Optimal Transportation and Curvature of Hamiltonian Systems

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INDAM meeting on Geometric Control and sub-Riemannian Geometry and Professor Agrachev's 60th Birthday

May 21, 2012

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Optimal Transportation Problem (OT)

Minimize the total cost

$$\int_M c(x, \varphi(x)) d\mu_{t_0}$$

among all Borel maps $\varphi : M \to M$ which pushes μ_{t_0} forward to μ_{t_1} (i.e. $\mu_{t_0}(\varphi^{-1}(U)) = \mu_{t_1}(U)$ for all Borel set U).

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- Minimizers of (*) are projections of curves $t \mapsto \Phi_{t,t_0}(x,p)$

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is a minimizer between μ_{t_0} and $(\varphi_{t,t_0})_*\mu_{t_0}$ for the cost c_{t,t_0}, where $t \in [t_0, t_1]$

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Relative Entropy Functional $E : \mathbb{R} \times \mathcal{W} \longrightarrow \mathbb{R}$

$$E(t,\mu) = \int_M \rho_t \log \rho_t d\mathfrak{m}_t,$$

where $\mu = \rho_t \mathfrak{m}_t$.

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Finsler version of the above result.

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Theorem: (Perelman 02', Topping 09', Lott 09', Brendle 10')

$$\frac{d^2}{dt^2}\bar{E}(t,\mu_t) + \frac{3}{2t}\frac{d}{dt}\bar{E}(t,\mu_t) \geq 0.$$

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Theorem: (Feldman-Ilmanen-Ni 05', Lott 09')

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- $H:\mathbb{R}\times T^*M\to\mathbb{R}$ fibrewise strictly convex Hamiltonian
- complete Hamiltonian vector field $\sum_i(H_{p_i}\partial_{x_i}-H_{x_i}\partial_{p_i})$
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- Vertical bundle $\mathcal{V}=\ker d\pi$, $\pi:T^*M\to M$ projection
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The operator $\mathbf{R}^{t,t_0}_{(x,p)} : J^{t,t_0}_{(x,p)} \to J^{t,t_0}_{(x,p)}$ with matrix representation $R(t)$ with respect to $E(t)$ is the curvature of $J_{(x,p)}$.

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Proposition:

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$$- \frac{d^2}{dt^2}\left(v^t(\Phi_{t,t_0}(x,df))\right) - b(t)\frac{d}{dt}\left(v^t(\Phi_{t,t_0}(x,df))\right)\Big) d\mu_{t_0}(\mathbf{x})$$

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Finsler version

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Let $c_1 k = -2$, $b = -\frac{\ddot{k}}{\dot{k}}$, $c_2 = \frac{2\dot{k}}{k} - \frac{\ddot{k}}{\dot{k}}$, $\mathfrak{m}_t = e^{-ku}\mathrm{vol}$, and $U = -\frac{c_1^2}{8}R$. Then

$$\frac{d^2}{dt^2}E(t,\mu_t) + b(t)\frac{d}{dt}E(t,\mu_t) + \frac{n\dot{c}_2}{2} + \frac{nc_2^2}{4} + \frac{nb^2}{4} \geq 0$$

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Let $k = Ct^m$, $c_1 = -\frac{2}{Ct^m}$, $c_2 = \frac{m+1}{t}$, $\mathfrak{m}_t = e^{-Ct^m u}\text{vol}$, and $U = -\frac{c_1^2}{8}R$. Then

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Theorem: (L. 12')

- $\mathfrak{m}_t$ a fixed family of measures
- μ_t a displacement interpolation
- $E(t,\mu) = \int_M \rho^q d\mathfrak{m}_t$, $\mu = \rho\mathfrak{m}_t$
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$$\frac{d^2}{dt^2}E(t,\mu_t) + (qb_1(t)+b_2(t))\frac{d}{dt}E(t,\mu_t)$$
$$\geq \int_M q\,(r^t_{\mathbf{x}})^q \Big[\mathbf{tr}(\mathbf{R}^{t,t_0}_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})}) - b_2(t)\frac{d}{dt}\left(v^t(\Phi_{t,t_0}(\mathbf{x},d\mathbf{f}))\right)$$
$$-\frac{d^2}{dt^2}\left(v^t(\Phi_{t,t_0}(\mathbf{x},d\mathbf{f}))\right) - \frac{q\,b_1(t)^2}{4} - \frac{nb_2(t)^2}{4}\Big]d\mu_{t_0}(\mathbf{x})$$

- $H(t,x,p) = \frac{1}{2}\sum_{i,j} g^{ij}(t,x)p_i p_j + U(t,x)$
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Let $c_1 k = -2$, $b_2 = -\frac{\ddot{k}}{\dot{k}}$, $c_2 = \frac{2\dot{k}}{k} - \frac{\ddot{k}}{\dot{k}}$, $\mathfrak{m}_t = e^{-ku}\mathrm{vol}$, and $U = -\frac{c_1^2}{8}R$. Then

$$\frac{d^2}{dt^2}E(t,\mu_t) + (q\,b_1(t) + b_2(t))\frac{d}{dt}E(t,\mu_t) + q\left(\frac{n\dot{c}_2}{2} + \frac{nc_2^2}{4} + \frac{qb_1^2}{4} + \frac{nb_2^2}{4}\right)E(t,\mu_t) \geq 0.$$

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$$\frac{d^2}{dt^2}E(t,\mu_t) + \frac{qC_2 - m + 1}{t}\frac{d}{dt}E(t,\mu_t) + \frac{q(2nm(m-1) + qC_2^2)}{4t^2}E(t,\mu_t) \geq 0$$

From Theorem to Corollary

Lemma:

Let $H(t,x,p) = \frac{1}{2}\sum_{i,j} g^{ij}(t,x)p_i p_j + U(t,x)$. Then

$$\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) = \text{Ric}_{\mathbf{x}}(\mathbf{p},\mathbf{p}) + \Delta U(\mathbf{x}) - \langle \nabla(\mathbf{tr}\,(\dot{g})), \mathbf{p}\rangle_{\mathbf{x}}$$
$$+ \mathbf{div}(\dot{g})_{\mathbf{x}}(\mathbf{p}) - \frac{1}{2}\mathbf{tr}\,(\ddot{g}_{\mathbf{x}}) + \frac{1}{4}|\dot{g}|^2_{\mathbf{x}}$$

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- $\frac{d}{dt}v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = k(t)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t} - U(t,\varphi_t)\right) + \dot{k}(t)u(t,\varphi_t)$
- $\frac{d^2}{dt^2}v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = -k(t)\Big(\dot{U}(t,\varphi_t) + 2\left\langle \nabla U, \nabla u\right\rangle_{\varphi_t} + \frac{1}{2}(c_1(t)\mathrm{Ric}_{\varphi_t}(\nabla u,\nabla u) + c_2(t)|\nabla u|^2_{\varphi_t})\Big) + 2\dot{k}(t)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t} - U(t,\varphi_t)\right) + \ddot{k}(t)u(t,\varphi_t)$

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- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} - \ddot{k}(t)u(t,\varphi_t(\mathbf{x})) + \left(c_2(t)k(t) - 2\dot{k}(t)\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$
- $\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \dot{k}(t)u(t,\varphi_t(\mathbf{x})) + k(t)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$

- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} - \ddot{k}(t)u(t,\varphi_t(\mathbf{x})) + \left(c_2(t)k(t) - 2\dot{k}(t)\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2} R(\varphi_t(\mathbf{x}))\right)$
- $\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \dot{k}(t)u(t,\varphi_t(\mathbf{x})) + k(t)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2} R(\varphi_t(\mathbf{x}))\right)$
- kill the term $u(t,\varphi_t)$ by setting $b = -\frac{\ddot{k}}{\dot{k}}$

- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} - \ddot{k}(t)u(t,\varphi_t(\mathbf{x})) + \left(c_2(t)k(t) - 2\dot{k}(t)\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$
- $\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \dot{k}(t)u(t,\varphi_t(\mathbf{x})) + k(t)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$
- kill the term $u(t,\varphi_t)$ by setting $b = -\frac{\ddot{k}}{\dot{k}}$
- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} - b(t)\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \left(c_2(t)k(t) - 2\dot{k}(t) + \frac{k(t)\ddot{k}(t)}{\dot{k}(t)}\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right) - \frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4}$

- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} - \ddot{k}(t)u(t,\varphi_t(\mathbf{x})) + \left(c_2(t)k(t) - 2\dot{k}(t)\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$
- $\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \dot{k}(t)u(t,\varphi_t(\mathbf{x})) + k(t)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$
- kill the term $u(t,\varphi_t)$ by setting $b = -\frac{\ddot{k}}{\dot{k}}$
- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} - b(t)\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \left(c_2(t)k(t) - 2\dot{k}(t) + \frac{k(t)\ddot{k}(t)}{\dot{k}(t)}\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right) - \frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4}$
- set $c_2(t)k(t) - 2\dot{k}(t) + \frac{k(t)\ddot{k}(t)}{\dot{k}(t)} = 0$

- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4} - \ddot{k}(t)u(t,\varphi_t(\mathbf{x})) + \left(c_2(t)k(t) - 2\dot{k}(t)\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$
- $\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \dot{k}(t)u(t,\varphi_t(\mathbf{x})) + k(t)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right)$
- kill the term $u(t,\varphi_t)$ by setting $b = -\frac{\ddot{k}}{\dot{k}}$
- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} - b(t)\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = \left(c_2(t)k(t) - 2\dot{k}(t) + \frac{k(t)\ddot{k}(t)}{\dot{k}(t)}\right)\left(\frac{1}{2}|\nabla u|^2_{\varphi_t(\mathbf{x})} + \frac{1}{2k(t)^2}R(\varphi_t(\mathbf{x}))\right) - \frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4}$
- set $c_2(t)k(t) - 2\dot{k}(t) + \frac{k(t)\ddot{k}(t)}{\dot{k}(t)} = 0$
- $\mathbf{tr}\left(\mathbf{R}^{t,t}_{(\mathbf{x},\mathbf{p})}\right) - \frac{d^2}{dt^2} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} - b(t)\frac{d}{dt} v^t_{\Phi_{t,t_0}(\mathbf{x},d\mathbf{f})} = -\frac{n\dot{c}_2}{2} - \frac{nc_2^2}{4}$

$\mathcal{THE\ \ END}$