Local optimality and structural stability of Pontryagin extremals with singular arcs

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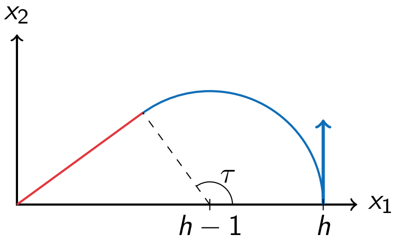
The Dubins–Dodgem car problem 1/2

$$T \to \min$$

$$\begin{aligned}
\dot{x}_1(t) &= \cos(x_3) & x_1(0) &= h & x_1(T) &= 0 \\
\dot{x}_2(t) &= \sin(x_3) & x_2(0) &= 0 & x_2(T) &= 0 \\
\dot{x}_3(t) &= u & x_3(0) &= \frac{\pi}{2} & x_3(T) &\in \mathbb{R}
\end{aligned}
\qquad u \in [-1, 1].$$

$$x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad f_0(x) = \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \end{pmatrix} \qquad f_1(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{x} = f_0(x) + u f_1(x) \qquad x(0) = \begin{pmatrix} h \\ 0 \\ \frac{\pi}{2} \end{pmatrix} \qquad x(T) \in \exp \mathbb{R} f_1(0)$$

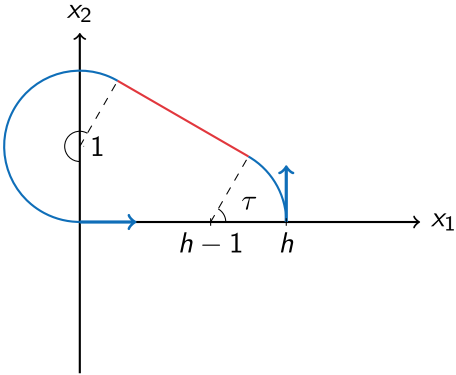


x_2
x_1
τ
$h-1$
h

The Dubins–Dodgem car problem 2/2

$$T \to \min$$

$$\begin{aligned}
\dot{x}_1(t) &= \cos(x_3) & x_1(0) &= h & x_1(T) &= 0 \\
\dot{x}_2(t) &= \sin(x_3) & x_2(0) &= 0 & x_2(T) &= 0 \qquad u \in [-1,1]. \\
\dot{x}_3(t) &= u & x_3(0) &= \frac{\pi}{2} & x_3(T) &\in 2\pi\mathbb{Z}
\end{aligned}$$


x_2
1
τ
x_1
$h-1$
h

The problem 1/2

minimise T subject to

$$\begin{aligned} &\dot{\xi}(t) = f_0(\xi(t)) + u(t) f_1(\xi(t)) && t \in [0, T] \\ &\xi(0) = x_0, \quad \xi(T) \in \mathcal{N}_f, && u(t) \in [-1, 1] \end{aligned}$$

Reference normal Pontryagin extremal $(\widehat{T}, \widehat{\xi}, \widehat{u})$ with adjoint covector

$$\widehat{\lambda} \colon [0, \widehat{T}] \to T^*\mathbb{R}^n$$

Aim

Look for second order conditions that ensure
strong local optimality of the triplet

The problem 2/2

minimise T subject to

$$\dot{\xi}(t) = f_0^r(\xi(t)) + u(t) f_1^r(\xi(t)) \quad t \in [0, T]$$
$$\xi(0) = x_0^r, \quad \xi(T) \in \mathcal{N}_f^r, \qquad u(t) \in [-1, 1]$$

Aim

Say the nominal problem corresponds to $r = 0$.
If $|r| < R$, does this perturbed problem have a strong local solution that is near $(\widehat{T}, \widehat{\xi}, \widehat{u})$?
Does it look like $(\widehat{T}, \widehat{\xi}, \widehat{u})$?
Is it – at least in some local sense – the unique solution?

Different kinds of strong local optimality

(time, state)–local optimality

there exist $\varepsilon > 0$ and a neighbourhood $\mathcal{V}$ of the graph of $\widehat{\xi}$ in $\mathbb{R} \times \mathbb{R}^n$ such that the triplet is optimal among all the triplets (T, ξ, u) such that

- $\left| T - \widehat{T} \right| < \varepsilon$
- $\Xi := \mathrm{Graph}(\xi) \in \mathcal{V}$

state–local optimality

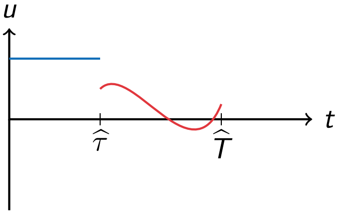
there exists a neighbourhood $\mathcal{U}$ of the range $\widehat{\xi}([0, \widehat{T}])$ of $\widehat{\xi}$ such that the triplet is optimal among all the triplets (T, ξ, u) such that

- $\xi([0, T]) \in \mathcal{U}$

Addressed cases: 1/2

$\mathcal{N}_f^r$ is an integral line of f_1^r & $\widehat{u}$ is bang–singular

- $\widehat{u}(t) \equiv u_1 \in \{-1, 1\}$ $\quad t \in [0, \widehat{\tau})$
- $\widehat{u}(t) \in (-1, 1)$ $\quad t \in (\widehat{\tau}, \widehat{T}]$

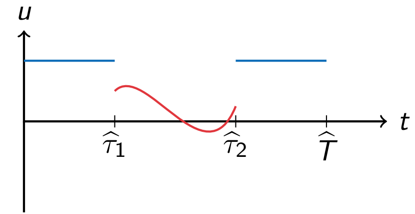


- $\mathcal{N}_f^r = \{\exp s f_1^r(y^r) \colon s \in \mathbb{R}\}$ W.l.o.g choose $y^0 = \widehat{x}_f := \widehat{\xi}(\widehat{T})$

Addressed cases: 2/2

$\mathcal{N}_f^r = \{y^r\}$ & $\widehat{u}$ is bang–singular–bang

- $\widehat{u}(t) \equiv u_1 \in \{-1, 1\}$ $\quad t \in [0, \widehat{\tau}_1)$
- $\widehat{u}(t) \in (-1, 1)$ $\quad t \in (\widehat{\tau}_1, \widehat{\tau}_2)$
- $\widehat{u}(t) \equiv u_2 \in \{-1, 1\}$ $\quad t \in (\widehat{\tau}_2, \widehat{T})$



- $\mathcal{N}_f = \{\widehat{x}_f\} := \{\widehat{\xi}(\widehat{T})\}$

Some quantities

- Reference vector field $\widehat{f}_t(x) := f_0(x) + \widehat{u}(t) f_1(x)$
- Reference Hamiltonian $\widehat{F}_t(\ell) := \langle \ell\,,\ \widehat{f}_t(\pi\ell) \rangle$
- $F_i(\ell) := \langle \ell\,,\ f_i(\pi\ell) \rangle \qquad i = 0, 1$
- Maximised Hamiltonian $F^{\max}(\ell) := \max_{u \in [-1,1]} (F_0(\ell) + u F_1(\ell))$
- $\Sigma := \{\ell \in T^*M \colon F_1(\ell) = 0\}$

Remark. $|u(t)| < 1$, $\forall t$ in the singular interval

$$\Longrightarrow \widehat{\lambda}(t) \in \Sigma \quad \forall t \text{ in the singular interval}$$

Hamiltonian approach to (time, state)–local optimality

- a suitable Hamiltonian (possibly time–dependent)

$$H\colon (t,\ell) \in [0,\widehat{T}] \times T^*\mathbb{R}^n \mapsto H_t(\ell) \in \mathbb{R}$$

$$\mathcal{H}_t(\Sigma) \subset \Sigma \quad \forall t \in [\widehat{\tau}, \widehat{T}]$$

$$H_t \geq F^{\max}, \quad H_t \circ \widehat{\lambda}(t) = \widehat{F}_t \circ \widehat{\lambda}(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\lambda}(t) = \overrightarrow{H}_t \circ \widehat{\lambda}(t)$$

- Let $\widehat{x}_1 := \widehat{\xi}(\widehat{\tau})$:
  find a smooth function $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ such that
  - $\mathrm{d}\alpha(\widehat{x}_1) = \widehat{\lambda}(\widehat{\tau})$
  - $\Lambda := \{(\mathrm{d}\alpha(x), x)\} \subset \Sigma$
  - Λ has some nice properties with respect to the flow $\mathcal{H}$

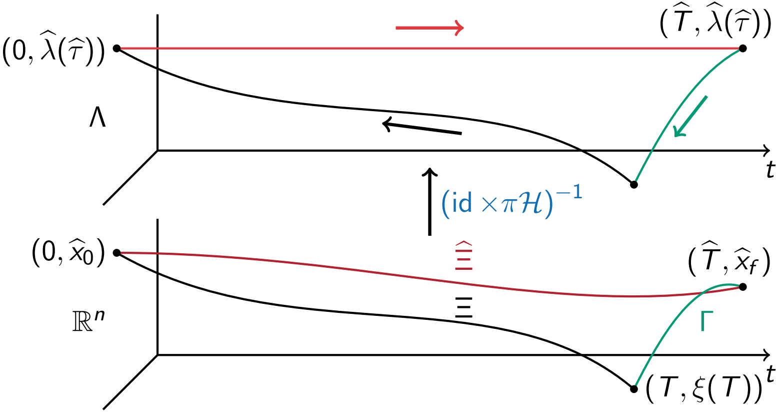
Hamiltonian approach to (time, state)–local optimality 2

- $$\begin{array}{ccc} \widehat{I} \times \Lambda & \xrightarrow{\mathrm{id} \times \mathcal{H}} & \widehat{I} \times T^*\mathbb{R}^n \\ & \nwarrow^{(\mathrm{id} \times \pi \mathcal{H})^{-1}} & \downarrow{\scriptstyle \mathrm{id} \times \pi} \\ & & \widehat{I} \times \mathbb{R}^n \end{array} \qquad \widehat{I} := [0, \widehat{T}]$$

Property

$\mathcal{H}^*(p\ \mathrm{d}q - H_t\ \mathrm{d}t)$ is exact on $[0, \widehat{T}] \times \Lambda$

Lifting trajectories



$$\Gamma(a) = \left(t(a) = T + a\left(\widehat{T} - T\right), \gamma(a) = \exp s(1-a) f_1(\widehat{x}_f) \right), \quad a \in [0,1]$$

$\mathcal{H}^*(p\,\mathrm{d}q - H_t\,\mathrm{d}t)$ is exact $\Longrightarrow$

$$0 = \oint \mathcal{H}^*(p\,\mathrm{d}q - H_t\,\mathrm{d}t) = \int_{(\mathrm{id}\times\pi\mathcal{H})^{-1}(\widehat{\Xi})} \mathcal{H}^*(p\,\mathrm{d}q - H_t\,\mathrm{d}t)$$

$$- \int_{(\mathrm{id}\times\pi\mathcal{H})^{-1}(\Gamma)} \mathcal{H}^*(p\,\mathrm{d}q - H_t\,\mathrm{d}t) - \int_{(\mathrm{id}\times\pi\mathcal{H})^{-1}(\Xi)} \mathcal{H}^*(p\,\mathrm{d}q - H_t\,\mathrm{d}t)$$

$$0 \le - \int_{(\mathrm{id}\times\pi\mathcal{H})^{-1}(\Gamma)} \mathcal{H}^*(p\,\mathrm{d}q - H_t\,\mathrm{d}t)$$

$\mathcal{H}^*(p\, \mathrm{d}q - H_t\, \mathrm{d}t)$ is exact $\implies$

$$0 \leq \int_0^1 \Big(- F_1\Big(\overbrace{\mathcal{H}_{t(a)}\left(\left(\pi\mathcal{H}_{t(a)}\right)^{-1}(\gamma(a))\right)}^{\in\Sigma}\Big)$$

$$+ H_{t(a)}\left(\mathcal{H}_{t(a)}\left(\left(\pi\mathcal{H}_{t(a)}\right)^{-1}(\gamma(a))\right)\right)\left(T - \widehat{T}\right)\Big)\, \mathrm{d}a$$

$$= \int_0^1 \left(1 + O\left(T - \widehat{T}\right)\right)\left(T - \widehat{T}\right) \mathrm{d}a$$

which implies $T \geq \widehat{T}$.

Hamiltonian approach to state–local optimality

- a suitable Hamiltonian

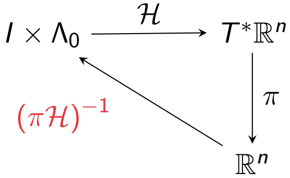
$$H\colon \ell \in T^*\mathbb{R}^n \mapsto H(\ell) \in \mathbb{R}$$

$$\mathcal{H}_t(\Sigma) \subset \Sigma \quad t \in [\widehat{\tau}, \widehat{T}]$$

$$H \geq F^{\max}, \quad H \circ \widehat{\lambda}(t) = \widehat{F}_t \circ \widehat{\lambda}(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\lambda}(t) = \overrightarrow{H} \circ \widehat{\lambda}(t)$$

- Let $\widehat{x}_1 := \widehat{\xi}(\widehat{\tau})$: find a smooth function $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ such that
  - $\mathrm{d}\alpha(\widehat{\xi}(\widehat{\tau})) = \widehat{\lambda}(\widehat{\tau})$
  - $\Lambda = \{(\mathrm{d}\alpha(x), x)\} \subset \Sigma$
  - Λ transverse to $\{H = 1\}$ in $\widehat{\lambda}(\tau)$
- $\Lambda_0 := \Lambda \cap \{H = 1\} \subset \Sigma$ is a $(n-1)$–dim manifold of $T^*\mathbb{R}^n$

Hamiltonian approach to state–local optimality 2

- 

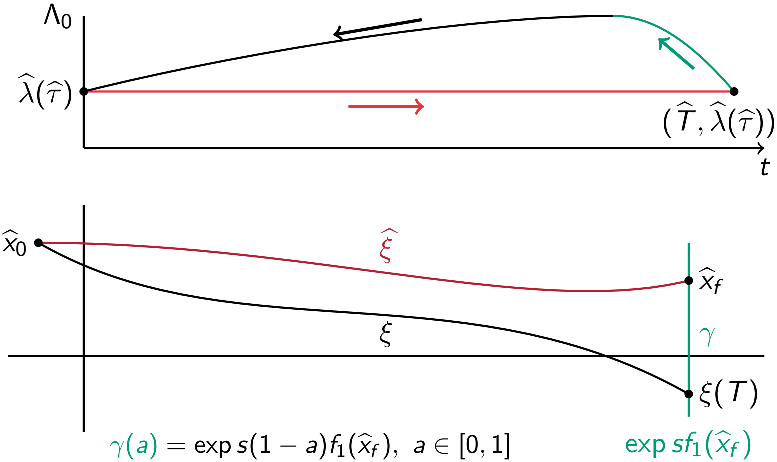
$$I := [-\delta, \widehat{T} + \delta]$$

Property

H does not depend on time

$\Longrightarrow$ $\mathcal{H}^*(p\, \mathrm{d}q)$ is exact on $[-\delta, \widehat{T} + \delta] \times \Lambda_0$

Lifting trajectories



$\mathcal{H}^*(p\ \mathrm{d}q)$ is exact $\Longrightarrow$

$$0 = \oint \mathcal{H}^*(p\ \mathrm{d}q) = \int_{(\pi\mathcal{H})^{-1}(\widehat{\xi})} \mathcal{H}^*(p\ \mathrm{d}q) - \int_{(\pi\mathcal{H})^{-1}(\gamma)} \mathcal{H}^*(p\ \mathrm{d}q) - \int_{(\pi\mathcal{H})^{-1}(\xi)} \mathcal{H}^*(p\ \mathrm{d}q)$$

$$= \int_0^{\widehat{T}} 1\ \mathrm{d}t - \int_0^s F_1 \circ \overbrace{\mathcal{H} \circ (\pi\mathcal{H})^{-1} \circ \gamma(a)}^{\in\Sigma}\ \mathrm{d}a$$

$$- \int_0^T \langle \mathcal{H} \circ (\pi\mathcal{H})^{-1}(\xi(t))\,,\ \dot{\xi}(t)\rangle\ \mathrm{d}t$$

$$\geq \widehat{T} - \int_0^T H(\mathcal{H} \circ (\pi\mathcal{H})^{-1}(\xi(t))\ \mathrm{d}t = \widehat{T} - T.$$

which implies $T \geq \widehat{T}$.

Necessary conditions

$$u_1 F_1(\widehat{\lambda}(t)) \geq 0 \qquad t \in [0, \widehat{\tau})$$

$$F_1(\widehat{\lambda}(t)) = 0 \quad F_0(\widehat{\lambda}(t)) = 1 \qquad t \in [\widehat{\tau}, \widehat{T}]$$

$$F_{101}(\widehat{\lambda}(t)) := \langle \widehat{\lambda}(t)\, ,\ [f_1, [f_0, f_1]](\widehat{\xi}(t)) \rangle \geq 0 \qquad t \in [\widehat{\tau}, \widehat{T}]$$

By differentiation

$$F_{01}(\widehat{\lambda}(t)) := \langle \widehat{\lambda}(t)\, ,\ [f_0, f_1](\widehat{\xi}(t)) \rangle = 0 \qquad t \in [\widehat{\tau}, \widehat{T}]$$

$$(F_{001} + \widehat{u}(t) F_{101})(\widehat{\lambda}(t)) = 0 \qquad t \in (\widehat{\tau}, \widehat{T})$$

where $F_{ijk}(p, q) := \langle p\, ,\ [f_i, [f_j, f_k]](q) \rangle$

The bang arc

Assumption: regularity along the bang arc

$$u_1 F_1(\widehat{\lambda}(t)) > 0 \qquad t \in [0, \widehat{\tau})$$

The singular arc

Strong generalised Legendre condition

$$F_{101}(\widehat{\lambda}(t)) > 0 \qquad t \in [\widehat{\tau}, \widehat{T}]$$

$$\Downarrow$$

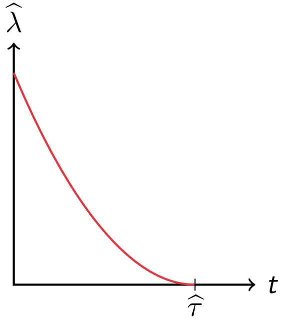
$$\widehat{u}(t) = \frac{-F_{001}}{F_{101}}(\widehat{\lambda}(t)) \qquad t \in (\widehat{\tau}, \widehat{T})$$

$$F^{\mathcal{S}} := F_0 - \frac{F_{001}}{F_{101}} F_1$$ Hamiltonian of singular extremals

$$F_{ijk}(p, q) := \langle p \, , \; [f_i, [f_j, f_k]](q) \rangle$$

PMP consequences at the junction point

$t \mapsto u_1 F_1(\widehat{\lambda}(t))$



$$u_1 F_1(\widehat{\lambda}(t)) > 0 \quad t \in [0, \widehat{\tau})$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u_1 F_1(\widehat{\lambda}(t))\Big|_{t=\widehat{\tau}} = u_1 F_{01}(\widehat{\lambda}(\widehat{\tau})) = 0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} u_1 F_1(\widehat{\lambda}(t))\Big|_{t=\widehat{\tau}-} =$$

$$= (u_1 F_{001} + F_{101})(\widehat{\lambda}(\widehat{\tau})) \geq 0$$

The junction point

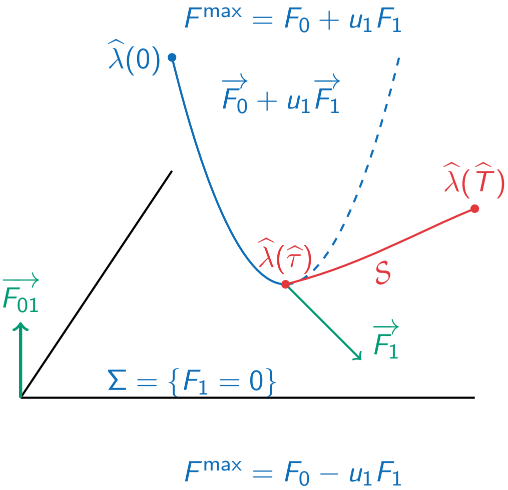
Regularity at the junction point

$$(u_1 F_{001} + F_{101})(\widehat{\lambda}(\widehat{\tau})) > 0$$

Equivalently

$\widehat{u}$ is discontinuous at $\widehat{\tau}$.

Geometric picture near the adjoint covector



$\overrightarrow{F_1}$ is tangent to Σ

$\overrightarrow{F_1}$ is transverse to $\mathcal{S}$

$\overrightarrow{F_{01}}$ is transverse to Σ

$\mathcal{S} := \{F_1 = F_{01} = 0\}$

Naive attempt

Choose $H_t = \begin{cases} H_1 := F_0 + u_1 F_1 & t \in [0, \widehat{\tau}) \\ F_0 + a(t,\ell) F_1 & t \in [\widehat{\tau}, \widehat{T}] \end{cases}$

for example: $a(t,\ell) = \widehat{u}(t)$, $a(t,\ell) = \frac{-F_{001}(\ell)}{F_{101}(\ell)} \dots$

and start at time $t = 0$ from points ℓ in a neighbourhood of $\widehat{\lambda}(0)$ with the flow of $\overrightarrow{H}_t$: at some time $t(\ell)$ the flow crosses Σ BUT

$$u_1 F_{01}(\exp t(\ell)(\overrightarrow{F_0} + u_1 \overrightarrow{F_1})(\ell)) < 0 \implies$$

$$u_1 F_1(\exp(t - t(\ell))(\overrightarrow{F_0} + a\overrightarrow{F_1}) \circ \exp t(\ell)(\overrightarrow{F_0} + u_1 \overrightarrow{F_1})(\ell)) < 0 \quad \text{for } t > t(\ell)$$

Does not work: H_t is not the maximised Hamiltonian along the flow of $\overrightarrow{H}_t$

Changing from $F_0 + u_1 F_1$ to $F_0 + |F_1|$ causes loss of invertibility

Change approach

$$t \in [\widehat{\tau}, \widehat{T}] \implies \widehat{\lambda}(t) \in \mathcal{S} := \{\ell \in T^*\mathbb{R}^n \colon F_1(\ell) = F_{01}(\ell) = 0\}$$

A new Hamiltonian (Stefani, 2004)

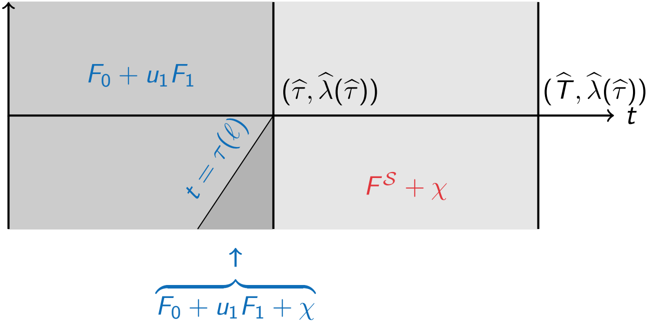
Regularity Assumptions $\implies$
$\exists\ \chi \colon T^*\mathbb{R}^n \to [0, +\infty)$ smooth such that

- $\chi(\ell) = 0 \iff \ell \in \mathcal{S}$
- $\overrightarrow{\chi}(\ell) = 0$ for any $\ell \in \mathcal{S}$
- for any $\nu \colon (t, \ell) \in [0, \widehat{T}] \times T^*\mathbb{R}^n \mapsto \nu_t(\ell) \in \mathbb{R}$, the vector field $\overrightarrow{F_0 + \nu_t F_1 + \chi}$ is tangent to Σ

The over–maximised Hamiltonian

$$H_t(\ell) = \begin{cases} F_0(\ell) + u_1 F_1(\ell) & u_1 F_{01}(\ell) \leq 0, \;\; t \in [0, \widehat{\tau}) \\ F_0(\ell) + u_1 F_1(\ell) + \chi(\ell) & u_1 F_{01}(\ell) > 0, \;\; t \in [0, \widehat{\tau}) \\ F_0(\ell) - \dfrac{F_{001}(\ell)}{F_{101}(\ell)} F_1(\ell) + \chi(\ell) & t \in [\widehat{\tau}, \widehat{T}] \end{cases}$$

The Hamiltonian



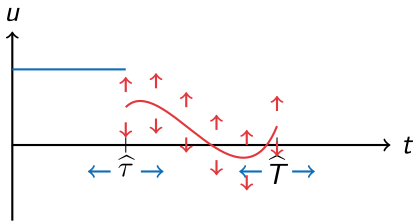
$$F^{\mathcal{S}} := F_0 - \frac{F_{001}}{F_{101}} F_1$$

The second variation

The dynamics is affine with respect to the control, so the classical second variation is completely degenerate $\Longrightarrow$
Proceed by perturbing:

- the switching time $\widehat{\tau}$;
- the final time $\widehat{T}$;
- the singular control $\widehat{u}|_{(\widehat{\tau},\widehat{T})}$

With the constraints $\xi(0) = \widehat{x}_0, \quad \xi(T) \in \mathcal{N}_f$



The second variation

Transform the problem into a Mayer one on the fixed time interval $[0, \widehat{T}]$ by reparametrising time

$$\text{minimise} \quad t(\widehat{T}) \quad \text{subject to}$$

$$\begin{aligned} &\dot{t}(s) = u_0(s), && t(0) = 0, \quad t(\widehat{T}) \in \mathbb{R} \\ &\dot{\xi}(s) = u_0(s)\,(f_0 + u(s) f_1)\,(\xi(s)) && \xi(0) = \widehat{x}_0, \quad \xi(\widehat{T}) \in \mathcal{N}_f \\ &u_0(s) > 0, \quad u(s) \in [-1, 1] \end{aligned}$$

reference extended controls: $u_0 \equiv 1, \; u \equiv \widehat{u}$

reference extended trajectory: $s \mapsto \left(s, \widehat{\xi}(s)\right)$

After a Goh's transformation the quadratic form is given by

$$
\begin{aligned}
&J''_{\mathrm{ext}}[(\gamma_0,\gamma_1,\varepsilon_0,\varepsilon_1,w)]^2\\
&\quad= \frac{1}{2}\big(\varepsilon_0(g^0_{\widehat{\tau}} + u_1 g^1_{\widehat{\tau}}) + \varepsilon_1(g^0_{\widehat{\tau}} + \widehat{u}|_{\widehat{\tau}+}\, g^1_{\widehat{\tau}}) + \gamma_0 g^1_{\widehat{\tau}}\big)^2 \cdot \beta_0(\widehat{x}_0) +\\
&\quad+ \frac{1}{2}\int_{\widehat{\tau}}^{\widehat{T}} \left(w^2(s)[\dot{g}^1_s, g^1_s] + 2w(s)\,\zeta(s)\cdot \dot{g}^1_s\right)\cdot \beta_0(\widehat{x}_0)\, ds
\end{aligned}
$$

which is required to be coercive on the 5-tuplets $(\gamma_0,\gamma_1,\varepsilon_0,\varepsilon_1,w) \in \mathbb{R}^4 \times L^2([\widehat{\tau},\widehat{T}])$ such that

$$
\begin{aligned}
&\dot{\zeta}(s) = w(s)\dot{g}^1_s(\widehat{x}_0)\\
&\zeta(\widehat{\tau}) = \varepsilon_0(g^0_{\widehat{\tau}} + u_1 g^1_{\widehat{\tau}})(\widehat{x}_0) + \varepsilon_1\,(g^0_{\widehat{\tau}} + \widehat{u}|_{\widehat{\tau}+}\, g^1_{\widehat{\tau}})(\widehat{x}_0) + \gamma_0 g^1_{\widehat{\tau}}(\widehat{x}_0)\\
&\zeta(\widehat{T}) = \gamma_1 g^1_{\widehat{T}}(\widehat{x}_0)
\end{aligned}
$$

admits a solution ζ.

The extended second variation cannot possibly be coercive: just choose the non null variation

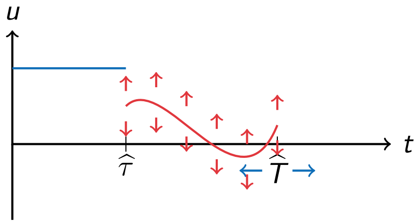
$$\varepsilon_0 = 1,\ \varepsilon_1 = -1,\ \gamma_0 = \widehat{u}|_{\widehat{\tau}+} - u_1,\ \gamma_1 = 0,\ w \equiv 0$$
$$\implies J''_{\mathrm{ext}} = 0$$

The second variation

In order to obtain a possibly coercive extended second variation,

- renounce to perturbing the switching time $\widehat{\tau}$;
- impose a stronger constraint on the final point:

$$\xi(T) = \widehat{\xi}(\widehat{T}) := \widehat{x}_f$$



The second variation

Transform the sub-problem into a Mayer one on the fixed time interval $[\widehat{\tau}, \widehat{T}]$ by reparametrising time

$$
\begin{aligned}
&\text{minimise} \quad t(\widehat{T}) \quad \text{subject to} \\
&\dot{t}(s) = u_0(s), \qquad\qquad t(\widehat{\tau}) = \widehat{\tau}, \quad t(\widehat{T}) \in \mathbb{R} \\
&\dot{\xi}(s) = u_0(s)\left(f_0 + u(s) f_1\right)(\xi(s)) \quad \xi(\widehat{\tau}) = \widehat{x}_1 \quad \xi(\widehat{T}) = \widehat{x}_f \\
&u_0(s) > 0, \quad u(s) \in [-1, 1]
\end{aligned}
$$

$\widehat{x}_1 := \widehat{\xi}(\widehat{\tau}), \qquad \widehat{\xi}(\widehat{T}) := \widehat{x}_f$

reference extended controls: $u_0 \equiv 1, \; u \equiv \widehat{u}$

reference extended trajectory: $s \mapsto \left(s, \widehat{\xi}(s)\right)$

Pull–back system

$$g_s^i := \widehat{S}_{t*}^{-1} f_i \circ \widehat{S}_s\,, \quad i = 0, 1$$

$$\widehat{g}_s := \widehat{S}_{s*}^{-1} \widehat{f}_s \circ \widehat{S}_s = g_s^0 + \widehat{u}(s) g_s^1$$

minimise $\tau(\widehat{T})$ subject to

$$\dot{\tau}(s) = u_0(s) - 1,$$

$$\dot{\eta}(s) = \left((u_0(s) - 1)\widehat{g}_s + u_0(s)(u(s) - \widehat{u}(s))\, g_s^1\right)(\eta(s))$$

$$\tau(\widehat{\tau}) = \widehat{\tau}, \quad \tau(\widehat{T}) \in \mathbb{R}, \quad \eta(\widehat{\tau}) = \widehat{x}_1, \quad \eta(\widehat{T}) = \widehat{x}_1$$

$$u_0(s) > 0, \quad |u(s)| \leq 1$$

constant reference trajectory $s \mapsto (\widehat{\tau}, \widehat{x}_1)$

2nd variation associated to $\widehat{\lambda}\Big|_{[\widehat{\tau},\widehat{T}]}$ (Agrachev-Stefani-Zezza 1998)

Let $\beta\colon \mathbb{R}^n \to \mathbb{R}$ such that $d\beta(\widehat{x}_1) = -\widehat{\lambda}(\widehat{\tau}) \in T^*_{\widehat{x}_1}\mathbb{R}^n$

$$J''[\delta u_0, \delta u]^2 = \frac{1}{2}\int_{\widehat{\tau}}^{\widehat{T}} \delta\eta(s)\cdot\left(\delta u_0(s)\,\widehat{g}_s + \delta u(s)\,g^1_s\right)\cdot\beta(\widehat{x}_1)\,ds$$

where δu_0, δu, $\delta\eta$ satisfy

$$\begin{aligned}
&\dot{\delta\eta}_0(s) = \delta u_0(s) && \delta\eta_0(\widehat{\tau}) = 0 && \delta\eta_0(\widehat{T}) \in \mathbb{R}\\
&\dot{\delta\eta}(s) = \delta u_0(s)\widehat{g}_s(\widehat{x}_1) + \delta u(s) g^1_s(\widehat{x}_1) && \delta\eta(\widehat{\tau}) = 0 && \delta\eta(\widehat{T}) = 0\\
&(\delta u_0, \delta u) \in L^2((\widehat{\tau},\widehat{T}),\mathbb{R}^2).
\end{aligned}$$

After a Goh's transformation the quadratic form is given by

$$J''_{\mathrm{ext}}[(\varepsilon_0, \varepsilon_1, w)]^2 = \frac{1}{2}(\varepsilon_0 f_0 + \varepsilon_1 f_1)^2 \cdot \beta(\widehat{x}_1) +$$

$$+ \frac{1}{2}\int_{\widehat{\tau}}^{\widehat{T}} \left(w^2(s)[\dot{g}_s^1, g_s^1] \cdot \beta(\widehat{x}_1) + 2w(s)\,\zeta(s) \cdot \dot{g}_s^1 \cdot \beta(\widehat{x}_1) \right) \mathrm{d}s$$

which is required to be coercive on the triplets
$(\varepsilon_0, \varepsilon_1, w) \in \mathbb{R}^2 \times L^2([\widehat{\tau}, \widehat{T}])$ such that

$$\dot{\zeta}(s) = w(s)\dot{g}_s^1(\widehat{x}_1)$$

$$\zeta(\widehat{\tau}) = \varepsilon_0 f_0(\widehat{x}_1) + \varepsilon_1\, f_1(\widehat{x}_1) \qquad \zeta(\widehat{T}) = 0$$

admits a solution ζ.
With a more appropriate choice of β the discrete part of J''_{ext} can be assumed to be null

Invertibility

The extended second variation

$$J''_{\text{ext}}[(\varepsilon_0, \varepsilon_1, w)]^2 = \frac{1}{2}\int_{\widehat{\tau}}^{\widehat{T}} \left(w^2(s)[\dot{g}_s^1, g_s^1] + 2w(s)\,\zeta(s)\cdot\dot{g}_s^1 \right)\cdot\beta(\widehat{x}_1)\,\mathrm{d}s$$

on the triplets $(\varepsilon_0, \varepsilon_1, w) \in \mathbb{R}^2 \times L^2([\widehat{\tau}, \widehat{T}])$ such that

$$\dot{\zeta}(s) = w(s)\dot{g}_s^1(\widehat{x}_1) \quad \zeta(\widehat{\tau}) = \varepsilon_0 f_0(\widehat{x}_1) + \varepsilon_1 f_1(\widehat{x}_1) \quad \zeta(\widehat{T}) = 0$$

admits a solution ζ.

Assumption

- for (time, state)–local optimality
  - The extended second variation restricted to $\varepsilon_0 = 0$ is coercive.
- for state–local optimality
  - The extended second variation is coercive.
  - The reference trajectory is not self–intersecting.

Consequences ((time, state)–local optimality)

- $f_1(\widehat{x}_1) \neq 0$
- $\exists \alpha \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\Lambda := \left\{ \ell \in T^*\mathbb{R} \colon \ell = \mathrm{d}\alpha(x), \quad x \in \mathcal{O}(\widehat{x}_1) \right\}$$

is a n–dimensional sub–manifold of Σ

$\Longrightarrow$ apply Hamiltonian methods

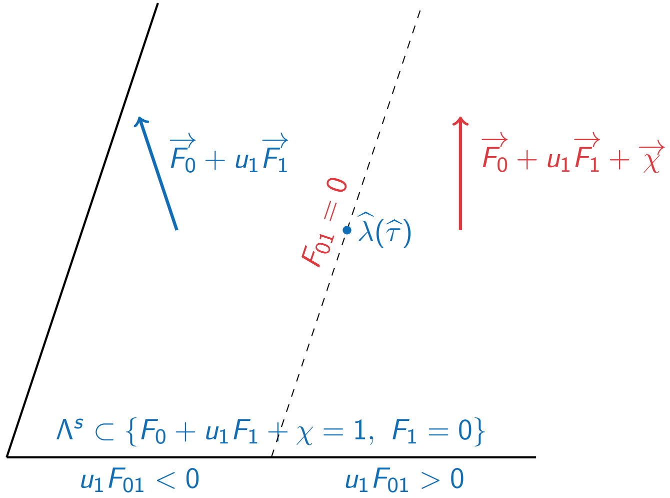
Consequences (state–local optimality)

- f_0 and f_1 are linearly independent at $\widehat{x}_1$
- $\exists \alpha_1 : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Lambda^s := \Big\{ \ell \in T^*\mathbb{R}^n : \ell = \mathrm{d}\alpha_1(x), \quad x \in \mathcal{O}(\widehat{x}_1), \\ \left(F_0 - \frac{F_{001}}{F_{101}} F_1 + \chi \right)(\ell) = 1 \Big\}$$

is a $(n-1)$–dimensional sub–manifold of Σ

Adjusting for exactness: $t < \widehat{\tau}$



Replace part of Λ^s

$f_1 \parallel \pi\Lambda^s$, $f_0(\widehat{x}_1)$ transverse to $\pi\Lambda^s \implies$

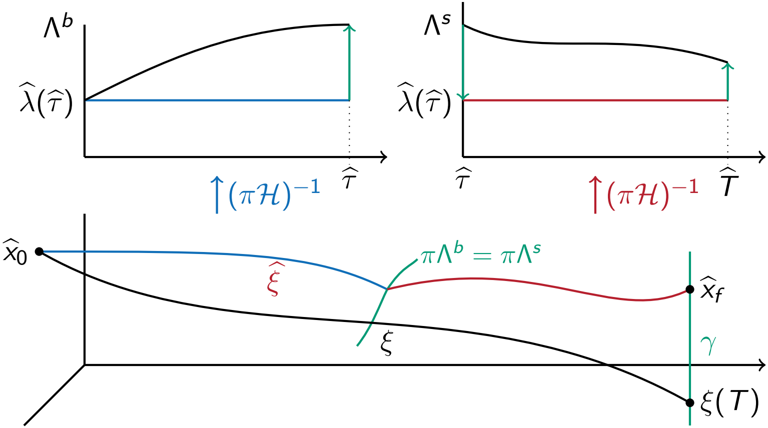
$$\begin{cases} (F_0 + u_1 F_1)(\,\mathrm{d}\gamma(x)) = \langle\, \mathrm{d}\gamma\,,\ f_0 + u_1 f_1 \rangle(x) = 1 \quad x \in \mathbb{R}^n \\ \gamma|_{\pi\Lambda^s} = \alpha_1|_{\pi\Lambda^s} \end{cases}$$

admits one and only one smooth solution

$$\gamma \colon x \in \mathcal{O}(\widehat{x}_1) \mapsto \gamma(x) \in \mathbb{R}$$

$$\Lambda^b = \left\{ \ell \in T^*\mathbb{R}^n \colon u_1 F_{01}(\ell) > 0, \quad \ell \in \Lambda^s \right\} \cup$$
$$\cup \left\{ \ell \in T^*\mathbb{R}^n \colon u_1 F_{01}(\ell) \leq 0, \quad \ell = \mathrm{d}\gamma(x),\ x \in \pi\Lambda^s \right\} \subset \Sigma$$

Lifting trajectories



$\gamma(a) = \exp s(1-a) f_1(\widehat{x}_1),\ a \in [0,1]$ $\qquad$ $\exp s f_1(\widehat{x}_1)$

The result

(time, state)–local optimality

Theorem Assume $(\widehat{T}, \widehat{\xi}, \widehat{u})$ is an admissible triplet for the given minimum time problem satisfying PMP in normal form and that $\widehat{u}$ has a bang–singular structure. If

- the regularity assumptions are satisfied
- the *restricted* extended second variation for the minimum time problem between the extrema of the singular arc is positive definite

then the triplet is a (time, state)–local optimiser.

The result

state–local optimality

Theorem Assume $(\widehat{T}, \widehat{\xi}, \widehat{u})$ is an admissible triplet for the given minimum time problem satisfying PMP in normal form and that $\widehat{u}$ has a bang–singular–bang structure. If

- the regularity assumptions are satisfied
- the extended second variation for the minimum time problem between the extrema of the singular arc is positive definite
- $\widehat{\xi}$ has no self–intersection

then the triplet is a state–local optimiser.

Perturbed problem

minimise T subject to

$$\dot{\xi}(t) = f_0^r(\xi(t)) + u(t) f_1^r(\xi(t)) \qquad t \in [0, T]$$

$$\xi(0) = x_0^r, \quad \xi(T) \in \mathcal{N}_f^r := \exp \mathbb{R} f_1^r(x_f^r), \quad u(t) \in [-1, 1]$$

The geometric picture

Under the smoothness assumption with respect to r and the regularity assumptions, the geometric picture in a neighbourhood of $\widehat{\lambda}$ remains the same:

- the level set $\Sigma_r := \{F_1^r = 0\}$ is an hypersurface in $T^*\mathbb{R}^n$
- any singular extremal of the perturbed system evolves on

$$\mathcal{S}^r := \{F_1^r = F_{01}^r = 0\}$$

- $F_{101}^r > 0$ in a neighbourhood of $\widehat{\lambda}\Big|_{[\widehat{\tau},\widehat{T}]}$, so we may define the Hamiltonian of singular extremals

$$F^{S,r} := \frac{-F_{001}^r}{F_{101}^r}$$

Find the times and the adjoint covector

$$\Phi\colon (r,\omega,\tau,T,s) \in (-R,R)\times(\mathbb{R}^n)^*\times\mathbb{R}^3 \mapsto$$
$$\exp(-sf_1^r)\pi\exp(T-\tau)\overrightarrow{F^{S,r}}\exp\tau(\overrightarrow{F_0^r}+u_1\overrightarrow{F_1^r})(\omega,x_0^r)-x_f^r \in \mathbb{R}^n$$

$$\Psi(r,\omega,\tau,T,s) = \Big(\Phi(r,\omega,\tau,T,s),$$
$$F_1^r\circ\exp\tau(\overrightarrow{F_0^r}+u_1\overrightarrow{F_1^r})(\omega,x_0^r),$$
$$F_{01}^r\circ\exp\tau\left(\overrightarrow{F_0^r}+u_1\overrightarrow{F_1^r}\right)(\omega,x_0^r),$$
$$F_0^r\circ\exp\tau\left(\overrightarrow{F_0^r}+u_1\overrightarrow{F_1^r}\right)(\omega,x_0^r)-1\Big)$$

$\widehat{\omega}_0 := \widehat{\lambda}(0) \Longrightarrow$

$$\Psi(0, \widehat{\omega}_0, \widehat{\tau}, \widehat{T}, 0) = (0,0,0,0)$$

$$\ker \left. \frac{\partial \Psi}{\partial(\omega, \tau, T, s)} \right|_{(0,\widehat{\omega}_0,\widehat{\tau},\widehat{T},0)} = ???$$

Assumption: controllability along $\widehat{\lambda}\big|_{[\widehat{\tau},\widehat{T}]}$

- $\widehat{\lambda}\big|_{[\widehat{\tau},\widehat{T}]}$ is the unique extremal associated to $\widehat{\xi}\big|_{[\widehat{\tau},\widehat{T}]}$
- equivalently:

$$\text{span}\left\{ f_0(\widehat{x}_1), f_1(\widehat{x}_1), \dot{g}_t^1(\widehat{x}_1),\ t \in [\widehat{\tau}, \widehat{T}] \right\} = \mathbb{R}^n$$

Under the regularity assumptions, the coercivity assumption and the controllability assumption

$$\ker \left. \frac{\partial \Psi}{\partial(\omega, \tau, T, s)} \right|_{(0,\widehat{\omega}_0,\widehat{\tau},\widehat{T},0)} = 0$$

$\Longrightarrow$ there exists an extremal $\lambda^r(t) := (\mu^r(t), \xi^r(t))$ such that

- λ^r is bang–singular
- the switching time τ^r and the final time T^r are near the switching time $\widehat{\tau}$ and the final time $\widehat{T}$ of $\widehat{\lambda}$
- the graph of λ^r is *near* the graph of $\widehat{\lambda}$ in $\mathbb{R} \times T^*\mathbb{R}^n$

Theorem

There exist $R > 0$, $\varepsilon > 0$ and a neighbourhood $\mathcal{V}$ of the graph of $\widehat{\lambda}$ in $\mathbb{R} \times T^*\mathbb{R}^n$ such that for any r, $\|r\| < R$, the extremal λ^r defined via the implicit function theorem is the only extremal of the perturbed problem whose graph is in $\mathcal{V}$ and whose final time is in $[\widehat{T} - \varepsilon, \widehat{T} + \varepsilon]$.

Is it a strong local minimiser?

The extended second variation along $\lambda^r|_{[\tau^r, T^r]}$ is coercive

- $\xi^r := \pi\lambda^r$ is a (time, state)–local optimiser
- if $\widehat{\xi}$ is simple, then also $\xi^r := \pi\lambda^r$ is simple and it is a state–local optimiser

The (badly perturbed) Dodgem car problem

$$T \to \min$$

$$\begin{aligned} \dot{x}_1(t) &= \cos(x_3) + ur & x_1(0) &= h & x_1(T) &= 0 \\ \dot{x}_2(t) &= \sin(x_3) & x_2(0) &= 0 & x_2(T) &= 0 \\ \dot{x}_3(t) &= u & x_3(0) &= \frac{\pi}{2} & x_3(T) &\in \mathbb{R} \end{aligned} \qquad u \in [-1, 1].$$

$$f_0^r(x) = f_0(x) \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \end{pmatrix} \qquad f_1^r(x) = \begin{pmatrix} r \\ 0 \\ 1 \end{pmatrix} = f_1(x) + \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\dot{x} = f_0(x) + u f_1(x) \qquad x(0) = \begin{pmatrix} h \\ 0 \\ \frac{\pi}{2} \end{pmatrix} \qquad x(T) \in \exp \mathbb{R} f_1(0)$$

$\Longrightarrow$ the extremal is b-s-b with the length of the second bang interval of order $\sqrt{r}$ (Felgenhauer 2011)