Rolling on a space form

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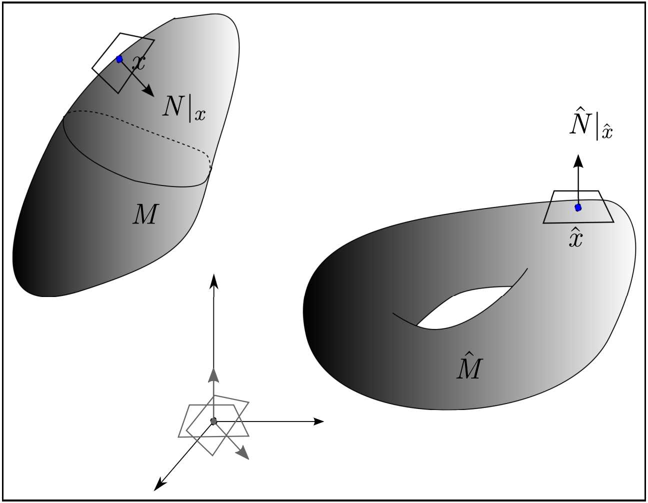
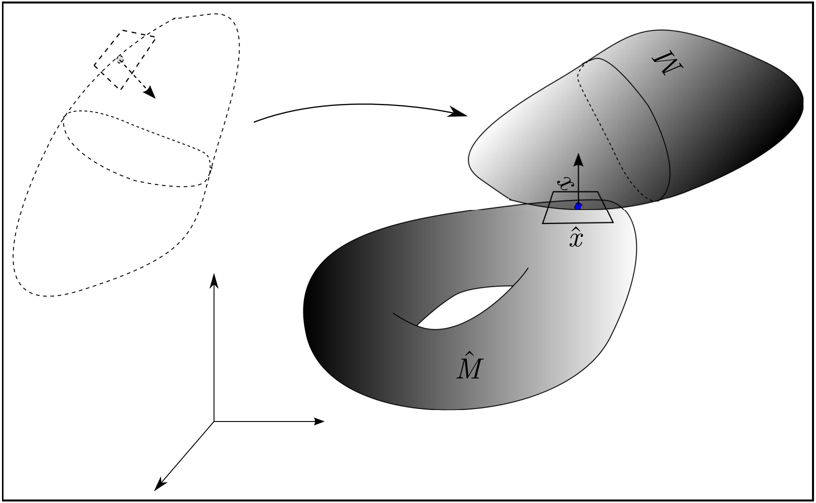
Agrachev's fest, Cortona, May 22nd, 2012



x
$N|_x$
M
$\hat{N}|_{\hat{x}}$
$\hat{x}$
$\hat{M}$



M
x
$\hat{x}$
$\hat{M}$

Classical Rolling of Surfaces - The State Space

- $M, \hat{M} \subset \mathbb{R}^3$ oriented surfaces with unit normal v.f. $N|_x$, $\hat{N}|_{\hat{x}}$
- Make M touch $\hat{M}$: contact points $x \in M$, $\hat{x} \in \hat{M}$ and rotation $U \in \mathrm{SO}(3)$ s.t. $UN|_x = \hat{N}|_{\hat{x}}$.
- "Physically" $U(M - x) + \hat{x}$, translating-and-rotating.
- U uniquely determined by isometry $A : T|_x M \to T|_{\hat{x}}\hat{M}$ of $\det = +1$. Indeed, $A = U|_{T|_x M}$.

Definition (State Space)

State space Q of orientations (contact configurations),

$$Q = \{A \in \mathcal{L}(T|_x M, T|_{\hat{x}}\hat{M}) \mid (x, \hat{x}) \in M \times \hat{M},\ A \text{ is isometry}, \det A = +1\}.$$

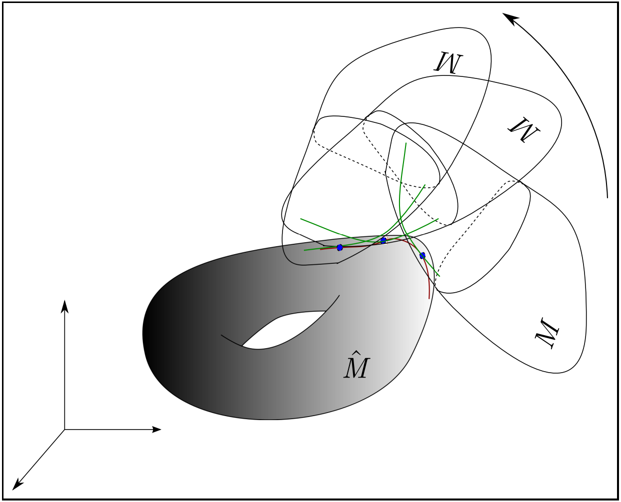
- Locally $Q \cong M \times \hat{M} \times S^1$, so $\dim Q = 5$.
- Notation: Write points of Q as $q = (x, \hat{x}; A) \in Q$.

Definition (Classical Rolling Kinematics)

M rolls on $\hat{M}$ along $\gamma : [0,1] \to M$ w/o slipping and w/o spinning if $\exists\ (\hat{\gamma}, U) : [0,1] \to \hat{M} \times \mathrm{SO}(3)$ s.t. $\forall t$,

$$
\begin{aligned}
&\text{(contact)} && U(t) N|_{\gamma(t)} = \hat{N}|_{\hat{\gamma}(t)} \\
&\text{(no-slip)} && U(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t) \\
&\text{(no-spin)} && \dot{U}(t)\, U(t)^{-1} \quad \text{has no } \hat{N}|_{\hat{\gamma}(t)} \times \text{-component}
\end{aligned}
$$

- $\forall q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and $\gamma : [0,1] \to M$ with $\gamma(0) = x_0$, $\exists$ unique $(\hat{\gamma}, U)$ as above s.t. $\hat{\gamma}(0) = \hat{x}_0$, $U(0)|_{T|_{x_0} M} = A_0$.
- Rolling curve: $q_R(\gamma, q_0)(t) := (\gamma(t), \hat{\gamma}(t); U(t)|_{T|_{\gamma(t)} M}) \in Q$


M
M
M
$\hat{M}$

Intrinsic n-dim. Rolling Model - State Space

- (M, g), $(\hat{M}, \hat{g})$ oriented conn. n-dim. Riemannian manifolds

Definition (State Space)

State space (of orientations) for rolling of M against $\hat{M}$,

$$Q = Q(M, \hat{M}) := \{A \in \mathcal{L}(T|_x M, T|_{\hat{x}} \hat{M}) \mid (x, \hat{x}) \in M \times \hat{M},\\ A \text{ isometry and } \det A = +1\}.$$

- Locally $Q \cong M \times \hat{M} \times \mathrm{SO}(n)$, so $\dim Q = n + n + \dfrac{n(n-1)}{2}$.
- Notation: Write points of Q as $q = (x, \hat{x}; A) \in Q$.
- Bundle: $\pi_Q : Q \to M \times \hat{M};\ (x, \hat{x}; A) \mapsto (x, \hat{x})$
  - Not principal in general!

Intrinsic n-dim. Rolling Model - Rolling Distribution

$(P^{\nabla})_0^t(\gamma)$: parallel transp. assoc. to ∇ linear connection along γ.

Definition (Rolling Lift and Distribution)

- Rolling lift: $q = (x, \hat{x}; A) \in Q$, $X \in T|_x M$,

$$\mathscr{L}_{\mathrm{R}}(X)|_q := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \big((P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma)\big) \in T|_q Q,$$

  where $\gamma, \hat{\gamma}$ any paths s.t. $\dot{\gamma}(0) = X$, $\dot{\hat{\gamma}}(0) = AX$.
- Rolling dist.: $\mathcal{D}_{\mathrm{R}}|_q := \mathscr{L}_{\mathrm{R}}(T|_x M)|_q \in T|_q Q$. $(\mathrm{rank} = n)$.

- $\mathcal{D}_{\mathrm{R}}$ Ehresmann conn. for $\pi_{Q,M} : Q \to M$; $(x, \hat{x}; A) \mapsto x$.
- $q(t)$ in Q rolling curve iff $\dot{q}(t) \in \mathcal{D}_{\mathrm{R}}$ $\forall t$, iff

$$\begin{cases} \dot{\hat{\gamma}}(t) = A(t)\dot{\gamma}(t), & \text{(no-slip)} \\ \overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(\cdot) = 0, & \text{(no-spin)} \end{cases}$$

where $\overline{\nabla} := \nabla \times \hat{\nabla}$ and $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$.

Rolling curvature and Lie brackets

Definition (Rolling curvature Rol of M over $\hat{M}$)

$X, Y \in VF(M)$, $\mathsf{Rol}(X, Y) : \pi_Q \to \pi_Q$,

$$\mathsf{Rol}(X, Y)(A) := AR(X, Y) - \hat{R}(AX, AY)A, \tag{1}$$

$$[\mathcal{L}_R(X), \mathcal{L}_R(Y)]|_q = \mathcal{L}_R([X, Y])|_q + \nu(\mathsf{Rol}(X, Y)(A))|_q.$$

- $\mathcal{D}_R$ involutive iff $\mathsf{Rol}(\cdot, \cdot) \equiv 0$
  iff (M, g) and $(\hat{M}, \hat{g})$ have both constant and equal curvature.
- Simpler proofs of basic results in Riem. Geo:
  - Cartan's lemma and Ambrose's theorem;
  - $\hat{M}$ complete Riem. manifold with cst. sect. curvature k and M_k space form (s. c.) with curvature k. Then, $\exists$ Riem. covering $\pi : M_k \to \hat{M}$.

2D case.

- General "surfaces": Agrachev-Sachkov, Bryant-Hsu, Marigo-Bicchi.
- Plate-ball problem ($M = S^2, \hat{M} = \mathbb{R}^2$), Jurdjevic...;

Results

- State-space $Q = 5$ dim $= (x, \hat{x}, \theta)$, (contact point on M and $\hat{M}$, plus relative orientation).
- CC $\Leftrightarrow$ M and $\hat{M}$ non isometric. (Still possibility of open orbits for isometric manifolds).
- Lie Algebraic Rank Condition at $(x, \hat{x}, \theta)$ if $K_M(x) \neq \hat{K}_{\hat{M}}(\hat{x})$.

Efficient motion-planning algorithms for rolling strictly convex surfaces on a plane (cf. e.g. Alouges-C.-Long; Marigo-Bicchi).

No useful "Ambrose-Singer" type of result in general

Theorem (Ambrose-Singer; Ozeki)

(N, g_N) smooth complete Riem. manifold. $p \in N$, $H|_p$ holonomy group at p with Lie algebra $\mathfrak{h}_p$. Then,

$$\begin{aligned}\mathfrak{h}_p \quad &= span\{P(\gamma)^{-1} \circ R(P(\gamma)x, P(\gamma)y) \circ P(\gamma); \\ &\qquad x, y \in T_pN, \gamma \text{ pw curve starting at } p\}.\end{aligned}$$

If (N, g_N) analytic,

$$\mathfrak{h}_p = span\{\nabla^k R(x, y; z_1, \cdots, z_k); k \geq 0, \quad x, y, z_1, \cdots, z_k \in T_pN\}.$$

Cov. derivation of Rol

$$\begin{aligned}\overline{\nabla}^k \mathsf{Rol}(X, Y; Z_1, \ldots, Z_k)(A) \qquad &:= A\nabla^k R(X, Y, (\cdot); Z_1, \ldots, Z_k) \\ - \quad &\hat{\nabla}^k \hat{R}(AX, AY, A(\cdot); AZ_1, \ldots, AZ_k).\end{aligned}$$

Riemannian Holonomy (Cartan)

"Quand on développe l'espace de Riemann sur l'espace euclidien tangent en x le long d'un cycle partant de x et y revenant, cet espace euclidien subit un déplacement et tous les déplacements correspondant aux différents cycles possibles forment un groupe, appelé ***groupe d'holonomie****.*

É. Cartan - La géometrie des espaces de Riemann, 1925

- Cartan's *groupe d'holonomie* :
  - Subgr. of $\mathrm{SE}(n)$ gen. by developing (rolling!) M onto $T|_x M$.
- $\mathcal{H}|_x = \{P_1^0(\gamma) \mid \gamma \in \Omega_x(M)\} \subset \mathrm{SO}(n)$, the linear hol. group.
- $\mathcal{A}|_x = \left\{ \left(P_1^0(\gamma), \int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s \right) \,\middle|\, \gamma \in \Omega_x(M) \right\} \subset \mathrm{SE}(n)$, the affine hol. group.

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- Let (M,g) roll against $(T|_x M, g|_{T|_x M})$. If $q_0 := (x, 0; \mathrm{Id})$, unique int. curve of $\mathcal{D}_{\mathrm{R}}$ through q_0 and above γ given by

$$q(t) = \Big(P_t^0(\gamma), \int_0^t P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s\Big).$$

- $\Longrightarrow$ fiber above x of the orbit $\mathcal{D}_{\mathrm{R}}$ through $q_0 = \mathcal{A}|_x$.
- **Remark:** All $\mathcal{D}_{\mathrm{R}}$-orbits are equivalent in this case.
- Conclusion: Cartan's *groupe d'holonomie* is $\mathcal{A}|_x$.
- Reasons to study $\mathcal{H}|_x$ instead of $\mathcal{A}|_x$?
  - de Rham decomposition theorem.
  - if $\mathcal{H}|_x$ irreducible $\Rightarrow \mathcal{A}|_x = \mathcal{H}|_x \ltimes T|_x M$ (Goto-Nomizu, 1955).
  - $\mathcal{H}|_x \subset \mathrm{SO}(n)$ (compact).
- Conclusion: Studying $\mathcal{H}|_x$ or $\mathcal{A}|_x$ equivalent.

Generalization of Holonomy by Rolling

Definition (Rolling Curve and Orbit)

$q_R(\gamma, q_0)(t) :=$ (unique) $\mathcal{D}_{\mathrm{R}}$-int. curve through q_0 above γ
$\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) :=$ orbit of $\mathcal{D}_{\mathrm{R}}$ through q (smooth $\subset Q$).
$\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)|_y :=$ fiber of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ over $y \in M$.

- If $(\hat{M}, \hat{g}) = T|_x M$, then $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)|_x \cong \mathcal{A}|_x$. Indep. of $q \in Q$.
- If $(\hat{M}, \hat{g})$ arbitrary Riem. manifold, $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)|_x$ should be considered as "generalization" of affine hol. group.
- In general, depends on $q \in Q$ and is not a group.

Exception

If $(\hat{M}, \hat{g})$ is any space form

(Generic) Necessary conds. for principal bundle structure

Theorem (C.-Godoy Molina-Kokkonen)

Under generic conds., if $\pi_{Q,M} : Q \to M$ principal bundle so that $\mathcal{D}_{\mathrm{R}}$ principal bundle connection then $(\hat{M}, \hat{g})$ is a space form.

Proof: Under generic conditions (RoI and R invertible)

$$\dim Iso(\hat{M}, \hat{g}) = \dim Sym_0(\mathcal{D}_{\mathrm{R}}) \leq \frac{n(n+1)}{2},$$

where $Sym_0(\mathcal{D}_{\mathrm{R}}) = \{S \in VF(Q), [S, \mathcal{D}_{\mathrm{R}}] \subset \mathcal{D}_{\mathrm{R}}, (\pi_{Q,M})_* S = 0\}$.

<u>1st =</u> {Killing field of $(\hat{M}, \hat{g})\} \to Sym_0(\mathcal{D}_{\mathrm{R}})$
$\hat{K} \mapsto \Big((x, \hat{x}, A) \mapsto (0, \hat{K}|_{\hat{x}}, \hat{\nabla}\hat{K}|_{\hat{x}} A)\Big)$ bijection.
<u>2nd ≤</u> $Sym_0(\mathcal{D}_{\mathrm{R}})|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)} \to T|_{\hat{x}}\hat{M} \times \mathfrak{so}(T|_{\hat{x}}\hat{M})$ injective.

$\mu : G \times Q \to Q$ principal bundle structure s.t. $\mu_* \mathcal{D}_{\mathrm{R}} = \mathcal{D}_{\mathrm{R}}$. Then $dim \mathfrak{g} = \frac{n(n+1)}{2}$ and $\forall X \in \mathfrak{g}$, $q \mapsto (\mu^q)_* X$ v.f. in $Sym_0(\mathcal{D}_{\mathrm{R}})$.
$\Rightarrow dim Iso(\hat{M}, \hat{g}) = \frac{n(n+1)}{2}$ Q.E.D.

Spherical and Hyperbolic Holonomies

- Riemannian holonomy = developing (rolling) (M, g) onto its tangent plane (Euclidean space).
- Let $(\hat{M}, \hat{g})$ be 1-conn. space form of const. curv. $c \neq 0$.

Proposition (C-Kokkonen, 2010)

Bundle $\pi_{Q,M} : Q \to M$ is principal and $\mathcal{D}_{\mathrm{R}}$ principal connection.

- **Reason:** $\mathrm{Iso}(\hat{M}, \hat{g})$ is maximal, of dim. $\frac{n(n+1)}{2}$.
- Note: $\mathrm{Iso} \cong \mathrm{O}(n+1)$ if $c > 0$ and $\mathrm{Iso} \cong \mathrm{O}^{+}(n, 1)$ if $c < 0$.

$\Rightarrow$ Orbits $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \cong$ Holonomy group of linear connection.
$\Rightarrow$ Study controllability WITHOUT Lie brackets computations.

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Spherical and Hyperbolic Holonomies

Proposition (C-Kokkonen, 2010)

Let $(\hat{M}, \hat{g})$ *be space form of curv.* $c \neq 0$. *Then for all* $q = (x, \hat{x}; A) \in Q$, *orbit* $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)|_x$ *diffeom. to holonomy group* $\mathcal{H}^c|_x$ *of connection* ∇^c *on* $\Pi : TM \oplus \mathbb{R} \to M$ *given by*

$$\nabla^c_X(Y, v) = (\nabla_X Y + vX, X(v) - cg(X, Y)).$$

Moreover, ∇^c *is metric w.r.t. to*

$$h_c((X, u), (Y, v)) := g(X, Y) + c^{-1}uv.$$

- Call ∇^c spherical (hyperbolic) connection if $c > 0$ ($c < 0$).
- Levi-Civita connection ∇ could then be parabolic.
- To study holonomies $\mathcal{H}^c$, enough to take $c \in \{-1, +1\}$.

de Rham type Theorem and Classification

Theorem (C- Godoy - Kokkonen, 2011)

Assume (M, g) *complete, simply connected and fix* $x \in M$.

(S) *If* $\mathcal{H}^1|_x$ *acts reducibly on the* h_1*-unit sphere of* $T|_x M \oplus \mathbb{R}$, *then* (M, g) *has const. curv.* $+1$.

(H1) $\mathcal{H}^{-1}|_x$ *acts reducibly on* $T|_x M \oplus \mathbb{R}$ *iff*

$$(M, g) = (\mathbb{H}^k \times M_1, \mathbf{h}_k \oplus_{\cosh(d(\cdot))} g_1)$$
$$\text{or} \quad (M, g) = (\mathbb{R} \times M_1, \mathrm{d}s \oplus_{e^{-s}} g_1),$$

where $(\mathbb{H}^k, \mathbf{h}_k)$ *is the* k*-dim. hyperbolic space and* d *is distance function from some point of it.*

(H2) $\mathcal{H}^{-1}|_x$ *acts irreducibly iff* $\mathcal{H}^{-1}|_x = \mathrm{SO}^0(n, 1)$ *(full).*

- Cases (S) and (H1) give de Rham type theorem for ∇^c.

Idea of the Proof

- Curvature of ∇^c is

$$R^c((X,u),(Y,v))(Z,w) = (R(X,Y)Z - cD(X,Y)Z, 0),$$

  where $D(X,Y)Z := g(Y,Z)X - g(X,Z)Y$.
- Suppose there are is $V|_x \oplus W|_x = T|_x M \oplus \mathbb{R}$ s.t. $V|_x$, $W|_x$ are $\mathcal{H}^c|_x$ invariant ($c \in \{-1,+1\}$).
- **Remark:** If $c = -1$ there is also degen. case which we omit.
- Def. subbund. $\mathcal{V}$, $\mathcal{W}$ of $TM \oplus \mathbb{R}$ by parallel trans. $V|_x$, $W|_x$.
- Split on M, $(0,1) = (A,a) + (B,b)$, $(A,a) \in \mathcal{V}$, $(B,b) \in \mathcal{W}$.
- Ambrose-Singer: $\mathcal{V}, \mathcal{W}$ are invariant by $R^c((X,u),(Y,v))$ and $R^c((X,u),(Y,v))(0,1) = 0$, so

$$R(X,Y)A = cD(X,Y)A, \quad R(X,Y)B = cD(X,Y)B.$$

- Let $\mathcal{V}_M$, $\mathcal{W}_M$ be images of $\mathcal{V}, \mathcal{W}$ on M by Π_*.
- They are smooth but singular (=drop rank)! Singular sets: $S_{\mathcal{V}} = \{x \mid (0,1) \in \mathcal{V}|_x\}$, $S_{\mathcal{W}} = \{x \mid (0,1) \in \mathcal{W}|_x\}$.
- $\mathcal{V}_M$, $\mathcal{W}_M$ are integrable on M and int. manif. are totally geodesic. Moreover, $S_{\mathcal{V}}$, $S_{\mathcal{W}}$ are int. manif. of $\mathcal{V}$, $\mathcal{W}$, resp.
- Distribution $\mathcal{V}_M^{\perp}$ (resp. $\mathcal{W}_M^{\perp}$) is const. rank and integrable on $M \backslash \mathcal{S}_{\mathcal{V}}$ (resp. $M \backslash \mathcal{S}_{\mathcal{W}}$). Int. manif. are spherical.

Theorem (Hiepko - 1979)

Given on complete, 1-connected (M, g) a distribution $\mathcal{D}$ s.t.

(i) *$\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable;*

(ii) *Int. manif. of $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) are totally geodesic (resp. spherical).*

Then (M, g) is a warped product (with specific warping function).

- $R(X, Y)A = cD(X, Y)A$ is then used to detect the w.f.