Max-plus algebra

. . . a guided tour

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Max-plus or tropical algebra

The max-plus semiring is the set $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$, equipped with

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So $2 + 3 = 3$, $2 \times 3 = 5$; $2^3 = ?\ 2 \times 3 = 6$.
$\sqrt{-1} = ?\ -0.5 \quad \text{for } (-0.5)^2 = -1$.

Variants

- min-plus semiring $\mathbb{R}_{\text{min}} := \mathbb{R} \cup \{+\infty\}$ with $a + b = \min(a, b)$, $ab = a + b$,
- max-times semiring $\mathbb{R}_{+}$ with $a + b = \max(a, b)$, $ab = a \times b$.
- boolean semiring $\{0, 1\}$, $a + b = a$ or b, $ab = a$ and b, subsemiring of the above

Studied by several researchers / schools including

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev ∼1965 ... Zimmerman, Butkovic; Optimization
- Maslov ∼ 80'- ... Kolokoltsov, Litvinov, Samborskii, Shpiz... Quasi-classic analysis, variations calculus
- Simon ∼ 78- ... Hashiguchi, Leung, Pin, Krob, ... Automata theory
- Gondran, Minoux ∼ 77 Operations research
- Cohen, Quadrat, Viot ∼ 83- ... Olsder, Baccelli, S.G., Akian initially discrete event systems, then optimal control, idempotent probabilities, combinatorial linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney ∼00- Optimal control
- Puhalskii ∼99- idempotent probabilities (large deviations)
- Viro; Mikhalkin, Passare, Sturmfels; Shustin, Itenberg, Kharlamov, Speyer, Develin , Joswig, Yu ... tropical geometry (emerged ∼ 02)

Like Monsieur Jourdain in Moliere's play "Le bourgeois gentilhomme", who was doing prose without knowing it. . .

every control theorist already knows max-plus

Lagrange problem / Lax-Oleinik semigroup

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S^t)_{t\geq 0}$, $S^t\phi := v(t,\cdot)$.

Superposition principle: $\forall \lambda \in \mathbb{R}$, $\forall \phi,\psi$,

$$\begin{aligned} S^t(\sup(\phi,\psi)) &= \sup(S^t\phi, S^t\psi) \\ S^t(\lambda+\phi) &= \lambda + S^t\phi \end{aligned}$$

So S^t is max-plus linear.

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The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0, \cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \sup_{y \in \mathbb{R}^n} tL(\frac{x - y}{t}) + \phi(y) \ .$$

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$$v(t,x) = \int G(x-y)\phi(y)dy \ .$$

Classical	Maxplus
Expectation	sup
Brownian motion	$L(\dot{\mathbf{x}}(s)) = (\dot{\mathbf{x}}(s))^2/2$
Heat equation:	Hamilton-Jacobi equation:
$\frac{\partial v}{\partial t} = -\frac{1}{2}\Delta v$	$\frac{\partial v}{\partial t} = \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$
$\exp(-\frac{1}{2}\|x\|^2)$	$-\frac{1}{2}\|x\|^2$
Fourier transform:	Fenchel transform:
$\int \exp(i\langle x, y\rangle) f(x) dx$	$\sup_x \langle x, y\rangle - f(x)$
convolution	inf or sup-convolution

See Akian, Quadrat, Viot, Duality & Opt. ...

Max-plus basis / finite-element method

Fleming, McEneaney 00-; Akian, Lakhoua, SG 04-

Approximate the value function by a linear comb. of "basis" functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

$$v(t,\cdot) \simeq \sum_{i\in[p]} \lambda_i(t) w_i$$

The w_i are given finite elements, to be chosen depending on the regularity of $v(t,\cdot)$

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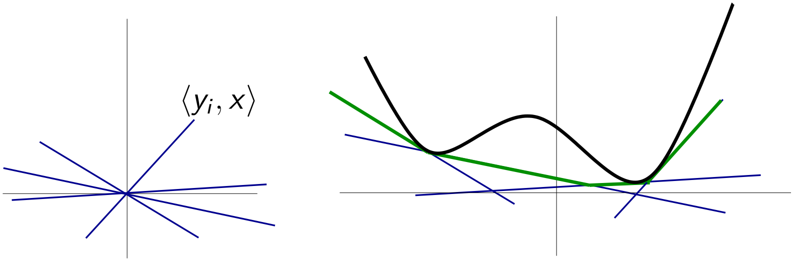
Best max-plus approximation

$$P(f) := \max\{g \leq f \mid g \text{ linear comb. of } w_i\}$$

linear forms $w_i : x \mapsto \langle y_i, x \rangle$

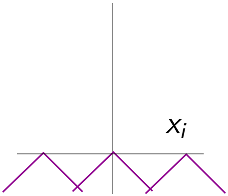


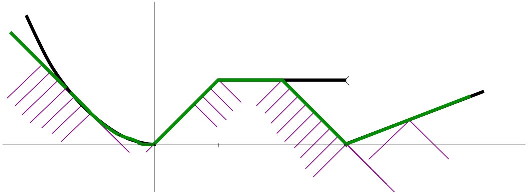
adapted if v is convex

Best max-plus approximation

$$P(f) := \max\{g \leq f \mid g \text{ linear comb. of } w_i\}$$

cone like functions $w_i : x \mapsto -C\|x - x_i\|$





adapted if v is C-Lip

Use max-plus linearity of S^h:

$$v^t = \sum_{i \in [p]} \lambda_i(t) w_i$$

and look for new coefficients $\lambda_i(t+h)$ such that

$$v^{t+h} \simeq \sum_{i \in [p]} \lambda_i(t+h) w_i$$

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Max-plus variational approach

Max-plus scalar product

$$\langle w, z \rangle := \int w(x) z(x) dx$$

For all test functions z_j, $j \in [q]$

$$\langle v^{t+h}, z_j \rangle = \sum_{i \in [p]} \lambda_i(t+h) \langle w_i, z_j \rangle$$
$$= \sum_{k \in [p]} \lambda_k(t) \langle S^h w_k, z_j \rangle$$

Max-plus variational approach

Max-plus scalar product

$$\langle w, z \rangle := \sup_x w(x) + z(x)$$

For all test functions z_j, $j \in [q]$

$$\sup_{i \in [p]} \lambda_i(t+h) + \langle w_i, z_j \rangle$$
$$= \sup_{k \in [p]} \lambda_k(t) + \langle S^h w_k, z_j \rangle$$

This is of the form

$$A\lambda(t+h) = B\lambda(t), \qquad A, B \in \mathbb{R}_{\max}^{q\times p}$$

The linear system $A\mu = b$ generically has no solution $\mu \in \mathbb{R}^p$, however, $A\mu \leq b$ has a maximal solution $A^\sharp b$ given by

$$(A^\sharp b)_j := \min_{i\in[q]} -A_{ij} + b_i \ .$$

Cohen, SG, Quadrat, LAA 04, Akian, SG, Kolokoltsov: Moreau conjugacies

So, the coeffs of $v(t+h)$ are recursively given:

$$\lambda(t+h) = A^\sharp B \lambda(t) \quad .$$

The global error is controlled by the projection errors of all the $v(t,\cdot)$. The method is efficient if $S^h w_i$ is evaluated by a high order scheme. Then, $A^\sharp B$ glues the characteristics in time h.

McEneaney's curse of dimensionality reduction

Suppose the Hamiltonian is a finite max of Hamiltonians arising from LQ problems

$$H = \sup_{i\in[r]} H_i, \;\; H_i = -(\frac{1}{2}x^* D_i x + x^* A_i^* p + \frac{1}{2}p^*\Sigma_i p)$$

(=LQ with switching). Let S^t and S_i^t denote the corresponding Lax-Oleinik semigroups, S_i^t is exactly known (Riccati!)

Want to solve $v = S^t v, \forall t \geq 0$

Choose a quadratic function ϕ such that $S^t\phi \to v$ as $t \to \infty$. Then, for $t = hk$ large enough,

$$v \simeq (S^h)^k\phi \ .$$

This is a sup of quadratic forms. Inessential terms are trimmed dynamically using Shor relaxation (SDP) $\to$ solution of a typical instance in dim 6 on a Mac in 30'

McEneaney, Desphande, SG; ACC 08

Choose a quadratic function ϕ such that $S^t\phi \to v$ as $t \to \infty$. Then, for $t = hk$ large enough,

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Ergodic control and games

Find a function ϕ and a scalar $\lambda \in \mathbb{R}$ such that

$$S^t\phi = \lambda t + \phi \quad \text{i.e.} \quad S^t\phi = \lambda^t\phi, \quad \forall t \geq 0 \ .$$

$$\lambda = H(x, \frac{\partial \phi}{\partial x}) \ .$$

The eigenfunction ϕ avoids the "horizon effect": it forces the player to act in finite horizon as if she would live forever

Some degenerate problems have been solved using max-plus techniques

- $L = T + V$, V potential with several points of minima Kolokoltsov, Maslov 92
- noncompact state space, max-plus Martin boundary / horoboundaries, extreme eigenvectors = limits of geodesics Akian, SG, Walsh, Doc. Math. to appear, Ishii Mitake: viscosity solution version
- related to Fathi weak-KAM theory

The PDE results are best understood by looking at the elementary finite dim case. . .

Finite dimensional spectral problem

Given $A \in \mathbb{R}_{\max}^{n\times n}$, find $u \in \mathbb{R}_{\max}^{n}$, $u \neq 0$, $\lambda \in \mathbb{R}_{\max}$, such that

$$Au = \lambda u$$

$G(A)$: arc $i \to j$ if $A_{ij} \neq -\infty$

Finite dimensional spectral problem

Given $A \in \mathbb{R}_{\max}^{n\times n}$, find $u \in \mathbb{R}_{\max}^{n}$, $u \not\equiv -\infty$, $\lambda \in \mathbb{R}_{\max}$, such that

$$\max_{j\in[n]} A_{ij} + u_j = \lambda + u_i$$

$G(A)$: arc $i \to j$ if $A_{ij} \neq -\infty$

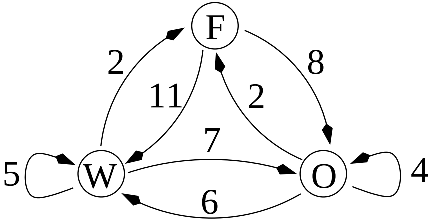
Theorem (Max-plus spectral theorem, Part I: Cuninghame-Green, 61)

Assume $G(A)$ is strongly connected. Then

- *the eigenvalue is unique:*

$$\rho_{\max}(A) := \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k}$$

Example: crop rotation



A_{ij} = reward of the year if crop j follows crop i

F=fallow (no crop), W=wheat, O=oat,

$\rho_{\max}(A) = 20/3$

N. Bacaer, C.R. Acad. d'Agriculture de France, 03

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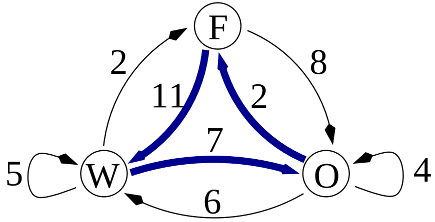

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Theorem (Max-plus spectral theorem, Part II: Gondran & Minoux 77, Cohen et al. 83)

- *The dim. of the eigenspace is the $\sharp$ of strongly connected components of the critical graph (union of the maximizing circuits) of A $\sim$ discrete Mather or Aubry set*

The max-plus spectral theorem looks like the Perron-Frobenius theorem ...

Metatheorem

What is known for (max-plus, positive) linear maps often carries over to

- *order preserving, sup-norm nonexpansive maps*
- *order preserving, positively homogeneous maps on cones*

Nussbaum, Akian, SG, Gunawardena, Lemmens...

Dynamic programming operators of zero-sum games

Every order preserving, sup-norm non-expansive map can be written as

$$f_i(x) = \inf_{a\in A(i)} \sup_{b\in B(i,a)} r_i^{ab} + P_i^{ab} x$$

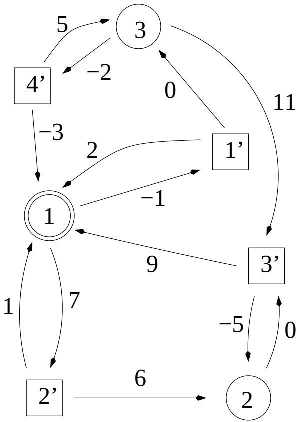
zero-sum, two player, infinite action spaces, $r_i^{ab} \in \mathbb{R}$, P_i^{ab} substochastic vector.
The game may be even assumed to be deterministic (P_i^{ab} degenerate), Rubinov, Singer.

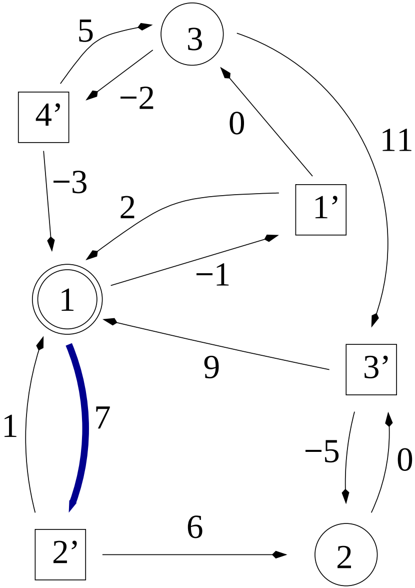
Combinatorial games with mean payoff

$G = (V, E)$ directed bipartite graph, r_{ij} weight of arc $(i, j) \in E$.
"Max" and "Min" move a pawn. Payments (made by Min to Max) correspond to moves.
The reward of Max (or the loss of Min) after k turns is

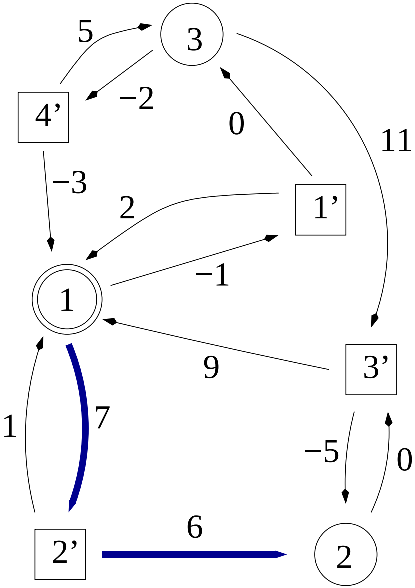
$$r_{i_0 i_1} + \cdots + r_{i_{2k-1} i_{2k}}$$

The circles (resp. squares) represent the nodes at which Max (resp. Min) can play.





If Max initially moves to $2'$



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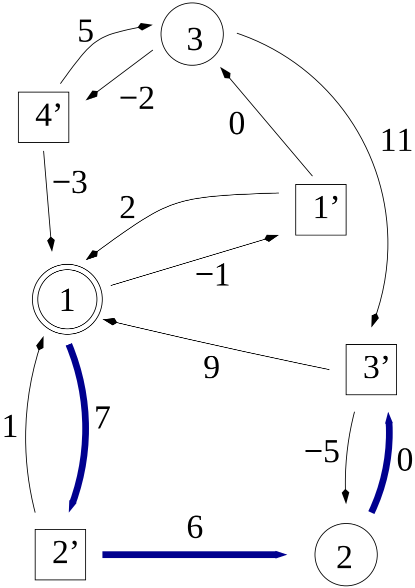

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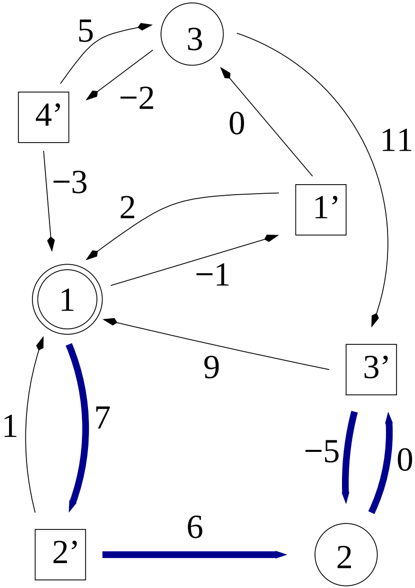
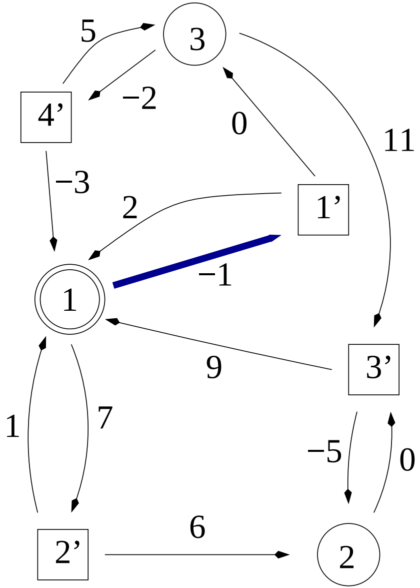
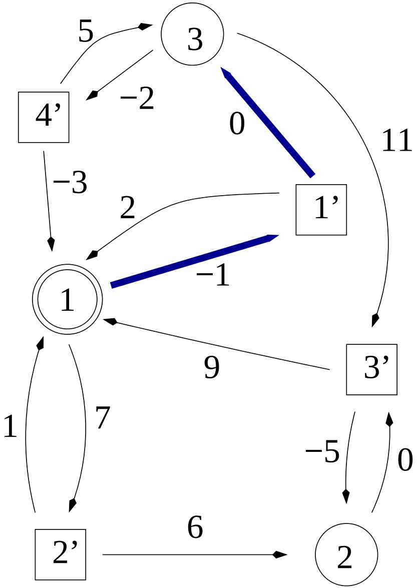


If Max initially moves to $2'$

he eventually looses 5 per turn.



But if Max initially moves to $1'$



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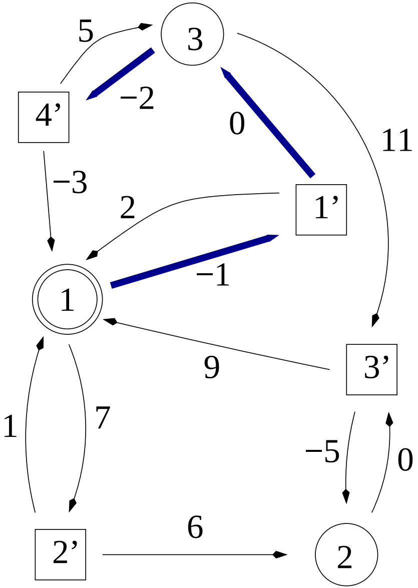

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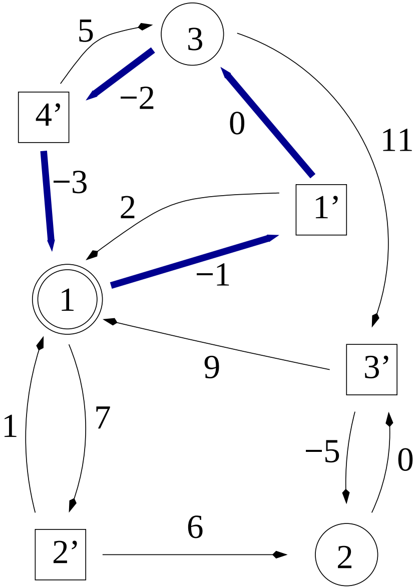


But if Max initially moves to $1'$

he only looses eventually $(1+0+2+3)/2 = 3$ per turn.

$v_j^N :=$ value of Max in horizon N, initial state j.
$v^N := (v_j^N)_{j\in[n]}$

$$v^N = f(v^{N-1}), \qquad v^0 = 0, \qquad f = g \circ g'$$

$$g : \mathbb{R}^p \to \mathbb{R}^n, \quad g_j(y) = \min_{(j,i)\in E} r_{ji} + y_i$$
$$g' : \mathbb{R}^n \to \mathbb{R}^p, \quad g'_i(x) = \max_{(i,j)\in E} r_{ij} + x_j$$

The mean payoff vector is

$$\chi(f) := \lim_{k\to\infty} f^k(0)/k$$

A theorem of Kohlberg (1980) implies that f admits an invariant half-line $t \mapsto u + t\eta$, $u, \eta \in \mathbb{R}^n$:

$$f(u + t\eta) = u + (t + 1)\eta \qquad t \text{ large}$$

Hence, $\chi(f) = \eta$.

Policy iteration for mean payoff games

A policy of Max is a map π which to each circle node associates a successor square node.

$f = \sup_\pi f^\pi$ where f^π is a min-plus linear operator (one player)

$$\forall x \in \mathbb{R}^n, \exists \pi, \qquad f(x) = f^\pi(x)$$

- for the current π, solve the one player game:

$$f^{\pi}(u^{\pi} + t\eta^{\pi}) = u^{\pi} + (t+1)\eta^{\pi}, \ t \text{ large}$$

- Check whether $f(u^{\pi} + t\eta^{\pi}) = u^{\pi} + (t+1)\eta^{\pi}$ for large t.
- Yes: π is optimal, mean payoff is η^{π}
- No: choose the new policy σ such that $f(u^{\pi} + t\eta^{\pi}) = f^{\sigma}(u^{\pi} + t\eta^{\sigma})$, t large

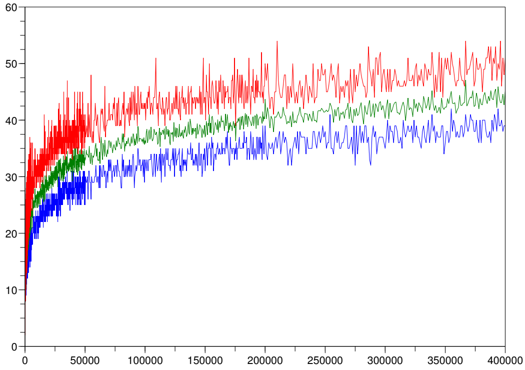
This is Newton, in which the tangent map is min-plus linear !

The idea goes back to Hofman and Karp 66: required irreducible stochastic matrices.

in the deterministic case, the bias vector u^π is not unique, naive policy iteration may cycle

Cochet, Gaubert, Gunawardena 99: use the max-plus spectral theorem, u^σ can be chosen to coincide with u^π on the critical graph of $\sigma \Rightarrow$ termination. Extended to zero-sum stochastic games Cochet, Gaubert CRAS 06

Works well in practice



but the complexity of mean payoff games is still an open problem (in NP $\cap$ coNP Condon).

Static analysis of programs by abstract interpretation

Cousot 77: finding invariants of a program reduces to computing the smallest fixed point of a monotone self-map of a complete lattice L

To each breakpoint i of the program, is associated a set $x^i \in L$ which is an overapproximation of the set of reachable values of the variables, at this breakpoint.

The best x is the smallest solution of $x = f(x)$

```
void main() {
  int x=0;          // 1
  while (x<100) {   // 2
    x=x+1;          // 3
  }                 // 4
}
```

Let $x_2^+ := \max x_2$. We arrive at

$$x_2^+ = \min(99, \max(0, x_2^+ + 1)) \ .$$

The smallest x_2^+ is 99.

Lattice of templates

S. Sankaranarayanan and H. Sipma and Z. Manna (VMCAI'05)

Polyhedra with limited degrees of freedom

A subset of $\mathbb{R}^n$ is coded by the discrete support function

$$\sigma_X : \mathcal{P} \to \mathbb{R}^n, \qquad \sigma_X(p) := \sup_{x \in X} p \cdot x$$

Max-plus finite elements again, X sublevel set of $\sup_{p \in \mathcal{P}} p \cdot x - \sigma_X(p)$.

Theorem (SG, Goubault, Taly, Zennou, ESOP'07)

When the arithmetics of the program is affine, *abstract interpretation over a lattice of templates reduces to finding the smallest fixed point of a map* $f : (\mathbb{R} \cup \{+\infty\})^n \to (\mathbb{R} \cup \{+\infty\})^n$

$$f_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} (r_i^{ab} + M_i^{ab} x)$$

with $M_i^{ab} := (M_{ij}^{ab})$, $M_{ij}^{ab} \geq 0$, *but possibly* $\sum_j M_{ij}^{ab} > 1$ (*negative discount rate!*)

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

$$i \leq +\infty$$
$$i \geq 1$$
$$j \leq 10$$
$$j \geq -\infty$$
$$i \leq j$$
$$i + 2j \leq 21$$
$$i + 2j \geq 21$$

$(i, j) \in [(1, 10), (7, 7)]$ (exact result).

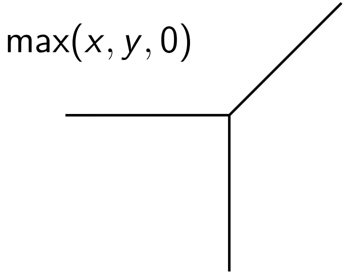
- solved by policy iteration for games
- often more accurate than value iteration with accelerations of convergence -widening/narrowing- used classically in the static analysis community

A glimpse of tropical algebra

A tropical line in the plane is the set of (x, y) such that the max in

$$ax + by + c$$

is attained at least twice.



A glimpse of tropical algebra

A tropical line in the plane is the set of (x, y) such that the max in

$$\max(a + x, b + y, c)$$

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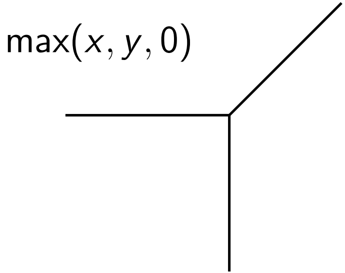
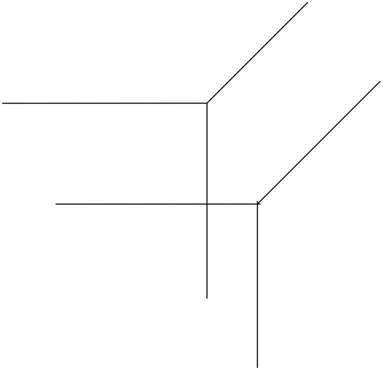
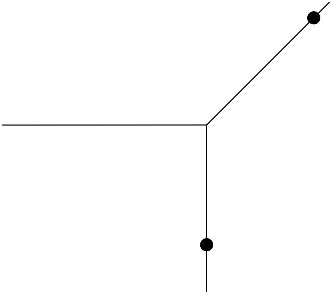


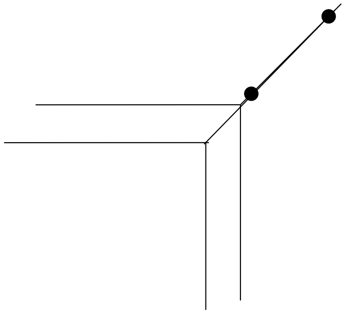
Two generic tropical lines meet at a unique point



By two generic points passes a unique tropical line

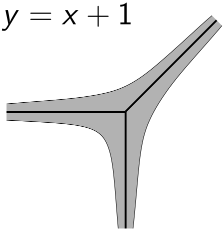


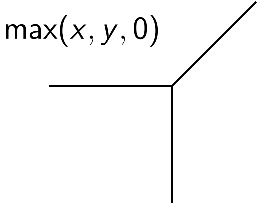
non generic case



Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \dots, \log|z_n|) \mid (z_1, \dots, z_n) \in V\}$$





If a sum of numbers is zero, then, two of them much have the same magnitude

See Passare & Rullgard, Duke Math. 04 for more information.

Nonarchimedean amoebas are simpler.

Here, $k := \mathbb{C}\{\{t\}\}$, e.g.

$s = 7t^{-1/2} + 1 + 8t + it^{3/2} + \cdots$

$v :=$ usual valuation, e.g. $v(s) = -1/2$

Theorem (Kapranov)

The amoeba of

$$\{z \in (k^*)^n \mid \sum_{i \in \mathbb{N}^n} a_i z_1^{i_1} \cdots z_n^{i_n} = 0\} \text{ is}$$

$$\{y \in \mathbb{R}^n \mid \min_{i \in \mathbb{N}^n} v(a_i) + \langle i, y \rangle \text{ attained twice}\}$$

The “fundamental theorem” of (maxplus) algebra

Cuninghame-Green & Meijer, 1980

A max-plus polynomial function can be factored uniquely

$$f(x) = a_d \prod_{1 \leq k \leq d} (x + \alpha_k)$$

The α_k are the tropical roots, can be computed in linear time and in a robust way.

The "fundamental theorem" of (maxplus) algebra

Cuninghame-Green & Meijer, 1980

A max-plus polynomial function can be factored uniquely

$$f(x) = a_d + \sum_{1 \leq k \leq d} \max(x, \alpha_k)$$

The α_k are the tropical roots, can be computed in linear time and in a robust way.

The (algebraic) tropical eigenvalues of a matrix $A \in \mathbb{R}_{\min}^{n\times n}$ are the roots of

$$\mathrm{per}(A + xI)$$

where

$$\mathrm{per}(M) := \sum_{\sigma\in S_n} \prod_{i\in[n]} M_{i\sigma(i)}$$

All geom. eigenvalues λ ($Au = \lambda u$) are algebraic eigenvalues, but the converse does not hold.

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Generalized Lidskii theorem

Theorem (Akian, SG, Bapat, CRAS 04)

Take $(\mathbb{C}\{\{\epsilon\}\}, v)$. *Let* $\mathcal{A} := a + \epsilon b$, $a, b \in \mathbb{C}^{n\times n}$, *let* $\lambda_1, \ldots, \lambda_n$ *denote the eigenvalues of* $\mathcal{A}$ *ordered by increasing valuation, and let* $\mu_1 \leq \cdots \leq \mu_n$ *denote the tropical eigenvalues of* $v(\mathcal{A})$. *Then,*

$$v(\lambda_1) \geq \mu_1, \; v(\lambda_1) + v(\lambda_2) \geq \mu_1 + \mu_2 \ldots$$

and $=$ *holds when certain minors do not vanish.*

Improving the accuracy of eigenvalue computations

Let $A_0 + A_1 x + A_2 x^2$ with $A_i \in \mathbb{C}^{n\times n}$, $\gamma_i := \|A_i\|$.
Fan, Lin, and Van Dooren proposed the scaling $x = \alpha^* y$, where $\alpha^* = \sqrt{\frac{\gamma_0}{\gamma_2}}$, then linearize+QZ.
The max-times polynomial $\max(\gamma_0, \gamma_1 x, \gamma_2 x^2)$ has one double root equal to α^* if $\gamma_1^2 \leq \gamma_0\gamma_2$, and two distinct roots otherwise:

$$\alpha^+ = \frac{\gamma_1}{\gamma_2}, \qquad \alpha^- = \frac{\gamma_0}{\gamma_1}$$

When $\alpha^+ \gg \alpha^-$, two different scalings $x = \alpha^{\pm} y$ are needed!

$$x^2 10^{-18} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + x \begin{pmatrix} -3 & 10 \\ 16 & 45 \end{pmatrix} + 10^{-18} \begin{pmatrix} 12 & 15 \\ 34 & 28 \end{pmatrix}$$

linearization+QZ: eigenvalues
-Inf,- 7.731e-19 , Inf, 3.588e-19

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Fan et al,

-Inf, -3.250e-19, Inf, 3.588e-19.

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tropical scaling
-7.250e-18 $\pm$ 9.744e-18 i, -2.102e+17 $\pm$ 7.387e+17 i

Matlab agrees up to 14 digits with PARI

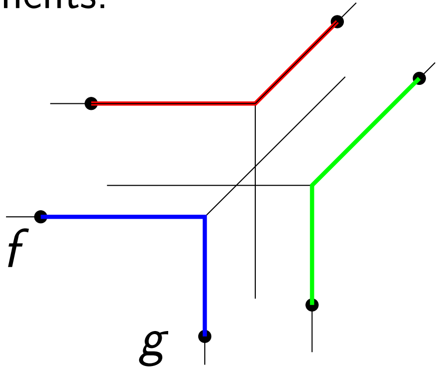
Theorem (SG, Sharify arXiv:0905.0121)

When $\alpha^+ \gg \alpha^-$, and when the A_i are well conditioned, the pencil has n eigenvalues of order α^+ and n eigenvalues of order α^-.

Experiments show that the tropical scaling reduces the backward error.

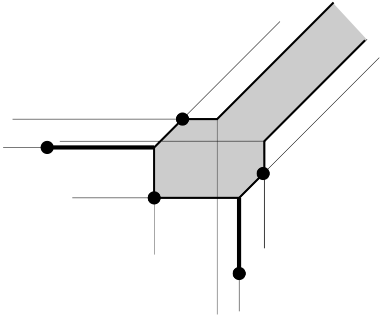
Tropical convexity

Tropical segments:



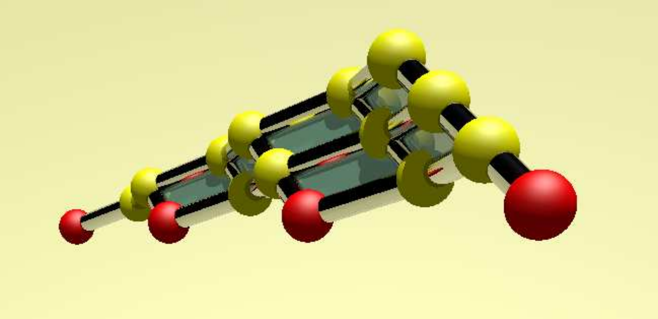
$$[f, g] := \{\sup(\lambda + f, \mu + g)\ \lambda, \mu \in \mathbb{R} \cup \{-\infty\},\ \max(\lambda, \mu) = 0\}.$$

Tropical convex set: $f, g \in C \implies [f, g] \in C$



The tropical cyclic polytope

(convex hull of p points on the moment curve $t \mapsto (t, t^2, \ldots, t^d)$, here $d = 3$, $p = 4$)



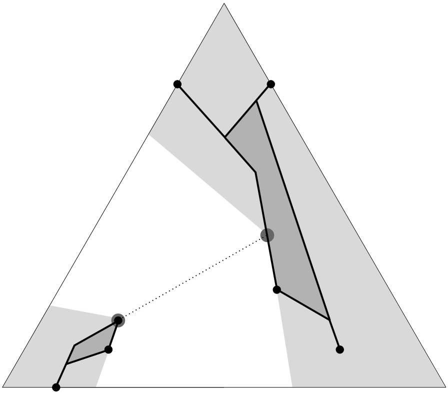
picture made with POLYMAKE (Gawrilow, Joswig)

Tropical convex geometry works

- Separation
- projection
- minimisation of distance
- Choquet theory (generation by extreme points)
- discrete convexity: Helly, Caratheodory , Minkowski, colorful Caratheodory, Tverberg

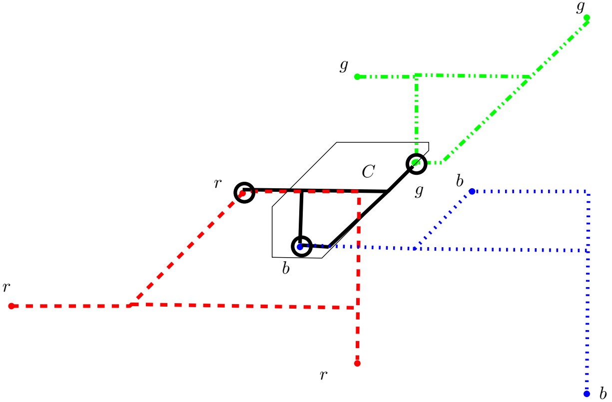
carry over !

Separation of two convex sets



SG & Sergeev arXiv:0706.3347 J. Math. Sci. (07)

Tropical version of Barany's Colorful Caratheodory Theorem



SG, Meunier, 09, arXiv:0804.1361 Disc. Comp. Geom, to appear

Tropical polyhedra can be efficiently handled

Allamigeon, SG, Goubault, arXiv:0904.3436

They have often fewer (and cannot have more) extreme points as usual polyhedra

Allamigeon, SG, Katz, arXiv:0906.3492

Tropical convexity has been applied. . .

- Discrete event systems Cohen, SG, Quadrat; more recently Katz
- Horoboundaries of metric spaces Walsh
- Static analysis (disjunctive invariants) Allamigeon, SG, Goubault SAS 08
- relations with tree metrics Develin, Sturmfels

Conclusion

Maxplus algebra

- is useful in applications (optimal control, discrete event systems, games)
- has proved to be a gold mine of counter examples and inspiration, for several other fields of mathematics (combinatorics, asymptotic analysis, geometry)
- is quite fun.

Some references available on http://minimal.inria.fr/gaubert/