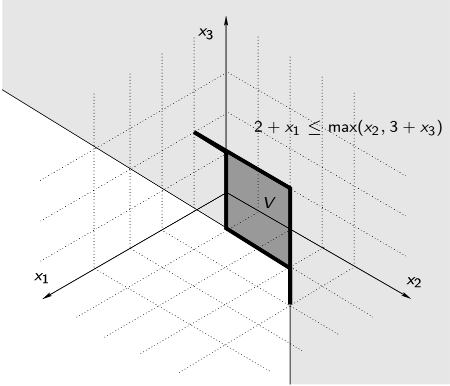

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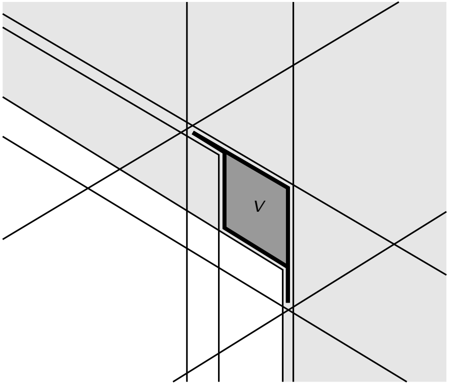

The Equivalence between tropical convexity and games. . .

Based on Akian, SG, Guterman arXiv:0912.2462 to appear in IJAC

Theorem (Equivalence, part I; Akian, SG, Guterman arXiv:0912.2462 → IJAC)

TFAE

- *C is a closed tropical convex cone*
- $C = \{u \mid u \leq T(u)\}$ *for some Shapley operator* T.

Recall $C \subset (\mathbb{R} \cup \{-\infty\})^n$ is a tropical convex cone if

$$u, v \in C, \lambda \in \mathbb{R} \cup \{-\infty\} \implies \sup(u, v) \in C, \quad \lambda + u \in C \quad .$$

The Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ extends continuously $\mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$,

$$T(x) = \inf_{y \geq x,\ y \in \mathbb{R}^n} T(y) \quad .$$

Easy implication: T order preserving and additively homogeneous $\implies \{u \mid u \leq T(u)\}$ is a closed tropical convex cone

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Easy implication: T order preserving and additively homogeneous $\implies \{u \mid u \leq T(u)\}$ is a closed tropical convex cone
Remark: $\{u \mid u \geq T(u)\}$ is a dual tropical (min-plus) cone.

Conversely, any closed tropical convex cone can be written as

$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \text{“}A_i x \leq B_i x\text{”}$$

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$$C = \bigcap_{i \in I} H_i$$

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$$H_i : \max_{1 \le j \le n} a_{ij} + x_j \le \max_{1 \le k \le n} b_{ik} + x_k , \qquad a_{ij}, b_{ij} \in \mathbb{R} \cup \{-\infty\}$$

$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \le k \le n} b_{ik} + x_k \ .$$

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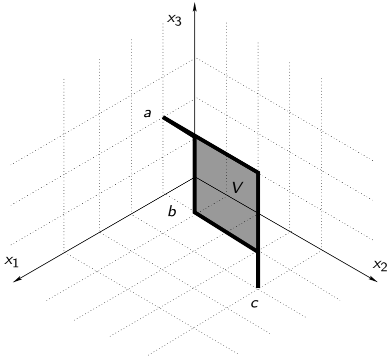
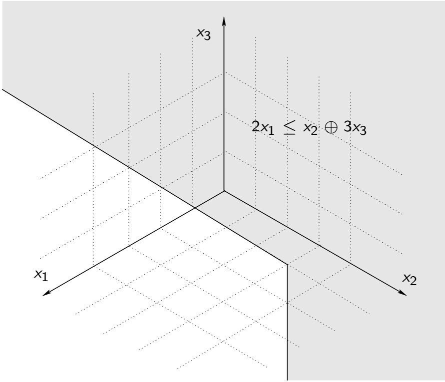
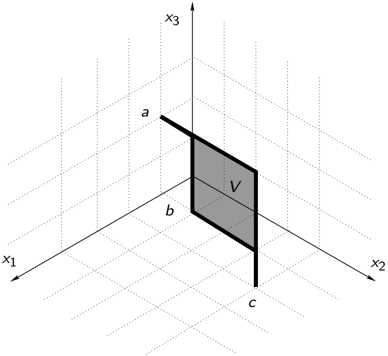
$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \le k \le n} b_{ik} + x_k \quad .$$

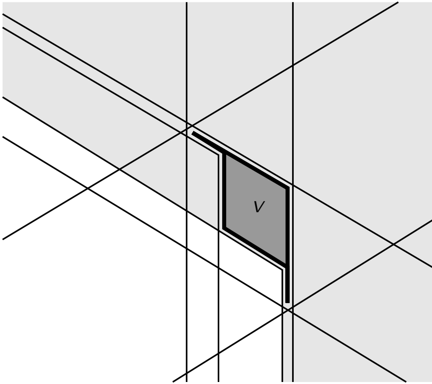
$$x \le T(x) \iff \max_{1 \le j \le n} a_{ij} + x_j \le \max_{1 \le k \le n} b_{ik} + x_k , \ \forall i \in I \quad .$$





$$2 + x_1 \leq \max(x_2, 3 + x_3)$$





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$$H_i : \max_{1\le j\le n} a_{ij} + x_j \le \max_{1\le k\le n} b_{ik} + x_k$$

$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\le k\le n} b_{ik} + x_k \quad .$$

Interpretation of the game

- State of MIN: variable x_j, $j \in \{1, \ldots, n\}$
- State of MAX: half-space H_i, $i \in I$
- In state x_j, Player MIN chooses a tropical half-space H_i with x_j in the LHS
- In state H_i, player MAX chooses a variable x_k at the RHS of H_i
- Payment $-a_{ij} + b_{ik}$.

Menu of the next lectures

- The mean payoff problem for repeated games
- Generalized Denjoy-Wolff theorem
- Deformation of Perron-Frobenius theory
- More combinatorics
- Extreme points of tropical polyhedra, Max-plus Martin Boundary
- Algorithms

Thank you!

Tropical methods for ergodic control and zero-sum games

Minilecture, Part II

Stephane.Gaubert@inria.fr

INRIA and CMAP, École Polytechnique

Dynamical Optimization in PDE and Geometry
Applications to Hamilton-Jacobi
Ergodic Optimization, Weak KAM
Université Bordeaux 1, December 12-21 2011

Today

From tropical algebra to nonexpansive mappings and back

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big) , \qquad i \in [n]$$

- $[n] := \{1, \ldots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$

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T is order preserving, additively homogeneous $\Rightarrow$ sup-norm nonexpansive:

$$x \leq y \implies T(x) \leq T(y)$$

$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$

$$\| T(x) - T(y) \| \leq \| x - y \|$$

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Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic) Gunawardena, Sparrow; Singer, Rubinov,

$$T_i(x) = \sup_{y \in \mathbb{R}} \Big(T_i(y) + \min_{1 \leq i \leq n} (x_i - y_i) \Big)$$

Repeated games

The value of the game in horizon k starting from state i is $(T^k(0))_i$.

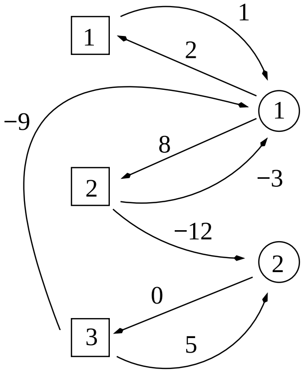
We are interested in the long term payment per time unit

$$\chi(T) := \lim_{k\to\infty} T^k(0)/k$$

Example: mean payoff games

$G = (V, E)$ bipartite graph. r_{ij} price of the arc $(i, j) \in E$. "Max" and "Min" move a token. The player receives the amount of the arc.

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



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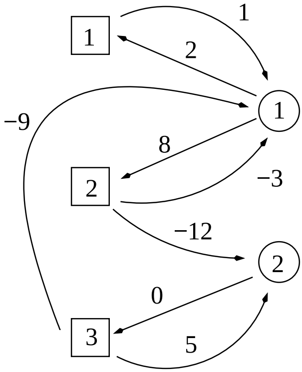


$$\chi(T) = \lim_k v^k/k = (-1,5)$$

$\chi(T) = \lim_k T^k(0)/k$ may not exist if the action spaces are infinite (Kohlberg, Neyman). Counter example in dimension 3.

However. Let v_α denote the discounted value

$$v_\alpha = T(\alpha v_\alpha), \qquad 0 < \alpha < 1 \ .$$

Theorem (Neyman 04 - book edited with Sorin)

If $\alpha \mapsto (1-\alpha)v_\alpha$ has bounded variation as $\alpha \to 1$, then

$$\lim_k T^k(0)/k = \lim_{\alpha \to 1^-} (1-\alpha)v_\alpha \qquad \textit{does exist}$$

Corollary (Neyman 04, Bewley and Kohlberg 76)

If the graph of T is semi-algebraic, then $\chi(T)$ does exists.

This is the case in particular if the action spaces are finite.

Original motivation. Von Neumann's value of a matrix game with imperfect information (rock-sissors-stone), given a $n \times p$ matrix M,

$$\operatorname{val} M = \min_{x \in \Sigma_n} \max_{y \in \Sigma_p} x^\top M y$$

where $\Sigma_n := \{x \in \mathbb{R}^n_+ \mid \sum_i x_i = 1\}$. The graph of $M \mapsto \operatorname{val} M$ is semi-algebraic.

Ingredients of the proof

A real Puiseux series in a parameter t is of the form

$$\sum_{k \geq K} a_k t^{k/r}, \qquad a_k \in \mathbb{R},\ r \geq 1,\ K \in \mathbb{Z}\ .$$

Eg., $-3/\sqrt{t} + 7 + 8\sqrt{t} + t + t^{3/2} + \dots$.

Can consider formal series, or converging series in $0 < |t| < D$ for D small enough.

Puiseux series constitute a real closed field (every square is nonnegative, every equation of odd degree has at least one root).

By Tarski's theorem, if the graph of T is semi-algebraic, the unique fixed point v_α of $x \mapsto T(\alpha x)$ has a converging Puiseux series expansion

$$v_\alpha = \sum_{k \geq K} a_k (1-\alpha)^{k/r}, \qquad r \geq 1 \ .$$

Use nonexpansiveness to deduce that $v_\alpha = O((1-\alpha)^{-1})$, the smallest exponent is ≥ -1.
Indeed,
$\|v_\alpha - T(0)\| = \|T(\alpha v_\alpha) - T(0)\| \leq \alpha \|v_\alpha\| \leq \alpha(\|v_\alpha - T(0)\| + \|T(0)\|)$, and so

$$\|v_\alpha - T(0)\| \leq \frac{\alpha}{1-\alpha} \|T(0)\| \ .$$

So bounded variation of $(1-\alpha)v_\alpha$ holds (converging, sign of derivative constant in a neighborhood of 1^-).

Classical Blackmailer example (Blackwell).

Blackmailer goes to see Victim.

Give me $p \in [0, 1]$

Victim pays, but with probability p^2, goes to see the police.

$$T : \mathbb{R} \to \mathbb{R}, \qquad T(x) = \max_{p \in [0,1]} p + (1 - p^2)x \ .$$

$$p^{\text{opt}} = 1/2x, \qquad T^k(0) \simeq \sqrt{k}, \qquad v_\alpha \simeq 1/\sqrt{1-\alpha} \ .$$

Even if

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_i} \left(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \right), \qquad i \in [n]$$

with compact action spaces, $(a, b) \mapsto r_i^{ab}$ and $(a, b) \mapsto P_{ij}^{ab}$ continuous;

it is not known whether $\lim_k T^k(0)/k$ exists.

See works by Sorin, Rosenberg, Vigeral,...

In general. . .

Escape rate

(X, d) metric space, $T : X \to X$,

$$d(T(x), T(y)) \le d(x, y) \ .$$

$$\rho(T) := \lim_{k\to\infty} \frac{d(x, T^k(x))}{k} = \inf_{k\geq 1} \frac{d(x, T^k(x))}{k}$$

(independent of $x \in X$ by nonexpansiveness, existence by subadditivity).

Taking $d(x,y) = \|y-x\|$, $\mathrm{t}(y-x)$, or $\mathrm{b}(y-x)$, we get that the following limits do exist and are independent of $x \in \mathbb{R}^n$

$$\lim_{k\to\infty} \frac{\|T^k(x)-x\|_\infty}{k} = \inf_{k\geq 1} \frac{\|T^k(x)-x\|_\infty}{k}$$

$$\overline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{t}(T^k(x)-x)}{k} = \inf_{k\geq 1} \frac{\mathrm{t}(T^k(x)-x)}{k}$$

$$\underline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{b}(T^k(x)-x)}{k} = \sup_{k\geq 1} \frac{\mathrm{b}(T^k(x)-x)}{k}$$

$$\mathrm{t}(z) := \max_i z_i, \qquad \mathrm{b}(z) := \min_i z_i \quad .$$

Theorem (Kohlberg & Neyman, Isr. J. Math., 81)

Assume $\rho(T) > 0$. Then, there exists a linear form $\varphi \in X^*$ of norm one such that for all $x \in X$,

$$\rho(T) = \lim_{k\to\infty} \varphi\big(T^k(x)/k\big) = \inf_{y\in X} \|T(y) - y\|$$

Actually, for all $x \in X$, there exists such a φ such that

$$\varphi(T^k(x)) \geq k\rho(T) + \varphi(x) \ .$$

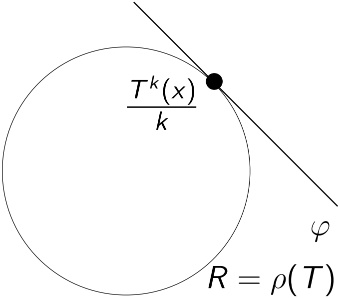
φ can be chosen in the weak-star closure of the set of extreme points of the dual unit ball

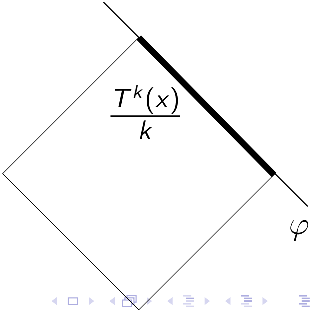
Corollary (Kohlberg & Neyman, Isr. J. Math., 81, extending Reich 73 and Pazy 71)

The limit

$$\lim_{k\to\infty} \frac{T^k(x)}{k}$$

exists in the weak (resp. strong) topology if X is reflexive and strictly convex (resp. if the norm of the dual space $X^\star$ is Frechet differentiable).





The special case of games

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \le j \le n} P_{ij}^{ab} x_j \Big), \qquad 1 \le i \le n$$

$$\rho(T) = \overline{\chi}(T) = \lim_{k \to \infty} \max_{1 \le j \le n} \frac{(T^k(x))_j}{k}$$

Theorem (SG, Gunawardena, TAMS 04)

For all $x \in \mathbb{R}^n$, there exists $1 \le i \le n$ such that

$$\big(T^k(x)\big)_i \ge x_i + k\rho(T), \qquad \forall k \in \mathbb{N} \ .$$

Initial state i guarantees the best reward per time unit.

Consider

$$F = \exp \circ T \circ \log, \qquad \mathbb{R}_+^n \to \mathbb{R}_+^n$$

which is order preserving, positively homogeneous, and continuous.

Theorem (non-linear Collatz-Wielandt, Nussbaum, LAA 86)

$$\begin{aligned}
\rho(F) &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad x \in \operatorname{Int} \mathbb{R}_+^n \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) = \mu v, v \in \mathbb{R}_+^n, v \neq 0\} \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in \mathbb{R}_+^n, v \neq 0\} \\
&= \inf\{\mu > 0 \mid F(v) \leq \mu v, v \in \operatorname{int} \mathbb{R}_+^n\}
\end{aligned}$$

Proof ingredients

The (reverse) Funk (hemi-)metric on a cone, Hilbert's and Thompson metric

C closed pointed cone, $X = \operatorname{int} C \neq \varnothing$,
$y \geq x \iff y - x \in C$,

$$\delta(x, y) = \mathrm{RFunk}(x, y) := \log \inf\{\lambda > 0 | \lambda x \geq y\}$$

$$\begin{aligned} \mathrm{RFunk}(x, y) &= \log \max_{\varphi \in C^* \setminus \{0\}} \frac{\varphi(y)}{\varphi(x)} \\ &= \log \max_{\varphi \in \mathrm{Extr}\, C^*} \frac{\varphi(y)}{\varphi(x)} \\ &= \log \max_{1 \leq i \leq n} \frac{y_i}{x_i} \qquad \text{if} \quad C = \mathbb{R}^n_+ \quad , \end{aligned}$$

Thompson's metric is the Finsler structure corresponding to the "weighted sup-norm" at point u,

$$\begin{aligned}\|x\|_u &= \inf\{\alpha \mid -\alpha \leq x \leq \alpha u\} \\ &= \max_i \frac{|x_i|}{u_i} \qquad \text{if } C = \mathbb{R}^n_+ \ .\end{aligned}$$

$$d_T(x,y) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} dt, \ \ \gamma(0) = x, \ \gamma(1) = y \ .$$

See Nussbaum.

Nonexpansiveness in the Funk metric

Lemma

$F : C \to C$ is order preserving and homogeneous of degree 1 iff

$$\mathrm{RFunk}(F(x), F(y)) \leq \mathrm{RFunk}(x, y), \qquad \forall x, y \in \mathrm{int}\, C \ .$$

[simple but useful: Gunawardena, Keane, Sparrow, Lemmens, Scheutzow, Walsh.]

So F is nonexpansive in the RFunk metric, in Thompson metric $d_T(x, y) = \mathrm{RFunk}(x, y) \vee \mathrm{RFunk}(y, x)$, and Hilbert's projective metric

$d_H(x, y) = \mathrm{RFunk}(y, x) - \mathrm{RFunk}(x, y)$.

Take q denote a positively homogeneous order preserving map from $\operatorname{Int} C$ to $(0,\infty)$, and let $u \in \operatorname{int} C$,

$$T_\epsilon(x) = T(x) + \epsilon q(x) u \enspace .$$

If C is normal, meaning that $0 \leq x \leq y \implies \|x\| \leq M\|y\|$ for some constant M, then $(\{x \in \operatorname{int} C, q(x) = 1\}, d_H)$ is complete, and T_ϵ is a strict contraction in d_H.
Then,

$$T_\epsilon(v_\epsilon) = \rho(T_\epsilon) v_\epsilon \enspace , v_\epsilon \in \operatorname{int} C \enspace .$$

Letting $\epsilon \to 0$, and taking an accumulation point v of v_ϵ, we get

$$T(v) = \mu v$$

so $\mu \leq \rho(T)$, but $\mu \geq \liminf_\epsilon \rho(T_\epsilon) \geq \rho(T)$.
This shows the Collatz-Wielandt theorem.

$$
\begin{aligned}
\rho(F) &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad x \in \operatorname{Int} C \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) = \mu v, v \in C, v \neq 0\} \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in C, v \neq 0\} \\
&= \inf\{\mu > 0 \mid F(v) \leq \mu v, v \in \operatorname{int} C\}
\end{aligned}
$$

Work of Akian, SG, Nussbaum: extension to the case of normal cones in Banach spaces, under compactness conditions (essential spectral radius).

Compare now Collatz-Wielandt

$$
\begin{aligned}
\rho(F) &= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in \mathbb{R}^n_+, v \neq 0\} \\
&= \inf\{\mu > 0 \mid F(w) \leq \mu w, w \in \operatorname{int} \mathbb{R}^n_+\} \\
&= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad \forall x \in \operatorname{int} \mathbb{R}^n_+
\end{aligned}
$$

and so

$$
\inf_{w \in \operatorname{int} \mathbb{R}^n_+} \max_{1\leq i\leq n} \frac{(F(w))_i}{w_i} = \rho(F) = \max_{\substack{v\in\mathbb{R}^n_+ \\ v\neq 0}} \min_{\substack{1\leq i\leq n \\ v_i\neq 0}} \frac{(F(v))_i}{v_i} \quad .
$$

with Kohlberg and Neyman

$$
\rho(T) := \lim_{k\to\infty} \left\| \frac{T^k(x)}{k} \right\| = \inf_{y\in X} \|T(y) - y\| = \lim_{k\to\infty} \varphi\big(T^k(x)/k\big) \quad .
$$

Is there an explanation of this analogy ?

Collatz-Wielandt and Kohlberg-Neyman are special cases of a general result.

Theorem (SG and Vigeral, Math. Proc. Phil. Soc. 11)

Let T be a nonexpansive self-map of a complete hemi-metric space (X, d) of non-positive curvature in the sense of Busemann. Let

$$\rho(T) := \lim_{k\to\infty} \frac{d(x, T^k(x))}{k}$$

Then, there exists a Martin function h such that

$$h(T(x)) \geq \rho(T) + h(x), \qquad \forall x$$

Moreover,

$$\rho(T) = \inf_{y\in X} d(y, T(y)) \ .$$

If in addition X is a metric space and $\rho(T) > 0$, then h is an horofunction.

It follows that

$$h(T^k(x)) \geq k\rho(T) + h(x), \qquad \forall x \in X \ .$$

Karlsson (Ergodic. Th. and Dyn. S., 01) proved an analogous result without nonpositive curvature, but with h depending on x (uses only subadditivity).

Indeed, $\rho(T) < \inf_{y \in X} d(x, T(y))$ may happen without nonpositive curvature (no strong duality).

See also works of Beardon, and Lins.

Corollary (SG and Vigeral)

Let T be a Shapley operator, $\mathcal{C}(K) \to \mathcal{C}(K)$, K compact. Then, for all $u \in \mathcal{C}(K)$, there exists a point $z \in K$ such that

$$[T^k(u)](z) \geq k\rho(T) + u(z)$$

where

$$\rho(T) = \lim_k \max_{y \in K} [T^k(u)](y)/k$$

E.g., Semigroup of Isaacs equation on a compact domain K, $v(s,y)$ value function at time s and point $y \in K$, there exists a state $z \in K$ such that

$$v(s,z) \geq s\rho(T) + v(0,z), \qquad \forall s \ .$$

No reachability assumption, only preserving $\mathcal{C}(K)$.

The formal analogue of the upper bound in Collatz-Wielandt for the Isaacs equation

$$v_t - H(x, Dv, D^2v) = 0, \qquad H(x, p, \cdot) \text{ order preserving}$$

is

$$\lambda = \lim_{s\to\infty} \sup v(s,\cdot)/s = \inf_{\phi\in\mathcal{C}^2} \sup_x H(x, D\phi(x), D^2\phi(x))$$

Let us explain the different notions appearing in this theorem . . .

Hemi-metric

δ is an hemi-metric on X if

- $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$
- $\delta(x,y) = \delta(y,x) = 0$ if and only if $x = y$.

Variant: weak metric of Papadoupoulos, Troyanov.

(X,δ) is complete if X is complete for the metric $d(x,y) := \max(\delta(x,y), \delta(y,x))$.

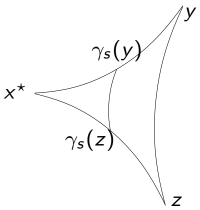
Example: the (reverse) Funk (hemi-)metric

$$\delta(x,y) = \mathrm{RFunk}(x,y) := \log\inf\{\lambda > 0 | \lambda x \geq y\}$$

Busemann convexity / nonpositive curvature condition

We say that (X, δ) is metrically star-shaped with center $x^\star$ if there exists a family of geodesics $\{\gamma_y\}_{y \in X}$, such that γ_y joins the center $x^\star$ to the point y, and

$$\delta\left(\gamma_y(s), \gamma_z(s)\right) \leq s\delta(y, z), \qquad \forall (y, z) \in X^2, \quad \forall s \in [0, 1]$$



Examples of nonpositively curved spaces. . .

X Banach space, with the choice of straight lines as goedesics

Busemann nonpositive curvature is weaker than CAT(0)

$X = \mathrm{int}\, S_n^+$, where S_n^+ is the cone of positive definite matrices, equipped with the hemi-metric

$$\delta(A,B) = \nu(\log \mathrm{Spec}(A^{-1}B)) \enspace ,$$

where $\nu : \mathbb{R}^n \to \mathbb{R}$ is symmetric, continuous, convex, positively homogeneous of degree 1 (so $\nu(\lambda x) = \lambda\nu(x)$, for all $\lambda \geq 0$), and such that $\nu(x) = \nu(-x) = 0 \implies x = 0$.
This coincides with the invariant Finsler structure

$$\delta(A,B) = \inf_{\gamma(0)=A,\, \gamma(1)=B} \int_0^1 \nu(\mathrm{Spec}(\gamma(s)^{-1}\dot\gamma(s))ds =$$

$$\Gamma_X(A) := XAX^*, \qquad \delta(\Gamma_X(A),\Gamma_X(B)) = \delta(A,B)$$

$\gamma_B(s) = P^{\frac{1}{2}}\left(P^{-\frac{1}{2}}BP^{-\frac{1}{2}}\right)^s P^{\frac{1}{2}}$ is a choice of geodesics with center P.

Riemannian metric

Cartan/Mostow, $\nu := \|\cdot\|_2$,

$$\delta(A,B) = \sqrt{\sum_i (\log \lambda_i)^2}, \quad \lambda_i \text{ eigen. of. } A^{-1}B$$

Thompson's part metric / Funk metric, (Nussbaum), $\nu := \|\cdot\|_\infty$, $\nu(x) = \max_i x_i$,

$$\begin{aligned} d_T(A,B) &= \max(\mathrm{RFunk}(A,B), \mathrm{RFunk}(A,B)) \\ &= \max_i |\log \lambda_i| \end{aligned}$$

ν symmetric jauge function: Bhatia.

Riemannian metric

Cartan/Mostow, $\nu := \|\cdot\|_2$,

$$\delta(A,B) = \sqrt{\sum_i (\log\lambda_i)^2}, \quad \lambda_i \text{ eigen. of. } A^{-1}B$$

Thompson's part metric / Funk metric, (Nussbaum), $\nu := \|\cdot\|_\infty$, $\nu(x) = \max_i x_i$,

$$\begin{aligned} d_T(A,B) &= \max(\text{RFunk}(A,B), \text{RFunk}(A,B)) \\ &= \max_i |\log\lambda_i| \end{aligned}$$

ν symmetric jauge function: Bhatia.
Nonpositive curvature corresponds to the log-majorization inequality: for all $U, V \in \text{int}\, V$, and for all $0 < s < 1$,

$$\log(\text{Spec}(U^s V^s)) \prec s \log(\text{Spec}(UV)) .$$

The discrete Riccati equation:

$$T(X) = A + M(B + X^{-1})^{-1}M^*, \qquad A, B \in S_n^+$$

is nonexpansive in any of these metrics!

Wojtowksi: Thompson metric; Bougerol: Riemanian metric; Lee and Lim: invariant Finsler metrics.

The horoboundary of a metric space

Defined by Gromov (81), see also Rieffel (Doc. Math. 02).
Fix a basepoint $\bar{x} \in X$.
$i : X \to \mathscr{C}(X)$,

$$i(x) : y \to [i(x)](y) := \delta(\bar{x}, x) - \delta(y, x).$$

so that

$$i(x)(\bar{x}) = 0, \quad \forall x \in X$$

Martin space: $\mathcal{M} := \overline{i(X)}$ (eg: product topology)
Boundary: $\mathcal{H} := \mathcal{M} \setminus i(X)$. An element of $\mathcal{H}$ is an horofunction.
A Busemann point is the limit $\lim_t i(x_t)$, where $(x_t)_{t \geq 0}$ is an infinite (almost) geodesic.
Busemann points $\subseteq$ boundary points, with equality for a polyhedral norm. See Walsh (boundary of normed space).

$i : X \to \mathscr{C}(X)$

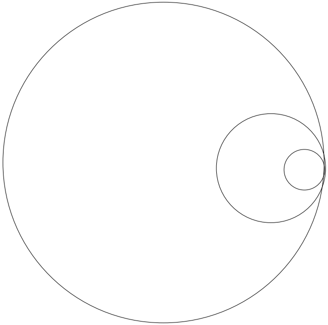
$$i(x) : y \to [i(x)](y) := \delta(\bar{x}, x) - \delta(y, x).$$

The boundary is usually defined as the closure of $i(X)$ in the topology of uniform convergence on bounded sets (Gromov, Ballman).
Indeed, i is continuous, and it is known to be an embedding if X is a complete geodesic space.
By Ascoli's theorem, on the equicontinuous set $i(X)$, the topology of uniform convergence on bounded sets and the pointwise convergence topology coincide if every closed ball is compact.

In general (infinite dimension), we need to take the pointwise convergence topology, then $i(X)$ is relatively compact, but i is no longer an embedding (the topology on $i(X)$ is too weak).

Exercise. Compute the horoboundary of $(\mathbb{R}, |\cdot|)$ and of $(\mathbb{R}^2, \|\cdot\|_1)$.

In the Poincare disk model, the level lines of horofunctions are horocircles



The Wolff-Denjoy theorem (1926) says that the orbits of a fixed point free analytic function leaving invariant the open disk converge to a boundary point (and that horodisks are invariants).

Theorem (SG and Vigeral, ibid.)

Let T be a nonexpansive self-map of a complete hemi-metric space (X, d) of non-positive curvature in the sense of Busemann. Then, there exists a Martin function h such that

$$h(T(x)) \geq \rho(T) + h(x), \qquad \forall x$$

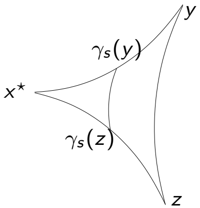
If in addition X is a metric space and $\rho(T) > 0$, then h is an horofunction.

Kohlberg-Neyman is a direct corollary. Since $h = \lim_\alpha -\| \cdot - x_\alpha \|$ modulo constants, h is concave. Take any $\varphi \in \partial h(x)$. Then,

$$\varphi(T^k(x) - x) \geq h(T^k(x)) - h(x) \geq k\rho(T) \ .$$

Proof idea. Fix a center $x^\star$, with geodesics $\gamma_y(0) = x^\star$, $\gamma_y(1) = y$. Busemann nonpositive curvature

$$\delta\left(\gamma_y(s), \gamma_z(s)\right) \leq s\delta(y, z), \qquad \forall (y, z) \in X^2, \quad \forall s \in [0, 1]$$



says that

$$r_\alpha(y) := \gamma_y(\alpha)$$

is a contraction of rate α.
The Martin function h is constructed as an accumulation point of $i(y_\alpha)$ as $\alpha \to 1^-$, where y_α is the fixed point of $T \circ r_\alpha$.

Collatz-Wielandt revisited

Let $F : C \to C$, where C is a symmetric cone (self-dual cone with a group of automorphisms acting transitively on it), say $C = \mathbb{R}^n_+$ or $C = S_n^+$.

Recall F is nonexpansive in RFunk iff it is order preserving and homogeneous of degree one.

Walsh (Adv. Geom. 08): the horoboundary of C in the (reverse) Funk metric is the Euclidean boundary: any Martin function h corresponds to some $u \in C \setminus \{0\}$:

$$h(x) = -\operatorname{RFunk}(x, u) + \operatorname{RFunk}(x^*, u) \ , \forall x \in \operatorname{int} C,$$

h is a horofunction iff $u \in \partial C \setminus \{0\}$.

Corollary (Collatz-Wielandt recovered, and more)

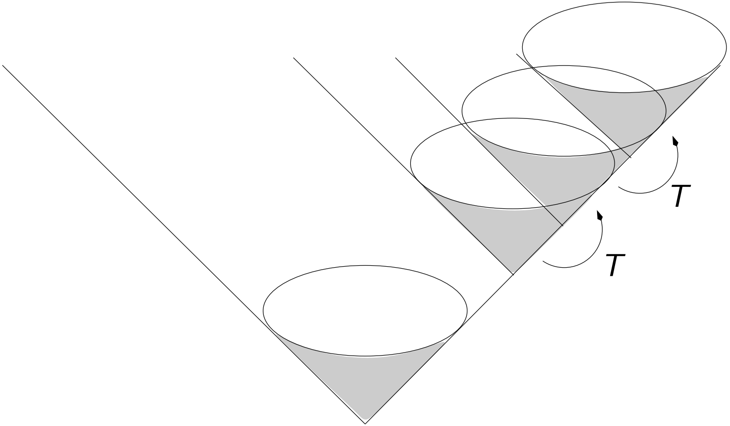
Let $T : \operatorname{int} C \to \operatorname{int} C$*, order-preserving and positively homogeneous,* C *symmetric cone. Then,*

$$\begin{aligned}
\rho(T) &:= \lim_{k\to\infty} \frac{\mathrm{RFunk}(x, T^k(x))}{k}, \qquad \forall x \in \operatorname{int} C \\
&= \inf_{y\in \operatorname{int} C} \mathrm{RFunk}(y, T(y)) \\
&= \log \inf\{\lambda > 0 \mid \exists y \in \operatorname{int} C,\ T(y) \leq \lambda y\} \\
&= \max_{u\in C\setminus\{0\}} -\mathrm{RFunk}(T(u), u) \\
&= \log \max\{\mu \geq 0 \mid \exists u \in C \setminus \{0\},\ T(u) \geq \mu u\}
\end{aligned}$$

and there is a generator w *of an extreme ray of* C *such that*

$$\log\left(w, T^k(x)\right) \geq \log\left(w, x\right) + k\rho(T), \qquad \forall k \in \mathbb{N}$$

Refines Gunawardena and Walsh, Kibernetica, 03.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T

Back to combinatorial games and tropical convexity.

Recall that $C \subset \mathbb{R}_{\max}^n$ is a tropical convex cone if

$$\lambda \in \mathbb{R}_{\max},\ u, v \in C \implies \sup\{u, v\} \in C,\ \lambda + u \in C \ .$$

Correspondence between tropical convexity and zero-sum games, part II

Theorem (Akian, SG, Guterman, arXiv:0912.2462 → IJAC)

TFAE:

- *C closed tropical convex cone*
- $C = \{u \in (\mathbb{R} \cup \{-\infty\})^n \mid u \leq T(u)\}$ *for some Shapley operator T*

and MAX has at least one winning state ($\overline{\chi}(T) \geq 0$) if and only if

$$C \neq \{(-\infty, \dots, -\infty)\} \ .$$

Proof of last statement. Think of T as a Perron-Frobenius operator in log-glasses:

$$F = \exp \circ T \circ \log, \qquad \mathbb{R}_+^n \to \mathbb{R}_+^n$$

$\overline{\chi}(T) \geq 0 \iff C \neq \{-\infty\}$ follows from Nussbaum's Collatz-Wielandt theorem, $F := \exp \circ T \circ \log$,

$$\begin{aligned} \overline{\chi}(T) &\geq 0 \\ \rho(F) &\geq 1 \\ \exists v \in \mathbb{R}_+^n, v \not\equiv 0, F(v) &\geq v \\ \exists u \not\equiv -\infty, T(u) &\geq u \end{aligned}$$

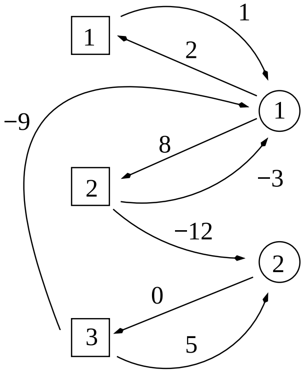
Polyhedral case

Theorem (Akian, SG, Guterman arXiv:0912.2462 → IJAC)

If the game is deterministic with finite action spaces (i.e. C is a tropical polyhedron), then the set of winning states is the support of C:

$$\{i \mid \exists u \in C,\ u_i \neq -\infty\} = \{i \mid \chi_i(T) \geq 0\}$$

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



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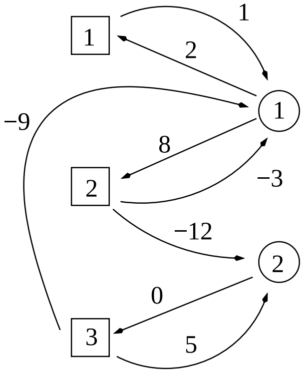

$$\chi(T) = (-1,5), x = (-\infty, 0) \text{ sol.}$$

Relies on Kohlberg's theorem 1980.

A nonexpansive piecewise affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ admits an invariant half-line

$$\exists v \in \mathbb{R}^n,\ \vec{\eta} \in \mathbb{R}^n, \quad T(v + s\vec{\eta}) = v + (s+1)\vec{\eta} \ .$$

The vector u such that $T(u) \geq u$ is obtained from v, η ($u_i = -\infty$ if $\eta_i < 0$, $u_i = v_i + s\vec{\eta}_i$ for large s otherwise).

Kohlberg's theorem uses vanishing discount

Proposition

If T is nonexpansive and piecewise affine $\mathbb{R}^n \to \mathbb{R}^n$, the discounted value $v_\alpha = T(\alpha v_\alpha)$ has a Laurent series expansion

$$v_\alpha = \frac{a_{-1}}{1-\alpha} + a_0 + (1-\alpha)a_1 + \dots, \, a_i \in \mathbb{R}^n$$

Nonexpansiveness $\implies$ 1 is necessarily a semisimple eigenvalue of $DT(x)$ at any point $x \in \mathbb{R}^n \implies$ pole of order ≤ 1.

$$T(v_\alpha - (1-\alpha)v_\alpha) = v_\alpha$$
$$T(sa_{-1} - a_0) = sa_{-1}, \qquad s \text{ large}$$

because T is piecewise affine.

Menu of tomorrow

- extreme points of tropical convex sets
- a bit more tropical geometry (tropical polynomials)
- some max-plus spectral theory
- max-plus Martin representation theorem
- deformation of Perron-Frobenius theory

Thank you !

Tropical methods for ergodic control and zero-sum games

Minilecture, Part III

Stephane.Gaubert@inria.fr

INRIA and CMAP, École Polytechnique

Dynamical Optimization in PDE and Geometry
Applications to Hamilton-Jacobi
Ergodic Optimization, Weak KAM
Université Bordeaux 1, December 12-21 2011

Today

Spectral theory

Algorithmic aspects

The max-plus spectral problem

Given $A = (A_{ij}) \in (\mathbb{R} \cup \{-\infty\})^{n \times n}$, find $v \in \mathbb{R} \cup \{-\infty\}^n$, $v \not\equiv -\infty$, $\lambda \in \mathbb{R}$, such that

$$\max_j A_{ij} + v_j = \lambda + v_i$$

$$\text{“}Av = \lambda v\text{”}$$

Among the oldest max-plus results.

Goes back to Cuninghame-Green 61, Vorobyev, Romanovski, Gondran and Minoux 77, Cohen, Dubois, Quadrat 83, . . . Some references in Akian, SG, Bapat: Handbook of linear algebra (finite dim) and Max-plus Martin boundary / discrete spectral theory (infinite dim).

Interpretation: dynamic programming, one player

Set of nodes $[d] := \{1, \ldots, d\}$, arc (i,j) with weight A_{ij}

$$A^k_{ij} = \sum_{m_1,\ldots,m_{k-1}\in[d]} A_{im_1}A_{m_1m_2}\cdots A_{m_{k-1}j}$$

Interpretation: dynamic programming, one player

Set of nodes $[d] := \{1, \ldots, d\}$, arc (i, j) with weight A_{ij}

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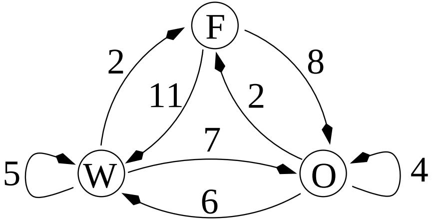
Interpretation: dynamic programming, one player

Set of nodes $[d] := \{1, \dots, d\}$, arc (i, j) with weight A_{ij}

$$
\begin{aligned}
A^k_{ij} &= \max_{m_1,\dots,m_{k-1}\in[d]} A_{im_1} + A_{m_1m_2} + \cdots + A_{m_{k-1}j} \\
&= \text{max weight path } i \to j \text{ length } k
\end{aligned}
$$

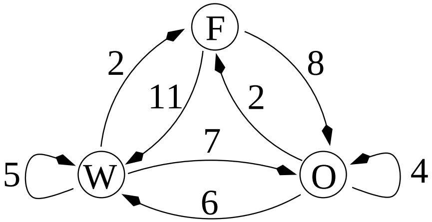
Crop rotation



A_{ij} = reward of the year if crop j follows crop i
F=fallow (no crop), W=wheat, O=oat,

$$(A^k v)_i = \sum_{j \in [d]} A^k_{ij} v_j$$

Crop rotation



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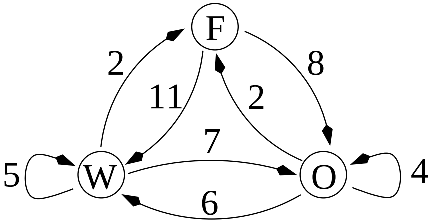

A_{ij} = reward of the year if crop j follows crop i
F=fallow (no crop), W=wheat, O=oat,

$$(A^k v)_i = \max_{j \in [d]} A^k_{ij} + v_j$$

= reward in k years, init. crop i; v_j term. reward

Eigenvector

Find $v \in \mathbb{R}_{\max}^d$, $v \not\equiv 0$, $\lambda \in \mathbb{R}_{\max}$, such that

$$Av = \lambda v$$

$$A^k v = \lambda^k v$$

Eigenvector

Find $v \in \mathbb{R}_{\max}^d$, $v \not\equiv -\infty$, $\lambda \in \mathbb{R}_{\max}$, such that

$$\max_{j \in [d]} A_{ij} + v_j = \lambda + v_i$$

$$A^k v = k\lambda + v$$

Theorem (Max-plus spectral theorem, Cuninghame-Green, 61, Gondran & Minoux 77, Cohen et al. 83)

Assume $G(A)$ is strongly connected. Then

- *the eigenvalue is unique:*

$$\rho_{\max}(A) := \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k}$$

- *Assume WLOG $\rho_{\max}(A) = 0$, then, $\exists \alpha_i \in \mathbb{R} \cup \{-\infty\}$,*

$$u = \max_{j \in \text{maximizing circuits}} \alpha_j + A^*_{\cdot,j}$$

$A^*_{ij} :=$ *max weight path arbitrary length $i \to j$.*

- "$A^{N+c} = \rho_{\max}(A)^c A^N$", $\exists N, c$

Theorem (Max-plus spectral theorem, Cuninghame-Green, 61, Gondran & Minoux 77, Cohen et al. 83)

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  $A^*_{ij} :=$ *max weight path arbitrary length $i \to j$.*

- $A^{N+c} = c\rho_{\max}(A) + A^N, \exists N, c$

The dual linear problem of

$$\min \lambda,\ A_{ij} + v_j \leq \lambda + u_i \qquad \forall i, j$$

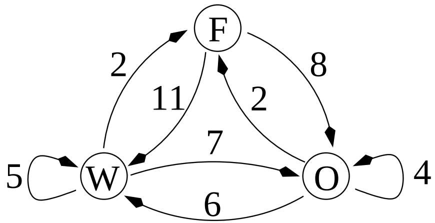
is

$$\rho(A) = \max_x \sum_{ij} A_{ij} x_{ij},\ x_{ij} \geq 0, \sum_j x_{ij} = \sum_j x_{ji}, \sum_{ij} x_{ij} = 1$$

The extreme points of the polytope of circulations are uniform measures supported by elementary circuits.

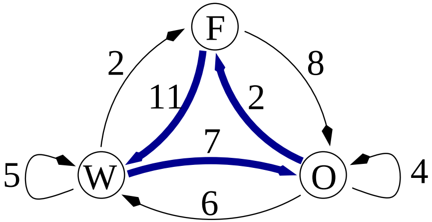
Complementary slackness shows that v, λ, x optimal iff $x_{ij}(\lambda + u_i - A_{ij} - v_j)$

Discrete version of maximizing measures.



F=fallow (no crop), W=wheat, O=oat, $\rho_{\max}(A) = 20/3$

N. Bacaer, C.R. Acad. d'Agriculture de France, 03



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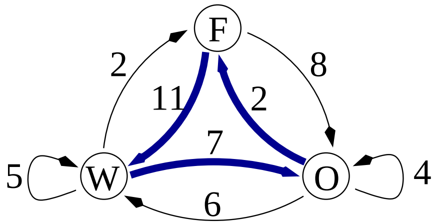


F=fallow (no crop), W=wheat, O=oat, $\rho_{\max}(A) = 20/3$

Actually, Bacaer showed that a memory of two years is needed to recover the different historical rotations

The critical graph $G^c(A)$ is the union of the maximizing circuits (analogue of Mather and Aubry sets - no difference between them in this discrete case).

Lemma

If i, j are in the same strongly connected component of the critical graph, then $A^*_{\cdot i}$ and $A^*_{\cdot j}$ are tropically proportional.

$$\text{``}A^* A^* = A^*\text{''}$$

$$\max_k A^*_{ik} + A^*_{kj} = A^*_{ij}$$

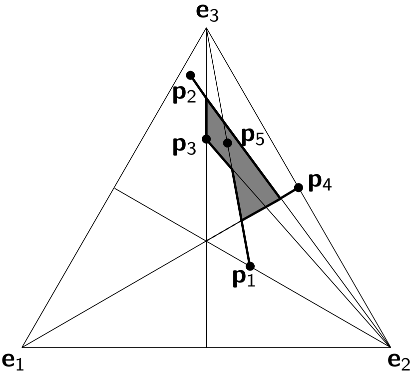
i, j in the same component means $A^*_{ij} + A^*_{ji} = 0$.

$$A^*_{kj} \geq A^*_{ki} + A^*_{ij} \geq A^*_{kj} + A^*_{ji} + A^*_{ij} = A^*_{kj}$$

A vector $u \in C$ is extreme if $u = \sup(v, w)$, $v, w \in C$ implies $u = v$ or $u = w$. I.e.,
$u \in [v, w], v, w \in C \implies u = v$ or $u = w$.

Theorem (Tropical Minkowski-Carathéodory, SG, Katz LAA07; Butkovič, Sergeev, Schneider LAA07; infinite dim Choquet Poncet thesis 11)

Every element of a closed tropical convex set of $\mathbb{R}^n_{\max}$ is the tropical convex combination of at most n extreme points.



Proof.

$$S_i(u) = \{x \in C \mid x \leq u \mid x_i = u_i\}$$

$$\text{Extr}\, C = \cup_i \text{Min}\, S_i$$

Proposition

Every $A^*_{\cdot j}$, $j \in G^c(A)$ *is* *extreme* *in the tropical cone* $\{v \mid Av = \lambda v\}$.

Cyclicity

WLOG: $\rho(A) = 0$.
The smallest c such that $A^{N+c} = A^N$ for some N (cylicity) is

$$c = \text{lcm}(\text{cyc}(K_1), \dots, \text{cyc}(K_s))$$

where $K_1, \dots, K_s$ are the strongly connected components of the critical graph, and the cyclicity of a strongly connected component is the gcd of the lengths of its circuits.

Cohen, Dubois, Quadrat, Viot 83, Nussbaum 88

Give example at the blackboard.

- If T is a nonexpansive mapping $\mathbb{R}^n \to \mathbb{R}^n$ with respect to a polyhedral norm, and if T has bounded orbits, then, any orbit converges to a periodic orbit of length bounded by a function of n and of the number of facets of the ball.
  Weller, Sine, Nussbaum, Verdyun-Lunel, Scheutzow, Lemmens, ...
- If T is a Shapley operator (order preserving, additively homogeneous) and convex (=1 player), possible orbits lengths are the orders of permutations Akian, SG 03.
- If T is a Shapley operator (2-player), the optimal bound on the length is $\binom{n}{\lfloor n/2 \rfloor}$, the size of a maximal antichain in $\{0,1\}^n$: Lemmens and Scheutzow, Ergodic Th. and Dyn. Sys.
- If T is sup-norm nonexpansive, Nussbaum conjectured the optimal length to be 2^n.

Spectral projector

WLOG $\rho(A) = 1$, $c = 1$.

$$A^N = A^{N+1} = \cdots = P, \qquad P = P^2, \qquad AP = PA$$

$$P_{ij} = \sup_k A^*_{ik} + A^*_{kj}$$

= Turnpike theorem (every long path goes through a maximizing circuit).

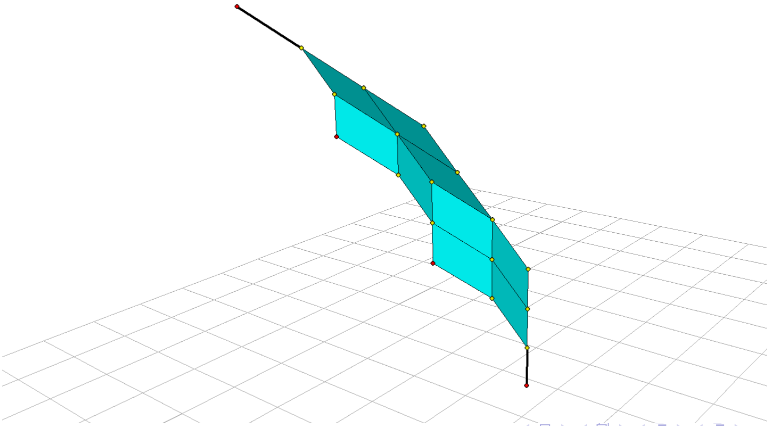
Let K denote the set of critical nodes, $E = \{u \mid Au = u\}$. The restriction $u \mapsto (u_i)_{i \in K}$ (trace on the projected Aubry set) is an isomorphism, with image

$$\{v \in \mathbb{R}^K \mid v_i - v_j \geq A^*_{ij}, \qquad \forall i, j \in K\}$$

$=$ Space of Lipschitz functions for the metric $-A^*$

Note all the tropical convex sets are images of linear projectors. The images of linear projectors arise precisely in this way.

For a Shapley operator (2 player), the tropical convex set $\{u \mid u \leq T(u)\}$ is a polyhedral complex Develin, Sturmfels Doc. Math. 04. Every cell of this complex corresponds to a strategy, and is the image of a linear projector.



The representation of the eigenspace carries over to the infinite dimensional setting.

- Generalizations to kernels appeared in works of Nussbaum and Mallet-Parret, under quasi-compactness conditions (essential spectral radius)

the existence of a continuous eigenvector is in general a difficult problem.

- Lax-Oleinik semigroups treated in book by Maslov and Kolokoltsov, Kluwer 97 (typically when the projected Aubry set is finite). Spectral projector written in this context. WKB asymptotics.

Here: abstract boundary theory

Martin boundary, discrete case (Dynkin)

Given P_{xy} Markov kernel, over a discrete infinite set E, find all nonnegative harmonic functions: $u = Pu$.
1) Define the Green kernel: $G = P^0 + P + P^2 + \cdots$
2) The Martin kernel is:

$$K_{xy} = \frac{G_{xy}}{G_{by}}$$

where $b \in E$ is a basepoint.
3) Let $\mathcal{K} := \{K_{\cdot y} \mid y \in E\}$
4) The Martin space $\mathcal{M}$ is the closure of $\mathcal{K}$ in the product topology.
5) The Martin boundary is $\mathcal{B} := \mathcal{M} \setminus \mathcal{K}$.

Theorem (classical Martin representation)

Every harmonic function u can be written as a positive linear combination of functions from the boundary:

$$u = \int_{\mathcal{B}} w \mu(dw) \ .$$

μ can be choosen to be supported by a subset of $\mathcal{B}$, the minimal Martin boundary. (We recognise Choquet's theorem!).

Computing the probabilistic Martin boundary is difficult, eg. Ney and Spitzer 65, boundary of random walk in $\mathbb{Z}^2$ is the circle, computing the tropical analogue is much easier!

The max-plus Martin boundary

Akian, SG, Walsh, CDC06, Doc. Math. 09 (Semigroup version), Ishii, Mitake 07 (PDE version).

Consider the eigenproblem over an arbitrary state space S

$$u_x = \sup_{y \in S} A_{xy} + u_y, \qquad \forall x \in S$$

The Martin kernel reads: $K_{xy} = A^*_{xy} - A^*_{by}$.
The Martin space $\mathcal{M}$ is the closure of $\mathcal{K} := \{K_{\cdot,y} \mid y \in S\}$ in the product topology (compact, Tychonoff). Martin boundary (set of horofunctions) is $\mathcal{B} = \mathcal{M} \setminus \mathcal{K}$.

When $A^*_{x,y} = -d(x,y)$ is the opposite of a metric, recover the construction of the horoboundary by Gromov.

The detour metric

$$A^* = \text{“}I + A^+\text{”}, \qquad A^+ = \text{“}A + A^2 + A^3 + \ldots\text{”}$$

$$A^+_{xy} = \sup(A_{xy}, A^2_{xy}, A^3_{xy}, \ldots)$$

$$H^\flat_{xy} = A^+_{bx} + A^+_{xy} - A^+_{by} \quad \text{detour penalty}$$

Extend $H^\flat$ to the whole Martin space

$$H^\flat(u,v) = \limsup_{x_d \to u} \liminf_{y_e \to v} H^\flat_{x_d, y_e}$$

where the limsup,inf are taken along nets x_d and y_e converging to u and v in the topology of the Martin space.

The Minimal Martin space is
$\mathcal{M}^m := \{w \in \mathcal{M} \mid H^\flat(w,w) = 0\}$.

Theorem (Max-plus Martin representation Akian, SG, Walsh, CDC06, Doc. Math. 09**)**

$\mathcal{M}^m$ is the set of extreme elements of $\{u \mid Au = u\}$. Any such u can be written as

$$u = \sup_{w \in \mathcal{M}_m} w + \mu(w), \qquad \mu : \mathcal{M}_m \to \mathbb{R} \cup \{-\infty\} \quad scs$$

$$\mu_u(w) := \limsup_{x_d \to w} A^*_{bx_d} + u(x_d)$$

Analogous to max-plus integral representations by Fathi, Siconolfi, Contreras, Ishii, Mitake, in different settings.

If the Martin space is metrisable, then $\mathcal{M}_m$ is precisely the set of Busemann points = limits of quasi-geodesics, i.e. of sequences $x_1, x_2, \dots$ such that there exists $\alpha \in \mathbb{R}$

$$A^*_{bx_k} \leq A^*_{bx_1} + A_{x_1 x_2} + \cdots + A_{x_{k-1} x_k} + \alpha, \qquad \forall k$$

Quasi geodesics correspond to almost-sure trajectories of the renormalized H-process of Dynkin.

Lax-Oleinik (continuous time) version in CDC06.

Linear quadratic control - nonquadratic solutions

Hamilton–Jacobi equation

$$\lambda = -|\mathbf{x}|^2 + \frac{1}{4}|\nabla w|^2$$

Maximise reward:

$$-\int_0^T (|\gamma(t)|^2 + |\dot{\gamma}(t)|^2 + \lambda)\, dt,$$

If $\lambda > 0$, solutions are

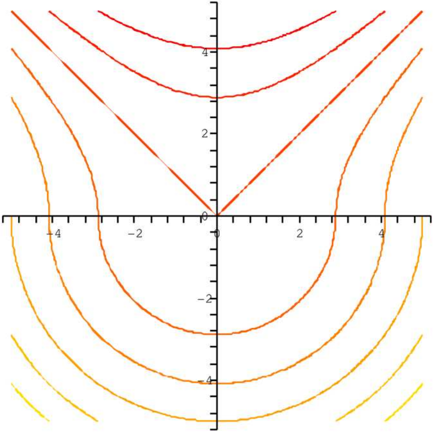
$$w(\mathbf{x}) = \sup_{\mathbf{n}}(\nu(\mathbf{n}) + h_{\mathbf{n}}(\mathbf{x})),$$

where ν is an upper semi–continuous map from the unit vectors to $\mathbb{R} \cup \{-\infty\}$.

When $\lambda = 0$, there is a horofunction for each direction $\mathbf{n}$:

$$h_{\mathbf{n}}(\mathbf{x}) = \begin{cases} -|\mathbf{x}|^2 + 2(\mathbf{x}\cdot\mathbf{n})^2, & \text{if } \mathbf{x}\cdot\mathbf{n} > 0, \\ -|\mathbf{x}|^2, & \text{otherwise.} \end{cases}$$

The function $-|\mathbf{x}|^2$ is also a horofunction.

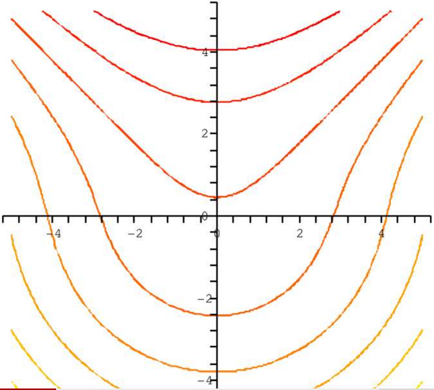


Horospheres of h_n with $n = (0, 1)$.

When $\lambda > 0$: for each direction $\mathbf{n}$,

$$h_{\mathbf{n}}(\mathbf{x}) = -\lambda \frac{|\mathbf{x}|^2}{R^2} + \mathbf{x} \cdot \mathbf{n} \frac{\lambda + 2|\mathbf{x}|^2}{R} - \lambda \log \frac{R}{\sqrt{\lambda}},$$

where $R := \sqrt{(\mathbf{x} \cdot \mathbf{n})^2 + \lambda} - \mathbf{x} \cdot \mathbf{n}$.



Back to finite dimension.

The max-plus spectral problem as a limit of the Perron-Frobenius problem

Deformation of the Perron root

Chain of spins (Ising)

$$Z = \sum_{\sigma_1,\dots,\sigma_n \in \Sigma^N} \exp(-\sum_{i=1}^{N} E(\sigma_i,\sigma_{i+1})/T), \qquad \sigma_{N+1} := \sigma_1$$

$-E(\sigma,\sigma') = H\sigma + J\sigma\sigma'$, $\sigma,\sigma' \in \{\pm 1\}$ (Ising)

$$Z_N = \operatorname{tr} M_T^N, \qquad (M_T)_{\sigma\sigma'} = \exp(-E(\sigma,\sigma')/T)$$

$F_N = N^{-1} T \log Z_N \sim T \log \rho(M_T)$ free energy per site,

$T \to 0$, ground state

$$\epsilon := \exp(-1/T), \qquad (M_T)_{\sigma,\sigma'} = \epsilon^{E(\sigma,\sigma')}$$

Similar to perturbation problems, but now, the “Puiseux series” have real exponents (Dirichlet series).

Kingman 61:

$$\log \circ \rho \circ \exp \qquad \text{convex [entrywise exp]}$$

Let $A, B \geq 0$, and $C = A^{(s)} \circ B^{(t)}$, with $s + t = 1, s, t \geq 0$ [entrywise product and exponent] then

$$\rho(C) \leq \rho(A)^s \rho(B)^t \enspace .$$

Indeed, $\log \rho(C) = \lim_m \log \|C^m\|/m$ is a pointwise limit of convex functions of $(\log C_{ij})$, for any monotone norm. □

So

$$\rho(A \circ B) \leq \rho(A^{(p)})^{1/p} \rho(B^{(q)})^{1/q} \qquad 1/p + 1/q = 1$$

$$\rho(B^{(q)})^{1/q} \to \max_{i_1,\dots,i_m} (B_{i_1 i_2} \cdots B_{i_{m-1} i_m})^{1/m} =: \rho_\infty(B)$$

Theorem (Friedland 86)

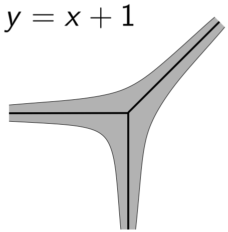
For all $A \in \mathbb{C}^{n \times n}$,

$$\rho(A) \leq \rho(pattern(A)) \rho_\infty(|A|) \leq n \rho_\infty(|A|)$$

and

$$\rho(A) \geq \rho_\infty(A) \qquad \textit{if } A_{ij} \geq 0 \ .$$

Explanation: approximation of an amoeba by its skeletton
$V \subset (\mathbb{C}^*)^n$, $A(V) = \{(\log|z_1|, \ldots, \log|z_n|) \mid x \in V\}$.



Cf. Gelfand, Kapranov, Zelevinsky; Passare, Rüllgaard; Purbhoo; Yger.

Limit of the Perron eigenvector. Consider $A^{(p)} = (A_{ij}^p)$, and let $U(p)$ denote the normalized Perron eigenvector of $A^{(p)}$.

Taking $p^{-1}\log$ / passing in the limit in

$$\lambda(p)U_i^p(p) = \sum_j A_{ij}^p U_j^p$$

we get that

$$\lambda + u_i = \max_j \log A_{ij} + u_j$$

where λ and u_j are accumulation points of $p^{-1}\log\lambda(p)$, $\log U_j(p)$, resp.

Which tropical eigenvector is selected?

WLOG $\lambda = \log \rho_\infty(A) = 0$.

Theorem (Akian, Bapat, SG CRAS 1998)

If there is only one SCC of the critical graph with maximal Perron root, then $u_i = (\log A)^*_{ij}$, *for any* j *in this class.*

Related work by Lopes, Mohr, Souza, Thieullen.

Give example at the blackboard.

Proof idea. Make diagonal scaling

$$B(p) = diag(\exp(-pu))A^p \,\mathrm{diag}(\exp(pu)) \enspace .$$

The matrix $B(p)$ has a limit in $[0,1]^{n\times n}$ as $p \to \infty$. We want $B(\infty)$ to have a positive eigenvector. A nonnegative matrix has a positive eigenvector iff the basic classes are exactly the final classes.
For the choice of eigenvector $u = (\log A)^*_{\cdot j}$, this is the case, because the saturation graph

$$\{(k,l) \mid \log A_{kl} + u_l = u_k\}$$

is a river network with sea $\mathrm{SCC}(j)$. Make drawing.

An application: perturbation of eigenvalues

Exercise.

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} ,$$

An application: perturbation of eigenvalues

Exercise.

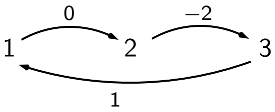
$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} ,$$

Show without computation that the eigenvalues have the following asymptotics as $\epsilon \to 0$

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} \quad , \qquad A = \begin{bmatrix} 1 & 0 & 4 \\ \infty & 1 & -2 \\ 1 & 2 & \infty \end{bmatrix} \quad .$$

We have $\gamma_1 = -1/3$, corresponding to the critical circuit:



Eigenvalues:

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

Assume that the entries of $\mathcal{A}_\varepsilon$ have Puiseux series expansions in ϵ, or even that $\mathcal{A}_\varepsilon = a + \epsilon b$, $a, b \in \mathbb{C}^{n\times n}$.

$\mathcal{L}_1, \dots, \mathcal{L}_n$ eigenvalues of $\mathcal{A}_\varepsilon$.

$v(s)$: opposite of the smallest exponent of a Puiseux series s.

$\gamma_1 \geq \cdots \geq \gamma_n$: tropical eigenvalues of $v(A_\epsilon)$.

Theorem (Akian, Bapat, SG CRAS04, arXiv:0402090)

$$v(\mathcal{L}_1) + \cdots + v(\mathcal{L}_n) \leq \gamma_1 + \cdots + \gamma_n$$

and equality holds under generic (Lidski-type) conditions.

The maximal tropical eigenvalue γ_1 coincides with the ergodic constant of the one-player game

$$\lambda + u_i = \max_{1 \leq j \leq n} \left(\mathrm{val}(A_\epsilon)_{ij} + u_j \right), \forall i$$

λ is the maximal circuit mean.

In general, tropical eigenvalues are non-differentiability points of a parametric optimal assignment problem = Legendre transform a the generic Newton polygon

The (algebraic) tropical eigenvalues of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ are the roots of

$$\text{“}\mathrm{per}(A + xI)\text{”}$$

where

$$\text{“}\mathrm{per}(M)\text{”} := \text{“}\sum_{\sigma\in S_n}\prod_{i\in[n]} M_{i\sigma(i)}\text{”}$$

All geom. eigenvalues λ (“$Au = \lambda u$”) are algebraic eigenvalues, but the converse does not hold. $\rho_{|max}(A)$ is the max algebraic eigenvalue.

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- All geom. eigenvalues λ (“$Au = \lambda u$”) are algebraic eigenvalues, but the converse does not hold. $\rho_{|max}(A)$ is the max algebraic eigenvalue.
- Trop. eigs. can be computed in $O(n)$ calls to an optimal assignment solver (Butkovič and Burkard) (not known whether the formal characteristic polynomial can be computed in polynomial time).

Theorem (Kapranov)

If $f(z) = \sum_k f_k z^k \in \mathbb{C}\{\{\epsilon\}\}[z_1, \ldots, z_n]$*, the closure of the image of* $f = 0$ *by* v *is the set of points* $x \in \mathbb{R}^n$ *at which the maximum*

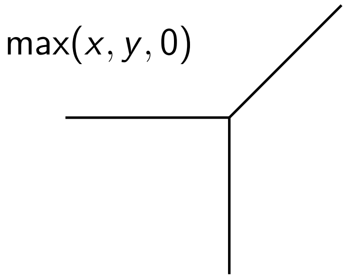
$$\max_k v(f_k) + \langle k, x \rangle$$

is attained at least twice.

Follows from Puiseux theorem when $n = 1$. Inclusion $\subset$ obvious. Converse: reduction to Puiseux.

When $n = 1$: the set of tropical roots is a zero-dimensional amoeba

Example. $y = x + 1$, $K = \mathbb{C}\{\{\epsilon\}\}$



Algorithms for games

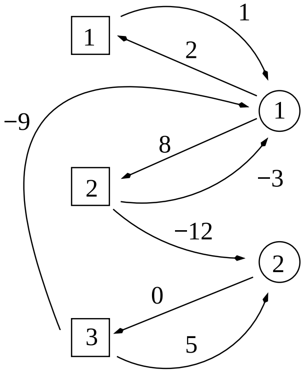
$$H_i : \max_{1\leq j\leq n} a_{ij} + x_j \leq \max_{1\leq k\leq n} b_{ik} + x_k$$

$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\leq k\leq n} b_{ik} + x_k \quad .$$

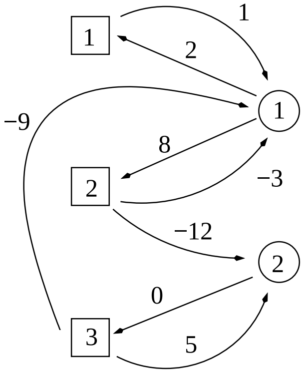
Interpretation of the game

- State of MIN: variable x_j, $j \in \{1, \dots, n\}$
- State of MAX: half-space H_i, $i \in I$
- In state x_j, Player MIN chooses a tropical half-space H_i with x_j in the LHS
- In state H_i, player MAX chooses a variable x_k at the RHS of H_i
- Payment $-a_{ij} + b_{ik}$.

$$A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}$$



$$\begin{aligned} 2 + x_1 &\leq 1 + x_1 \\ 8 + x_1 &\leq \max(-3 + x_1, -12 + x_2) \\ x_2 &\leq \max(-9 + x_1, 5 + x_2) \end{aligned}$$



$$\chi(T) = \lim_k v^k/k = (-1, 5)$$

Proposition

If T is nonexpansive and piecewise affine $\mathbb{R}^n \to \mathbb{R}^n$, the discounted value $v_\alpha = T(\alpha v_\alpha)$ has a Laurent series expansion

$$v_\alpha = \frac{a_{-1}}{1-\alpha} + a_0 + (1-\alpha)a_1 + \dots, a_i \in \mathbb{R}^n$$

This is the case for a stochastic game with perfect information and finite action spaces.

Then

$$\chi(T) = \lim_k T^k(0)/k = a_{-1} \ .$$

- Strategy of MAX $\sigma : \{H_1, \dots, H_m\} \to \{x_1, \dots, x_n\}$, in state H_i choose coordinate $x_{\sigma(i)}$

Duality theorem (coro of Kohlberg) If finite action spaces, then

$$\chi(T) = \max_{\sigma} \chi(T^{\sigma}) = \min_{\pi} \chi(T_{\pi}) \enspace .$$

- Strategy of MAX $\sigma : \{H_1, \dots, H_m\} \to \{x_1, \dots, x_n\}$, in state H_i choose coordinate $x_{\sigma(i)}$
- Strategy of MIN $\pi : \{1, \dots, n\} \to \{1, \dots, m\}$, in state x_j choose hyperplane $H_{\pi(j)}$

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- Strategy of MIN $\pi : \{1, \dots, n\} \to \{1, \dots, m\}$, in state x_j choose hyperplane $H_{\pi(j)}$
- One player Shapley operators

$$[T^\sigma(x)]_j = \inf_{1 \le i \le m} -a_{ij} + b_{i\sigma(i)} + x_{\sigma(i)} \enspace .$$

$$[T_\pi(x)]_j = -a_{\pi(j)j} + \max_{1 \le k \le n} b_{\pi(j)k} + x_k \enspace .$$

Duality theorem (coro of Kohlberg) If finite action spaces, then

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Duality theorem (coro of Kohlberg) If finite action spaces, then

$$\chi(T) = \max_\sigma \chi(T^\sigma) = \min_\pi \chi(T_\pi) \enspace .$$

Every $\chi(T^\sigma)$ and $\chi(T_\pi)$ can be computed in polynomial time.

Proof: Blackwell optimality

For all $x \in \mathbb{R}^n$, we have a selection

$$\exists \sigma, \pi, \; T(x) = T^\sigma(x) = T_\pi(x) \; .$$

So for all $0 < \alpha < 1$, the discounted value $v_\alpha = T(\alpha v_\alpha)$ satsifies

$$v_\alpha(T) = \max_\sigma v_\alpha(T^\sigma) = \min_\pi v_\alpha(T_\pi) \; .$$

Since χ is the first coefficient of the Laurent series

$$\chi(T) = \max_\sigma \chi(T^\sigma) = \min_\pi \chi(T_\pi) \; .$$

σ, π are Blackwell optimal if optimal for all $\alpha \in (\bar{\alpha}, 1)$ (exist because the zeros of a Laurent series cant accumulate at 1^-).

Corollary (Condon 92, Zwick and Paterson, TCS 96)

Mean payoff games are in NP $\cap$ co-NP.

- I can convince you that $\chi_i(T) \geq 0$ (initial state i is winning) by giving you a strategy σ of MAX such that $\chi_i(T^\sigma) \geq 0$. You can check that in polynomial time by solving a one player game.
- I can convince you that the opposite is true by giving you a strategy π of MIN such that $\chi_i(T_\pi) < 0$. You can also check this in polynomial time.

The class NP $\cap$ co-NP captures the good characterizations of Edmonds. Evidence that the problem is not NP-complete.

- "$Ax \leq Bx$" unfeasible iff $\exists \pi$, $\overline{\chi}(T_\pi) < 0$.

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- "$Ax \leq Bx$" feasible iff $\exists \sigma$, $\overline{\chi}(T^\sigma) \geq 0$.

- “$Ax \leq Bx$” unfeasible iff $\exists \pi$, $\overline{\chi}(T_\pi) < 0$.
- “$Ax \leq Bx$” feasible iff $\exists \sigma$, $\overline{\chi}(T^\sigma) \geq 0$.
- $\exists x \in \mathbb{R}^n_{\max}$, $Ax \leq Bx$? is in NP $\cap$ co-NP

Corollary

Feasibiliby in tropical linear programming, i.e.,

$$\exists ?u \in (\mathbb{R} \cup \{-\infty\})^n, \ \max_j a_{ij} + u_j \leq \max_j b_{ij} + u_j, \ 1 \leq i \leq p$$

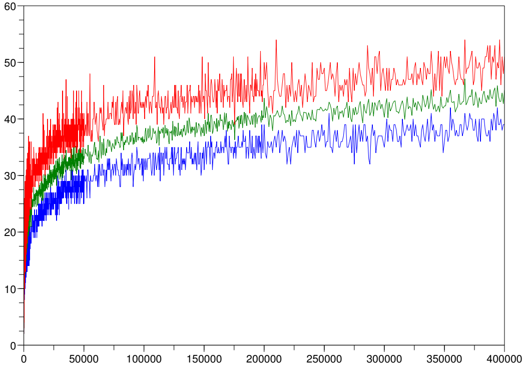
is polynomial-time equivalent to mean payoff games.

are in NP $\cap$ coNP: Zwick, Paterson 96.

Tropical convex sets are log-limits of classical convex sets: polynomial time solvability of mean payoff games might follow from a strongly polynomial-time algorithm in linear programming (Schewe).

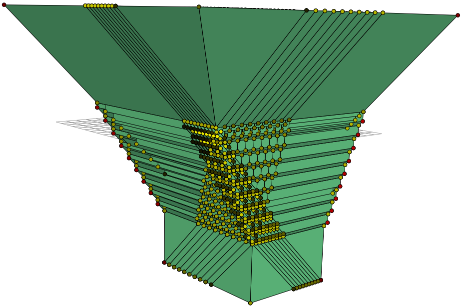
Several pseudo-polynomial algorithms exist for (deterministic) mean payoff games: Zwick, Paterson TCS96. No pseudo-polynomial algorithm seems to be known for stochastic mean payoff game. However, Policy iteration works (Cochet,SG 06), - based on a tropical idea = spectral projectors - ; alternative algorithm by Boros, Gurvich, Elbassioni, Makino, . . .

Policy iteration for games scales well in practice. ♯ iterations / ♯ nodes



However, Friedmann LICS 10 showed that policy iteration for games can be exponential.

Intersection of 10 affine tropical hyperplanes in dimension 3, only 24 vertices, but 1215 pseudo-vertices.



Tropical double description Allamigeon, SG, Goubault.
Efficient implementation in TPLib/caml by Allamigeon.

Concluding remarks

- Tropical algebra $\sim$ discrete version of Weak KAM
- Tropical convex cones arises when considering spaces of weak KAM solutions (1-player), or sub/super solutions.
- Combinatorial properties in the discrete case (lenghts of periodic orbits)
- Thinking tropical brings “complex” perspective on Lax-Oleinik semigroups (not just one eigenvalue)
- Relation between ergodic problem and optimal assignment appears in the discrete case (the eigenvalues are nondifferentiability points of an optimal assignment problem), is there a PDE analogue (relation with mass transport problem)?
- Tropical algebra is fun!

Thank you